GLOBAL STRONG SOLUTION TO THE DENSITY-DEPENDENT INCOMPRESSIBLE VISCOELASTIC FLUIDS

XIANPENG HU AND DEHUA WANG

Abstract. The existence and uniqueness of the global strong solution with small initial data to the three-dimensional density-dependent incompressible viscoelastic fluids is established. The local existence and uniqueness of the global strong solution with small initial data to the three-dimensional compressible viscoelastic fluids is also obtained. A new method is developed to estimate the solution with weak regularity. Moreover, as a byproduct, we show the global existence and uniqueness of strong solution to the density-dependent incompressible Navier-Stokes equations using a different technique from [8]. All the results apply to the two-dimensional case.

Contents

Date: April 29, 2009.

2000 *Mathematics Subject Classification.* 35A05, 76A10,76D03.

Key words and phrases. Density-dependent incompressible viscoelastic fluids, strong solution, global existence, uniqueness.

1. Introduction

Elastic solids and viscous fluids are two extremes of material behavior. Viscoelastic fluids show intermediate behavior with some remarkable phenomena due to their "elastic" nature. These fluids exhibit a combination of both fluid and solid characteristics, keep memory of their past deformations, and their behaviour is a function of these old deformations. Viscoelastic fluids have a wide range of applications and hence have received a great deal of interest. Examples and applications of viscoelastic fluids include from oil, liquid polymers, mucus, liquid soap, toothpaste, clay, ceramics, gels, some types of suspensions, to bioactive fluids, coatings and drug delivery systems for controlled drug release, scaffolds for tissue engineering, and viscoelastic blood flow flow past valves; see [10, 13, 35] for more applications. For the viscoelastic materials, the competition between the kinetic energy and the internal elastic energy through the special transport properties of their respective internal elastic variables makes the materials more untractable in understanding their behavior, since any distortion of microstructures, patterns or configurations in the dynamical flow will involve the deformation tensor. For classical simple fluids, the internal energy can be determined solely by the determinant of the deformation tensor; however, the internal energy of complex fluids carries all the information of the deformation tensor. The interaction between the microscopic elastic properties and the macroscopic fluid motions leads to the rich and complicated rheological phenomena in viscoelastic fluids, and also causes formidable analytic and numerical challenges in mathematical analysis. The equations of the density-dependent incompressible viscoelastic fluids of Oldroyd type ([27, 28]) in three spatial dimensions take the following form [12, 23, 29]:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1a}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla P(\rho) = \operatorname{div}(\rho\mathtt{F}\mathtt{F}^\top), \tag{1.1b}$$

$$\mathtt{F}_t + \mathbf{u}\cdot\nabla\mathtt{F} = \nabla\mathbf{u}\,\mathtt{F}, \tag{1.1c}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{1.1d}$$

where ρ stands for the density, $\mathbf{u}\in\mathbb{R}^3$ the velocity, and $\mathtt{F}\in M^{3\times3}$ (the set of 3×3 matrices) the deformation gradient. The viscosity coefficient $\mu>0$ is a constant. The increasing convex function $P(\rho)=A\rho^\gamma$ is the pressure, where $\gamma>1$ and $A>0$ are constant. Without loss of generality, we set $A=1$ in this paper. The symbol $\otimes$ denotes the Kronecker tensor product and $\mathtt{F}^\top$ means the transpose matrix of $\mathtt{F}$. As usual we call equation (1.1a) the continuity equation. For system (1.1), the corresponding elastic energy is chosen to be the special form of the Hookean linear elasticity:

$$W(\mathtt{F}) = \frac{1}{2}|\mathtt{F}|^2,$$

which, however, does not reduce the essential difficulties for analysis. The methods and results of this paper can be applied to more general cases.

In this paper, we consider equations (1.1) subject to the initial condition:

$$(\rho,\mathbf{u},E)|_{t=0} = (\rho_0(x),\mathbf{u}_0(x),E_0(x)), \quad x\in\mathbb{R}^3, \tag{1.2}$$

and we are interested in the global existence and uniqueness of strong solution to the initial-value problem (1.1)-(1.2) near its equilibrium state in the three dimensional space $\mathbb{R}^3$. Here the equilibrium state of the system (1.1) is defined as: ρ is a positive constant (for simplicity, $\rho = 1$), $\mathbf{u} = 0$, and $\mathtt{F} = I$ (the identity matrix in $M^{3\times 3}$). We introduce a new unknown variable E by setting

$$\mathtt{F} = I + E.$$

Then, (1.1c) reads

$$E_t + \mathbf{u} \cdot \nabla E = \nabla \mathbf{u} E + \nabla \mathbf{u}. \tag{1.3}$$

By a *strong solution*, we mean a triple $(\rho, \mathbf{u}, \mathtt{F})$ satisfying (1.1) almost everywhere with initial condition (1.2), in particular, $(\rho, \mathbf{u}, \mathtt{F})(\cdot, t) \in W^{2,q}$, $q \in (3, 6]$ for almost all $t > 0$ in this paper.

When density ρ is a constant, system (1.1) governs the homogeneous incompressible viscoelastic fluids, and there exist rich results in the literature for the global existence of classical solutions (namely in H^3 or other functional spaces with much higher regularity); see [5, 6, 15, 16, 18, 19, 20, 23] and the references therein. When density ρ is not a constant, the question related to existence becomes much more complicated and not much has been done. In [17] the authors considered the global existence of classical solutions in H^3 of small perturbation near its equilibrium for the compressible viscoelastic fluids without the pressure term. One of the main difficulties in proving the global existence is the lacking of the dissipative estimate for the deformation gradient. To overcome this difficulty, the authors in [16] introduced an auxiliary function to obtain the dissipative estimate, while the authors in [18] directly deal with the quantities such as $\Delta\mathbf{u} + \mathrm{div}\mathtt{F}$. Those methods can provide them with some good estimates, partly because of their high regularity of $(\mathbf{u}, \mathtt{F})$. However, in this paper, we deal with the strong solution with much less regularity in $W^{2,q}$, $q \in (3, 6]$, hence those methods do not apply. Thus, we need a new method to overcome this obstacle, and we find that a combination between the velocity and the convolution of the divergence of the deformation gradient with the fundamental solution of Laplace operator will develop some good dissipative estimates required for the global existence. The global existence of strong solution to the initial-value problem (1.1)-(1.2) is established based on the local existence and global uniform estimates. The local existence is obtained using a fixed point theorem without incompressible condition (1.1d), that is, the local existence holds for both the incompressible viscoelastic fluids (1.1a)-(1.1d) and the compressible viscoelastic fluids (1.1a)-(1.1c). The global existence and uniqueness of strong solution also holds for the density-dependent incompressible Navier-Stokes equations when the deformation gradient does not appear, which, as a byproduct, gives a similar result to [8] but through a different technique.

The viscoelastic fluid system (1.1) can be regarded as a combination of the inhomogeneous incompressible Navier-Stokes equation with the source term $\mathrm{div}(\rho\mathtt{F}\mathtt{F}^\top)$ and the equation (1.1c). For the global existence of classical solutions with small perturbation near an equilibrium for the compressible Navier-Stokes equations, we refer the reader to [24, 25, 26, 30] and the references cited therein. We remark that, for the nonlinear inviscid elastic systems, the existence of solutions was established by Sideris-Thomases in [33] under the null condition; see also [31] for a related discussion.

The existence of global weak solutions with large initial data of (1.1) is still an outstanding open question. In this direction for the homogeneous incompressible viscoelastic fluids, when the contribution of the strain rate (symmetric part of $\nabla\mathbf{u}$) in the constitutive equation is neglected, Lions-Masmoudi in [22] proved the global existence of weak solutions with large initial data for the Oldroyd model. Also Lin-Liu-Zhang in [19] proved the existence of global weak solutions with large initial data for the incompressible viscoelastic fluids when the velocity satisfies the Lipschitz condition. When dealing with the global existence of weak solutions of the viscoelastic fluid system (1.1) with large data, the rapid oscillation of the density and the non-compatibility between the quadratic form and the weak convergence are two of the major difficulties.

The rest of the paper is organized as follows. In Section 2, we recall briefly the density-dependent incompressible viscoelastic fluids from some basic mechanics and conservation laws. In Section 3, we state our main results, including the local and global existence and uniqueness of the strong solution to the equations of the viscoelastic fluids, as well as to the incompressible Navier-Stokes equations with small data. In Section 4, we prove the local existence via a fixed-point theorem. In Section 5, we prove the uniqueness of the solution obtained in Section 4. In Section 6, we establish some global *a priori* estimates, especially on the dissipation of the deformation gradient and gradient of the density. In Section 7, we first prove some energy estimates uniform in time and some refined estimates on the density and the deformation gradient, and then give the proof of the global existence.

2. Background of Mechanics for Viscoelastic Fluids

To provide a better understanding of system (1.1), we recall briefly some background of viscoelastic fluids from mechanics in this section.

First, we discuss the deformation gradient $\mathtt{F}$. The dynamics of a velocity field $\mathbf{u}(x,t)$ in mechanics can be described by the flow map or particle trajectory $x(t,X)$, which is a time dependent family of orientation preserving diffeomorphisms defined by:

$$\begin{cases} \frac{d}{dt}x(t,X) = \mathbf{u}(t,x(t,X)), \\ x(0,X) = X, \end{cases} \tag{2.1}$$

where the material point X (Lagrangian coordinate) is deformed to the spatial position $x(t,X)$, the reference (Eulerian) coordinate at time t. The deformation gradient $\widetilde{\mathtt{F}}$ is defined as

$$\widetilde{\mathtt{F}}(t,X) = \frac{\partial x}{\partial X}(t,X),$$

which describes the change of configuration, amplification or pattern during the dynamical process, and satisfies the following equation by changing the order of differentiation:

$$\frac{\partial \widetilde{\mathtt{F}}(t,X)}{\partial t} = \frac{\partial \mathbf{u}(t,x(t,X))}{\partial X}. \tag{2.2}$$

In the Eulerian coordinate, the corresponding deformation gradient $\mathtt{F}(t,x)$ is defined as

$$\mathtt{F}(t,x(t,X)) = \widetilde{\mathtt{F}}(t,X).$$

Equation (2.2), combined with the chain rule and (2.1), gives

$$\begin{aligned} \partial_t\mathtt{F}(t,x(t,X)) + \mathbf{u}\cdot\nabla\mathtt{F}(t,x(t,X)) &= \partial_t\mathtt{F}(t,x(t,X)) + \frac{\partial\mathtt{F}(t,x(t,X))}{\partial x}\cdot\frac{\partial x(t,X)}{\partial t} \\ &= \frac{\partial\widetilde{\mathtt{F}}(t,X)}{\partial t} = \frac{\partial\mathbf{u}(t,x(t,X))}{\partial X} = \frac{\partial\mathbf{u}(t,x(t,X))}{\partial x}\frac{\partial x}{\partial X} \\ &= \frac{\partial\mathbf{u}(t,x(t,X))}{\partial x}\widetilde{\mathtt{F}}(t,X) = \nabla\mathbf{u}\cdot\mathtt{F}, \end{aligned}$$

which is exactly equation (1.1c). Here, and in what follows, we use the conventional notations:

$$(\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}, \quad (\nabla \mathbf{u}\,\mathtt{F})_{i,j} = (\nabla \mathbf{u})_{ik}\mathtt{F}_{kj}, \quad (\mathbf{u}\cdot\nabla\mathtt{F})_{ij} = u_k\frac{\partial \mathtt{F}_{ij}}{\partial x_k},$$

and summation over repeated indices will always be well understood. In viscoelastic fluids, (1.1c) can also be interpreted as the consistency of the flow maps generated by the velocity field $\mathbf{u}$ and the deformation gradient $\mathtt{F}$.

The difference between fluids and solids lies in the fact that, in fluids, such as Navier-Stokes equations [26], the internal energy can be determined solely by the determinant part of $\mathtt{F}$ (equivalently the density ρ, and hence, (1.1c) can be disregarded); while in elasticity, the energy depends on all information of $\mathtt{F}$.

In the continuum physics, if we assume the material is homogeneous, the conservation laws of mass and of momentum become [7, 16, 31]

$$\partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0, \tag{2.3}$$

and

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla P(\rho) = \mathrm{div}((\det\mathtt{F})^{-1}S\mathtt{F}^\top), \tag{2.4}$$

with $\mathrm{div}\mathbf{u} = 0$ where

$$\rho\det\mathtt{F} = 1, \tag{2.5}$$

and

$$S_{ij}(\mathtt{F}) = \frac{\partial W}{\partial \mathtt{F}_{ij}}. \tag{2.6}$$

Here S, $\rho S\mathtt{F}^\top$, $W(\mathtt{F})$ denote *Piola-Kirchhoff stress*, *Cauchy stress*, and the elastic energy of the material, respectively. Recall that the condition (2.6) implies that the material is hyperelastic [23]. In the case of Hookean (linear) elasticity [15, 16, 20],

$$W(\mathtt{F}) = \frac{1}{2}|\mathtt{F}|^2 = \frac{1}{2}tr(\mathtt{F}\mathtt{F}^\top), \tag{2.7}$$

where the notation "tr" stands for the trace operator of a matrix, and hence,

$$S(\mathtt{F}) = \mathtt{F}. \tag{2.8}$$

Combining equations (2.1)-(2.8) together, we obtain system (1.1).

If the viscoelastic system (1.1) satisfies

$$\mathrm{div}(\rho_0\mathtt{F}_0^\top) = 0,$$

initially at $t = 0$ with $\mathtt{F}_0 = I + E_0$, it was verified in [18] (see Proposition 3.1) that this condition will insist in time, that is,

$$\mathrm{div}(\rho(t)\mathtt{F}(t)^\top) = 0, \quad \text{for} \quad t \geq 0. \tag{2.9}$$

Another hidden, but important, property of the viscoelastic fluids system (1.1) is concerned with the curl of the deformation gradient (see [15, 16]). Formally, the fact that the Lagrangian derivatives commute and the definition of the deformation gradient imply

$$\partial_{X_k}\widetilde{\mathtt{F}}_{ij} = \frac{\partial^2 x_i}{\partial X_k\partial X_j} = \frac{\partial^2 x_i}{\partial X_j\partial X_k} = \partial_{X_j}\widetilde{\mathtt{F}}_{ik},$$

which is equivalent to, in the Eulerian coordinates,

$$\widetilde{\mathtt{F}}_{lk}\nabla_l\mathtt{F}_{ij}(t, x(t,X)) = \widetilde{\mathtt{F}}_{nj}\nabla_n\mathtt{F}_{ik}(t, x(t,X)),$$

that is,

$$\mathtt{F}_{lk}\nabla_l\mathtt{F}_{ij}(t,x)=\mathtt{F}_{nj}\nabla_n\mathtt{F}_{ik}(t,x),$$

which means that, using $\mathtt{F}=I+E$,

$$\nabla_k E_{ij}+E_{lk}\nabla_l E_{ij}=\nabla_j E_{ik}+E_{nj}\nabla_n E_{ik}. \tag{2.10}$$

According to (2.10), it is natural to assume that the initial condition of E in the viscoelastic fluids system (1.1) should satisfy the compatibility condition

$$\nabla_k E(0)_{ij}+E(0)_{lk}\nabla_l E(0)_{ij}=\nabla_j E(0)_{ik}+E(0)_{nj}\nabla_n E(0)_{ik}. \tag{2.11}$$

Finally, if the density ρ is a constant, (1.1) becomes its corresponding homogeneous (density-independent) incompressible form (see [6, 15, 16, 18, 19, 20] and references therein)

$$\begin{cases} \mathrm{div}\mathbf{u}=0,\\ \partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\mu\Delta\mathbf{u}+\nabla P=\mathrm{div}(\mathtt{F}\mathtt{F}^\top),\\ \partial_t\mathtt{F}+\mathbf{u}\cdot\nabla\mathtt{F}=\nabla\mathbf{u}\,\mathtt{F}. \end{cases} \tag{2.12}$$

For more discussions on viscoelastic fluids and related models, see [4, 5, 7, 11, 12, 14, 19, 22, 23, 29, 34] and the references cited therein.

3. Main Results

In this Section, we state our main results. As usual, the global existence is built on the local existence and global uniform estimates.

In this paper, the standard notations for Sobolev spaces $W^{s,q}$ and Besov spaces B^s_{pq} ([3]) will be used. Throughout this paper, the real interpolation method ([3]) will be adopted and the following interpolation spaces will be needed

$$X_p^{2(1-\frac1p)}=\big(L^q(\mathbb{R}^3),W^{2,q}(\mathbb{R}^3)\big)_{1-\frac1p,p}=B_{qp}^{2(1-\frac1p)},$$

and

$$Y_p^{1-\frac1p}=\big(L^q(\mathbb{R}^3),W^{1,q}(\mathbb{R}^3)\big)_{1-\frac1p,p}=B_{qp}^{1-\frac1p}.$$

Now we introduce the following functional spaces to which the solution and initial conditions of the system (1.1) will belong. Given $1\le p,q\le\infty$ and $T>0$, we set $Q_T=\mathbb{R}^3\times(0,T)$, and

$$\mathcal{W}^{p,q}(0,T):=\big\{\mathbf{u}\in W^{1,p}(0,T;(L^q(\mathbb{R}^3))^3)\cap L^p(0,T;(W^{2,q}(\mathbb{R}^3))^3):\mathrm{div}\mathbf{u}=0\big\}$$

with the norm

$$\|\mathbf{u}\|_{\mathcal{W}^{p,q}(0,T)}:=\|\mathbf{u}\|_{W^{1,p}(0,T;L^q(\mathbb{R}^3))}+\|\mathbf{u}\|_{L^p(0,T;W^{2,q}(\mathbb{R}^3))},$$

as well as

$$V_0^{p,q}:=\left(X_p^{2(1-\frac1p)}\cap Y_p^{1-\frac1p}\right)^3\times\big(W^{1,q}(\mathbb{R}^3)\big)^{10}$$

with the norm

$$\|(f,g)\|_{V_0^{p,q}}:=\|f\|_{X_p^{2(1-\frac1p)}}+\|f\|_{Y_p^{1-\frac1p}}+\|g\|_{W^{1,q}(\mathbb{R}^3)}.$$

We denote

$$\mathcal{W}(0,T)=\mathcal{W}^{p,q}(0,T)\cap\mathcal{W}^{2,2}(0,T),$$

and

$$V_0 = V_0^{p,q} \cap V_0^{2,2}.$$

Our first result is the following local existence:

Theorem 3.1 (Local existence for viscoelastic fluids)**.** *Let $T_0 > 0$ be given and $(\mathbf{u}_0, \rho_0, E_0) \in V_0$ with $p \in [2, \infty), q \in (3, \infty)$. There exists a positive constant $\delta_0 < 1$, depending on T_0, such that if*

$$\|(\mathbf{u}_0, \rho_0 - 1, E_0)\|_{V_0} \le \delta_0,$$

then the initial-value problem (1.1a)-(1.1c) *with* (1.2) *as well as the initial-value problem* (1.1)-(1.2) *have a unique strong solution on $\mathbb{R}^3 \times (0, T_0)$, satisfying*

$$(\mathbf{u}, \rho, \mathtt{F}) \in \mathcal{W}(0, T_0) \times \left(W^{1,p}(0, T_0; L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)) \cap L^p(0, T_0; W^{1,q}(\mathbb{R}^3) \cap W^{1,2}(\mathbb{R}^3))\right)^{10}.$$

Remark 3.1. The solution constructed in Section 4 later does not require the incompressible condition (1.1d), thus we have the local existence for both the compressible and incompressible cases. The solutions in Theorem 3.1 is local in time since $\delta_0 = \delta_0(T_0)$ implies that T_0 is finite for a given $\delta_0 \ll 1$.

Remark 3.2. An interesting case is the case $q \le p$. Indeed, by the real interpolation method and Theorem 6.4.4 in [3], we have

$$W^{2(1-\frac{1}{p}),q} \subset B_{qp}^{2(1-\frac{1}{p})} = X_p^{2(1-\frac{1}{p})},$$

and

$$W^{1-\frac{1}{p},q} \subset B_{qp}^{1-\frac{1}{p}} = Z_p^{1-\frac{1}{p}}.$$

Then, if we replace the functional space $V_0^{p,q}$ in Theorem 3.1 by

$$\mathcal{V}_0^{p,q} := \left((W^{2(1-\frac{1}{p}),q}(\mathbb{R}^3))^3 \cap (W^{1-\frac{1}{p},q}(\mathbb{R}^3))^3\right) \times (W^{1,q}(\mathbb{R}^3))^{10},$$

Theorem 3.1 is still valid.

The above local existence, with the aid of global estimates and the suitable choice of the smallness of the initial data, will result in the following global existence:

Theorem 3.2 (Global existence for viscoelastic fluids)**.** *Assume that*

- *$p = 2$ and $q \in (3, 6]$;*
- *There exists a $\delta_0 > 0$, such that, for any δ with $0 < \delta \le \delta_0 \ll 1$, the initial data satisfies*
$$\|(\mathbf{u}_0, \rho_0 - 1, E_0)\|_{V_0} \le \delta^2, \tag{3.1}$$
with compatibility condition (2.11)*, $\mathrm{div}\mathbf{u}_0 = 0$, and*
$$\mathrm{div}(\rho_0 \mathtt{F}_0^\top) = 0; \tag{3.2}$$
- *In addition, the initial data satisfies*
$$\int_{\mathbb{R}^3} \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{1}{2}\rho_0|E_0|^2 + \frac{1}{\gamma - 1}(\rho_0^\gamma - \gamma\rho_0 + \gamma - 1)\right) dx \le \delta^4, \tag{3.3}$$
and
$$\|\nabla \mathbf{u}_0\|_{L^2} + \|\mathbf{u}_0 \cdot \nabla \mathbf{u}_0\|_{L^2} + \|\Delta \mathbf{u}_0\|_{L^2} + \|\nabla \rho_0\|_{L^2} + \|\nabla E_0\|_{L^2 \cap L^q} \le \delta^4. \tag{3.4}$$

Then, there exists a $\mu_0 > 0$ depending only on q (and determined by (6.18)), such that if $0 < \mu \le \mu_0$, the initial-value problem (1.1)-(1.2) *has a unique strong solution defined on $\mathbb{R}^3 \times (0, \infty)$ with*

$$(\mathbf{u}, \rho, \mathtt{F}) \in \mathcal{W}(0,T) \times \left(W^{1,p}(0,T; L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)) \cap L^p(0,T; W^{1,q}(\mathbb{R}^3) \cap W^{1,2}(\mathbb{R}^3))\right)^{10},$$

for each $T > 0$. Furthermore, the solution satisfies

$$\sup_{t\in[0,\infty)} \|(\mathbf{u}(t), \rho(t) - 1, E(t))\|_{V_0^{p,q}} < \sqrt{\delta_0}. \tag{3.5}$$

Remark 3.3. Notice that if $q > 3$, then by Theorem 5.15 in [1], the imbedding $W^{1,q}(\mathbb{R}^3) \hookrightarrow C_B^0(\mathbb{R}^3)$ is continuous. Here, the notation $C_B^0(\mathbb{R}^3)$ means the spaces of bounded, continuous functions in $\mathbb{R}^3$. Hence the condition (3.1) implies that, if we choose δ sufficiently small, by Sobolev's imbedding theorem, there exists a positive constant C_0 such that

$$\rho_0 \ge C_0 > 0, \quad \text{for a.e.} \quad x \in \mathbb{R}^3. \tag{3.6}$$

Remark 3.4. Under assumption (3.2), the authors in [17, 18] showed that the property will insist in time, that is, for all $t \ge 0$,

$$\operatorname{div}(\rho \mathtt{F}^\top) = 0.$$

Remark 3.5. If the density ρ is a constant, for simplicity, $\rho = 1$, Theorems 3.1 and 3.2 become the analogous results of the homogeneous incompressible viscoelastic fluids (2.12). In other words, following our argument in this paper, we can recover the global existence of strong solutions, or even classical solutions, of the homogeneous incompressible viscoelastic fluids near its equilibrium.

An important consequence of Theorem 3.2 is the case as $E = 0$ and disregarding the equation (1.1c) . In this case, one has the global existence of density-dependent incompressible Navier-Stokes equations, since the term on the right-hand side of (1.1b) can be incorporated into the pressure. We state the result without proof as follows.

Corollary 3.1 (Global existence for Navier-Stokes equations)**.** *Assume that*

- *$p = 2$ and $q \in (3, 6]$;*
- *There exists a $\delta_0 > 0$, such that, for any δ with $0 < \delta \le \delta_0 \ll 1$, the initial data satisfies*
$$\|(\mathbf{u}_0, \rho_0 - 1)\|_{V_0} \le \delta^2,$$
with $\operatorname{div}\mathbf{u}_0 = 0$*;*
- *In addition, the initial data satisfies*
$$\int_{\mathbb{R}^3} \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{1}{\gamma - 1}(\rho_0^\gamma - \gamma\rho_0 + \gamma - 1)\right) dx \le \delta^4.$$

Then, the initial-value problem for the density-dependent incompressible Navier-Stokes equations has a unique strong solution defined on $\mathbb{R}^3 \times (0, \infty)$ such that

$$(\mathbf{u}, \rho) \in \mathcal{W}(0,T) \times \left(W^{1,p}(0,T; L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)) \cap L^p(0,T; W^{1,q}(\mathbb{R}^3) \cap W^{1,2}(\mathbb{R}^3))\right),$$

for each $T > 0$. Furthermore, the solution satisfies

$$\sup_{t\in[0,\infty)} \|(\mathbf{u}(t), \rho(t) - 1)\|_{V_0^{p,q}} < \sqrt{\delta_0}.$$

Remark 3.6. The similar result to Corollary 3.1 was shown in [8]. It is worthy of noticing that there is no assumption on the amplitude of the viscosity μ or condition (3.4) because those two conditions are only useful when dealing with the dissipation of the transformation gradient $\mathtt{F}$. Actually, as seen from the argument later in this paper, Corollary 3.1 still holds if $p \geq 2$ and $q > 3$.

4. Local Existence

In this section, we prove the local existence of strong solution in Theorem 3.1. To this end, we introduce the following new variables by scaling

$$s := \nu^2 t, \quad y := \nu x, \quad v(y,s) := \frac{1}{\nu}\mathbf{u}(x,t), \quad r(y,s) := \rho(x,t), \quad G(y,s) := E(x,t),$$

where $\nu > 0$ will be determined later. Then, system (1.1), with (1.1c) replaced by (1.3), becomes

$$r_t + \mathrm{div}(rv) = 0, \tag{4.1a}$$

$$(rv)_t + \mathrm{div}\,(rv \otimes v) - \mu\Delta v + \nu^{-2}\nabla P = \nu^{-2}\mathrm{div}\left(r(I+G)(I+G)^\top\right), \tag{4.1b}$$

$$G_t + v \cdot \nabla G = \nabla v G + \nabla v, \tag{4.1c}$$

$$\mathrm{div} v = 0. \tag{4.1d}$$

From (2.10), one has

$$\nabla_k G_{ij} + G_{lk}\nabla_l G_{ij} = \nabla_j G_{ik} + G_{nj}\nabla_n G_{ik}. \tag{4.2}$$

Thus, if we denote by G_i the i-th row of the matrix G (or the i-th component of the vector G), then (4.2) becomes

$$\mathrm{curl}\ G_i = G_{nj}\nabla_n G_{ik} - G_{lk}\nabla_l G_{ij}. \tag{4.3}$$

The proof of local existence of strong solution with small initial data will be carried out through three steps by using a fixed point theorem. Instead of working on (1.1) directly, we will work on (4.1). We note that (4.1) is just a scaling version of (1.1). It can be seen from the argument below that we only need to verify the local existence in $\mathcal{W}^{p,q}(0,T)$, $0 < T \leq T_0$, while initial data belongs to $V_0^{p,q}$.

4.1. Solvability of the density with a fixed velocity.

Let $A_j(x,t)$, $j = 1, ..., n$, be symmetric $m \times m$ matrices in $\mathbb{R}^n \times (0,T)$, $f(x,t)$ and $\upsilon_0(x)$ be m-dimensional vector functions defined in $\mathbb{R}^n \times (0,T)$ and $\mathbb{R}^n$, respectively.

For the following initial-value problem:

$$\begin{cases} \partial_t \upsilon + \displaystyle\sum_{i=1}^n A_j(x,t)\partial_j \upsilon + B(x,t)\upsilon = f(x,t), \\ \upsilon(x,0) = \upsilon_0(x), \end{cases} \tag{4.4}$$

we have

Lemma 4.1. *Assume that*

$$A_j \in \left[C(0,T;H^s(\mathbb{R}^n)) \cap C^1(0,T;H^{s-1}(\mathbb{R}^n))\right]^{m\times m},\ j=1,...,n,$$

$$B \in C((0,T),H^{s-1}(\mathbb{R}^n))^{m\times m}, \quad f \in C((0,T),H^s(\mathbb{R}^n))^m, \quad v_0 \in H^s(\mathbb{R}^n)^m,$$

with $s > \frac{n}{2}+1$ *is an integer. Then there exists a unique solution to* (4.4)*, i.e, a function*

$$v \in \left[C([0,T),H^s(\mathbb{R}^n)) \cap C^1((0,T),H^{s-1}(\mathbb{R}^n))\right]^m$$

satisfying (4.4) *pointwise.*

Proof. This lemma is a direct consequence of Theorem 2.16 in [26] with $A_0(x,t) = I$. □

To solve the density with respect to the fixed velocity, we have

Lemma 4.2. *Under the same conditions as Theorem 3.1, there is a unique strictly positive function*

$$r := \mathcal{S}(v) \in W^{1,p}(0,T;L^q(\mathbb{R}^3)) \cap L^\infty(0,T;W^{1,q}(\mathbb{R}^3))$$

which satisfies the continuity equation (4.1a) *and* $r-1 \in L^\infty(0,T;L^q(\mathbb{R}^3))$*. Moreover, the density satisfies*

$$\|\nabla r\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} \le C(T,\|v\|_{\mathcal{W}(0,T)})\left(\|\nabla r_0\|_{L^q(\mathbb{R}^3)}+1\right), \tag{4.5}$$

and the norm $\|\mathcal{S}(v)-1\|_{W^{1,q}(\mathbb{R}^3)}(t)$ *is a continuous function in time.*

Here, and in what follows, C stands for a generic positive constant, and in some case, we will specify its dependence on parameters by the notation $C(\cdot)$.

Proof. For the proof of the first part of this lemma, we refer the reader to Theorem 9.3 in [26], or the first part of the proof for Lemma 4.3 below. The positivity of density follows directly from the observations: by writing (4.1a) along characteristics $\frac{d}{dt}X(t) = v$,

$$\frac{d}{dt}r(t,X(t)) = -r(t,X(t))\mathrm{div}v(t,X(t)), \quad X(0)=x,$$

and with the help of Gronwall's inequality,

$$\begin{aligned}(\inf_x \rho_0)\exp\left(-\int_0^t \|\mathrm{div}v(t)\|_{L^\infty(\mathbb{R}^3)}dx\right) &\le r(t,x)\\ &\le (\sup_x \rho_0)\exp\left(\int_0^t \|\mathrm{div}v(t)\|_{L^\infty(\mathbb{R}^3)}dx\right).\end{aligned}$$

Now, we can assume that the continuity equation holds pointwise in the following form:

$$\partial_t r + r\mathrm{div}v + v\cdot\nabla r = 0.$$

Taking the gradient in both sides of the above identity, multiplying by $|\nabla r|^{q-2}\nabla r$ and then integrating over $\mathbb{R}^3$, we get, by Young's inequality

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\nabla r\|^q_{L^q(\mathbb{R}^3)} &\le \int_{\mathbb{R}^3}|\nabla r|^q|\mathrm{div}v|dx + \int_{\mathbb{R}^3} r|\nabla r|^{q-1}|\nabla\mathrm{div}v|dx\\ &\quad + \int_{\mathbb{R}^3}|\nabla v||\nabla r|^q dx - \frac{1}{q}\int_{\mathbb{R}^3} v\nabla|\nabla r|^q dx\\ &\le \|\nabla r\|^q_{L^q(\mathbb{R}^3)}\left(\|\nabla v\|_{L^\infty(\mathbb{R}^3)} + \|r\|_{L^\infty(\mathbb{R}^3)}\|\nabla\mathrm{div}v\|_{L^q(\mathbb{R}^3)}\right)\\ &\quad + \frac{1}{q}\int_{\mathbb{R}^3}\mathrm{div}v|\nabla r|^q dx + \|r\|_{L^\infty(\mathbb{R}^3)}\|\nabla\mathrm{div}v\|_{L^q(\mathbb{R}^3)}\\ &\le C\|\nabla r\|^q_{L^q(\mathbb{R}^3)}\|v\|_{W^{2,q}(\mathbb{R}^3)} + \|r\|_{L^\infty(\mathbb{R}^3)}\|\nabla\mathrm{div}v\|_{L^q(\mathbb{R}^3)},\end{aligned} \tag{4.6}$$

since $q > 3$. Then (4.5) follows from Gronwall's inequality.

Finally, noting from (4.6) and (4.5) that $\frac{d}{dt}\|\nabla r\|^q_{L^q(\mathbb{R}^3)} \in L^1(0,T)$, and hence

$$\frac{d}{dt}\|\nabla(r-1)\|^q_{L^q(\mathbb{R}^3)} \in L^1(0,T),$$

which together with (4.5) implies that $\|\nabla(r-1)\|^q_{L^q(\mathbb{R}^3)}(t)$ is continuous in time, and hence, $\|\nabla(r-1)\|_{L^q(\mathbb{R}^3)}(t)$ is continuous in time. Similarly, from the continuity equation, we know that

$$\partial_t(r-1) = -\mathrm{div}((r-1)v) - \mathrm{div}v \in L^p(0,T;L^q(\mathbb{R}^3)),$$

which, together with the fact $r-1 \in L^\infty(0,T;L^q(\mathbb{R}^3))$, yields $r-1 \in C([0,T];L^q(\mathbb{R}^3))$. Hence, the quantity $\|r-1\|_{W^{1,q}(\mathbb{R}^3)}(t)$ is continuous in time. The proof of Lemma 4.2 is complete. □

4.2. **Solvability of the deformation gradient with a fixed velocity.** Due to the hyperbolic structure of (4.1c), we can apply Lemma 4.1 again to solve the deformation gradient G in terms of the given velocity. For this purpose, we have

Lemma 4.3. *Under the same conditions as Theorem 3.1, there is a unique function*

$$G := \mathcal{T}(v) \in W^{1,p}(0,T;L^q(\mathbb{R}^3)) \cap L^\infty(0,T;W^{1,q}(\mathbb{R}^3))$$

which satisfies the equation (4.1c). *Moreover, the deformation gradient satisfies*

$$\|\nabla G\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} \le C(T,\|v\|_{\mathcal{W}(0,T)})\left(\|\nabla G(0)\|_{L^q(\mathbb{R}^3)} + 1\right), \tag{4.7}$$

and, the norm $\|G\|_{W^{1,q}(\mathbb{R}^3)}(t)$ is a continuous function in time.

Proof. First, we assume that $v \in C^1(0,T;C_0^\infty(\mathbb{R}^3))$, $G_0 \in C_0^\infty(\mathbb{R}^3)$. Then, we can rewrite (4.1c) in the component form as

$$\partial_t G_j + v\cdot\nabla G_j = \nabla v\, G_j + \nabla v_j, \quad \text{for all} \quad 1 \le j \le 3.$$

Applying Lemma 4.1 successively with $A_k(x,t) = v_k(x,t)I$ for all $1 \le k \le 3$, $B(x,t) = \nabla v$, and $f(x,t) = \nabla v_j$, we get a solution

$$G \in \cap_{l=1}^\infty\left\{C^1(0,T,H^{l-1}(\mathbb{R}^3)) \cap (0,T;H^l(\mathbb{R}^3))\right\},$$

which implies, by the Sobolev imbedding theorem,

$$G \in \cap_{k=1}^\infty C^1(0,T;C^k(\mathbb{R}^3)) = C^1(0,T;C^\infty(\mathbb{R}^3)). \tag{4.8}$$

Next, for $v \in \mathcal{W}^{p,q}(0,T)$, there are two sequences:

$$v_n \in C^1(0,T;C_0^\infty(\mathbb{R}^3)), \quad G_0^n \in C_0^\infty(\mathbb{R}^3),$$

such that

$$v_n \to v \text{ in } \mathcal{W}(0,T), \quad G_0^n \to G_0 \text{ in } W^{1,q}(\mathbb{R}^3),$$

thus $v_n \to v$ in $C(B(0,a)\times(0,T))$ for all $a>0$ where $B(0,a)$ denotes the ball with radius a and centered at the origin. According to the previous result (4.8), there are a sequence of functions $\{G_n\}_{n=1}^\infty \subset C^1(0,T;C^\infty(\mathbb{R}^3))$ satisfying

$$\partial_t G_n + v_n\cdot\nabla G_n = \nabla v_n\, G_n + \nabla v_n, \tag{4.9}$$

with $G_n(0)=G_0^n$. Multiplying (4.9) by $|G_n|^{q-2}G_n$, and integrating over $\mathbb{R}^3$, using integration by parts and Young's inequality, we obtain,

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^3}|G_n|^q dx\\
&= -\frac{1}{q}\int_{\mathbb{R}^3} v_n\cdot\nabla|G_n|^q dx + \int_{\mathbb{R}^3}\nabla v_n|G_n|^{q-2}G_n^2 dx + \int_{\mathbb{R}^3}\nabla v_n|G_n|^{q-2}G_n dx\\
&\le \frac{1+q}{q}\|G_n\|_{L^q}^q(\|\nabla v_n\|_{L^\infty}+\|\nabla v_n\|_{L^q}) + \|\nabla v_n\|_{L^q}.
\end{aligned}$$

From Gronwall's inequality, one obtains,

$$\begin{aligned}
&\int_{\mathbb{R}^3}|G_n|^q dx\\
&\le \left(\int_{\mathbb{R}^3}|G_n(0)|^q dx + q\int_0^t \|\nabla v_n\|_{L^q}\exp\left(-\int_0^t (q+1)(\|\nabla v_n\|_{L^\infty}+\|\nabla v_n\|_{L^q})d\tau\right)ds\right)\\
&\qquad\times\exp\left(\int_0^t (q+1)(\|\nabla v_n\|_{L^\infty}+\|\nabla v_n\|_{L^q})ds\right)\\
&\le \left(\int_{\mathbb{R}^3}|G_n(0)|^q dx + q\int_0^t \|\nabla v_n\|_{L^q} ds\right)\exp\left(\int_0^t (q+1)(\|\nabla v_n\|_{L^\infty}+\|\nabla v_n\|_{L^q})ds\right).
\end{aligned}$$

Thus,

$$\|G_n\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} \le C(T,\|v\|_{L^p(0,T;W^{2,q}(\mathbb{R}^3))})\left(\|G(0)\|_{L^q(\mathbb{R}^3)}+1\right) < \infty. \tag{4.10}$$

Hence, up to a subsequence, we can assume that the sequence $\{v_n\}$ was chosen so that

$$G_n \to G \quad \text{weak-* in} \quad L^\infty(0,T;L^q(\mathbb{R}^3)).$$

Taking the gradient in both sides of (4.9), multiplying by $|\nabla G_n|^{q-2}\nabla G_n$ and then integrating over $\mathbb{R}^3$, we get, with the help of Hölder's inequality and Young's inequality,

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\|\nabla G_n\|^q_{L^q(\mathbb{R}^3)} &\le \int_{\mathbb{R}^3}|\nabla G_n|^q|\nabla v_n|dx + \int_{\mathbb{R}^3}|G_n||\nabla G_n|^{q-1}|\nabla\nabla v_n|dx\\
&\quad + \int_{\mathbb{R}^3}|\nabla v_n||\nabla G_n|^q dx - \frac{1}{q}\int_{\mathbb{R}^3} v_n\nabla|\nabla G_n|^q dx\\
&\quad + \int_{\mathbb{R}^3}|\nabla\nabla v_n||\nabla G_n|^{q-1}dx\\
&\le \int_{\mathbb{R}^3}|\nabla G_n|^q|\nabla v_n|dx + \int_{\mathbb{R}^3}|G_n||\nabla G_n|^{q-1}|\nabla\nabla v_n|dx\\
&\quad + \int_{\mathbb{R}^3}|\nabla v_n||\nabla G_n|^q dx + \frac{1}{q}\int_{\mathbb{R}^3}|\nabla v_n||\nabla G_n|^q dx\\
&\quad + \int_{\mathbb{R}^3}|\nabla\nabla v_n||\nabla G_n|^{q-1}dx\\
&\le C\|\nabla G_n\|^q_{L^q(\mathbb{R}^3)}\|v_n\|_{W^{2,q}(\mathbb{R}^3)} + (\|G_n\|^q_{L^\infty(\mathbb{R}^3)}+1)\|v_n\|_{W^{2,q}(\mathbb{R}^3)}\\
&\le C\|\nabla G_n\|^q_{L^q(\mathbb{R}^3)}\|v_n\|_{W^{2,q}(\mathbb{R}^3)} + (\|G_n\|^q_{L^q(\mathbb{R}^3)}+1)\|v_n\|_{W^{2,q}(\mathbb{R}^3)},
\end{aligned} \tag{4.11}$$

since $q > 3$. Using Gronwall's inequality and (4.10), we conclude from (4.11) that

$$\|\nabla G_n\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} \le C(T, \|v_n\|_{\mathcal{W}(0,T)})\left(\|\nabla G_n(0)\|_{L^q(\mathbb{R}^3)}+1\right),$$

and hence,

$$\begin{aligned}
\|\nabla G\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} &\le \liminf_{n\to\infty}\|\nabla G_n\|_{L^\infty(0,T;L^q(\mathbb{R}^3))}\\
&\le C(T, \|v\|_{\mathcal{W}(0,T)})\left(\|\nabla G(0)\|_{L^q(\mathbb{R}^3)}+1\right).
\end{aligned} \tag{4.12}$$

Therefore,

$$\|G_n\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3))} \le C(T, \|v\|_{L^p(0,T;W^{2,q}(\mathbb{R}^3))}, \|G(0)\|_{W^{1,q}(\mathbb{R}^3)}) < \infty.$$

Furthermore, since $q > 3$, we deduce $G \in L^\infty(Q_T)$ and

$$\|G\|_{L^\infty(Q_T)} \le C(T, \|v\|_{L^p(0,T;W^{2,q}(\mathbb{R}^3))}, \|G(0)\|_{W^{1,q}(\mathbb{R}^3)}) < \infty.$$

Passing to the limit as $n \to \infty$ in (4.9), we show that (4.1c) holds at least in the sense of distributions. Therefore,

$$\partial_t G \in L^p(0,T;L^q(\mathbb{R}^3)),$$

then $G \in W^{1,p}(0,T;L^q(\mathbb{R}^3))$, and hence $G \in C([0,T];L^q(\mathbb{R}^3))$.

Finally, to show that the quantity $\|G\|_{W^{1,q}(\mathbb{R}^3)}(t)$ is continuous in time, it suffices to show that $\|\nabla G\|_{L^q(\mathbb{R}^3)}$ is continuous in time. Indeed, from (4.11), we know that

$$\frac{d}{dt}\|\nabla G\|^q_{L^q(\mathbb{R}^3)}(t) \in L^p(0,T),$$

which, with (4.12), implies that $\|\nabla G\|_{L^q(\mathbb{R}^3)} \in C([0,T])$. The proof of Lemma 4.3 is complete. □

4.3. **Local existence via the fixed-point theorem.** In order to solve locally system (4.1), we need to use the following fixed point theorem (cf. 1.4.11.6 in [26]):

Theorem 4.1 (Tikhonov Theorem)**.** *Let M be a nonempty bounded closed convex subset of a separable reflexive Banach space X and let $F : M \mapsto M$ be a weakly continuous mapping (i.e. if $x_n \in M, x_n \to x$ weakly in X, then $F(x_n) \to F(x)$ weakly in X as well). Then F has at least one fixed point in M.*

Now, let us consider the following operator

$$L\omega := \frac{d\omega}{dt} - \mu\Delta\omega, \quad \omega \in \mathcal{W}^{p,q}(0,T).$$

One has the following theorem by the maximal regularity of parabolic equations; see Theorem 9.2 in [26], or equivalently Theorem 4.10.7 and Remark 4.10.9 in [2] (page 188).

Theorem 4.2. *Given $1 < p < \infty$, $\omega_0 \in V_0^{p,q}$ and $f \in L^p(0,T;L^q(\mathbb{R}^3)^3)$, the Cauchy problem*

$$L\omega = f, \quad t \in (0,T); \quad \omega(0) = \omega_0,$$

has a unique solution $\omega := L^{-1}(\omega_0, f) \in \mathcal{W}^{p,q}(0,T)$, and

$$\|\omega\|_{\mathcal{W}^{p,q}(0,T)} \le C\left(\|f\|_{L^p(0,T;L^q(\mathbb{R}^3))} + \|\omega_0\|_{V_0^{p,q}}\right),$$

where C is independent of ω_0, f and T. Moreover, there exists a positive constant c_0 independent of f and T such that

$$\|\omega\|_{\mathcal{W}^{p,q}(0,T)} \ge c_0 \sup_{t\in(0,T)} \|\omega(t)\|_{V_0^{p,q}}.$$

Notice that Theorem 4.2 implies that the operator L is invertible. Thus we define the operator $\mathcal{H}(v) : \mathcal{W}^{p,q}(0,T) \mapsto \mathcal{W}^{p,q}(0,T)$ by

$$\begin{aligned}\mathcal{H}(v) := L^{-1}\Big(v_0,\ &\partial_t((1-\mathcal{S}(v))v) - \mathrm{div}(\mathcal{S}(v)v\otimes v) + \nu^{-2}\nabla(P(1) - P(\mathcal{S}(v)))\\ &+ \nu^{-2}\mathrm{div}(\mathcal{S}(v)(I+\mathcal{T}(v))(I+\mathcal{T}(v))^\top)\Big).\end{aligned} \tag{4.13}$$

Then, solving system (4.1) is equivalent to solving

$$v = \mathcal{H}(v). \tag{4.14}$$

To solve (4.14), we define

$$B_R(0) := \{v \in \mathcal{W}^{p,q}(0,T) :\ \|v\|_{\mathcal{W}^{p,q}(0,T)} \le R\}.$$

Then, we prove first the following claim:

Lemma 4.4. *There are $\nu, T > 0$, and $0 < R < 1$ such that*

$$\mathcal{H}(B_R(0)) \subset B_R(0).$$

Proof. Let $T > 0$, $0 < R < 1$ and $v \in B_R(0)$. Since $\mathcal{S}(v)$ solves (4.1a), we can rewrite operator $\mathcal{H}$ as

$$\begin{aligned}\mathcal{H}(v) = L^{-1}\Big(v_0,\ &(1-\mathcal{S}(v))\partial_t v - \mathcal{S}(v)v\cdot\nabla v + \nu^{-2}\nabla(P(1) - P(\mathcal{S}(v)))\\ &+ \nu^{-2}\mathrm{div}(\mathcal{S}(v)(I+\mathcal{T}(v))(I+\mathcal{T}(v))^\top)\Big).\end{aligned} \tag{4.15}$$

Thus, it suffices to prove that the terms in the above expression are small in the norm of $L^p(0,T;(L^q(\mathbb{R}^3))^3)$.

First of all, we begin to deal with the first term by letting $\bar{r} := \mathcal{S}(v) - 1$. Thus, $\bar{r}$ satisfies the equations

$$\begin{cases} \partial_t \bar{r} + \operatorname{div}(\bar{r}v) = 0, \\ \bar{r}(x,0) = r_0 - 1. \end{cases}$$

Repeating the argument in Section 4.2 again, we obtain

$$\begin{aligned} \|\bar{r}\|_{L^\infty(Q_T)} &\le C\|\bar{r}\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3))} \\ &\le C\|r_0 - 1\|_{W^{1,q}(\mathbb{R}^3)} C(T, \|v\|_{\mathcal{W}(0,T)}) \\ &\le \|r_0 - 1\|_{W^{1,q}(\mathbb{R}^3)} C(T,R) \le C(T)R, \end{aligned}$$

where, by the formula of change of variables, we deduce that

$$\|r_0 - 1\|_{L^q(\mathbb{R}^3)} \le \nu^{\frac{3}{q}} \|\rho_0 - 1\|_{L^q(\mathbb{R}^3)} \le R,$$

and

$$\|\nabla r_0\|_{L^q(\mathbb{R}^3)} \le \nu^{\frac{3-q}{q}} \|\nabla \rho_0\|_{L^q(\mathbb{R}^3)} \le R,$$

if $\|\rho_0 - 1\|_{L^q(\mathbb{R}^3)}$ is small enough and $\nu > 1$ is large enough. Hence, due to the assumption $v \in B_R(0)$, we obtain

$$\|(1 - \mathcal{S}(v))\partial_t v\|_{L^p(0,T;L^q(\mathbb{R}^3))} \le C(T)R^2. \tag{4.16}$$

Secondly, by the Sobolev imbedding,

$$\int_{\mathbb{R}^3} |v\nabla v|^q dx \le \|v\|^q_{L^q(\mathbb{R}^3)} \|\nabla v\|^q_{L^\infty(\mathbb{R}^3)} \le C\|v\|^q_{L^q(\mathbb{R}^3)} \|v\|^q_{W^{2,q}(\mathbb{R}^3)},$$

and thus, since $W^{1,p}(0,T;L^q(\mathbb{R}^3)) \hookrightarrow C([0,T];L^q(\mathbb{R}^3))$, we deduce

$$\begin{aligned} \int_0^T \left(\int_{\mathbb{R}^3} |v\nabla v|^q dx \right)^{\frac{p}{q}} ds &\le C \int_0^T \|v\|^p_{L^q(\mathbb{R}^3)} \|v\|^p_{W^{2,q}(\mathbb{R}^3)} ds \\ &\le C\|v\|^p_{L^\infty(0,T;L^q(\mathbb{R}^3))} \|v\|^p_{L^p(0,T;W^{2,q}(\mathbb{R}^3))} \\ &\le \|v\|^{2p}_{\mathcal{W}(0,T)} \le CR^{2p}. \end{aligned}$$

Therefore, we get

$$\|\mathcal{S}(v)(v\cdot\nabla)v\|_{L^p(0,T;L^q(\mathbb{R}^3))} \le CR^2. \tag{4.17}$$

Thirdly, for the term $\nabla P(\mathcal{S}(v))$, we can estimate it as follows

$$\begin{aligned} &\|\nabla P(\mathcal{S}(v))\|_{L^p(0,T;L^q(\mathbb{R}^3))} \\ &\le C(T) \sup\left\{P'(\eta) : C(T)^{-1} \le \eta \le C(T)\right\} (\|\nabla r_0\|_{L^q(\mathbb{R}^3)} + 1). \end{aligned} \tag{4.18}$$

Fourthly, for the term $\operatorname{div}(\mathcal{S}(v)(I + \mathcal{T}(v))(I + \mathcal{T}(v))^\top$, we have

$$|\operatorname{div}(\mathcal{S}(v)(I + \mathcal{T}(v))(I + \mathcal{T}(v))^\top)| \le |\nabla\mathcal{S}(v)||I + \mathcal{T}(v)|^2 + 2\mathcal{S}(v)|\nabla\mathcal{T}(v)||I + \mathcal{T}(v)|,$$

and hence,

$$\begin{aligned} &\|\operatorname{div}(\mathcal{S}(v)(I + \mathcal{T}(v))(I + \mathcal{T}(v))^\top)\|_{L^p(0,T;L^q)} \\ &\le \||\nabla\mathcal{S}(v)||I + \mathcal{T}(v)|^2\|_{L^p(0,T;L^q)} + 2\|\mathcal{S}(v)|\nabla\mathcal{T}(v)||I + \mathcal{T}(v)|\|_{L^p(0,T;L^q)} \\ &\le C(T)M, \end{aligned} \tag{4.19}$$

with

$$M = \max\left\{\|G_0\|_{W^{1,q}} + 1, \|G_0\|_{L^\infty(\mathbb{R}^3)} + 1, \|r_0\|_{W^{1,q}} + 1, \|r_0\|_{L^\infty(\mathbb{R}^3)} + 1\right\}^3 < \infty.$$

Combining together (4.16), (4.17), (4.18), (4.19), using the Theorem 4.2, and assuming parameter ν sufficiently large and $R < 1$ sufficiently small, we get

$$\|\mathcal{H}(v)\|_{\mathcal{W}(0,T)} \le C(T)(R^2 + \nu^{-2}) \le R.$$

The proof of Lemma 4.4 is complete. □

Thus, it is only left to show the following:

Lemma 4.5. *The operator $\mathcal{H}$ is weakly continuous from $\mathcal{W}^{p,q}(0,T)$ into itself.*

Proof. Assume that $v_n \to v$ weakly in $\mathcal{W}^{p,q}(0,T)$, and set $r_n := \mathcal{S}(v_n)$, $G_n := \mathcal{T}(v_n)$, then $\{r_n\}_{n=1}^\infty$ and $\{G_n\}_{n=1}^\infty$ are uniformly bounded in $L^\infty(0,T;W^{1,q}(\mathbb{R}^3)) \cap W^{1,p}(0,T;L^q(\mathbb{R}^3))$ by Lemmas 4.1 and 4.3. Hence, up to a subsequence, we can assume that $r_n \to r$ and $G_n \to G$ weakly* in $L^\infty(0,T;W^{1,q}(\mathbb{R}^3)) \cap W^{1,p}(0,T;L^q(\mathbb{R}^3))$ and then strongly in $C((0,T)\times B(0,a))$ for all $a > 0$. And at least the same convergence holds for v_n. Thus, (4.1a) and (4.1c) follow easily from above convergence.

Since $r_n \to r$ weakly* in $L^\infty(0,T;W^{1,q}(\mathbb{R}^3)) \cap W^{1,p}(0,T;L^q(\mathbb{R}^3))$, we can assume that $P'(\mathcal{S}(v_n))\nabla\mathcal{S}(v_n) \to P'(\mathcal{S}(v))\nabla\mathcal{S}(v)$ weakly in $L^p(0,T;L^q(\mathbb{R}^3))$ and hence,

$$L^{-1}\left(0, \nabla P(\mathcal{S}(v_n))\right) \to L^{-1}\left(0, \nabla P(\mathcal{S}(v))\right) \quad \text{weakly in} \quad \mathcal{W}(0,T),$$

since the strong continuity of L^{-1} from $L^p(0,T;L^q(\mathbb{R}^3))$ into $\mathcal{W}(0,T)$ and the linearity of the operator L imply also the weak continuity in these spaces.

Similarly, since $\partial_t v_n \to \partial_t v$ weakly in $L^p(0,T;L^q(\mathbb{R}^3))$ and $r_n \to r$ in $C((0,T)\times B(0,a))$ for all $a > 0$, we have $(r_e - r_n)\partial_t v_n \to (r_e - r)\partial_t v$ weakly in $L^p(0,T;L^q(\mathbb{R}^3))$ and consequently

$$L^{-1}\left(0, (r_e - r_n)\partial_t v_n\right) \to L^{-1}\left(0, (r_e - r)\partial_t v\right) \quad \text{weakly in} \quad \mathcal{W}(0,T).$$

Since $\nabla v_n \to \nabla v$ weakly in $W^{1,p}(0,T;W^{-1,q}(\mathbb{R}^3)) \cap L^p(0,T;W^{1,q}(\mathbb{R}^3))$ which is compactly imbedded in to $C([0,T];L^q(B(0,a)))$ for all $a > 0$, we can assume that $v_n \to v$ strongly in $L^\infty(0,T;L^q(B(0,a)))$ for all $a > 0$, and then

$$\mathcal{S}(v_n)(v_n\cdot\nabla)v_n \to \mathcal{S}(v)(v\cdot\nabla)v$$

weakly in $L^p(0,T;L^q(\mathbb{R}^3))$. Hence

$$L^{-1}\left(0, \mathcal{S}(v_n)(v_n\cdot\nabla)v_n\right) \to L^{-1}\left(0, \mathcal{S}(v)(v\cdot\nabla)v\right) \quad \text{weakly in} \quad \mathcal{W}(0,T).$$

Finally, due to the facts that $r_n \to r$ and $G_n \to G$ weakly* in $L^\infty(0,T;W^{1,q}(\mathbb{R}^3)) \cap W^{1,p}(0,T;L^q(\mathbb{R}^3))$ and strongly in $C((0,T)\times B(0,a))$ for all $a > 0$, we deduce that

$$\operatorname{div}(\mathcal{S}(v_n)(I + \mathcal{T}(v_n))(I + \mathcal{T}(v_n))^\top) \to \operatorname{div}(\mathcal{S}(v)(I + \mathcal{T}(v))(I + \mathcal{T}(v))^\top)$$

weakly in $L^p(0,T;L^q(\mathbb{R}^3))$. Therefore,

$$L^{-1}\left(0, \operatorname{div}(\mathcal{S}(v_n)(I + \mathcal{T}(v_n))(I + \mathcal{T}(v_n))^\top)\right) \to L^{-1}\left(0, \operatorname{div}(\mathcal{S}(v)(I + \mathcal{T}(v))(I + \mathcal{T}(v))^\top)\right)$$

weakly in $\mathcal{W}(0,T)$.

Thus, we can conclude that

$$\mathcal{H}(v_n) \to \mathcal{H}(v) \quad \text{weakly in} \quad \mathcal{W}(0,T),$$

due to the weak continuity of map $L^{-1}(v, 0)$. The proof of Lemma 4.5 is complete. □

Therefore, by Theorem 4.1, there exists at least a fixed point

$$v = \mathcal{H}(v) \in B_R(0) \subset \mathcal{W}(0,T), \tag{4.20}$$

of mapping $\mathcal{H}$. The fixed point v provides a local in time solution $(\rho, \mathbf{u}, \mathtt{F})$ of system (1.1) near its equilibrium through the scaling with ν sufficiently large.

The proof of the local existence in Theorem 3.1 is complete. The uniqueness will be proved in the next section.

5. Uniqueness

In this section, we prove the uniqueness of the local solution found in the previous section. Notice that, the argument in Section 4 yields that

$$\partial_t v \in L^2(0,T;L^2(\mathbb{R}^3)), \quad \nabla r \in L^2(0,T;L^2(\mathbb{R}^3)), \quad \nabla G \in L^2(0,T;L^2(\mathbb{R}^3)).$$

Hence, using the interpolation, we deduce that

$$\partial_t v \in L^{p_0}(0,T;L^3(\mathbb{R}^3)), \quad \nabla r \in L^{p_0}(0,T;L^3(\mathbb{R}^3)), \quad \nabla G \in L^{p_0}(0,T;L^3(\mathbb{R}^3)),$$

where

$$\frac{1}{p_0} = \frac{\theta}{2} + \frac{1-\theta}{p}, \quad \frac{1}{3} = \frac{\theta}{2} + \frac{1-\theta}{q},$$

for some $\theta \in [0,1]$. Now, assume that v_1, v_2 satisfying (4.20) for some $T > 0$. Let

$$r := \mathcal{S}(v_1) - \mathcal{S}(v_2), \quad v := v_1 - v_2, \quad G := \mathcal{T}(v_1) - \mathcal{T}(v_2),$$

with a little abuse of notations (however, there should be no confusion in the rest of this section). Then, we have

$$\begin{cases} \partial_t r + v_1 \cdot \nabla r + v \cdot \nabla \mathcal{S}(v_2) + r\mathrm{div} v_1 + \mathcal{S}(v_2)\mathrm{div} v = 0, \\ r(0) = 0. \end{cases} \tag{5.1}$$

Multiplying (5.1) by r, and integrating over $\mathbb{R}^3$, we get

$$\frac{1}{2}\frac{d}{dt}\|r\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}^3} |r|^2 \mathrm{div} v_1 dx + \int_{\mathbb{R}^3} \left(v\nabla\mathcal{S}(v_2) r + |r|^2\mathrm{div} v_1 + r\mathcal{S}(v_2)\mathrm{div} v\right) dx = 0,$$

which yields

$$\begin{aligned} \frac{d}{dt}\|r\|_{L^2(\mathbb{R}^3)}^2 &\le \|\mathrm{div} v_1\|_{L^\infty}\|r\|_{L^2}^2 + \varepsilon\|\nabla v\|_{L^2}^2 + C(\varepsilon)\|\nabla\mathcal{S}(v_2) r\|_{L^{\frac{6}{5}}}^2 \\ &\quad + \varepsilon\|\nabla v\|_{L^2(\mathbb{R}^3)}^2 + C(\varepsilon)\|\mathcal{S}(v_2)\|_{L^\infty}^2\|r\|_2^2 \\ &\le \|\mathrm{div} v_1\|_{L^\infty}\|r\|_{L^2}^2 + \varepsilon\|\nabla v\|_{L^2}^2 + C(\varepsilon)\|\nabla\mathcal{S}(v_2)\|_{L^3}^2\|r\|_{L^2}^2 \\ &\quad + \varepsilon\|\nabla v\|_{L^2(\mathbb{R}^3)}^2 + C(\varepsilon)\|\mathcal{S}(v_2)\|_{L^\infty}^2\|r\|_2^2 \\ &\le \eta_1(\varepsilon)\|r\|_{L^2}^2 + 2\varepsilon\|\nabla v\|_{L^2(\mathbb{R}^3)}^2, \end{aligned} \tag{5.2}$$

where $\varepsilon > 0$, $\eta_1(\varepsilon) = \|\mathrm{div} v_1\|_{L^\infty} + C(\varepsilon)\left(\|\nabla\mathcal{S}(v_2)\|_{L^3}^2 + \|\mathcal{S}(v_2)\|_{L^\infty}^2\right)$.

Similarly, from (4.1c), we obtain

$$\begin{cases} \partial_t G + v_1 \cdot \nabla G + v \cdot \nabla G_2 = \nabla v_1 G + \nabla v G_2 + \nabla v, \\ G(0) = 0. \end{cases} \tag{5.3}$$

Multiplying (5.3) by G, and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2-\frac{1}{2}\int_{\mathbb{R}^3}|G|^2\mathrm{div}v_1dx+\int_{\mathbb{R}^3}v\cdot\nabla\mathcal{T}(v_2):Gdx\\&=\int_{\mathbb{R}^3}|G|^2\nabla v_1dx+\int_{\mathbb{R}^3}\nabla v\mathcal{T}(v_2):Gdx+\int_{\mathbb{R}^3}\nabla v:Gdx,\end{aligned}$$

which yields

$$\begin{aligned}\frac{d}{dt}\|G\|_{L^2(\mathbb{R}^3)}^2&\le\|\mathrm{div}v_1\|_{L^\infty}\|G\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\|\nabla\mathcal{T}(v_2)G\|_{L^{\frac{6}{5}}}^2\\&\quad+\varepsilon\|\nabla v\|_{L^2(\mathbb{R}^3)}^2+C(\varepsilon)\|\mathcal{T}(v_2)\|_{L^\infty}^2\|G\|_2^2+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\|G\|_{L^2}^2\\&\le\|\mathrm{div}v_1\|_{L^\infty}\|G\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\|\nabla\mathcal{T}(v_2)\|_{L^3}^2\|G\|_{L^2}^2\\&\quad+\varepsilon\|\nabla v\|_{L^2(\mathbb{R}^3)}^2+C(\varepsilon)\|\mathcal{T}(v_2)\|_{L^\infty}^2\|G\|_2^2+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\|G\|_{L^2}^2\\&\le\eta_2(\varepsilon)\|G\|_{L^2}^2+3\varepsilon\|\nabla v\|_{L^2(\mathbb{R}^3)}^2,\end{aligned}\tag{5.4}$$

where $\eta_2(\varepsilon)=\|\mathrm{div}v_1\|_{L^\infty}+C(\varepsilon)\left(\|\nabla\mathcal{T}(v_2)\|_{L^3}^2+\|\mathcal{T}(v_2)\|_{L^\infty}^2+1\right)$.

For each v_j, $j=1,2$, we deduce from (4.1b) that

$$\begin{cases}\mathcal{S}(v_j)\partial_tv_j-\mu\Delta v_j\\\qquad=-\mathcal{S}(v_j)(v_j\cdot\nabla)v_j-\nabla P(\mathcal{S}(v_j))+\mathrm{div}(\mathcal{S}(v_j)(I+\mathcal{T}(v_j))(I+\mathcal{T}(v_j))^\top),\\v_j(0)=v_0.\end{cases}$$

Subtracting these equations, we obtain,

$$\begin{aligned}&\mathcal{S}(v_1)\partial_tv_1-\mathcal{S}(v_2)\partial_tv_2-\mu\Delta v\\&=-\mathcal{S}(v_1)(v_1\cdot\nabla)v_1+\mathcal{S}(v_2)(v_2\cdot\nabla)v_2-\nabla P(\mathcal{S}(v_1))+\nabla P(\mathcal{S}(v_2))\\&\quad+\mathrm{div}(\mathcal{S}(v_1)(I+\mathcal{T}(v_1))(I+\mathcal{T}(v_1))^\top)-\mathrm{div}(\mathcal{S}(v_2)(I+\mathcal{T}(v_2))(I+\mathcal{T}(v_2))^\top).\end{aligned}\tag{5.5}$$

Since

$$\begin{aligned}&-\mathcal{S}(v_1)(v_1\cdot\nabla)v_1+\mathcal{S}(v_2)(v_2\cdot\nabla)v_2\\&=-\mathcal{S}(v_1)(v\cdot\nabla)v_1-(\mathcal{S}(v_1)-\mathcal{S}(v_2))(v_2\cdot\nabla)v_1-\mathcal{S}(v_2)(v_2\cdot\nabla)v,\end{aligned}$$

and

$$\begin{aligned}&\mathcal{S}(v_1)(I+\mathcal{T}(v_1))(I+\mathcal{T}(v_1))^\top-\mathcal{S}(v_2)(I+\mathcal{T}(v_2))(I+\mathcal{T}(v_2))^\top\\&=\mathcal{S}(v_1)G(I+\mathcal{T}(v_1))^\top+r(I+\mathcal{T}(v_2))(I+\mathcal{T}(v_1))^\top+\mathcal{S}(v_2)(I+\mathcal{T}(v_2))G^\top,\end{aligned}$$

we can rewrite (5.5) as

$$\begin{aligned}&\mathcal{S}(v_1)\partial_tv-\mu\Delta v\\&=-r\partial_tv_2-\mathcal{S}(v_1)(v\cdot\nabla)v_1-\mathcal{S}(v_2\cdot\nabla)v_1-\mathcal{S}(v_2)(v_2\cdot\nabla)v-\nabla P(\mathcal{S}(v_1))+\nabla P(\mathcal{S}(v_2))\\&\quad+\mathrm{div}(\mathcal{S}(v_1)G(I+\mathcal{T}(v_1))^\top+r(I+\mathcal{T}(v_2))(I+\mathcal{T}(v_1))^\top+\mathcal{S}(v_2)(I+\mathcal{T}(v_2))G^\top).\end{aligned}\tag{5.6}$$

Multiplying (5.6) by v, using the continuity equation (1.1a) and integrating over $\mathbb{R}^3$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\mathcal{S}(v_1)|v|^2dx+\int_{\mathbb{R}^3}\mu|\nabla v|^2dx\\
&=\int_{\mathbb{R}^3}\frac{1}{2}\mathcal{S}(v_1)(v_1\cdot\nabla)v\cdot v-r\partial_tv_2v-\mathcal{S}(v_1)(v\cdot\nabla)v_1v-\mathcal{S}(v_2\cdot\nabla)v_1v\\
&\quad-\mathcal{S}(v_2)(v_2\cdot\nabla)vv-\nabla P(\mathcal{S}(v_1))v+\nabla P(\mathcal{S}(v_2))v\\
&\quad-(\mathcal{S}(v_1)G(I+\mathcal{T}(v_1))^\top+r(I+\mathcal{T}(v_2))(I+\mathcal{T}(v_1))^\top+\mathcal{S}(v_2)(I+\mathcal{T}(v_2))G^\top)\nabla vdx\\
&\le\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\|\mathcal{S}(v_1)\|_{L^\infty}^2\|v_1\|_{L^\infty}^2\|v\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\|\partial_tv_2\|_{L^3}^2\|r\|_{L^2}^2\\
&\quad+\|\mathcal{S}(v_1)\|_{L^\infty}\|\nabla v_1\|_{L^\infty}\|v\|_{L^2}^2+2\|v_2\|_{L^\infty}\|\nabla v_1\|_{L^\infty}(\|r\|_{L^2}^2+\|v\|_{L^2}^2)\\
&\quad+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\|\mathcal{S}(v_2)\|_{L^\infty}^2\|v_2\|_{L^\infty}^2\|v\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2\\
&\quad+C(\varepsilon)(\sup\{P'(\eta):C(T)^{-1}\le\eta\le C(T)\})^2\|r\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2\\
&\quad+C(\varepsilon)(\|\mathcal{S}(v_1)\|_{L^\infty}^2(1+\|\mathcal{T}(v_1)\|_{L^\infty}^2)\|G\|_{L^2}^2\\
&\quad+\|\mathcal{S}(v_2)\|_{L^\infty}^2(1+\|\mathcal{T}(v_2)\|_{L^\infty}^2)\|G\|_{L^2}^2+\|r\|_{L^2}^2(1+\|\mathcal{T}(v_1)\|_{L^\infty}^2)(1+\|\mathcal{T}(v_2)\|_{L^\infty}^2))\\
&\le5\varepsilon\|\nabla v\|_{L^2}^2+\eta_3(\varepsilon)(\|r\|_{L^2}^2+\|v\|_{L^2}^2+\|G\|_{L^2}^2)
\end{aligned}\tag{5.7}$$

with

$$\begin{aligned}
\eta_3(\varepsilon)=&\,C(\varepsilon)\|\mathcal{S}(v_1)\|_{L^\infty}^2\|v_1\|_{L^\infty}^2+C(\varepsilon)\|\partial_tv_2\|_{L^3}^2+\|\mathcal{S}(v_1)\|_{L^\infty}\|\nabla v_1\|_{L^\infty}\\
&+2\|v_2\|_{L^\infty}\|\nabla v_1\|_{L^\infty}+C(\varepsilon)\|\mathcal{S}(v_2)\|_{L^\infty}^2\|v_2\|_{L^\infty}^2\\
&+C(\varepsilon)(\sup\{P'(\eta):C(T)^{-1}\le\eta\le C(T)\})^2\\
&+C(\varepsilon)(\|\mathcal{S}(v_1)\|_{L^\infty}^2(1+\|\mathcal{T}(v_1)\|_{L^\infty}^2)+\|\mathcal{S}(v_2)\|_{L^\infty}^2(1+\|\mathcal{T}(v_2)\|_{L^\infty}^2)\\
&+(1+\|\mathcal{T}(v_1)\|_{L^\infty}^2)(1+\|\mathcal{T}(v_2)\|_{L^\infty}^2)).
\end{aligned}$$

Summing up (5.2), (5.4), and (5.7), by taking $\varepsilon=\frac{\mu}{20}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}(\mathcal{S}(v_1)|v|^2+|r|^2+|G|^2)dx+\mu\int_{\mathbb{R}^3}|\nabla v|^2dx\\
&\le2(\eta_3(\varepsilon)+\eta_2(\varepsilon)+\eta_1(\varepsilon))(\|v\|_{L^2}^2+\|r\|_{L^2}^2+\|G\|_{L^2}^2)\\
&\le2\eta(\varepsilon,t)\int_{\mathbb{R}^3}(\mathcal{S}(v_1)|v|^2+|r|^2+|G|^2)dx,
\end{aligned}\tag{5.8}$$

with

$$\eta(\varepsilon,t)=\frac{\eta_3(\varepsilon)+\eta_2(\varepsilon)+\eta_1(\varepsilon)}{\min\{\min_{x\in\mathbb{R}^3}\mathcal{S}(v_1)(x,t),1\}}.$$

It is a routine matter to establish the integrability with respect to t of the function $\eta(\varepsilon,t)$ on the interval $(0,T)$. This is a consequence of the regularity of $v_1,v_2\in\mathcal{W}(0,T)$ and the estimates in Lemmas 4.2 and 4.3 for $\mathcal{S}(v_i)$, $\mathcal{T}(v_i)$ with $i=1,2$. Therefore, (5.8), combining with Gronwall's inequality, implies

$$\int_{\mathbb{R}^3}\left(\mathcal{S}(v_1)|v|^2+|r|^2+|G|^2\right)dx=0,\quad\text{for all}\quad t\in(0,T),\tag{5.9}$$

and consequently

$$v \equiv 0, \quad r \equiv 0, \quad G \equiv 0.$$

Thus, the uniqueness in Theorem 3.1 is established.

6. Global *A Priori* Estimates

Up to now, we prove that for any given T_0, we can find a unique solution to the scaling system (4.1). That is, we have proved the local existence of solution to the viscoelastic fluid system (1.1) and its uniqueness. In order to establish the global existence for the unique solution we constructed in the previous sections, we need to obtain some uniform *a priori* estimates which are independent of the time T. To simplify the presentation, we will focus on the case $\nu = 1$, that is, system (1.1).

We introduce the new variable:

$$\sigma := \nabla \ln \rho.$$

Then, we have

Lemma 6.1. *Function σ satisfies*

$$\partial_t \sigma + \nabla(\mathbf{u} \cdot \sigma) = 0, \tag{6.1}$$

in the sense of distributions. Moreover, the norm $\|\sigma(t)\|_{L^q(\mathbb{R}^3)}$ is continuous in time.

Proof. We follow the argument in [26] (Section 9.8) by denoting $\sigma_\varepsilon = S_\varepsilon \sigma$, where S_ε is the standard mollifier in the spatial variables. Then, we have

$$\partial_t \sigma_\varepsilon + \nabla(\mathbf{u} \cdot \sigma_\varepsilon) = \mathcal{R}_\varepsilon,$$

with

$$\begin{aligned} \mathcal{R}_\varepsilon &= \nabla(\mathbf{u} \cdot \sigma_\varepsilon) - S_\varepsilon \nabla(\mathbf{u} \cdot \sigma) \\ &= (\mathbf{u} \cdot \nabla \sigma_\varepsilon - S_\varepsilon(\mathbf{u} \cdot \nabla \sigma)) + (\sigma_\varepsilon \nabla \mathbf{u} - S_\varepsilon(\sigma \cdot \nabla \mathbf{u})) \\ &=: \mathcal{R}_\varepsilon^1 + \mathcal{R}_\varepsilon^2. \end{aligned} \tag{6.2}$$

Since $\sigma \in L^\infty(0,T; L^q(\mathbb{R}^3))$ and $\mathbf{u} \in L^p(0,T; W^{1,\infty}(\mathbb{R}^3))$, we deduce from Lemma 6.7 in [26] (cf. Lemma 2.3 in [21]) that $\mathcal{R}_\varepsilon^1 \to 0$ as $\varepsilon \to 0$. Moreover,

$$\|(\sigma_\varepsilon - \sigma)\nabla \mathbf{u}\|_{L^1(0,T;L^q(\mathbb{R}^3))} \le \|\sigma - \sigma_\varepsilon\|_{L^{\frac{p}{p-1}}(0,T;L^q(\mathbb{R}^3))} \|\nabla \mathbf{u}\|_{L^p(0,T;L^\infty(\mathbb{R}^3))} \to 0,$$

and $S_\varepsilon(\sigma \cdot \nabla \mathbf{u}) \to \sigma \cdot \nabla \mathbf{u}$ in $L^1(0,T; L^q(\mathbb{R}^3))$ since $\sigma \cdot \nabla \mathbf{u} \in L^p(0,T; L^q(\mathbb{R}^3))$. Thus, we have $\mathcal{R}_\varepsilon^2 \to 0$ in $L^p(0,T; L^q(\mathbb{R}^3))$. Then, taking the limit as $\varepsilon \to 0$ in (6.2), we get (6.1).

Multiplying (6.1) by $|\sigma|^{q-2}\sigma$, and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned} \frac{1}{q}\left|\frac{d}{dt}\|\sigma\|_{L^q(\mathbb{R}^3)}^q\right| &= \left|\int_{\mathbb{R}^3} (-\partial_j \mathbf{u}_k \sigma_j \sigma_k |\sigma|^{q-2} - \frac{1}{q}\mathrm{div}\mathbf{u}|\sigma|^q) dx\right| \\ &\le \|\nabla \mathbf{u}\|_{L^\infty} \|\sigma\|_{L^q}^q + \frac{1}{q}\|\mathrm{div}\mathbf{u}\|_{L^\infty} \|\sigma\|_{L^q}^q \\ &\le C\|\mathbf{u}\|_{W^{2,q}} \|\sigma\|_{L^q}^q. \end{aligned}$$

Dividing the above inequality by $\|\sigma\|_{L^q}^{q-1}$, we obtain

$$\left|\frac{d}{dt}\|\sigma\|_{L^q}\right| \le C\|\mathbf{u}\|_{W^{2,q}} \|\sigma\|_{L^q}.$$

Since $\sigma \in L^\infty(0,T;L^q(\mathbb{R}^3))$, $\frac{d}{dt}\|\sigma\|_{L^q} \in L^p(0,T)$. Thus, $\|\sigma\|_{L^q} \in C(0,T)$. The proof of Lemma 6.1 is complete. $\square$

For a given $R = \delta_0 \ll 1$ as in Section 4, if the initial data satisfies $\|\mathbf{u}(0), \rho_0 - 1, E(0)\|_{V_0} \leq \delta^2$ with $0 < \delta \ll \min\{\frac{1}{3}, \delta_0\}$, let $T(R)$ be the maximal time T such that there is a solution of the equation $\mathbf{u} = \mathcal{H}(\mathbf{u})$ in $B_R(0)$. By virtue of Lemma 4.2, Lemma 4.3 and Lemma 6.1, we know that $\|\mathcal{S}(\mathbf{u}) - 1\|_{W^{1,q}(\mathbb{R}^3)}$, $\|\sigma\|_{L^q}$ and $\|\mathcal{T}(\mathbf{u})\|_{W^{1,q}}$ are continuous in the interval $[0, T(R))$. On the other hand, under the assumptions on initial data and Remark 3.3, we know, if δ is sufficiently small, then

$$\|\sigma(0)\|_{L^q(\mathbb{R}^3)} \leq \frac{1}{C_0}\|\nabla\rho(0)\|_{L^q(\mathbb{R}^3)} \leq \delta^{\frac{3}{2}} \ll 1.$$

Hence, there exists a maximum positive number T_1 such that

$$\max\left\{\|\mathcal{S}(\mathbf{u}) - 1\|_{W^{1,q}}(t), \|\sigma\|_{L^q}(t), \|\mathcal{T}(\mathbf{u})\|_{W^{1,q}}(t)\right\} \leq \sqrt{R} \ll 1 \quad \text{for all} \quad t \in [0, T_1]. \tag{6.3}$$

Now, we denote $T = \min\{T(R), T_1\}$. Without loss of generality, we assume that $T < \infty$.

Since $q > 3$, we have

$$\|\rho - 1\|_{L^\infty(\mathbb{R}^3)} \leq C\|\rho - 1\|_{W^{1,q}(\mathbb{R}^3)} \leq C\sqrt{R} < \frac{1}{2},$$

if R is sufficiently small. Hence, one obtains

$$\frac{1}{2} \leq \rho \leq \frac{3}{2}.$$

On the other hand, for any given $t \in (0, T)$, we can write

$$\begin{aligned}
\|\mathbf{u}(t)\|_{L^q}^p &= \|\mathbf{u}(0)\|_{L^q}^p + \int_0^t \frac{d}{ds}\|\mathbf{u}(s)\|_{L^q}^p ds \\
&= \|\mathbf{u}_0\|_{L^q}^p + \frac{p}{q}\int_0^t \left(\|\mathbf{u}(t)\|_{L^q}^{p-q}\int_{\mathbb{R}^3}|\mathbf{u}(s)|^{q-2}\mathbf{u}(s)\partial_s\mathbf{u}(t)dx\right) dt \\
&\leq \|\mathbf{u}_0\|_{L^q}^p + \frac{p}{q}\int_0^t \|\mathbf{u}(s)\|_{L^q}^{p-1}\|\partial_s\mathbf{u}\|_{L^q} ds \\
&\leq \delta^{2p} + \frac{p}{q}\left(\int_0^t \|\mathbf{u}\|_{L^q}^p ds\right)^{\frac{p-1}{p}}\left(\int_0^t \|\partial_s\mathbf{u}\|_{L^q}^p ds\right)^{\frac{1}{p}} \\
&\leq \delta^{2p} + \frac{p}{q}R^p,
\end{aligned}$$

and consequently,

$$\|\mathbf{u}\|_{L^\infty(0,t;L^q)} \leq \left(\delta^{2p} + \frac{p}{q}R^p\right)^{\frac{1}{p}} \leq CR, \quad t \in (0, T). \tag{6.4}$$

Similarly, we have, for all $t \in [0, T]$,

$$\|\mathbf{u}\|_{L^\infty(0,t;L^2)} \leq CR.$$

6.1. **Dissipation of the deformation gradient.** The main difficulty of the proof of Theorem 3.2 lies in the lack of estimates on the dissipation of the deformation gradient. This is partly because of the transport structure of equation (1.1c). It is worthy of pointing out that it is extremely difficult to directly deduce the dissipation of the deformation gradient. Fortunately, for the viscoelastic fluids system (1.1), as we can see in [6, 15, 16, 17, 18, 19, 20], some sort of combinations between the gradient of the velocity and the deformation gradient indeed induce some dissipation. To make this statement more precise, we rewritten the momentum equation (1.1b) as, using (1.1a)

$$\begin{aligned}\partial_t \mathbf{u} - \mu\Delta\mathbf{u} - \mathrm{div}E &= -\rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla P(\rho) + \mathrm{div}(\rho(I+E)^\top) \\ &\quad + \mathrm{div}((\rho-1)E) + \mathrm{div}(\rho E E^\top) + (1-\rho)\partial_t\mathbf{u},\end{aligned} \tag{6.5}$$

and prove the following estimate:

Lemma 6.2.

$$\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))} \le C(p,q,\mu)\left(R + \sqrt{R}\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))}\right). \tag{6.6}$$

Proof. Now we introduce the function $Z_1(x,t)$ as

$$Z_1 := \mathcal{E} * \mathrm{div}E = \int_{\mathbb{R}^3} \mathcal{E}(x-y)\mathrm{div}E dy, \tag{6.7}$$

where $\mathcal{E}$ is the fundamental solution of the Laplacian $-\Delta$ in $\mathbb{R}^3$. Then, (6.5) becomes

$$\partial_t\mathbf{u} - \mu\Delta\left(\mathbf{u} - \frac{1}{\mu}Z_1\right) = \mathcal{F}_1, \tag{6.8}$$

where, with the help of Remark 3.4,

$$\mathcal{F}_1 = -\rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla P(\rho) + \mathrm{div}((\rho-1)E) + \mathrm{div}(\rho E E^\top) + (1-\rho)\partial_t\mathbf{u}.$$

Also, from (1.3), we have

$$\begin{aligned}\frac{\partial Z_1}{\partial t} &= \int_{\mathbb{R}^3} \mathcal{E}(x-y)\mathrm{div}\frac{\partial E}{\partial t} dy \\ &= \int_{\mathbb{R}^3} \mathcal{E}(x-y)\mathrm{div}(\nabla\mathbf{u} + \nabla\mathbf{u}E - (\mathbf{u}\cdot\nabla)E) dy.\end{aligned} \tag{6.9}$$

From (6.8) and (6.9), we deduce, denoting $Z = \mathbf{u} - \frac{1}{\mu}Z_1$,

$$\partial_t Z - \mu\Delta Z = \mathcal{F} := \mathcal{F}_1 - \mathcal{F}_2, \tag{6.10}$$

where

$$\mathcal{F}_2 = -\frac{1}{\mu}\mathbf{u} + \frac{1}{\mu}\mathcal{E} * \mathrm{div}(\nabla\mathbf{u}E - (\mathbf{u}\cdot\nabla)E).$$

Equation (6.10) with Theorem 4.2 implies that

$$\begin{aligned}\|Z\|_{\mathcal{W}(0,T)} &\le C(p,q)\left(\|Z(0)\|_{X_p^{2(1-\frac{1}{p})}} + \|\mathcal{F}\|_{L^p(0,T;L^q(\mathbb{R}^3))}\right) \\ &\le C(p,q)\left(R + \|\mathcal{F}\|_{L^p(0,T;L^q(\mathbb{R}^3))}\right).\end{aligned} \tag{6.11}$$

Next, we estimate $\|\mathcal{F}_i\|_{L^p(0,T;L^q(\mathbb{R}^3))}$, $i=1,2$, term by term. Indeed, for $\mathcal{F}_1$, using (6.4), we have

$$\begin{aligned}\|\mathcal{F}_1\|_{L^p(0,T;L^q(\mathbb{R}^3))} \leq\ & \|\rho\|_{L^\infty(Q_T)}\|\mathbf{u}\|_{L^\infty(0,T;L^q(\mathbb{R}^3))}\|\nabla\mathbf{u}\|_{L^p(0,T;L^\infty(\mathbb{R}^3))} \\ & + \alpha\|\nabla E\|_{L^p(0,T;L^q(\Omega))} + \alpha\|\sigma\|_{L^p(0,T;L^q(\Omega))}\|E\|_{L^\infty(Q_T)} \\ & + \|\rho-1\|_{L^\infty(Q_T)}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))} \\ & + \|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))}\|E\|_{L^\infty(Q_T)} + \|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))}\|E\|^2_{L^\infty(Q_T)} \\ & + \|E\|_{L^\infty(Q_T)}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))} \\ & + \|\rho-1\|_{L^\infty(Q_T)}\|\partial_t v\|_{L^p(0,T;L^q(\mathbb{R}^3))} \\ \leq\ & 2R\|\mathbf{u}\|_{L^p(0,T;W^{2,q}(\mathbb{R}^3))} + \alpha\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))} \\ & + \alpha\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))}\|E\|_{L^\infty(Q_T)} + \sqrt{R}\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))} \\ & + R\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))} + \sqrt{R}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))} + R^{\frac{3}{2}} \\ \leq\ & R^{\frac{3}{2}} + \alpha\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))} + \sqrt{R}\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))} \\ & + \sqrt{R}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}.\end{aligned} \tag{6.12}$$

Here, $\alpha = \sup\left\{xP'(x) : \frac{1}{2} \leq x \leq \frac{3}{2}\right\}$ and in the first inequality, we used the identity

$$\nabla\rho = -\rho\operatorname{div}E^\top - \nabla\rho E^\top$$

due to Remark 3.4. And, for $\mathcal{F}_2$, noting that $|\nabla\mathcal{E}| \leq C\frac{1}{|x|^2}$, and from integrating by parts, we have

$$|\mathcal{F}_2| \leq \frac{1}{\mu}|\mathbf{u}| + \frac{C}{\mu}\frac{1}{|x|^2} * (\nabla\mathbf{u}E - (\mathbf{u}\cdot\nabla)E),$$

with

$$\begin{aligned}\|\nabla\mathbf{u}E - (\mathbf{u}\cdot\nabla)E\|_{L^p(0,T;L^{\frac{3q}{q+3}}(\mathbb{R}^3))} \leq\ & \|\nabla\mathbf{u}\|_{L^p(0,T;L^3(\mathbb{R}^3))}\|E\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} \\ & + \|\mathbf{u}\|_{L^p(0,T;L^3(\mathbb{R}^3))}\|\nabla E\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} \\ \leq\ & R^{\frac{3}{2}}.\end{aligned}$$

Hence, one can estimate, by $L^p - L^q$ estimate of Riesz potential,

$$\begin{aligned}\|\mathcal{F}_2\|_{L^p(0,T;L^q(\mathbb{R}^3))} \leq\ & \frac{1}{\mu}\|\mathbf{u}\|_{L^p(0,T;L^q(\mathbb{R}^3))} + \frac{C}{\mu}\left\|\frac{1}{|x|^2} * (\nabla\mathbf{u}E - (\mathbf{u}\cdot\nabla)E)\right\|_{L^p(0,T;L^q(\mathbb{R}^3))} \\ \leq\ & \frac{1}{\mu}R + \frac{C}{\mu}\|\nabla\mathbf{u}E - (\mathbf{u}\cdot\nabla)E\|_{L^p(0,T;L^{\frac{3q}{3+q}}(\mathbb{R}^3))} \\ \leq\ & \frac{1}{\mu}(R + CR^{\frac{3}{2}}).\end{aligned} \tag{6.13}$$

Therefore, from (6.12) and (6.13), we obtain

$$\begin{aligned}\|\mathcal{F}\|_{L^p(0,T;L^q(\mathbb{R}^3))} \leq\ & R^{\frac{3}{2}} + \frac{1}{\mu}R + \alpha\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))} + \sqrt{R}\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))} \\ & + \sqrt{R}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}.\end{aligned} \tag{6.14}$$

Inequalities (6.11) and (6.14) imply that

$$\begin{aligned}
&\|Z\|_{L^p(0,T;W^{2,q}(\mathbb{R}^3))}\\
&\le C(p,q)\Big(2R+\frac{1}{\mu}R+\alpha\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}+\sqrt{R}\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))}\\
&\qquad\qquad+\sqrt{R}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}\Big).
\end{aligned}\tag{6.15}$$

Hence, we have, from (6.7)

$$\begin{aligned}
&\|\mathrm{div}E\|_{L^p(0,T;L^q(\mathbb{R}^3))}\\
&\le \mu\left(\|Z\|_{L^p(0,T;W^{2,q}(\mathbb{R}^3))}+\|\mathbf{u}\|_{L^p(0,T;W^{2,q}(\mathbb{R}^3))}\right)\\
&\le C(p,q)\mu\Big(3R+\frac{1}{\mu}R+\alpha\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}+\sqrt{R}\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))}\\
&\qquad\qquad+\sqrt{R}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}\Big).
\end{aligned}\tag{6.16}$$

On the other hand, from the identity (4.3), we deduce that

$$\begin{aligned}
\|\mathrm{curl}\ E_i\|_{L^p(0,T;L^q(\mathbb{R}^3))}&\le 2\|E\|_{L^\infty(Q_T)}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}\\
&\le C\|E\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3))}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}\\
&\le C\sqrt{R}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}.
\end{aligned}\tag{6.17}$$

Combining together (6.16) and (6.17), we obtain

$$\begin{aligned}
\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}\le C(p,q)\mu\Big(&3R+\frac{1}{\mu}R+\alpha\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}+\sqrt{R}\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))}\\
&+\sqrt{R}\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}\Big),
\end{aligned}$$

and hence, by choosing $\sqrt{R}\ll\frac{1}{2}$ and the assumption

$$C(p,q)\mu\alpha<1,\tag{6.18}$$

one obtains (6.6). The proof of Lemma 6.2 is complete. $\square$

Remark 6.1. Notice that, in view of the above argument, estimate (6.6) is actually valid for all $t\in[0,T]$, that is, for all $t\in[0,T]$,

$$\|\nabla E\|_{L^p(0,t;L^q(\mathbb{R}^3))}\le C(p,q,\mu)\left(R+\sqrt{R}\|\sigma\|_{L^p(0,t;L^q(\mathbb{R}^3))}\right).$$

6.2. **Dissipation of the gradient of the density.** To make Theorem 3.2 valid, we need further the uniform estimate on the dissipation of the gradient of the density.

Lemma 6.3. *For any* $t\in(0,T)$,

$$\|\sigma\|_{L^p(0,t;L^q(\mathbb{R}^3))}\le C(p,q,\mu)R.\tag{6.19}$$

Proof. Multiplying (1.1b) by $\sigma|\sigma|^{q-2}$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{\mu+\lambda}{q}\frac{d}{dt}\|\sigma\|_{L^q}^q+\int_{\mathbb{R}^3}\rho P'(\rho)|\sigma|^q dx\\
&=\mu\int_{\mathbb{R}^3}\Delta\mathbf{u}\cdot\sigma|\sigma|^{q-2}dx-\int_{\mathbb{R}^3}\rho\partial_t\mathbf{u}\cdot\sigma|\sigma|^{q-2}dx\\
&\quad-\int_{\mathbb{R}^3}\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\sigma|\sigma|^{q-2}dx-(\mu+\lambda)\int_{\mathbb{R}^3}\nabla(\mathbf{u}\cdot\sigma)\cdot\sigma|\sigma|^{q-2}dx\\
&\quad+\int_{\mathbb{R}^3}\operatorname{div}(\rho(I+E)(I+E)^\top)\cdot\sigma|\sigma|^{q-2}dx.
\end{aligned}\tag{6.20}$$

We estimate the right-hand side of (6.20) term by term,

$$\left|\int_{\mathbb{R}^3}\Delta\mathbf{u}\cdot\sigma|\sigma|^{q-2}dx\right|\le\|\Delta\mathbf{u}\|_{L^q}\|\sigma\|_{L^q}^{q-1};$$

$$\left|\int_{\mathbb{R}^3}\rho\partial_t\mathbf{u}\cdot\sigma|\sigma|^{q-2}dx\right|\le\|\partial_t\mathbf{u}\|_{L^q}\|\sigma\|_{L^q}^{q-1};$$

$$\begin{aligned}
\left|\int_{\mathbb{R}^3}\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\sigma|\sigma|^{q-2}dx\right|&\le\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^q}\|\sigma\|_{L^q}^{q-1}\\
&\le\|\mathbf{u}\|_{L^q}\|\mathbf{u}\|_{W^{2,q}}\|\sigma\|_{L^q}^{q-1};
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\mathbb{R}^3}\nabla(\mathbf{u}\cdot\sigma)\cdot\sigma|\sigma|^{q-2}dx\right|&=\left|\int_{\mathbb{R}^3}\partial_j\mathbf{u}_k\sigma_k\sigma_j|\sigma|^{q-2}dx+\int_{\mathbb{R}^3}\mathbf{u}_k\partial_j\partial_k(\ln\rho)\partial_j\ln\rho|\sigma|^{q-2}dx\right|\\
&=\left|\int_{\mathbb{R}^3}\partial_j\mathbf{u}_k\sigma_k\sigma_j|\sigma|^{q-2}dx+\frac{1}{2}\int_{\mathbb{R}^3}\mathbf{u}_k\partial_k|\sigma|^2|\sigma|^{q-2}dx\right|\\
&=\left|\int_{\mathbb{R}^3}\partial_j\mathbf{u}_k\sigma_k\sigma_j|\sigma|^{q-2}dx+\int_{\mathbb{R}^3}\mathbf{u}_k\partial_k|\sigma||\sigma|^{q-1}dx\right|\\
&=\left|\int_{\mathbb{R}^3}\partial_j\mathbf{u}_k\sigma_k\sigma_j|\sigma|^{q-2}dx+\frac{1}{q}\int_{\mathbb{R}^3}\mathbf{u}_k\partial_k|\sigma|^q dx\right|\\
&=\left|\int_{\mathbb{R}^3}\partial_j\mathbf{u}_k\sigma_k\sigma_j|\sigma|^{q-2}dx-\frac{1}{q}\int_{\mathbb{R}^3}|\sigma|^q\operatorname{div}\mathbf{u}dx\right|\\
&\le C\|\nabla\mathbf{u}\|_{L^\infty}\|\sigma\|_{L^q}^q\le C\|\mathbf{u}\|_{W^{2,q}}\|\sigma\|_{L^q}^q,
\end{aligned}$$

and, due to (2.9), we can rewrite

$$\begin{aligned}
\left(\operatorname{div}(\rho(I+E)(I+E)^\top)\right)_i&=\frac{\partial(\rho(e_i+E_i)(e_j+E_j))}{\partial x_j}\\
&=(e_i+E_i)\frac{\partial(\rho(e_j+E_j))}{\partial x_j}+\rho(e_j+E_j)\frac{\partial(e_i+E_i)}{\partial x_j}\\
&=\rho(e_j+E_j)\frac{\partial E_i}{\partial x_j},
\end{aligned}$$

then one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\operatorname{div}(\rho(I+E)(I+E)^\top)\cdot\sigma|\sigma|^{q-2}dx\right|=\left|\int_{\mathbb{R}^3}\rho(e_j+E_j)\frac{\partial E_i}{\partial x_j}\sigma_i|\sigma|^{q-2}dx\right|\\
&\le\|\nabla E\|_{L^q}\|I+E\|_{L^\infty}\|\sigma\|_{L^q}^{q-1}\le 2\|\nabla E\|_{L^q(\mathbb{R}^3)}\|\sigma\|_{L^q}^{q-1}.
\end{aligned}$$

On the other hand, we have

$$\rho P'(\rho) = P'(1) + (\rho-1)\int_0^1 \big(P'(\eta(\rho-1)+1) + (\eta(\rho-1)+1)P''(\eta(\rho-1)+1)\big)\, d\eta,$$

for ρ close to 1 and consequently

$$\begin{aligned}\|\rho P'(\rho) - P'(1)\|_{L^\infty} &\le \|\rho-1\|_{L^\infty} \sup\left\{|f(x)| : \frac{1}{2} \le x \le \frac{3}{2}\right\}\\ &\le C\sqrt{R}\sup\left\{|f(x)| : \frac{1}{2} \le x \le \frac{3}{2}\right\} \le C\sqrt{R},\end{aligned}$$

where $f(x) = P'(x) + xP''(x)$. Thus, from (6.20), we obtain

$$\begin{aligned}&\frac{\mu+\lambda}{q}\frac{d}{dt}\|\sigma\|_{L^q}^q + P'(1)\|\sigma\|_{L^q}^q\\ &\le \|\sigma\|_{L^q}^{q-1}\Big(\|\Delta\mathbf{u}\|_{L^q} + \|\partial_t\mathbf{u}\|_{L^q} + \|\mathbf{u}\|_{W^{2,q}}\|\mathbf{u}\|_{L^q}\\ &\qquad + C\|\mathbf{u}\|_{W^{2,q}}\|\sigma\|_{L^q} + C\|\nabla E\|_{L^q(\mathbb{R}^3)} + \sqrt{R}\|\sigma\|_{L^q}\Big)\\ &\le C\|\sigma\|_{L^q}^{q-1}\Big(\|\Delta\mathbf{u}\|_{L^q} + \|\partial_t\mathbf{u}\|_{L^q} + \|\mathbf{u}\|_{W^{2,q}}\|\mathbf{u}\|_{L^q}\\ &\qquad + \|\mathbf{u}\|_{W^{2,q}}\|\sigma\|_{L^q} + \|\nabla E\|_{L^q(\mathbb{R}^3)} + \sqrt{R}\|\sigma\|_{L^q}\Big),\end{aligned}$$

and hence, by assuming that $R \ll 1$, one obtains

$$\begin{aligned}&\frac{\mu+\lambda}{q}\frac{d}{dt}\|\sigma\|_{L^q}^q + \frac{1}{2}P'(1)\|\sigma\|_{L^q}^q\\ &\le C\|\sigma\|_{L^q}^{q-1}\big(\|\Delta\mathbf{u}\|_{L^q} + \|\partial_t\mathbf{u}\|_{L^q} + \|\mathbf{u}\|_{W^{2,q}}\|\mathbf{u}\|_{L^q} + \|\mathbf{u}\|_{W^{2,q}}\|\sigma\|_{L^q} + \|\nabla E\|_{L^q}\big).\end{aligned} \tag{6.21}$$

Multiplying (6.21) by $\|\sigma\|_{L^q}^{p-q}$, we obtain

$$\begin{aligned}&\frac{\mu+\lambda}{p}\frac{d}{dt}\|\sigma\|_{L^q}^p + \frac{1}{2}P'(1)\|\sigma\|_{L^q}^p\\ &\le C\|\sigma\|_{L^q}^{p-1}\big(\|\Delta\mathbf{u}\|_{L^q} + \|\partial_t\mathbf{u}\|_{L^q} + \|\mathbf{u}\|_{W^{2,q}}\|\mathbf{u}\|_{L^q} + \|\mathbf{u}\|_{W^{2,q}}\|\sigma\|_{L^q} + \|\nabla E\|_{L^q}\big).\end{aligned}$$

Integrating the above inequality over the interval $(0,t)$, one obtains, by using (6.6),

$$\begin{aligned}
&\frac{\mu+\lambda}{p}\|\sigma(t)\|_{L^q}^p+\frac{1}{2}P'(1)\int_0^t\|\sigma\|_{L^q}^p ds\\
&\le\frac{\mu+\lambda}{p}\|\sigma(0)\|_{L^q}^p+C\left(\int_0^t\|\sigma\|_{L^q}^p ds\right)^{\frac{p-1}{p}}\left(\left(\int_0^t\|\partial_t\mathbf{u}\|_{L^q}^p ds\right)^{\frac{1}{p}}\right.\\
&\qquad+\left(\|\mathbf{u}\|_{L^\infty(0,t;L^q(\mathbb{R}^3))}+\|\sigma\|_{L^\infty(0,t;L^q(\mathbb{R}^3))}+1\right)\left(\int_0^t\|\mathbf{u}\|_{W^{2,q}}^p ds\right)^{\frac{1}{p}}\\
&\qquad\left.+\|\nabla E\|_{L^p(0,t;L^q(\mathbb{R}^3))}\right)\\
&\le\frac{\mu+\lambda}{p}\|\sigma(0)\|_{L^q}^p+C(p,q,\mu)\sqrt{R}\|\sigma\|_{L^p(0,T;L^q(\mathbb{R}^3))}^p\\
&\qquad+C(p,q,\mu)R\left(\int_0^t\|\sigma\|_{L^q}^p ds\right)^{\frac{p-1}{p}}\left(1+\|\sigma\|_{L^\infty(0,t;L^q(\mathbb{R}^3))}+\|\mathbf{u}\|_{L^\infty(0,t;L^q(\mathbb{R}^3))}\right).
\end{aligned}$$

and hence, by letting R be so small such that $C(p,q,\mu)\sqrt{R}<\frac{1}{4}$, one obtains

$$\begin{aligned}
&\frac{\mu+\lambda}{p}\|\sigma(t)\|_{L^q}^p+\frac{1}{4}P'(1)\int_0^t\|\sigma\|_{L^q}^p ds\\
&\le\frac{\mu+\lambda}{p}\|\sigma(0)\|_{L^q}^p+C(p,q,\mu)R\left(\int_0^t\|\sigma\|_{L^q}^p ds\right)^{\frac{p-1}{p}}\left(1+\|\sigma\|_{L^\infty(0,t;L^q)}+\|\mathbf{u}\|_{L^\infty(0,t;L^q)}\right).
\end{aligned}\tag{6.22}$$

Plugging (6.4) into (6.22), we obtain

$$\begin{aligned}
&\frac{\mu+\lambda}{p}\|\sigma(t)\|_{L^q}^p+\frac{1}{4}P'(1)\int_0^t\|\sigma\|_{L^q}^p ds\\
&\le\frac{\mu+\lambda}{p}\|\sigma(0)\|_{L^q}^p+C(p,q,\mu)R\left(\int_0^t\|\sigma\|_{L^q}^p ds\right)^{\frac{p-1}{p}}\left(1+\|\sigma\|_{L^\infty(0,t;L^q(\mathbb{R}^3))}\right).
\end{aligned}$$

Then, Young's inequality yields

$$\begin{aligned}
&\frac{\mu+\lambda}{p}\|\sigma(t)\|_{L^q}^p+\frac{1}{8}P'(1)\int_0^t\|\sigma\|_{L^q}^p ds\\
&\le\frac{\mu+\lambda}{p}\delta^{\frac{3}{2}p}+C(p,q,\mu)R^p(1+\|\sigma\|_{L^\infty(0,t;L^q(\mathbb{R}^3))})^p,
\end{aligned}\tag{6.23}$$

for all $0\le t<T$.

Now, we let R be so small that

$$C(p,q,\mu)^{\frac{1}{p}}\sqrt{R}\left(1+\sqrt{R}\right)<\frac{1}{2}.$$

Due to the fact that $\|\sigma(0)\|_{L^q(\mathbb{R}^3)}\le\delta^{\frac{3}{2}}$, we can assume that $\|\sigma(t)\|_{L^q}<\frac{1}{2}\sqrt{R}$ in some maximal interval $(0,t_{\max})\subset(0,T)$. If $t_{\max}<T$, then, $\|\sigma(t_{\max})\|_{L^q}=\frac{1}{2}\sqrt{R}$ and by (6.23),

$$\frac{1}{2}\sqrt{R}=\|\sigma(t_{\max})\|_{L^q}\le C(p,q,\mu)^{\frac{1}{p}}R(1+\sqrt{R})<\frac{1}{2}\sqrt{R},$$

which is a contradiction. Hence, $t_{\max} = T$ and

$$\|\sigma\|_{L^q} \leq \frac{1}{2}\sqrt{R}, \quad \text{for all} \quad t \in [0,T]. \tag{6.24}$$

Thus, by (6.23), one obtains (6.19). The proof of Lemma 6.3 is complete. □

We remark that, from (6.6) and (6.19), one has

$$\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))} \leq C(p,q,\mu)R. \tag{6.25}$$

7. Global Existence

In this section, we prove the global existence in Theorem 3.2. Define

$$\begin{aligned} T_{\max} := \sup\Big\{ T > 0 :\ & \exists\, \mathbf{u} \in \mathcal{W}(0,T) \text{ with } \mathbf{u} = \mathcal{H}(\mathbf{u}), \text{ such that, } \|\mathbf{u}\|_{\mathcal{W}(0,T)} \leq R, \\ & \|\mathcal{S}(\mathbf{u}) - 1\|_{L^\infty(0,T;W^{1,q})} \leq \sqrt{R},\ \|\sigma\|_{L^\infty(0,T;L^q)} \leq \sqrt{R}, \text{ and} \\ & \|\mathcal{T}(\mathbf{u})\|_{L^\infty(0,T;W^{1,q})} \leq \sqrt{R} \Big\}, \end{aligned}$$

where R was constructed in the previous section.

If $T_{\max} = \infty$, we are done. From now on, we assume that $T_{\max} < \infty$.

7.1. **Uniform estimates in time.** We now establish some estimates which are uniform in time T. First we prove the following energy estimates:

Lemma 7.1. *Under the same assumptions as Theorem 3.1, we have*

$$\|\nabla \mathbf{u}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq CR^2, \tag{7.1}$$

$$\|\mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq CR^2, \tag{7.2}$$

$$\|E\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq CR^2, \tag{7.3}$$

$$\|\rho - 1\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq CR^2, \tag{7.4}$$

where C is a constant independent of $T \in (0, T_{\max})$.

Proof. First we recall that

$$\mathbf{u} \in W^{1,2}(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;W^{2,2}(\mathbb{R}^3))$$

and

$$\rho, E \in W^{1,2}(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\Omega)).$$

Multiplying equation (1.1b) by $\mathbf{u}$, and integrating over $\mathbb{R}^3$, we obtain, using the conservation of mass (1.1a),

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{\gamma-1}(\rho^\gamma + \gamma - 1)\right)dx + \int_{\mathbb{R}^3}\mu|\nabla\mathbf{u}|^2 dx \\ &= -\int_{\mathbb{R}^3}\rho\mathtt{F}\mathtt{F}^\top : \nabla\mathbf{u}\,dx. \end{aligned}$$

Here, the notation $A:B$ means the dot product between two matrices. Thus, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho|\mathbf{u}|^2+\frac{1}{\gamma-1}(\rho^\gamma+\gamma-1)\right)dx+\int_0^t\int_{\mathbb{R}^3}\mu|\nabla\mathbf{u}|^2dxds\\&=\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2+\frac{1}{\gamma-1}(\rho_0^\gamma+\gamma-1)\right)dx-\int_0^t\int_{\mathbb{R}^3}\rho\mathtt{F}\mathtt{F}^\top:\nabla\mathbf{u}dxds.\end{aligned}$$

From the conservation of mass (1.1a), one has

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho|\mathbf{u}|^2+\frac{1}{\gamma-1}(\rho^\gamma-\gamma\rho+\gamma-1)\right)dx+\int_0^t\int_{\mathbb{R}^3}\mu|\nabla\mathbf{u}|^2dxds\\&=\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2+\frac{1}{\gamma-1}(\rho_0^\gamma-\gamma\rho_0+\gamma-1)\right)dx-\int_0^t\int_{\mathbb{R}^3}\rho\mathtt{F}\mathtt{F}^\top:\nabla\mathbf{u}dxds.\end{aligned}\tag{7.5}$$

On the other hand, due to equations (1.1c) and (1.1a), we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho|\mathtt{F}|^2\right)&=\frac{\partial\rho}{\partial t}|\mathtt{F}|^2+2\rho\mathtt{F}:\frac{\partial\mathtt{F}}{\partial t}\\&=\frac{\partial\rho}{\partial t}|\mathtt{F}|^2+2\rho\mathtt{F}:(\nabla\mathbf{u}\,\mathtt{F}-\mathbf{u}\cdot\nabla\mathtt{F})\\&=\frac{\partial\rho}{\partial t}|\mathtt{F}|^2+2\rho\mathtt{F}:(\nabla\mathbf{u}\,\mathtt{F})-\rho\mathbf{u}\cdot\nabla|\mathtt{F}|^2\\&=\frac{\partial\rho}{\partial t}|\mathtt{F}|^2+2\rho\mathtt{F}:(\nabla\mathbf{u}\,\mathtt{F})+\mathrm{div}(\rho\mathbf{u})|\mathtt{F}|^2-\mathrm{div}(\rho\mathbf{u}|\mathtt{F}|^2)\\&=2\rho\mathtt{F}:(\nabla\mathbf{u}\,\mathtt{F})-\mathrm{div}(\rho\mathbf{u}|\mathtt{F}|^2).\end{aligned}\tag{7.6}$$

Integrating (7.6) over $\mathbb{R}^3$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho|\mathtt{F}|^2dx=\int_{\mathbb{R}^3}\rho\mathtt{F}:(\nabla\mathbf{u}\,\mathtt{F})dx.\tag{7.7}$$

Since

$$\int_0^t\int_{\mathbb{R}^3}\rho\mathtt{F}:(\nabla\mathbf{u}\,\mathtt{F})dxds=\int_0^t\int_{\mathbb{R}^3}\rho\mathtt{F}\mathtt{F}^\top:\nabla\mathbf{u}dxds,$$

we finally obtain, by summing (7.5) and (7.6),

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho|\mathbf{u}|^2+\frac{1}{2}\rho|\mathtt{F}|^2+\frac{1}{\gamma-1}(\rho^\gamma-\gamma\rho+\gamma-1)\right)dx+\int_0^t\int_{\mathbb{R}^3}\mu|\nabla\mathbf{u}|^2dxds\\&=\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2+\frac{1}{2}\rho_0|\mathtt{F}_0|^2+\frac{1}{\gamma-1}(\rho_0^\gamma-\gamma\rho_0+\gamma-1)\right)dx.\end{aligned}\tag{7.8}$$

Thanks to Remark 3.4, we have

$$\rho(I+E^\top):\nabla\mathbf{u}=0.$$

Hence, from (1.1c) and (1.1a), we have

$$\partial_t(\rho\,\mathrm{tr}E)=0.\tag{7.9}$$

Therefore, from (7.8), (7.9) and the conservation of mass (1.1a), we finally arrive at

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho|\mathbf{u}|^2+\frac{1}{2}\rho|E|^2+\frac{1}{\gamma-1}(\rho^\gamma-\gamma\rho+\gamma-1)\right)dx+\int_0^t\int_{\mathbb{R}^3}\mu|\nabla\mathbf{u}|^2dxds\\&=\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2+\frac{1}{2}\rho_0|E_0|^2+\frac{1}{\gamma-1}(\rho_0^\gamma-\gamma\rho_0+\gamma-1)\right)dx\le R^4.\end{aligned}\tag{7.10}$$

Since $\mu>0$ is a constant and $\rho\in[\frac{1}{2},\frac{3}{2}]$, then inequalities (7.1)-(7.3) follow from (7.10), and inequality (7.4) follows from (7.10) and the following straightforward inequalities: for some $\eta>0$, we have

$$x^\gamma-1-\gamma(x-1)\ge\begin{cases}\eta|x-1|^2, & \text{if}\quad \gamma\ge 2,\\ \eta|x-1|^2, & \text{if}\quad |x|<2\quad\text{and}\quad \gamma<2.\end{cases}$$

The proof of Lemma 7.1 is complete. □

Based on the uniform estimates from Section 6, we have

Lemma 7.2. *Under the same assumptions as Theorem 3.2,*

$$\|\mathcal{S}(\mathbf{u})-1\|_{L^\infty(0,T;W^{1,q})}<\sqrt{R},\quad \|\sigma\|_{L^\infty(0,T;L^q)}<\sqrt{R},\tag{7.11}$$

for any $T\in[0,T_{\max}]$.

Proof. According to (6.24), it is obvious to see that

$$\max_{t\in[0,T]}\|\sigma\|_{L^q}(t)<\sqrt{R}.$$

Hence, we are only left to show

$$\max_{t\in[0,T]}\|\mathcal{S}(\mathbf{u})-1\|_{W^{1,q}}(t)<\sqrt{R}.$$

Indeed, for any $t\in(0,T)$, we have, by using (1.1a) and (6.19),

$$\begin{aligned}&\|\mathcal{S}(\mathbf{u})(t)-1\|_{L^q}^\alpha\\&=\|\rho_0-1\|_{L^q}^\alpha+\int_0^t\frac{d}{ds}\|\mathcal{S}(\mathbf{u})(s)-1\|_{L^q}^\alpha ds\\&=\|\rho_0-1\|_{L^q}^\alpha+\frac{\alpha}{q}\int_0^t\Big(\|\mathcal{S}(\mathbf{u})(s)-1\|_{L^q}^{\alpha-q}\\&\qquad\qquad\times\int_{\mathbb{R}^3}|\mathcal{S}(\mathbf{u})(s)-1|^{q-2}(\mathcal{S}(\mathbf{u})(s)-1)\partial_s\mathcal{S}(\mathbf{u})(s)dx\Big)ds\\&\le\|\rho_0-1\|_{L^q}^\alpha+\frac{\alpha}{q}\int_0^t\|\mathcal{S}(\mathbf{u})(s)-1\|_{L^q}^{\alpha-1}\|\partial_s\mathcal{S}(\mathbf{u})\|_{L^q}ds\\&\le\delta^{2\alpha}+\frac{\alpha}{q}\left(\int_0^t\|\mathcal{S}(\mathbf{u})(s)-1\|_{L^q}^{\frac{(5q-6)p}{3q-6}}ds\right)^{\frac{p-1}{p}}\left(\int_0^t\|\partial_s\mathcal{S}(\mathbf{u})\|_{L^q}^p ds\right)^{\frac{1}{p}},\end{aligned}\tag{7.12}$$

where

$$\alpha=\frac{(5q-6)(p-1)}{3q-6}+1.$$

From (1.1a) and (1.1d), we obtain

$$\begin{aligned}\|\partial_t\rho\|_{L^p(0,T;L^q(\mathbb{R}^3))} &= \|\nabla\rho\cdot\mathbf{u}\|_{L^p(0,T;L^q(\mathbb{R}^3))}\\ &\le 2\|\sigma\|_{L^\infty(0,T;L^q(\mathbb{R}^3))}\|\mathbf{u}\|_{L^p(0,T;L^\infty(\mathbb{R}^3))}\le CR^2.\end{aligned}\tag{7.13}$$

On the other hand, from the Gagliardo-Nirenberg inequality, we have

$$\|\rho-1\|_{L^q(\mathbb{R}^3)}\le C\|\rho-1\|^\theta_{L^2(\mathbb{R}^3)}\|\nabla(\rho-1)\|^{1-\theta}_{L^q(\mathbb{R}^3)}\le C\|\rho-1\|^\theta_{L^2(\mathbb{R}^3)}\|\sigma\|^{1-\theta}_{L^q(\mathbb{R}^3)},$$

with $\theta=\frac{2q}{5q-6}$. Thus, by Hölder's inequality, (7.4), and (6.19), one has

$$\|\rho-1\|_{L^{\frac{(5q-6)p}{3q-6}}(0,T;L^q(\mathbb{R}^3))}\le C\|\rho-1\|^\theta_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|\sigma\|^{1-\theta}_{L^p(0,T;L^q(\mathbb{R}^3))}\le CR,$$

which, together (7.12) and (7.13), yields

$$\|\mathcal{S}(\mathbf{u})(t)-1\|_{L^q}\le CR.$$

Hence, according to (6.19), we obtain, by letting R be sufficiently small,

$$\max_{t\in[0,T]}\max\left\{\|\mathcal{S}(v)(t)-1\|_{W^{1,q}(\mathbb{R}^3)},\|\sigma(t)\|_{L^q(\mathbb{R}^3)}\right\}<\sqrt{R}.\tag{7.14}$$

The proof of Lemma 7.2 is complete. □

Lemma 7.3. *For each $1\le l\le 3$, $\frac{\partial E}{\partial x_l}$ satisfies*

$$\partial_t\frac{\partial E}{\partial x_l}+\mathbf{u}\cdot\nabla\frac{\partial E}{\partial x_l}=-\frac{\partial\mathbf{u}}{\partial x_l}\cdot\nabla E+\nabla\left(\frac{\partial\mathbf{u}}{\partial x_l}\right)E+\nabla\mathbf{u}\frac{\partial E}{\partial x_l}+\nabla\frac{\partial\mathbf{u}}{\partial x_l}\tag{7.15}$$

in the sense of distributions, that is, for all $\psi\in C_0^\infty(Q_T)$, we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3}\frac{\partial E}{\partial x_l}\partial_t\psi dxdt+\int_0^T\int_{\mathbb{R}^3}\mathrm{div}(\mathbf{u}\psi)\frac{\partial E}{\partial x_l}\\&=-\int_0^T\int_{\mathbb{R}^3}\left(-\frac{\partial\mathbf{u}}{\partial x_l}\cdot\nabla E+\nabla\left(\frac{\partial\mathbf{u}}{\partial x_l}\right)E+\nabla\mathbf{u}\frac{\partial E}{\partial x_l}+\nabla\frac{\partial\mathbf{u}}{\partial x_l}\right)\psi dxdt,\end{aligned}$$

for any $T\in(0,T_{\max})$.

Proof. The proof is a direct application of the regularization. Indeed, one easily obtains, using (1.3),

$$\begin{aligned}\partial_t(S_\varepsilon E)+\mathbf{u}\cdot\nabla(S_\varepsilon E)&=S_\varepsilon(\partial_tE+\mathbf{u}\cdot\nabla E)+\mathbf{u}\cdot\nabla(S_\varepsilon E)-S_\varepsilon(\mathbf{u}\cdot\nabla E)\\&=S_\varepsilon(\nabla\mathbf{u}E+\nabla\mathbf{u})+\mathbf{u}\cdot\nabla(S_\varepsilon E)-S_\varepsilon(\mathbf{u}\cdot\nabla E).\end{aligned}\tag{7.16}$$

Differentiate (7.16) with respect to x_l, we get

$$\begin{aligned}&\partial_t\left(\frac{\partial S_\varepsilon E}{\partial x_l}\right)+\mathbf{u}\cdot\nabla\left(\frac{\partial S_\varepsilon E}{\partial x_l}\right)\\&=S_\varepsilon\left(\frac{\partial}{\partial x_l}(\nabla\mathbf{u}E+\nabla\mathbf{u})\right)+\frac{\partial}{\partial x_l}\Big(\mathbf{u}\cdot\nabla(S_\varepsilon E)-S_\varepsilon(\mathbf{u}\cdot\nabla E)\Big)-\frac{\partial\mathbf{u}}{\partial x_l}\cdot\nabla S_\varepsilon E.\end{aligned}\tag{7.17}$$

Notice that

$$\begin{aligned}\frac{\partial}{\partial x_l}\Big(\mathbf{u}\cdot\nabla(S_\varepsilon E)-S_\varepsilon(\mathbf{u}\cdot\nabla E)\Big)&=\frac{\partial\mathbf{u}}{\partial x_l}\cdot\nabla S_\varepsilon E-S_\varepsilon\left(\frac{\partial\mathbf{u}}{\partial x_l}\cdot\nabla E\right)\\&\quad+\mathbf{u}\cdot\nabla S_\varepsilon\left(\frac{\partial E}{\partial x_l}\right)-S_\varepsilon\left(\mathbf{u}\cdot\nabla\frac{\partial E}{\partial x_l}\right).\end{aligned}$$

According to Lemma 6.7 in [26] (cf. Lemma 2.3 in [21]), we know that

$$\frac{\partial \mathbf{u}}{\partial x_l} \cdot \nabla S_\varepsilon E - S_\varepsilon \left(\frac{\partial \mathbf{u}}{\partial x_l} \cdot \nabla E \right) \to 0,$$

and

$$\mathbf{u} \cdot \nabla S_\varepsilon \left(\frac{\partial E}{\partial x_l} \right) - S_\varepsilon \left(\mathbf{u} \cdot \nabla \frac{\partial E}{\partial x_l} \right) \to 0,$$

in $L^1(0, T; L^q(\mathbb{R}^3))$ as $\varepsilon \to 0$. Hence,

$$\frac{\partial}{\partial x_l} \Big(\mathbf{u} \cdot \nabla (S_\varepsilon E) - S_\varepsilon (\mathbf{u} \cdot \nabla E) \Big) \to 0$$

in $L^1(0, T; L^q(\mathbb{R}^3))$. Thus, letting $\varepsilon \to 0$ in (7.17), we deduce

$$\partial_t \frac{\partial E}{\partial x_l} + \mathbf{u} \cdot \nabla \frac{\partial E}{\partial x_l} = -\frac{\partial \mathbf{u}}{\partial x_l} \cdot \nabla E + \nabla \left(\frac{\partial \mathbf{u}}{\partial x_l} \right) E + \nabla \mathbf{u} \frac{\partial E}{\partial x_l} + \nabla \frac{\partial \mathbf{u}}{\partial x_l},$$

in the sense of weak solutions. The proof of Lemma 7.3 is complete. $\square$

Using (7.15), formally we have,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \partial_t \left(\frac{\partial E}{\partial x_l} \right) \left| \frac{\partial E}{\partial x_l} \right|^{q-2} \frac{\partial E}{\partial x_l} dx \\
&= \int_{\mathbb{R}^3} \left(-\mathbf{u} \cdot \nabla \frac{\partial E}{\partial x_l} - \frac{\partial \mathbf{u}}{\partial x_l} \cdot \nabla E + \nabla \left(\frac{\partial \mathbf{u}}{\partial x_l} \right) E + \nabla \mathbf{u} \frac{\partial E}{\partial x_l} + \nabla \frac{\partial \mathbf{u}}{\partial x_l} \right) \left| \frac{\partial E}{\partial x_l} \right|^{q-2} \frac{\partial E}{\partial x_l} dx \\
&\le C \Big(\|\nabla \mathbf{u}\|_{L^\infty} \|\nabla E\|^q_{L^q(\mathbb{R}^3)} + \|E\|_{L^\infty(Q_T)} \|\mathbf{u}\|_{W^{2,q}} \|\nabla E\|^{q-1}_{L^q(\mathbb{R}^3)} + \|\mathbf{u}\|_{W^{2,q}} \|\nabla E\|^{q-1}_{L^q} \Big) \\
&\le C \Big(\|\nabla \mathbf{u}\|_{L^\infty} \|\nabla \mathbf{u}\|^q_{L^q(\mathbb{R}^3)} + \sqrt{R} \|\mathbf{u}\|_{W^{2,q}} \|\nabla E\|^{q-1}_{L^q(\mathbb{R}^3)} + \|\mathbf{u}\|_{W^{2,q}} \|\nabla E\|^{q-1}_{L^q} \Big).
\end{aligned} \tag{7.18}$$

We remark that the rigorous argument for the above estimate involves a tedious regularization procedure as in DiPerna-Lions [9], thus we omit the details and refer the reader to

[9]. Using (7.18), one obtains

$$
\begin{aligned}
&\left\|\frac{\partial E}{\partial x_l}(t)\right\|_{L^q}^p\\
&=\left\|\frac{\partial E(0)}{\partial x_l}\right\|_{L^q}^p+\int_0^t \frac{d}{ds}\left\|\frac{\partial E}{\partial x_l}(s)\right\|_{L^q}^p ds\\
&=\left\|\frac{\partial E(0)}{\partial x_l}\right\|_{L^q}^p+\frac{p}{q}\int_0^t\left[\left\|\frac{\partial E}{\partial x_l}\right\|_{L^q}^{p-q}\int_{\mathbb{R}^3}\left|\frac{\partial E}{\partial x_l}\right|^{q-2}\left(\frac{\partial E}{\partial x_l}\right)\partial_s\left(\frac{\partial E}{\partial x_l}(s)\right)dx\right]ds\\
&\leq\left\|\frac{\partial E(0)}{\partial x_l}\right\|_{L^q}^p+C\left(\frac{p}{q}\right)\int_0^t\left\|\frac{\partial E}{\partial x_l}\right\|_{L^q}^{p-q}\Big[\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla E\|_{L^q(\mathbb{R}^3)}^q\\
&\quad+\sqrt{R}\|\mathbf{u}\|_{W^{2,q}}\|\nabla E\|_{L^q(\mathbb{R}^3)}^{q-1}+\|\mathbf{u}\|_{W^{2,q}}\|\nabla E\|_{L^q}^{q-1}\Big]ds\\
&\leq\left\|\frac{\partial E(0)}{\partial x_l}\right\|_{L^q}^p+C\left(\frac{p}{q}\right)\int_0^t\|\nabla E\|_{L^q}^{p-1}[\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla E\|_{L^q(\mathbb{R}^3)}+(1+\sqrt{R})\|\mathbf{u}\|_{W^{2,q}}]ds\\
&\leq\delta^{2p}+C\left(\frac{p}{q}\right)\left(\int_0^t\|\nabla E\|_{L^q}^p ds\right)^{\frac{p-1}{p}}\left(\int_0^t\|\mathbf{u}\|_{W^{2,q}}^p ds\right)^{\frac{1}{p}}\left(\max_{t\in[0,T]}\|\nabla E\|+\sqrt{R}+1\right)\\
&\leq\delta^{2p}+C\left(\frac{p}{q}\right)R^p\left(\max_{t\in[0,T]}\|\nabla E\|+\sqrt{R}+1\right).
\end{aligned}
\tag{7.19}
$$

Taking the summation over l in (7.19) and taking the maximum over the time t, one has,

$$\max_{t\in[0,T]}\|\nabla E\|^p\leq\delta^{2p}+C\left(\frac{p}{q}\right)R^p\left(\max_{t\in[0,T]}\|\nabla E\|+\sqrt{R}+1\right),$$

and hence, by letting R,δ be sufficiently small and using (6.25), we obtain,

$$\max_{t\in[0,T]}\|\nabla E\|^p\leq\delta^{2p}+CR^p<(\sqrt{R})^p. \tag{7.20}$$

We are now left to deal with the quantity $\|E\|_{L^q(\mathbb{R}^3)}$. To this end, from the Gagliardo-Nirenberg inequality, we have

$$\|E\|_{L^q(\mathbb{R}^3)}\leq C\|E\|_{L^2(\mathbb{R}^3)}^\theta\|\nabla E\|_{L^q(\mathbb{R}^3)}^{1-\theta},$$

with $\theta=\frac{2q}{5q-6}$. Thus, by Hölder's inequality, (7.3), and (6.25)

$$\|E\|_{L^{\frac{(5q-6)p}{3q-6}}(0,T;L^q(\mathbb{R}^3))}\leq C\|E\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}^\theta\|\nabla E\|_{L^p(0,T;L^q(\mathbb{R}^3))}^{1-\theta}\leq CR. \tag{7.21}$$

Hence, we have the following estimate:

Lemma 7.4. *Under the same assumptions as Theorem 3.2, it holds*

$$\|E\|_{L^\infty(0,T;L^q(\mathbb{R}^3))}<\sqrt{R}, \tag{7.22}$$

for any $T\in[0,T_{\max}]$.

Proof. By (1.3), (6.25) and (7.20), and letting

$$\alpha=\frac{(5q-6)(p-1)}{3q-6}+1,$$

one obtains,

$$
\begin{aligned}
&\|E(t)\|^\alpha_{L^q}\\
&= \|E(0)\|^\alpha_{L^q} + \int_0^t \frac{d}{ds}\|E(s)\|^\alpha_{L^q} ds\\
&= \|E(0)\|^\alpha_{L^q} + \frac{\alpha}{q}\int_0^t \left(\|E(s)\|^{\alpha-q}_{L^q}\int_{\mathbb{R}^3} |E(s)|^{q-2}E(s)\partial_s E(s)dx\right) ds\\
&= \|E(0)\|^\alpha_{L^q} + \frac{\alpha}{q}\int_0^t \left(\|E(s)\|^{\alpha-q}_{L^q}\int_{\mathbb{R}^3} |E(s)|^{q-2}E(s)\Big[\nabla\mathbf{u}E + \nabla\mathbf{u} - \mathbf{u}\cdot\nabla E\Big]dx\right) ds\\
&\le \|E(0)\|^\alpha_{L^q} + \frac{\alpha}{q}\int_0^t \left(\|E(s)\|^{\alpha-1}_{L^q}\Big[2\|\nabla\mathbf{u}\|_{L^\infty}\|E\|_{L^q} + \|\nabla\mathbf{u}\|_{L^q}\Big]\right) ds\\
&\le \|E(0)\|^\alpha_{L^q} + \frac{\alpha}{q}\left(\int_0^t \|E(s)\|^{\frac{(5q-6)p}{3q-6}}_{L^q} dt\right)^{\frac{p-1}{p}} \|\mathbf{u}\|_{L^p(0,T;W^{2,q}(\mathbb{R}^3))}\\
&\qquad\qquad\qquad\qquad\qquad \times\left(2\sup_{t\in(0,T_{\max})}\|E(t)\|_{L^q(\mathbb{R}^3)} + 1\right)\\
&\le \|E(0)\|^\alpha_{L^q} + \left(\frac{2\alpha}{q}\right) R\left(\int_0^t \|E(s)\|^{\frac{(5q-6)p}{3q-6}}_{L^q} dt\right)^{\frac{p-1}{p}}\\
&\le \|E(0)\|^\alpha_{L^q} + CR\|E\|^{\alpha-1}_{L^{\frac{(5q-6)p}{3q-6}}(0,T;L^q(\mathbb{R}^3))}.
\end{aligned}
\tag{7.23}
$$

Then, according to (7.21), one has, for all $t\in[0,T_{\max}]$,

$$\|E(t)\|^\alpha_{L^q} \le \delta^{2\alpha} + CR^\alpha < \sqrt{R}^\alpha, \tag{7.24}$$

if R is sufficiently small. Thus, (7.22) follows from (7.24). The proof of Lemma 7.4 is complete. □

Lemma 7.4, together with (7.20) and Lemma 7.2, gives

$$\max_{t\in[0,T]}\max\left\{\|\mathcal{S}(\mathbf{u})-1\|_{W^{1,q}}(t), \|\sigma\|_{L^q}(t), \|T(\mathbf{u})\|_{W^{1,q}}(t)\right\} \le CR < \sqrt{R}. \tag{7.25}$$

Similarly, we can obtain

$$\max_{t\in[0,T]}\max\left\{\|\mathcal{S}(\mathbf{u})-1\|_{W^{1,2}}(t), \|\sigma\|_{L^2}(t), \|T(\mathbf{u})\|_{W^{1,2}}(t)\right\} \le CR < \sqrt{R}. \tag{7.26}$$

7.2. **Refined estimates on $\nabla\rho$ and ∇E.** In order to prove Theorem 3.2, we need some refined estimates on $\|\nabla\rho\|_{L^2(0,T;L^q(\mathbb{R}^3))}$ and $\|\nabla E\|_{L^2(0,T;L^q(\mathbb{R}^3))}$.

Lemma 7.5.

$$\|\nabla\rho\|_{L^2(0,T;L^q(\mathbb{R}^3))} \le R^2, \tag{7.27}$$

for any $T\in(0,T_{\max})$.

Proof. Taking the divergence in (1.1b), and using $\mathrm{div}\mathbf{u}=0$, one obtains

$$\Delta P(\rho) = \mathrm{div}(\mathrm{div}(\rho EE^\top)) + \mathrm{div}(\mathrm{div}(\rho E)) - \mathrm{div}(\rho\mathbf{u}\cdot\nabla\mathbf{u}) - \mathrm{div}((\rho-1)\partial_t\mathbf{u}). \tag{7.28}$$

Since, $\mathrm{div}(\rho(I+E)^\top)=0$, we get

$$\mathrm{div}(\mathrm{div}(\rho E))=\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(\rho E_{ij})=\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(\rho E_{ij})=-\Delta\rho$$

in the sense of distributions. Hence, (7.28) becomes

$$\Delta P(\rho)+\Delta\rho=\mathrm{divdiv}(\rho EE^\top)-\mathrm{div}(\rho\mathbf{u}\cdot\nabla\mathbf{u})-\mathrm{div}((\rho-1)\partial_t\mathbf{u}). \tag{7.29}$$

Hence, one obtains, using L^q theory of elliptic equations and Taylor's formula,

$$\begin{aligned}
\|\nabla\rho\|_{L^2(0,T;L^q(\mathbb{R}^3))}\le{}& C\Big(\|\rho\mathbf{u}\nabla\mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))}+\|\mathrm{div}(\rho EE^\top)\|_{L^2(0,T;L^q(\mathbb{R}^3))}\\
&+\|(\rho-1)\partial_t\mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))}\Big)\\
\le{}& C\Big(\|\rho\|_{L^\infty(Q_T)}\|\nabla\mathbf{u}\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\|\mathbf{u}\|_{L^\infty(0,T;L^q(\mathbb{R}^3))}\\
&+\|\nabla\rho\|_{L^2(0,T;L^q(\mathbb{R}^3))}\|E\|^2_{L^\infty(Q_T)}\\
&+\|\rho\|_{L^\infty(Q_T)}\|\nabla E\|_{L^2(0,T;L^q(\mathbb{R}^3))}\|E\|_{L^\infty(Q_T)}\\
&+\|\rho-1\|_{L^\infty(Q_T)}\|\partial_t\mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))}\Big)\\
\le{}& C\Big(\|\rho\|_{L^\infty(Q_T)}\|\mathbf{u}\|_{L^2(0,T;W^{2,q}(\mathbb{R}^3))}\|\mathbf{u}\|_{L^\infty(0,T;L^q(\mathbb{R}^3))}\\
&+\|\nabla\rho\|_{L^2(0,T;L^q(\mathbb{R}^3))}\|E\|^2_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3))}\\
&+\|\rho\|_{L^\infty(Q_T)}\|\nabla E\|_{L^2(0,T;L^q(\mathbb{R}^3))}\|E\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3)))}+R^2\Big)\\
\le{}& CR^2.
\end{aligned}$$

The proof of Lemma 7.5 is complete. □

Ir order to refine $\|\nabla E\|_{L^2(0,T;L^q(\mathbb{R}^3))}$, we need the following estimate:

Lemma 7.6.

$$\|\partial_t\mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))}\le CR^{\frac{3}{2}}, \tag{7.30}$$

for any $T\in(0,T_{\max})$.

Proof. We first notice that, by the Gagliardo-Nirenberg inequality, for $q\in(3,6]$,

$$\|\mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))}\le C\|\nabla\mathbf{u}\|^\theta_{L^2(0,T;L^2(\mathbb{R}^3))}\|\mathbf{u}\|^{1-\theta}_{L^2(0,T;L^2(\mathbb{R}^3))}\le CR^{1+\theta}, \tag{7.31}$$

with $\theta=\frac{3(q-2)}{2q}\in(\frac12,1]$. Next, we multiply (1.1b) by $\partial_t\mathbf{u}$ and integrate over $\mathbb{R}^3\times(0,t)$ to deduce

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\rho|\partial_t\mathbf{u}|^2dxds+\mu\int_{\mathbb{R}^3}|\nabla\mathbf{u}(t)|^2dx\\
&=\mu\int_{\mathbb{R}^3}|\nabla\mathbf{u}_0|^2dx-\int_0^t\int_{\mathbb{R}^3}\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\partial_t\mathbf{u}dxds-\int_0^t\int_{\mathbb{R}^3}\nabla P\partial_t\mathbf{u}dxds\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\mathrm{div}(\rho EE^\top)\partial_t\mathbf{u}dxds+\int_0^t\int_{\mathbb{R}^3}\mathrm{div}(\rho E)\partial_t\mathbf{u}dxds\\
&:=\mu\int_{\mathbb{R}^3}|\nabla\mathbf{u}_0|^2dx+\sum_{i=1}^4 I_i,
\end{aligned} \tag{7.32}$$

with the following estimates on I_i $(i = 1...4)$: recalling $Q_T = \mathbb{R}^3 \times (0, T)$,

$$\begin{aligned}
|I_1| &\le \|\sqrt{\rho}\partial_t \mathbf{u}\|_{L^2(Q_T)}\|\sqrt{\rho}\|_{L^\infty(Q_T)}\|\mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|\nabla \mathbf{u}\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\\
&\le \frac{1}{8}\int_0^T\int_{\mathbb{R}^3}\rho|\partial_t\mathbf{u}|^2 dxds + CR^4;
\end{aligned}$$

$$\begin{aligned}
|I_2| &\le C\|\nabla\rho\|_{L^2(Q_T)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(Q_T)} \le \frac{1}{8}\int_0^T\int_{\mathbb{R}^3}\rho|\partial_t\mathbf{u}|^2 dxds + C\|\nabla\rho\|^2_{L^2(Q_T)}\\
&\le \frac{1}{8}\int_0^T\int_{\mathbb{R}^3}\rho|\partial_t\mathbf{u}|^2 dxds + CR^4;
\end{aligned}$$

$$\begin{aligned}
|I_3| &\le C\|\nabla\rho\|_{L^2(Q_T)}\|E\|^2_{L^\infty(Q_T)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(Q_T)} + C\|E\|_{L^\infty(Q_T)}\|\nabla E\|_{L^2(Q_T)}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2(Q_T)}\\
&\le \frac{1}{8}\int_0^T\int_{\mathbb{R}^3}\rho|\partial_t\mathbf{u}|^2 dxds + CR^4;
\end{aligned}$$

$$\begin{aligned}
|I_4| &\le \left|\int_0^t\int_{\mathbb{R}^3}\partial_t(\rho E)\nabla\mathbf{u}dxds\right| + \left|\int_{\mathbb{R}^3}\rho_0E_0\nabla\mathbf{u}_0 dx\right| + \left|\int_{\mathbb{R}^3}\rho(T)E(T)\nabla\mathbf{u}(T)dx\right|\\
&\le (\|\rho\|_{L^\infty(Q_T)}\|\partial_t E\|_{L^2(Q_T)} + \|E\|_{L^\infty(Q_T)}\|\partial_t\rho\|_{L^2(Q_T)})\|\nabla\mathbf{u}\|_{L^2(Q_T)}\\
&\quad + CR^3 + (\|\nabla\rho(T)\|_{L^2(\mathbb{R}^3)}\|E(T)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla E(T)\|_{L^2(\mathbb{R}^3)}\|\rho(T)\|_{L^\infty(\mathbb{R}^3)})\|\mathbf{u}(T)\|_{L^2(\mathbb{R}^3)}\\
&\le CR^3,
\end{aligned}$$

where, for the estimate I_4, we used equations (1.1a), (1.1c), Lemma 7.1 and estimate (7.31). Thus, from (7.32), one obtains

$$\|\partial_t\mathbf{u}\|_{L^2(Q_T)} \le CR^{\frac{3}{2}}, \tag{7.33}$$

and

$$\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le CR^{\frac{3}{2}}. \tag{7.34}$$

Now, we differentiate (1.1b) with respect to t, multiply the resulting equation by $\partial_t\mathbf{u}$, and integrate it over $\mathbb{R}^3$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho|\partial_t\mathbf{u}|^2dx + \mu\int_{\mathbb{R}^3}|\nabla\partial_t\mathbf{u}|^2dx\\
&= \frac{1}{2}\int_{\mathbb{R}^3}\partial_t\rho|\partial_t\mathbf{u}|^2dx - \int_{\mathbb{R}^3}\partial_t\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\partial_t\mathbf{u}dx\\
&\quad - \int_{\mathbb{R}^3}\rho\partial_t\mathbf{u}\cdot\nabla\mathbf{u}\cdot\partial_t\mathbf{u}dx - \int_{\mathbb{R}^3}\rho\mathbf{u}\cdot\nabla\partial_t\mathbf{u}\cdot\partial_t\mathbf{u}dx - \int_{\mathbb{R}^3}\nabla\partial_t P\partial_t\mathbf{u}dx\\
&\quad - \int_{\mathbb{R}^3}\partial_t(\rho EE^\top)\nabla\partial_t\mathbf{u}dx - \int_{\mathbb{R}^3}\partial_t(\rho E)\nabla\partial_t\mathbf{u}dx\\
&:= \sum_{j=1}^7 J_j,
\end{aligned}\tag{7.35}$$

where using (7.34), we can control J_j $(j = 1...7)$ as follows:

$$|J_1| = \left|\int_{\mathbb{R}^3}\nabla\rho\mathbf{u}|\partial_t\mathbf{u}|^2dx\right| \le \|\partial_t\mathbf{u}\|^2_{L^6}\|\nabla\rho\|_{L^3}\|\mathbf{u}\|_{L^3} \le CR^2\|\nabla\partial_t\mathbf{u}\|^2_{L^2};$$

$$\begin{aligned}|J_2| &\le \|\partial_t\mathbf{u}\|_{L^6}\|\nabla\rho\|_{L^3}\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^6} \le R\|\nabla\partial_t\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2\|\Delta\mathbf{u}\|_{L^2}\\ &\le R^2\|\nabla\partial_t\mathbf{u}\|_{L^2}^2 + R^3\|\Delta\mathbf{u}\|_{L^2}^2;\end{aligned}$$

$$|J_3| \le \|\rho\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^{\frac{3}{2}}}\|\partial_t\mathbf{u}\|_{L^6}^2 \le \|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\partial_t\mathbf{u}\|_{L^2}^2 \le CR^{\frac74}\|\nabla\partial_t\mathbf{u}\|_{L^2}^2;$$

$$|J_4| \le \|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^3}\|\nabla\partial_t\mathbf{u}\|_{L^2}\|\partial_t\mathbf{u}\|_{L^6} \le CR\|\nabla\partial_t\mathbf{u}\|_{L^2}^2;$$

$$\begin{aligned}|J_5| &\le C\|\partial_t\rho\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2} \le C\|\nabla\rho\mathbf{u}\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\\ &\le C\|\nabla\rho\|_{L^3}\|\mathbf{u}\|_{L^6}\|\nabla\partial_t\mathbf{u}\|_{L^2} \le C\|\nabla\rho\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\\ &\le \frac{\mu}{4}\|\nabla\partial_t\mathbf{u}\|_{L^2}^2 + CR^2\|\nabla\mathbf{u}\|_{L^2}^2;\end{aligned}$$

$$\begin{aligned}|J_6| &\le \|\partial_t\rho\|_{L^2}\|E\|_{L^\infty}^2\|\nabla\partial_t\mathbf{u}\|_{L^2} + \|\rho\|_{L^\infty}\|E\|_{L^\infty}\|\partial_tE\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\\ &\le R^2\|\nabla\rho\|_{L^3}\|\mathbf{u}\|_{L^6}\|\nabla\partial_t\mathbf{u}\|_{L^2} + R\|\nabla\mathbf{u}E - \mathbf{u}\cdot\nabla E + \nabla\mathbf{u}\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\\ &\le R^2\|\nabla\rho\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\\ &\quad + R\left(\|\nabla\mathbf{u}\|_{L^3}\|E\|_{L^6} + \|\nabla E\|_{L^3}\|\mathbf{u}\|_{L^6} + \|\nabla\mathbf{u}\|_{L^2}\right)\|\nabla\partial_t\mathbf{u}\|_{L^2}\\ &\le R^6\|\nabla\mathbf{u}\|_{L^2}^2 + \frac{\mu}{4}\|\nabla\partial_t\mathbf{u}\|_{L^2}^2 + R^2(R^2\|\nabla\mathbf{u}\|_{L^3}^2 + \|\nabla\mathbf{u}\|_{L^2}^2);\end{aligned}$$

and

$$\begin{aligned}|J_7| &\le \|\rho\|_{L^\infty}\|\partial_tE\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2} + \|E\|_{L^\infty}\|\partial_t\rho\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\\ &\le \frac{\mu}{4}\|\nabla\partial_t\mathbf{u}\|_{L^2}^2 + R^2\|\nabla\mathbf{u}\|_{L^3}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 + R^2\|\nabla\rho\mathbf{u}\|_{L^2}^2\\ &\le \frac{\mu}{4}\|\nabla\partial_t\mathbf{u}\|_{L^2}^2 + R^2\|\nabla\mathbf{u}\|_{L^3}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 + R^4\|\nabla\mathbf{u}\|_{L^2}^2.\end{aligned}$$

We remark that in the above estimates, we used several times the interpolation inequality:

$$\|f\|_{W^{2,3}(\mathbb{R}^3)} \le \|f\|_{W^{2,2}(\mathbb{R}^3)}^\theta\|f\|_{W^{2,q}(\mathbb{R}^3)}^{1-\theta}$$

for some $\theta\in(0,1)$. These estimates and (7.35) imply that, for R sufficiently small,

$$\frac12\frac{d}{dt}\int_{\mathbb{R}^3}\rho|\partial_t\mathbf{u}|^2dx + \frac{\mu}{8}\int_{\mathbb{R}^3}|\nabla\partial_t\mathbf{u}|^2dx \le R^3\|\Delta\mathbf{u}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2 + R^2\|\nabla\mathbf{u}\|_{L^3}^2. \tag{7.36}$$

Integrating (7.36) over $(0,T)$, we obtain that, using (7.1),

$$\|\nabla\partial_t\mathbf{u}\|_{L^2(Q_T)} \le CR^{\frac32}. \tag{7.37}$$

Here we used the estimate

$$\|\rho_0\partial_t\mathbf{u}(0)\|_{L^2} \le C\|\mathbf{u}_0\cdot\nabla\mathbf{u}_0\|_{L^2} + \|\Delta\mathbf{u}_0\|_{L^2} + \|\nabla\rho_0\|_{L^2} + \|\nabla E_0\|_{L^2} \le \delta^4$$

by letting $t=0$ in (1.1b). Thus, by (7.33), (7.37) and the Gagliardo-Nirenberg inequality, we obtain

$$\|\partial_t\mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))} \le \|\partial_t\mathbf{u}\|_{L^2(Q_T)}^\theta\|\nabla\partial_t\mathbf{u}\|_{L^2(Q_T)}^{1-\theta} \le CR^{\frac32},$$

for some $\theta\in(\frac12,1]$. The proof of Lemma 7.6 is complete. □

With (7.30) in hand, we can now get the estimate for $\|\nabla E\|_{L^2(0,T;L^q(\mathbb{R}^3))}$.

Lemma 7.7.

$$\|\nabla E\|_{L^2(0,T;L^q(\mathbb{R}^3))} \le CR^{\frac32}, \tag{7.38}$$

for any $T\in(0,T_{\max})$.

Proof. Substituting the following two facts

$$\partial_t \mathrm{div} E = \mathrm{div}(-\mathbf{u}\cdot\nabla E + \nabla\mathbf{u}E) + \Delta\mathbf{u},$$

and

$$\mathrm{div}(\rho E) = \mathrm{div}((\rho-1)E) + \mathrm{div}E,$$

into (1.1b), multiplying the resulting equation by $|\mathrm{div}E|^{q-2}\mathrm{div}E$ and integrating it over $\mathbb{R}^3$, we can obtain

$$\begin{aligned}
&\frac{\mu}{q}\frac{d}{dt}\|\mathrm{div}E\|_{L^q}^q + \|\mathrm{div}E\|_{L^q}^q\\
&\le \left|\int_{\mathbb{R}^3}\rho\partial_t\mathbf{u}|\mathrm{div}E|^{q-2}\mathrm{div}E dx\right| + \left|\int_{\mathbb{R}^3}\rho\mathbf{u}\nabla\mathbf{u}|\mathrm{div}E|^{q-2}\mathrm{div}E dx\right|\\
&\quad + \left|\int_{\mathbb{R}^3}\nabla P|\mathrm{div}E|^{q-2}\mathrm{div}E dx\right| + \left|\int_{\mathbb{R}^3}\mathrm{div}(\rho EE^\top)|\mathrm{div}E|^{q-2}\mathrm{div}E dx\right|\\
&\quad + \left|\int_{\mathbb{R}^3}\mathrm{div}((\rho-1)E)|\mathrm{div}E|^{q-2}\mathrm{div}E dx\right|\\
&\quad + \left|\int_{\mathbb{R}^3}\mathrm{div}(\nabla\mathbf{u}E - \mathbf{u}\cdot\nabla E)|\mathrm{div}E|^{q-2}\mathrm{div}E dx\right|\\
&:= \sum_{m=1}^{6} M_m,
\end{aligned} \tag{7.39}$$

where

$$M_1 \le \|\rho\|_{L^\infty}\|\partial_t\mathbf{u}\|_{L^q}\|\mathrm{div}E\|_{L^q}^{q-1};$$

$$M_2 \le \|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^q}\|\nabla\mathbf{u}\|_{L^\infty}\|\mathrm{div}E\|_{L^q}^{q-1} \le R\|\mathbf{u}\|_{W^{2,q}(\mathbb{R}^3)}\|\mathrm{div}E\|_{L^q}^{q-1};$$

$$M_3 \le C\|\nabla\rho\|_{L^q}\|\mathrm{div}E\|_{L^q}^{q-1};$$

$$\begin{aligned}
M_4 &\le \|\nabla\rho\|_{L^q}\|E\|_{L^\infty}^2\|\mathrm{div}E\|_{L^q}^{q-1} + \|\rho\|_{L^\infty}\|E\|_{L^\infty}\|\nabla E\|_{L^q}\|\mathrm{div}E\|_{L^q}^{q-1}\\
&\le (R^2\|\nabla\rho\|_{L^q} + R\|\nabla E\|_{L^q})\|\mathrm{div}E\|_{L^q}^{q-1};
\end{aligned}$$

$$\begin{aligned}
M_5 &\le \|\rho-1\|_{L^\infty}\|\nabla E\|_{L^q}\|\mathrm{div}E\|_{L^q}^{q-1} + \|\nabla\rho\|_{L^q}\|E\|_{L^\infty}\|\mathrm{div}E\|_{L^q}^{q-1}\\
&\le R(\|\nabla E\|_{L^q} + \|\nabla\rho\|_{L^q})\|\mathrm{div}E\|_{L^q}^{q-1};
\end{aligned}$$

$$M_6 \le (\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla E\|_{L^q} + \|\Delta\mathbf{u}\|_{L^q}\|E\|_{L^\infty})\|\mathrm{div}E\|_{L^q}^{q-1} \le R\|\mathbf{u}\|_{W^{2,q}}\|\mathrm{div}E\|_{L^q}^{q-1}.$$

Here in M_6, we used $\mathrm{div}\mathbf{u} = 0$. With those estimates in hand, we multiply (7.39) by $|\mathrm{div}E|_{L^q}^{2-q}$ to deduce that, using Young's inequality,

$$\frac{\mu}{2}\frac{d}{dt}\|\mathrm{div}E\|_{L^q}^2 + \|\mathrm{div}E\|_{L^q}^2 \le C\|\partial_t\mathbf{u}\|_{L^q}^2 + R^2\|\mathbf{u}\|_{W^{2,q}}^2 + \|\nabla\rho\|_{L^q}^2 + R^2\|\nabla E\|_{L^q}^2. \tag{7.40}$$

On the other hand, we still have

$$\|\mathrm{curl}E\|_{L^q}^2 \le \|E\|_{L^\infty}^2\|\nabla E\|_{L^q}^2 \le CR^2\|\nabla E\|_{L^q}^2.$$

Hence, substituting this into (7.40), we get

$$\frac{\mu}{2}\frac{d}{dt}\|\mathrm{div}E\|_{L^q}^2 + \|\nabla E\|_{L^q}^2 \le C\|\partial_t\mathbf{u}\|_{L^q}^2 + R^2\|\mathbf{u}\|_{W^{2,q}}^2 + \|\nabla\rho\|_{L^q}^2 + CR^2\|\nabla E\|_{L^q}^2. \tag{7.41}$$

Integrating (7.41) over $(0,T)$ and using the estimates (7.27), (7.30), we obtain

$$\|\nabla E\|_{L^2(0,T;L^q(\mathbb{R}^3))} \le CR^{\frac{3}{2}}.$$

The proof of Lemma 7.7 is complete. □

7.3. **Proof of Theorem 3.2.** First we have the following estimate on $\|\mathbf{u}\|_{\mathcal{W}(0,T)}$:

Lemma 7.8.

$$\|\mathbf{u}\|_{\mathcal{W}(0,T)} < R, \tag{7.42}$$

for all $T \in [0, T_{\max}]$, if R is small enough.

Proof. First from Theorem 4.1, we have

$$\begin{aligned}
\|\mathbf{u}\|_{\mathcal{W}(0,T)} &= \|\mathcal{H}(\mathbf{u})\|_{\mathcal{W}(0,T)} \\
&\le C(q)\Big(\|\mathbf{u}_0\|_{V_0^{2,q}} + \|(1-\mathcal{S}(\mathbf{u}))\partial_t \mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))} \\
&\qquad + \|\mathcal{S}(\mathbf{u})(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))} + \|\nabla P(\mathcal{S}(\mathbf{u}))\|_{L^2(0,T;L^q(\mathbb{R}^3))} \\
&\qquad + \|\operatorname{div}(\mathcal{S}(\mathbf{u})(I+\mathcal{T}(\mathbf{u}))(I+\mathcal{T}(\mathbf{u}))^\top)\|_{L^2(0,T;L^q(\mathbb{R}^3))}\Big).
\end{aligned}$$

From the previous computations in Lemma 7.2, we have

$$\begin{aligned}
\|(1-\mathcal{S}(\mathbf{u}))\partial_t \mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))} &\le \|\mathcal{S}(\mathbf{u}) - 1\|_{L^\infty(Q_T)}\|\mathbf{u}\|_{\mathcal{W}(0,T)} \\
&\le C\|\mathcal{S}(\mathbf{u}) - 1\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3))}\|\mathbf{u}\|_{\mathcal{W}(0,T)} \\
&\le C(q)R^{\frac{3}{2}}.
\end{aligned}$$

Similarly, one has

$$\|\mathcal{S}(\mathbf{u})(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))} \le \|\mathcal{S}(\mathbf{u})\|_{L^\infty(Q_T)}\|\mathbf{u}\|^2_{\mathcal{W}(0,T)} \le C(q)R^2;$$

$$\|\nabla P(\mathcal{S}(\mathbf{u}))\|_{L^2(0,T;L^q(\mathbb{R}^3))} = \|P'(\mathcal{S}(\mathbf{u}))\nabla\rho\|_{L^2(0,T;L^q(\mathbb{R}^3))} \le CR^2;$$

and

$$\begin{aligned}
&\|\operatorname{div}(\mathcal{S}(\mathbf{u})(I+\mathcal{T}(\mathbf{u}))(I+\mathcal{T}(\mathbf{u}))^\top)\|_{L^2(0,T;L^q(\mathbb{R}^3))} \\
&\le \|\nabla\rho\|_{L^2(0,T;L^q(\mathbb{R}^3))}\left(1 + \|E\|^2_{L^\infty(Q_T)}\right) \\
&\quad + \|\nabla E\|_{L^2(0,T;L^q(\mathbb{R}^3))}\|\rho\|_{L^\infty(Q_T)}\left(1 + \|E\|_{L^\infty(Q_T)}\right) \\
&\le C(q)R^{\frac{3}{2}}.
\end{aligned}$$

Thus, we obtain,

$$\|\mathbf{u}\|_{\mathcal{W}(0,T)} \le C(q)(R^2 + R^{\frac{3}{2}}) < R, \quad \text{for all} \quad T \in [0, T_{\max}],$$

if R is small enough. The proof of Lemma 7.8 is complete. □

Finally we are in the position to give the proof of Theorem 3.2 as follows.

Let $T_n \nearrow T_{\max}$ be an increasing sequence with limit equal to $T_{\max}$. Then, it is necessary to have, according to Theorem 4.2,

$$\sup_{n\in N} \|\mathbf{u}(T_n)\|_{X_p^{2(1-\frac{1}{p})}} \le \frac{C(q)}{c_0}R,$$

which, combining with the definition of $T_{\max}$, implies

$$\|(\mathbf{u}(T_n), \mathcal{S}(\mathbf{u})(T_n) - 1, \mathcal{T}(\mathbf{u})(T_n))\|_{V_0^{p,q}} \leq C(q, c_0)R.$$

Therefore, if we take each triple $(\mathbf{u}(T_n), \mathcal{S}(\mathbf{u})(T_n) - 1, \mathcal{T}(\mathbf{u})(T_n))$ as a new initial condition, then according to Theorem 3.1, we know the solution can be extended to the interval $(0, T_n + T_0)$, for some $T_0 > 0$. Hence, the solution can be extended to the interval $(0, T_{\max} + T_0)$, which, together with (7.25) and (7.26), is a contradiction with the maximality of $T_{\max}$. So it cannot be true that $T_{\max} < \infty$, but $T_{\max} = \infty$, i.e. $\mathbf{u}$, $\mathcal{S}(\mathbf{u})$, and $\mathcal{T}(\mathbf{u})$ are well-defined on $\mathbb{R}^3 \times (0, \infty)$. The proof of Theorem 3.2 is complete.

Acknowledgments

Xianpeng Hu's research was supported in part by the National Science Foundation grant DMS-0604362 and by the Mellon Predoctoral Fellowship of the University of Pittsburgh. Dehua Wang's research was supported in part by the National Science Foundation under Grant DMS-0604362, and by the Office of Naval Research under Grant N00014-07-1-0668.

References

[1] Admas, R. A., *Sobolev spaces.* Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.
[2] Amann, H., *Linear and quasilinear parabolic problems.* Vol. I. Abstract linear theory. Birkhúser Boston, Inc., Boston, 1995.
[3] Bergh, J.; Löfström, J., *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin-New York, 1976.
[4] Bird, R. B.; Armstrong, R. C.; Hassager, O., *Dynamics of polymeric liquids. Vol. 1. Fluid mechanics.* Wiley Interscience, New York, 1987.
[5] Chemin, J.; Masmoudi, N., *About lifespan of regular solutions of equations related to viscoelastic fluids.* SIAM J. Math. Anal. 33 (2001), 84–112.
[6] Chen, Y.; Zhang, P., *The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions.* Comm. Partial Differential Equations 31 (2006), 1793–1810.
[7] Dafermos, C. M., *Hyperbolic conservation laws in continuum physics.* Second edition. Grundlehren der Mathematischen Wissenschaften, 325. Springer-Verlag, Berlin, 2005.
[8] Danchin, R., *Density-dependent incompressible fluids in bounded domains.* J. Math. Fluid Mech. 8 (2006), 333–381.
[9] DiPerna, R. J.; Lions, P.-L., *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math. 98 (1989), 511–547.
[10] Goktekin, G. B.; Bargteil, A. W.; O'Brien, J. F., *A method for animating viscoelastic fluids*, ACM Transactions on Graphics 23 (2004), 463-468.
[11] Gurtin, M. E., *An introduction to Continuum Mechanics.* Mathematics in Science and Engineering,158. Academic Press, New YorkLondon, 1981.
[12] Joseph, D., *Fluid dynamics of viscoelastic liquids.* Applied Mathematical Sciences, 84. Springer-Verlag, New York, 1990.
[13] Kunisch, K.; Marduel, M., *Optimal control of non-isothermal viscoelastic fluid flow*, J. Non-Newtonian Fluid Mechanics 88 (2000), 261-301.
[14] Larson, R. G., *The Structure and Rheology of Complex Fluids.* Oxford University Press, New York, 1995.
[15] Lei, Z.; Liu, C.; Zhou, Y., *Global existence for a 2D incompressible viscoelastic model with small strain.* Commun. Math. Sci. 5 (2007), 595–616.
[16] Lei, Z.; Liu, C.; Zhou, Y., *Global solutions for incompressible viscoelastic fluids.* Arch. Ration. Mech. Anal. 188 (2008), 371–398.

[17] Lei, Z.; Liu, C.; Zhou, Y., *Global solutions for compressible viscoelastic fluids with small initial data.* Preprint.

[18] Lei, Z.; Zhou, Y., *Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit.* SIAM J. Math. Anal. 37 (2005), 797–814.

[19] Lin, F; Liu, C.; Zhang, P., *On hydrodynamics of viscoelastic fluids.* Comm. Pure Appl. Math. 58 (2005), 1437–1471.

[20] Lin, F; Zhang, P., *On the initial-boundary value problem of the incompressible viscoelastic fluid system.* Comm. Pure Appl. Math. 61 (2008), 539–558.

[21] Lions, P. L., *Mathematical Topics in Fluid Mechanics. Vol. 1. Incompressible Models.* Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[22] Lions, P. L.; Masmoudi, N., *Global solutions for some Oldroyd models of non-Newtonian flows.* Chinese Ann. Math. Ser. B 21 (2000), 131–146.

[23] Liu, C.; Walkington, N. J., *An Eulerian description of fluids containing visco-elastic particles.* Arch. Ration. Mech. Anal. 159 (2001), 229–252.

[24] Matsumura, A.; Nishida, T., *The initial-value problem for the equations of motion of viscous and heat-conductive gases.* J. Math. Kyoto Univ. 20 (1980), 67–104.

[25] Matsumura, A.; Nishida, T., *Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids.* Comm. Math. Phys. 89 (1983), 445–464.

[26] Novotný, A.; Straškraba, I., *Introduction to the mathematical theory of compressible flow.* Oxford Lecture Series in Mathematics and its Applications, 27. Oxford University Press, Oxford, 2004.

[27] Oldroyd, J. G., *On the formation of rheological equations of state*, Proc. Roy. Soc. London, Series A 200 (1950), 523-541.

[28] Oldroyd, J. G., *Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids*, Proc. Roy. Soc. London, Series A 245 (1958), 278-297.

[29] Renardy, M.; Hrusa, W. J.; Nohel, J. A., *Mathematical Problems in Viscoelasticity.* Longman Scientic and Technical; copublished in the US with John Wiley, New York, 1987.

[30] Salvi, R.; Straškraba, I., *Global existence for viscous compressible fluids and their behavior as $t \to \infty$.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 40 (1993), 17–51.

[31] Sideris, T. C., *Nonlinear hyperbolic systems and elastodynamics.* Phase space analysis of partial differential equations. Vol. II, 451–485, Pubbl. Cent. Ric. Mat. Ennio Giorgi, Scuola Norm. Sup., Pisa, 2004.

[32] Sideris, T. C., *Personal notes on viscoelasticity*, 2007.

[33] Sideris, T. C.; Thomases, B., *Global existence for three-dimensional incompressible isotropic elastodynamics via the incompressible limit.* Comm. Pure Appl. Math. 58 (2005), 750–788.

[34] Slemrod, M., *Constitutive relations for Rivlin-Ericksen uids based on generalized rational ap- proximation.* Arch. Ration. Mech. Anal. 146 (1999), 73-93.

[35] Yu, J. D.; Sakai, S.; Sethian, J. A., *Two-phase viscoelastic jetting* , J. Computational Physics 220 (2007), 568-585.

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA.
E-mail address: `xih15@pitt.edu`

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA.
E-mail address: `dwang@math.pitt.edu`