# Chebychev Optimized Approximate Deconvolution Models of Turbulence

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**Abstract**

If the Navier-Stokes equations are averaged with a local, spacial convolution type filter, $\overline{\boldsymbol{\phi}} = g_\delta * \boldsymbol{\phi}$, the resulting system is not closed due to the filtered nonlinear term $\overline{\mathbf{u}\mathbf{u}}$. An approximate deconvolution operator $D$ is a bounded linear operator which satisfies

$$\mathbf{u} = D(\overline{\mathbf{u}}) + O(\delta^\alpha),$$

where $\delta$ is the filter width and $\alpha \geq 2$. Using a deconvolution operator as an approximate filter inverse, yields the closure

$$\overline{\mathbf{u}\mathbf{u}} = \overline{D(\overline{\mathbf{u}})D(\overline{\mathbf{u}})} + O(\delta^\alpha).$$

The residual stress of this model (and related models) depends directly on the deconvolution error, $\mathbf{u} - D(\overline{\mathbf{u}})$. This report derives deconvolution operators satisfying the necessary conditions of [24] yielding an effective turbulence model, which minimize the deconvolution error for velocity fields with finite kinetic energy. We also give a convergence theory of deconvolution as $\delta \to 0$, an ergodic theorem as the deconvolution order $N \to \infty$, and estimate the increase in accuracy obtained by parameter optimization. The report concludes with numerical illustrations.

## 1 Introduction

Various turbulence models are used for simulations seeking to predict flow statistics or averages. In large eddy simulation (LES) the evolution of local, spatial averages is sought. The accuracy of a model measured in a chosen norm, $||\cdot||$, i.e.

$$||\textit{averaged NSE solution} - \textit{LES solution}||,$$

can be assessed in several experimental and analytical ways. One important analytical approach is to optimize the model's *consistency error/residual stress* as a function of the

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averaging radius $\delta$ and the Reynolds number $Re$ and, most importantly, model parameters. One approach is to optimize model parameters for special flows, such as boundary layers or homogeneous, isotropic turbulence. The complement (considered herein) is to optimize over general velocity fields with finite kinetic energy. We analyze the residual stress in the model and give an analytic and numerical comparison of the deconvolution error of two different optimization strategies: for special vs. general velocities.

Numerical simulation of complex flows present many challenges. Often, simulations are based on various regularizations of the Navier-Stokes equations (NSE) rather than the NSE themselves, [10], [12], [22]. The oldest example was proposed by Leray in 1934, [19]:

$$\mathbf{v}_t + \overline{\mathbf{v}} \cdot \nabla \mathbf{v} - \nu \triangle \mathbf{v} + \nabla p = \mathbf{f}, \text{ and } \nabla \cdot \mathbf{v} = 0, \tag{1.1}$$

where $\overline{\mathbf{v}} = G\mathbf{v}$ is a smoothed/averaged velocity. Herein, we select the differential filter of $\mathbf{v}$, introduced by Germano, [9], and given by $G = (-\delta^2 \triangle + I)^{-1}$, i.e.

$$-\delta^2 \triangle \overline{\mathbf{v}} + \overline{\mathbf{v}} = \mathbf{v}. \tag{1.2}$$

This combination is sometimes called the Leray-alpha regularization, [4], [5], [11]. The Leray regularization's solution is smoother, more stable, and possesses (marginally) fewer scales than the NSE's solution. Still, the resulting error, even with a high accuracy numerical method, cannot be better than the error committed in the first step, replacing $\mathbf{v}$ by $\overline{\mathbf{v}}$ in (1.1). From (1.2) the error is $\mathbf{v} - \overline{\mathbf{v}} = O(\delta^2)$ at best. Experiments in [15] have also shown that, due to its low accuracy, (1.1) with the filter (1.2) can have catastrophic error growth and not adequately conserve physically important integral invariants. The experiments in [15] also indicate that the increase in accuracy resulting from using deconvolution models (replacing (1.1) with (1.3)) decreases error growth and improves conservation properties.

Approximate deconvolution operators, $D : L^2(\Omega) \to L^2(\Omega)$, have the property that

$$D(\overline{\mathbf{v}}) = \text{higher order approximation of } \mathbf{v}.$$

The van Cittert deconvolution procedure (studied herein and defined precisely in Section 2) gives a family ($D = D_N$, where N=0,1,2,...) of deconvolution operators with accuracy

$$\mathbf{e}(\mathbf{u}) := \mathbf{u} - D_N(\overline{\mathbf{u}}) = O(\delta^{2N+2}), \text{ for smooth } \mathbf{u}.$$

More accurate regularization of the NSE, which surpass (1.1) and related models for numerical simulations include:

1. The Leray deconvolution family [14], [15]:

$$\mathbf{v}_t + D(\overline{\mathbf{v}}) \cdot \nabla \mathbf{v} - \nu \triangle \mathbf{v} + \nabla p = \mathbf{f} \text{ and } \nabla \cdot \mathbf{v} = 0. \tag{1.3}$$

2. The time relaxation regularization of Stolz, Adam, and Kleiser [23], [21], [16]:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \triangle \mathbf{v} + \nabla p + \chi (I - DG)^2 \mathbf{v} = \mathbf{f} \text{ and } \nabla \cdot \mathbf{v} = 0. \tag{1.4}$$

3. The deconvolution $\alpha$-regularization [20] (enhancing NS-$\alpha$ accuracy, e.g [4], [5], [11]):

$$\mathbf{v}_t + (\nabla \times \mathbf{v}) \times D(\overline{\mathbf{v}}) - \nu \triangle \mathbf{v} + \nabla P = \mathbf{f} \text{ and } \nabla \cdot \mathbf{v} = 0. \tag{1.5}$$

4. The Approximate Deconvolution LES Models [21], [3], [13], [7]:

$$\mathbf{v}_t + \overline{D(\mathbf{v}) \cdot \nabla D(\mathbf{v})} - \nu \triangle \mathbf{v} + \nabla p + \chi (I - DG)\mathbf{v} = \mathbf{f} \text{ and } \nabla \cdot \mathbf{v} = 0. \tag{1.6}$$

5. The NS-omega deconvolution models [18]:

$$\mathbf{v}_t + \mathbf{v} \times \nabla D(\overline{\mathbf{v}}) - \nu \triangle \mathbf{v} + \nabla P = \mathbf{f} \text{ and } \nabla \cdot \mathbf{v} = 0. \tag{1.7}$$

For all these (and others as well) the modelling error is dominated by the *deconvolution error*

$$\mathbf{e}(\mathbf{u}) := \mathbf{u} - D(\overline{\mathbf{u}}).$$

This report considers minimizations of the deconvolution error for general (non-smooth) velocity fields $\mathbf{u}$. Since these (and other) models exist only to be used on a basis for numerical simulations of under-resolved flows, we minimize the deconvolution error over the resolved scales (i.e. over the scales that can be represented on a computational mesh). We begin by reviewing the van Cittert deconvolution operator, in Section 2, and give, for completeness, a convergent result as $\delta \to 0$ for fixed $N$ (standard). Section 2 also considers convergence as $N \to \infty$ for fixed $\delta$, a highly singular limit since van Cittert is an asymptotic rather than convergent approximation. We prove an ergodic theorem for the deconvolution iterates for a general filter: the large scales of the averages of iterates converge as $N \to \infty$. In Sections 3 and 4, we show how to optimize the van Cittert procedure to substantially increase its accuracy with no increase in computational cost. Section 3 reduces optimization to a Chebychev optimization problem. From this reduction we recover the optimal van Cittert parameters and show that the model's error is $O(\delta^{2/3} \mathrm{e}^{-1.24N})$, Section 4. Section 1.2 below considers, as an example, one of the above regularizations and gives the analysis of the model error in terms of the deconvolution error (addressed in Section 3). Finally, Section 5 closes with a few illustrations of the optimized method.

### 1.1 The formulation

Underlying all regularizations (1.3) - (1.7) are the true Navier-Stokes equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \triangle \mathbf{u} + \nabla p = \mathbf{f} \text{ and } \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \tag{1.8}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $\mathbf{f}$ is the body force, and $\Omega = (0, L)^n$ ($n = 2$ or 3) is the flow domain. We consider the case of $L$-periodic boundary conditions

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j, t) = \mathbf{u}(\mathbf{x}, t), \; j = 1, ..., n.$$

The Navier-Stokes equations are supplemented by the initial condition, the usual normalization condition in the periodic case of zero mean velocity and pressure, and the assumption that all data are square integrable with zero mean

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}) \text{ and } \int_Q \mathbf{u}\, d\mathbf{x}=\int_Q p\, d\mathbf{x}=0,$$
$$\int_Q |\mathbf{u}_0(\mathbf{x},t)|^2 d\mathbf{x}<\infty,\ \int_Q |\mathbf{f}(\mathbf{x},t)|^2 d\mathbf{x}<\infty,\ \text{and } \int_Q \mathbf{f}(\mathbf{x},t)d\mathbf{x}=0,\ \text{for } 0\le t. \tag{1.9}$$

## 1.2 The connection between deconvolution error and model error

Consider, as an example, the time relaxation regularization (1.4). The true NSE can be rewritten as $\nabla\cdot\mathbf{u}=0$ and

$$\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}-\nu\triangle\mathbf{u}+\nabla p+\chi(I-DG)^2\mathbf{u}=\mathbf{f}+\chi(I-DG)^2\mathbf{u}. \tag{1.10}$$

An equation for the model error,

$$\mathbf{e}_{model}=\mathbf{u}_{NSE}-\mathbf{v}_{model}, \tag{1.11}$$

is driven by the deconvolution error, $\chi(I-DG)^2\mathbf{u}$ and is obtained by subtracting the model (1.4) from (1.10)

$$\begin{aligned}&\frac{\partial}{\partial t}\mathbf{e}_{model}+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{v}\cdot\nabla\mathbf{v}-\nu\triangle\mathbf{e}_{model}\\ &\qquad+\nabla(p-p_{model})+\chi(I-DG)^2\mathbf{e}_{model}=\chi(I-DG)^2\mathbf{u}\\ &\mathbf{e}_{model}(0)=0.\end{aligned} \tag{1.12}$$

From (1.12) it is clear that zero deconvolution error trivially implies zero model error. It is thus reasonable to hope that *small* deconvolution error (i.e. small $\left\|\chi(I-DG)^2\mathbf{u}\right\|$ on the RHS) translates to small model error. For strong solutions this is indeed the case.

**Proposition 1.1.** *Consider the NSE with periodic boundary conditions. If $\nabla\mathbf{u}\in L^4(0,T;L^2(\Omega))$, then the error in the Time Relaxation Regularization Model (1.4) satisfies*

$$\begin{aligned}&\sup_{[0,T]}\|\mathbf{e}_{model}\|^2+\int_0^T\left(\nu\|\nabla\mathbf{e}_{model}\|^2+\chi\left\|(I-DG)^2\mathbf{e}_{model}\right\|^2\right)\,dt\\ &\qquad\le e^{C(\mathbf{u})\nu^{-3}T}\int_0^T\chi\left\|(I-DG)^2\mathbf{u}\right\|^2\,dt.\end{aligned}$$

*Proof.* Taking the inner product of (1.12) with $\mathbf{e}_{model}$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{e}_{model}\|^2+(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{v}\cdot\nabla\mathbf{v},\mathbf{e}_{model})+\nu\|\nabla\mathbf{e}_{model}\|^2+\chi\|\mathbf{e}_{model}-D(\overline{\mathbf{e}}_{model})\|^2\\ =\chi(\mathbf{u}-D(\overline{\mathbf{u}}),\mathbf{e}_{model}-D(\overline{\mathbf{e}}_{model})).\end{aligned} \tag{1.13}$$

The standard splitting

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{v}\cdot\nabla\mathbf{v},\mathbf{e}_{model}) &= (\mathbf{e}_{model}\cdot\nabla\mathbf{u},\mathbf{e}_{model})+(\mathbf{v}\cdot\nabla\mathbf{e}_{model},\mathbf{e}_{model}) \\
&= (\mathbf{e}_{model}\cdot\nabla\mathbf{u},\mathbf{e}_{model}) \qquad (1.14)
\end{aligned}
$$

and the Cauchy Schwarz inequality give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{e}_{model}\|^2+\nu\|\nabla\mathbf{e}_{model}\|^2+\frac{\chi}{2}\|\mathbf{e}_{model}-D(\overline{\mathbf{e}}_{model})\|^2 \\
&\qquad\le -(\mathbf{e}_{model}\cdot\nabla\mathbf{u},\mathbf{e}_{model})+\frac{\chi}{2}\|\mathbf{u}-D(\overline{\mathbf{u}})\|^2. \qquad (1.15)
\end{aligned}
$$

We have

$$
\begin{aligned}
|(\mathbf{e}_{model}\cdot\nabla\mathbf{u},\mathbf{e}_{model})| &\le \|\mathbf{e}_{model}\|^{1/2}\|\nabla\mathbf{e}_{model}\|^{3/2}\|\nabla\mathbf{u}\| \\
&\le \frac{\nu}{2}\|\nabla\mathbf{e}_{model}\|^2+C\nu^{-3}\|\nabla\mathbf{u}\|^4\|\mathbf{e}_{model}\|^2.
\end{aligned}
$$

Using this in the RHS of (1.15) and then applying Gronwall's inequality we deduce

$$
\begin{aligned}
&\sup_{[0,T]}\|\mathbf{e}_{model}\|^2+\int_0^T\left(\nu\|\nabla\mathbf{e}_{model}\|^2+\chi\left\|(I-DG)^2\mathbf{e}_{model}\right\|^2\right)\,dt \\
&\qquad\le e^{C(\|\nabla\mathbf{u}\|)\nu^{-3}T}\int_0^T\chi\left\|(I-DG)^2\mathbf{u}\right\|^2\,dt.
\end{aligned}
$$

□

The model's error is bounded by the deconvolution error $\mathbf{e}=\mathbf{u}-D(\overline{\mathbf{u}})$ evaluated at the true solution of the NSE. Since analogous bounds can be proven for the regularizations and models (1.3) through (1.7) we consider, we turn to minimizing the deconvolution error.

## 2 Approximate Deconvolution Methods.

The basic problem in deconvolution is to find $\mathbf{u}$ from $\overline{\mathbf{u}}$, in other words:

$$
\text{given } \mathbf{u}\ (\,+\,noise) \text{ solve } G\mathbf{u}=\overline{\mathbf{u}},\ \text{ for } \mathbf{u}. \qquad (2.1)
$$

If the averaging operator is smoothing, the deconvolution problem will be not stably invertible due to small divisor problems.

**Definition 2.1.** *An approximate deconvolution operator, $D: L^2(\Omega)\to L^2(\Omega)$ is an approximate inverse of $G$ satisfying:*
*(i) $D: L^2(\Omega)\to L^2(\Omega)$ is a bounded linear operator and*
*(ii)$D(\overline{\boldsymbol{\phi}})=\boldsymbol{\phi}+O(\delta^\alpha)$, for some $\alpha\ge 2$ and sufficiently smooth $\boldsymbol{\phi}$.*

This section considers the van Cittert approximate deconvolution algorithm, [2]. The approximation $D_N(\overline{\mathbf{u}})$, for the operator equation (2.1), is computed by $N$ steps of first order Richardson iteration. Each step of van Cittert requires only one filtering step.

**Algorithm 2.1.** [*The van Cittert Algorithm*]: *Choose*

$$\mathbf{u}_0 = \overline{\mathbf{u}}.$$

*For* $n = 0, 1, 2, ..., N-1$*, perform*

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \{\overline{\mathbf{u}} - G\mathbf{u}_n\}.$$

*Set* $D_N(\overline{\mathbf{u}}) := \mathbf{u}_N$.

For example, for N=0, 1, and 2 the deconvolution operator $D_N$ is

$$\begin{aligned} D_0(\overline{\mathbf{u}}) &= \overline{\mathbf{u}}, \quad \overline{\mathbf{u}} \simeq D_0(\overline{\mathbf{u}}) + O(\delta^2), \\ D_1(\overline{\mathbf{u}}) &= 2\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}}, \quad \overline{\mathbf{u}} \simeq D_1(\overline{\mathbf{u}}) + O(\delta^4), \\ D_2(\overline{\mathbf{u}}) &= 3\overline{\mathbf{u}} - 3\overline{\overline{\mathbf{u}}} + \overline{\overline{\overline{\mathbf{u}}}}, \quad \overline{\mathbf{u}} \simeq D_2(\overline{\mathbf{u}}) + O(\delta^6). \end{aligned}$$

For the Cauchy problem, $\Omega = \mathbb{R}^n$, the transfer function of $D_N$ (for N = 0, 1, 2) is

$$\widehat{D}_0(k) = 1, \quad \widehat{D}_1(k) = 2 - \frac{1}{k^2+1} = \frac{2k^2+1}{k^2+1} \quad \text{and} \quad \widehat{D}_2(k) = 1 + \frac{1}{k^2+1} + \left(\frac{k^2}{k^2+1}\right)^2.$$

These three and the transfer function of exact deconvolution $(k^2+1)$ are plotted in Figure 1. The graphs of the transfer functions have high order contact near 0. Thus $D_N$ leads to a very accurate solution of the deconvolution problem.

There are two convergence issues that arise immediately:

1. Convergence as $\delta \to 0$ for fixed $N$ and general $\mathbf{u} \in L^2(\Omega)$ (see Theorem 2.2).
2. Convergence as $N \to \infty$ for $\delta$ fixed (possibly true for some specific filters, but likely not true in general, see Theorem 2.3 and [13]).

**Theorem 2.2.** [*Convergence as* $\delta \to 0$ *for general velocities* ] *Suppose that* $\overline{\mathbf{u}} \to \mathbf{u}$ *in* $L^2(\Omega)$ *as* $\delta \to 0$*, for all* $\mathbf{u} \in L^2(\Omega)$*. Then, for* $N$ *fixed we have* $D_N(\overline{\mathbf{u}}) \to \mathbf{u}$ *in* $L^2(\Omega)$ *as* $\delta \to 0$*.*

*Proof.* As $\delta \to 0$, we have $\mathbf{u}_0 = \overline{\mathbf{u}} \to \mathbf{u}$ and thus

$$\mathbf{u}_1 = \mathbf{u}_0 + \{\overline{\mathbf{u}} - \overline{\mathbf{u}}_0\} \to \mathbf{u} + \{\mathbf{u} - \mathbf{u}\} = \mathbf{u}.$$

Similarly, each $\mathbf{u}_n \to \mathbf{u}$ and $D_N(\overline{\mathbf{u}}) \to \mathbf{u}$. □

Since the deconvolution problem is ill posed, convergence of $D_N(\overline{\mathbf{u}})$ to $\mathbf{u}$ as $N \to \infty$ is not expected. Nevertheless, it is possible to prove a type of ergodic theorem for averages predicted by the van Cittert algorithm for very general operators $G$. Sharper convergence theorems depend upon specific choices of the averaging operator $G$, see [14] for an example.

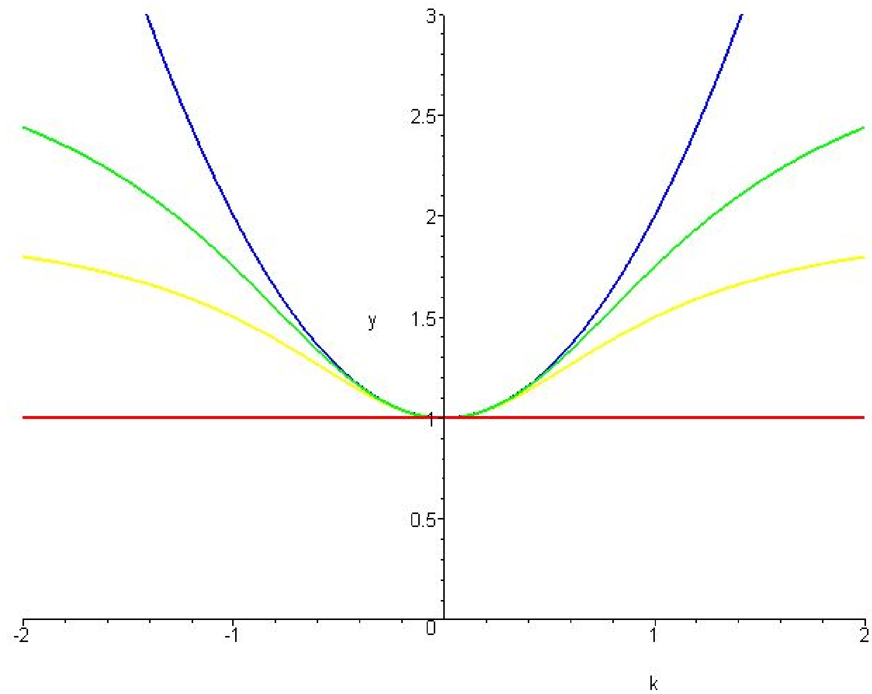


Figure 1: Exact and van Cittert Approximate Deconvolution Operators (N=0,1,2)

**Theorem 2.3.** [*Convergence as* $N \to \infty$] *Let* $X$ *be a Hilbert space and suppose that the averaging operator* $G : X \to X$ *is a bounded linear operator with* $\|I - G\|_{\mathcal{L}(X \to X)} \leq 1$*. For* $\mathbf{u} \in X$*, consider the van Cittert iteration*

$$\mathbf{u}_0 = G\mathbf{u}, \quad \mathbf{u}_{n+1} = \mathbf{u}_n + \{G\mathbf{u} - G\mathbf{u}_n\}.$$

*Let*

$$\mathbf{v}_N = \frac{\mathbf{u}_0 + \mathbf{u}_1 + ... + \mathbf{u}_N}{N+1}.$$

*Then* $G^2(\mathbf{u} - \mathbf{v}_N) \to 0$ *in* $X$ *as* $N \to \infty$*, specifically,*

$$\sup_{\mathbf{u} \in X} \frac{||G^2(\mathbf{u} - \mathbf{v}_N)||}{||G\mathbf{u}||} \quad \leq \quad \frac{2}{N+1}.$$

*Proof.* Let $B = I - G$. Then,

$$G\mathbf{u}_N = G\mathbf{u} - B^{N+1}G\mathbf{u} \;\text{ or }\; G\mathbf{e}_N = B^{N+1}G\mathbf{u},$$

where $\mathbf{e}_N = \mathbf{u} - \mathbf{u}_N = \mathbf{u} - D_N(\overline{\mathbf{u}})$. Consider $G(\mathbf{u} - \mathbf{v}_N)$ . A similar algebraic calculation gives

$$\begin{aligned} G^2(\mathbf{u}-\mathbf{v}_N) &= \frac{1}{N+1}G^2(\mathbf{e}_0+\mathbf{e}_1+\cdots+\mathbf{e}_N) = \\ &= \frac{1}{N+1}G\sum_{n=0}^{N}B^{n+1}G\mathbf{u} = \\ &= \frac{1}{N+1}\{BG\mathbf{u}-B^{N+2}G\mathbf{u}\} \to 0. \end{aligned}$$

as $N \to \infty$, since $\|B\| \leq 1$. Taking norms of both sides completes the proof. □

For LES, convergence of the van Cittert approximation $D_N(\overline{\mathbf{u}})$ to $\mathbf{u}$ as $N \to \infty$ is not as significant as convergence of $D_N(\overline{\mathbf{u}})$ to $\mathbf{u}$ as $\delta \to 0$ and the asymptotic order of accuracy as $\delta \to 0$ for fixed $N$. When the averaging is given by a differential filter, the accuracy of $D_N(\overline{\mathbf{u}})$ as an approximation to $\mathbf{u}$ for smooth functions was addressed by Stolz and Adams [21], Dunca [6], Dunca and Epshteyn [7].

**Lemma 2.1.** *Let the averaging operator be given by the differential filter* $G\mathbf{v} := (-\delta^2\triangle + I)^{-1}\mathbf{v}$. *For any* $\mathbf{v} \in L^2(\Omega)$,

$$\mathbf{v} - D_N(\overline{\mathbf{v}}) = (-1)^{N+1}\delta^{2N+2}\triangle^{N+1}G^{N+1}\mathbf{v}.$$

*Thus, if* $\triangle^{2N+2}\mathbf{v} \in L^2(\Omega)$ *we have*

$$\|\mathbf{v} - D_N(\overline{\mathbf{v}})\| \leq \delta^{2N+2}\left\|\triangle^{2N+2}\mathbf{v}\right\|.$$

*Proof.* For the proof, see for example [7]. □

The use of van Cittert as an asymptotic, rather than iterative, approximation of an ill-posed, rather than non-singular, linear problem as well as the associated convergence theory is very different than that of first order Richardson. However, the form of the iteration is the same. Exploiting this algorithmic similarity, relaxation parameters can be introduced at *no additional computational cost.* We shall optimize these parameters, for deconvolution of fluid velocities, in Section 3. In Algorithm 2.1 it is also clear that, at no extra cost, the parameters can be chosen to have different values in different regions. In fact, we expect different optimal values near walls (still an open problem), away from walls in free turbulence (in [17]), and for general velocities (considered herein).

**Algorithm 2.4.** [*Accelerated van Cittert Algorithm*]*: Given relaxation parameters* $\omega_n$, *choose*

$$\mathbf{u}_0 = \overline{\mathbf{u}}.$$

*For* $n = 0, 1, 2, ..., N-1$ *perform*

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \omega_n\{\overline{\mathbf{u}} - G\mathbf{u}_n\}.$$

*Set* $D_N^{\omega}(\overline{\mathbf{u}}) := \mathbf{u}_N$.

The operator $D_N^\omega$ is the *Accelerated van Cittert* deconvolution operator. The key to optimization is the following recursion formula for the operator $D_N^\omega$.

**Lemma 2.2.** *For N=0, 1, 2, ... , we have:*

$$D_{N+1}^\omega = D_N^\omega + \omega_N(I - D_N^\omega G). \tag{2.2}$$

*Proof.* First, we note that $D_0^\omega = I$ on $L^2(\Omega)$ and $D_1^\omega$ is a linear combination of the identity and $G$. It follows that $D_1^\omega$ commutes with differentiation, since both $I$ and $G$ do. Using an induction argument we deduce that $D_N^\omega$ commutes with $G$ for N=0,1,2,... . Furthermore, for any positive integer N we have

$$\begin{aligned} D_{N+1}^\omega(\overline{\mathbf{u}}) = \mathbf{u}_{N+1} &= \mathbf{u}_N + \omega_N\{\overline{\mathbf{u}} - G\mathbf{u}_N\} \\ &= D_N^\omega(\overline{\mathbf{u}}) + \omega_N \ \{ \ \overline{\mathbf{u}} - GD_N^\omega(\overline{\mathbf{u}}) \ \} \\ &= ( \ D_N^\omega + \omega_N \ (I - D_N^\omega G) \ ) \, \overline{\mathbf{u}}. \end{aligned}$$

Thus, $D_{N+1}^\omega = D_N^\omega + \omega_N \ (I - D_N^\omega G)$ for every positive integer N. □

Next, we analyze in more detail properties of the Accelerated van Cittert deconvolution operator, $D_N^\omega$.

**Lemma 2.3.** *Let the averaging operator be the differential filter* $G\mathbf{v} := (-\delta^2\triangle + I)^{-1}\mathbf{v}$. *If the relaxation parameters* $\omega_i$, $i = 0, 1, ..., N$, *are positive, then the Accelerated van Cittert deconvolution operator* $D_N^\omega : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ *is symmetric positive definite.*

*Proof.* The operator $G$ is bounded, compact and self adjoint. Indeed, multiplying (1.2) by $\overline{\mathbf{v}}$ and integrating over $\Omega$ we get

$$0 \leq \|G\mathbf{v}\|^2 \leq \|\mathbf{v}\|^2 .$$

This shows that $G$ is bounded and $\|G\| \leq 1$. To show $G$ is self-adjoint and positive definite, note that for every $\mathbf{v} \in \mathrm{L}^2(\Omega)$ we have

$$0 \leq \delta^2 \|\nabla\overline{\mathbf{v}}\|^2 + \|\overline{\mathbf{v}}\|^2 = (\mathbf{v}, \overline{\mathbf{v}}) = (\mathbf{v}, G\mathbf{v}).$$

Both $D_0^\omega$ and $D_1^\omega$ are symmetric, as linear combinations of $I$ and $G$. Proceeding by mathematical induction, assume $D_n^\omega$ is symmetric, for a positive integer $n$. From Lemma 2.2, we know

$$D_{n+1}^\omega = D_n^\omega + \omega_n(I - D_n^\omega G).$$

Thus $D_{n+1}^\omega$ is symmetric as linear combination of symmetric operators $I$, $G$, and $D_n^\omega$. To show $D_1^\omega$ is bounded, we apply the Spectral Mapping Theorem. We have

$$\|D_1^\omega\| = \lambda(D_1^\omega) = \lambda( \ \ D_0^\omega + \omega_0(I - D_0^\omega G) \ \ ) = 1 + \omega_0(1 - \lambda(G)).$$

Since $\|G\| \leq 1$, we deduce that

$$1 \leq \|D_1^\omega\| \leq 1 + \omega_0.$$

Proceeding by induction, it is easy to see that for every positive integer $n$ we have

$$1 \leq \|D_n^\omega\| \leq 1 + \sum_{i=0}^{n-1} \omega_i.$$

This concludes the proof. □

## 3 Chebychev Optimized Deconvolution

This section calculates relaxation parameters $\omega_i$ which minimize the deconvolution error

$$\mathbf{e}_N = \mathbf{u} - D_N^\omega(\overline{\mathbf{u}}),$$

for general (non-smooth) velocity fields $\mathbf{u}$. To begin, we give a recursion formula for the deconvolution error $\mathbf{e}_N$.

**Lemma 3.1.** *The deconvolution error* $\mathbf{e}_N$*, satisfies* $\mathbf{e}_0 = \mathbf{u} - \overline{\mathbf{u}}$ *and for all positive integers* $N$ *we have*

$$\mathbf{u} - D_N^\omega \overline{\mathbf{u}} = \prod_{i=0}^{N-1} (I - \omega_i G)\mathbf{e}_0. \tag{3.1}$$

*Proof.* We will use mathematical induction. Note that the conclusion holds true for $N = 1$:

$$\mathbf{e}_1 = (I - \omega_0 G)\mathbf{u} - (I - \omega_0 G)\overline{\mathbf{u}} = (I - \omega_0 G)\mathbf{e}_0,$$

since $\overline{\mathbf{u}} = G\mathbf{u}$. Assuming $\mathbf{e}_n = \prod_{i=0}^{n-1}(I - \omega_i G)\mathbf{e}_0$ for any integer $n \geq 1$, let us prove

$$\mathbf{e}_{n+1} = \prod_{i=0}^{n}(I - \omega_i G)\mathbf{e}_0.$$

Since $\mathbf{e}_{n+1}$ can be rewritten as $\mathbf{e}_{n+1} = (I - \omega_n G)\mathbf{u} - (I - \omega_n G)\mathbf{u}_n$, applying the induction hypothesis we obtain

$$\mathbf{e}_{n+1} = \prod_{i=0}^{n}(I - \omega_i G)\mathbf{e}_0, \text{ for all integers } n \geq 1. \tag{3.2}$$

Therefore (3.1) holds true. □

Expand the velocity field $\mathbf{u}(\mathbf{x},t)$ in Fourier series

$$\mathbf{u}(\mathbf{x},t)=\sum_{k}\sum_{|\mathbf{k}|=k}\widehat{\mathbf{u}}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}},\ \text{where } \widehat{\mathbf{u}}(\mathbf{k},t)=\frac{1}{L^3}\int_\Omega \mathbf{u}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} \tag{3.3}$$

and $\mathbf{k}=\frac{2\pi}{L}\mathbf{n}$ ($\mathbf{n}\in\mathbb{Z}^3$) is the wave number.

**Definition 3.1.** *Let $\delta$ be the filter's averaging radius. The resolved scales are span$\{e^{i\mathbf{k}\cdot\mathbf{x}} \mid |\mathbf{k}|\leq \pi/\delta\}$. If $\mathbf{u}$ is given by (3.3), its projection onto the resolved scales, $P_{RS}\mathbf{u}$, is*

$$P_{RS}\mathbf{u}=\sum_{k\leq\pi/\delta}\sum_{|\mathbf{k}|=k}\widehat{\mathbf{u}}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.4}$$

We denote by $k_{\min}$, and $k_{\max}$ the smallest and the largest wave number of $P_{RS}\mathbf{u}$

$$0\ <\ k_{\min}\ \leq\ k\ \leq\ k_{\max}\ =\ \pi/\delta\ <\ \infty.$$

The total kinetic energy at point $\mathbf{x}$ in space and at time $t$ is $E(\mathbf{u})(t)$. Using Parceval's equality, we deduce $\widehat{E}(k,t)$, the kinetic energy at wave number $\mathbf{k}$, and also

$$E(P_{RS}\mathbf{u})(t)=\frac{2\pi}{L}\sum_{k\leq\pi/\delta}\widehat{E}(k,t),\ \text{where } \widehat{E}(k,t)=\frac{L}{2\pi}\sum_{|\mathbf{k}|=k}\frac{1}{2}|\widehat{\mathbf{u}}(\mathbf{k},t)|^2. \tag{3.5}$$

**Lemma 3.2.** *Let $\mathbf{u}\in L^2(\Omega)$. For any positive integer $N$, the deconvolution error $\mathbf{e}_N=\mathbf{u}-D_N^\omega\overline{\mathbf{u}}$ satisfies*

$$\|P_{RS}(\mathbf{u}-D_N^\omega\overline{\mathbf{u}})\|^2=\sum_{k\leq\pi/\delta}\sum_{|\mathbf{k}|=k}\prod_{i=0}^{N-1}(1-\omega_i\frac{1}{\delta^2k^2+1})^2(1-\frac{1}{\delta^2k^2+1})^2|\widehat{\mathbf{u}}(\mathbf{k},t)|^2. \tag{3.6}$$

*Proof.* From (1.2) and (3.3) we deduce

$$\widehat{\overline{\mathbf{u}}}(\mathbf{k},t)=\frac{1}{1+\delta^2k^2}\widehat{\mathbf{u}}(\mathbf{k},t). \tag{3.7}$$

With this, using Parceval's equality and Lemma 3.1, we have

$$\|P_{RS}(\mathbf{u}-D_1^\omega\overline{\mathbf{u}})\|^2\ =\ \sum_{k\leq\pi/\delta}\sum_{|\mathbf{k}|=k}(1-\omega_i\frac{1}{\delta^2k^2+1})^2(1-\frac{1}{\delta^2k^2+1})^2|\widehat{\mathbf{u}}(\mathbf{k},t)|^2\ .$$

Using mathematical induction, we prove (3.6), for any positive integer $N$. $\square$

Using Lemma 3.2, the optimization problem reduces to minimizing the expression:

$$\sum_{k_{\min}\le k\le \pi/\delta}\sum_{|\mathbf{k}|=k}\prod_{i=0}^{N-1}(1-\omega_i\frac{1}{\delta^2k^2+1})^2(1-\frac{1}{\delta^2k^2+1})^2|\widehat{\mathbf{u}}(\mathbf{k},t)|^2. \tag{3.8}$$

Consider thus the function $F_N:\mathbb{R}_+^N\to\mathbb{R}_+$, where

$$F_N(\omega_0,...,\omega_{N-1})=\sum_{k_{\min}\le k\le \pi/\delta}\sum_{|\mathbf{k}|=k}\prod_{i=0}^{N-1}(1-\omega_i\frac{1}{\delta^2k^2+1})^2(1-\frac{1}{\delta^2k^2+1})^2|\widehat{\mathbf{u}}(\mathbf{k},t)|^2. \tag{3.9}$$

We are seeking for relaxation parameters $\omega_i$ to minimize the error in deconvolution. In another words, we want to find $\min_{\omega_i}F_N(\omega_0,...,\omega_{N-1})$. We have

$$\begin{aligned}\min_{\omega_i}F_N(\omega_0,...,\omega_{N-1}) \ \le\ \min_{\omega_i}\max_{k_{\min}\le k\le k_{\max}}\prod_{i=0}^{N-1}(1-\omega_i\frac{1}{\delta^2k^2+1})^2\\ \sum_{k_{\min}\le k\le \pi/\delta}\sum_{|\mathbf{k}|=k}(1-\frac{1}{\delta^2k^2+1})^2|\widehat{\mathbf{u}}(\mathbf{k},t)|^2.\end{aligned}$$

Thus, minimizing the deconvolution error for a general velocity field leads to the problem of minimizing, with respect to $\omega_i$, the expression

$$\min_{\omega_i}\max_{k_{\min}\le k\le k_{\max}}\prod_{i=0}^{N-1}(1-\omega_i\frac{1}{\delta^2k^2+1}). \tag{3.10}$$

The change of variable, $x\leftarrow\frac{1}{\delta^2k^2+1}$ gives that for $k_{\min}\le k\le k_{\max}$ we have

$$0\ <\ a:=\frac{1}{\delta^2k_{\max}^2+1}\ =\frac{1}{\pi^2+1}\le\ x\ \le\ \frac{1}{\delta^2k_{\min}^2+1}=:b\ <\ 1.$$

Then, (3.10) leads to

$$\min_{\omega_i}\max_{k_{\min}\le k\le k_{\max}}\prod_{i=0}^{N-1}(1-\omega_i\frac{1}{\delta^2k^2+1})\ \ \le\ \ \min_{\omega_i}\max_{a\le x\le b}\prod_{i=0}^{N-1}(1-\omega_i x). \tag{3.11}$$

To proceed, we denote by $\Pi_N^1$, the set of all polynomial functions of degree less than or equal to $N$, which are 1 at the origin, i.e.

$$\Pi_N^1=\{p(x)|p(0)=1\}.$$

Note that $\prod_{i=0}^{N-1}(1-\omega_i x)\in\Pi_N^1$ and thus

$$\min_{\omega_i}\max_{a\le x\le b}|\prod_{i=0}^{N-1}(1-\omega_i x)|\le\min_{\Pi_N^1}\max_{x_1\le x\le x_2}|\prod_{i=0}^{N-1}(1-\omega_i x)|. \tag{3.12}$$

It is well known, see for example Axelsson, [1] (page 180), that the least maximum is achieved by the Chebychev polynomials, namely

$$\min_{\Pi_N^1} \max_{a \le x \le b} | \prod_{i=0}^{N-1} (1 - \omega_i x)| = \max_{a \le x \le b} \frac{T_N\left(\frac{b+a-2x}{b-a}\right)}{T_N\left(\frac{b+a}{b-a}\right)} = \frac{1}{T_N\left(\frac{b+a}{b-a}\right)}, \tag{3.13}$$

where $T_N(x) = \cosh\left(N \cosh^{-1}(x)\right)$ is the $N^{th}$ Chebychev polynomial, for all $x \ge 1$.

**Remark 3.1.** *Following [1], further calculations in (3.13) show that*

$$\min_{\Pi_N^1} \max_{a \le x \le b} | \prod_{i=0}^{N-1} (1 - \omega_i x)| = 2\frac{\sigma^N}{1+\sigma^{2N}}, \text{ where } \sigma = \frac{1 - \sqrt{a/\ b}}{1 + \sqrt{a/\ b}}. \tag{3.14}$$

**Corollary 3.1.** *For* $\mathbf{u} \in L^2(\Omega)$ *we have*

$$\|P_{RS}(\mathbf{u} - D_1^\omega \overline{\mathbf{u}})\|^2 \le \frac{1}{T_N^2\left(\frac{b+a}{b-a}\right)} \sum_{k_{\min} \le k \le \pi/\delta} \sum_{|\mathbf{k}|=k} (1 - \frac{1}{\delta^2 k^2 + 1})^2 |\hat{\mathbf{u}}(\mathbf{k}, t)|^2. \tag{3.15}$$

*Proof.* This follows easily from Lemma 3.2 and (3.13). □

**Proposition 3.1.** *The parameters* $\omega_j$ *solving the min-max (3.10) problem are given by*

$$\omega_j = \frac{1}{\frac{b-a}{2} cos\left(\frac{2j+1}{2N}\pi\right) + \frac{b+a}{2}}, \tag{3.16}$$

*for all positive integers* $N$ *and* $j = 0, 1, ..., N-1$.

*Proof.* From (3.13), the optimal parameters for the optimization problem are given by the inverses of the zeros of $T_N$. So, (3.16) holds true. □

Further *useful* progress depends on $a > 0$, i.e. on either $k_{\max} < \infty$ or on restricting to the error in the resolved scales (for which $k_{\max} = \pi/\delta$).

## 3.1 Expected accuracy increase for turbulent flows

The $\omega_i$ in (3.15) optimize deconvolution models over general velocities fields. It is useful to compare the resulting errors to the case when $\omega_i$ are optimized over special velocities with a $k^{-5/3}$ energy spectrum. If the comparison is done for velocities with a $k^{-5/3}$ energy spectrum, it can be done exactly analytically (and will be most favorable for latter case). Indeed, let $< \cdot >$ denote time averaging

$$< \boldsymbol{\phi} > (x) := \limsup_{T \to \infty} \frac{1}{T} \int_0^T \boldsymbol{\phi}(x, t) dt. \tag{3.17}$$

Let $\widehat{E}(k) =< \widehat{E}(k,t) >$. For homogeneous, isotropic turbulence the Kolmogorov K-41 theory predicts that over

$$0 \quad < \quad k_{\min} \ := U\nu^{-1} \quad \leq k \leq \quad \varepsilon^{1/4}\nu^{-3/4} =: k_{\max} \quad < \quad \infty,$$

we have

$$\widehat{E}(k) \simeq \alpha\varepsilon^{2/3}k^{5/3}, \tag{3.18}$$

where $\alpha$ (in the range 1.4 to 1.7) is the universal Kolmogorov constant, $U$ and $L$ are, respectively, a global reference velocity and and length scales and $\varepsilon$ denotes the energy dissipation rate of the particular flow

$$\varepsilon :=< \frac{\nu}{L^3}\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 d\mathbf{x} > . \tag{3.19}$$

The K-41 theory predicts that outside the inertial range, we still have $\widehat{E}(k) \simeq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$, since $\widehat{E}(k) \simeq 0$ for $k \geq k_{\max}$ and since $\widehat{E}(k) \simeq \widehat{E}(k_{\min})$ for $k \leq k_{\min}$.

Consider Chebychev optimized deconvolution. Time averaging (3.6) and using Parceval's equality, we obtain

$$< \frac{1}{L^3} \|P_{RS}(\mathbf{u} - D_N^{\omega}\overline{\mathbf{u}})\|^2 >= 2\frac{2\pi}{L} \sum_{k_{\min}<k<\pi/\delta} \prod_{i=0}^{N-1} (1-\omega_i \frac{1}{\delta^2k^2+1})^2 (1-\frac{1}{\delta^2k^2+1})^2 \widehat{E}(k).$$

In [17], the authors considered

$$< \frac{1}{L^3} \ \|P_{RS}(\mathbf{u} - D_N^{\omega}\overline{\mathbf{u}})\|^2 \ >, \text{ subject to } \widehat{E}(k) \simeq \alpha\varepsilon^{2/3}k^{-5/3}. \tag{3.20}$$

We compare (3.15) to this case. For (3.15) we calculate the deconvolution error as

$$< \ \frac{1}{L^3} \ \|P_{RS}(\mathbf{u} - D_N^{\omega}\overline{\mathbf{u}})\|^2 \ \ >\leq \ \frac{4\pi}{L} \sum_{0<k\leq\pi/\delta} \ \prod_{i=0}^{N-1} (1-\omega_i \frac{1}{\delta^2k^2+1})^2 (\frac{\delta^2k^2}{\delta^2k^2+1})^2 \widehat{E}(k). \tag{3.21}$$

Additionally, using (3.13), for Chebychev optimized deconvolution, we obtain

$$\begin{aligned} < \ \frac{1}{L^3} \ \|P_{RS}(\mathbf{u} - D_N^{\omega}\overline{\mathbf{u}})\|^2 \ > \ &\leq \ \ ( \min_{\omega_i} \max_{0\leq \ k\leq\frac{\pi}{\delta}} \prod_{i=0}^{N-1} (1-\omega_i \frac{1}{\delta^2k^2+1}) \ )^2 \\ &\qquad \alpha \ \epsilon^{2/3} \ \frac{4\pi}{L} \sum_{0<k\leq\pi/\delta} (\frac{\delta^2k^2}{\delta^2k^2+1})^2 k^{-5/3} \\ &\leq \ \frac{1}{\cosh^2(0.62 \ N)} \ \alpha \ \epsilon^{2/3} \ \ \frac{4\pi}{L} \int_0^{\frac{\pi}{\delta}} (\frac{\delta^2k^2}{\delta^2k^2+1})^2 k^{-5/3} dk \\ &= \ \frac{1}{\cosh^2(0.62 \ N)} \ \ \alpha \ \epsilon^{2/3} \ \delta^{2/3} \ \frac{4\pi}{L} \ 0.54. \end{aligned} \tag{3.22}$$

Since $\frac{1}{\cosh(x)} \le e^{-x}$, for $x \ge 0$, we obtain the bound for the time average deconvolution error

$$< \frac{1}{L^3} \|P_{RS}(\mathbf{u} - D_N^{\omega}\overline{\mathbf{u}})\|^2 > \quad \le \quad 2.16\ \alpha\ \epsilon^{2/3}\ \delta^{2/3}\ \frac{\pi}{L}\ e^{-1.24N}. \tag{3.23}$$

**Remark 3.2.** *The unoptimized case of $\omega_i \equiv 1$ was studied in [13] with result*

$$< \frac{1}{L^3}\ ||\mathbf{u} - D_N\overline{\mathbf{u}}||^2 > \le (\frac{3}{2} + \frac{1}{4N + \frac{10}{3}})\ \alpha\ \ \epsilon^{2/3}\ \delta^{2/3}. \tag{3.24}$$

## 4 Comparison of Attained Accuracy

There are three versions of van Cittert to be compared: unoptimized, Chebychev-optimized for a general flow field (herein), and previous work [17] in which the optimality problem was formulated for special velocity fields with the exact energy spectrum $\widehat{E}(k) \sim k^{-5/3}$. We will refer to this last possibility as "K-41 optimization".

To compare the three, we consider the case of velocity fields with energy spectrum of $k^{-5/3}$. Using (3.16), we first compute the values of the Chebychev parameters. These and K-41 optimized relaxation parameters (from [17]) are in Tables 1 and 2 respectively.

| $N$ | $\omega_0$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|---|---|---|---|---|---|
| 1 | 1.83 | - | - | - | - |
| 2 | 1.15 | 4.44 | - | - | - |
| 3 | 1.06 | 1.83 | 6.54 | - | - |
| 4 | 1.03 | 1.38 | 2.68 | 7.90 | - |
| 5 | 1.02 | 1.23 | 1.83 | 3.58 | 8.75 |

Table 1: Chebychev optimized parameters

| $N$ | $\omega_0$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|---|---|---|---|---|---|
| 1 | 2.10 | - | - | - | - |
| 2 | 2.02 | 2.02 | - | - | - |
| 3 | 1.44 | 4.91 | 1.44 | - | - |
| 4 | 1.49 | 1.49 | 5.83 | 1.49 | - |
| 5 | 1.53 | 1.53 | 6.52 | 1.53 | 1.53 |

Table 2: K-41 optimized parameters

Because of the form of the RHS of estimates (3.23) and (3.24), we normalize the errors calculated by $\alpha\ \epsilon^{2/3}\ \delta^{2/3}$. Thus, using Lemma 3.2, we give in Table 3

$$\frac{< \frac{1}{L^3} \|P_{RS}(\mathbf{u} - D_N^{\omega}\overline{\mathbf{u}})\|^2 \ >}{\alpha\ \epsilon^{2/3}\ \delta^{2/3}}, \text{ when } N = 0, 1, 2, 3, 4, 5$$

for the three cases $\omega_i \equiv 1$, $\omega_i$ from Table 1, and $\omega_i$ from Table 2. Table 3 shows that both optimizations reduce the error over standard van Cittert significantly. Figure 2 gives a plot of (normalized) deconvolution error vs. wave number, for $N = 2$ for all three cases of standard van Cittert, K-41 optimized and Chebychev optimized van Cittert. Figure 2 shows that both optimized van Citter improve the error in deconvolution for irregular velocities, while the unoptimized van Cittert is more accurate for very smooth velocity fields.

| N | K-41 optimized $\omega_i$ | Chebychev optimized $\omega_i$ | $\omega_i = 1$ |
|---|---|---|---|
| 1 | 0.150 | 0.157 | 0.258 |
| 2 | 0.068 | 0.066 | 0.155 |
| 3 | 0.017 | 0.022 | 0.101 |
| 4 | 0.007 | 0.006 | 0.070 |
| 5 | 0.003 | 0.002 | 0.049 |

Table 3: Normalized deconvolution error

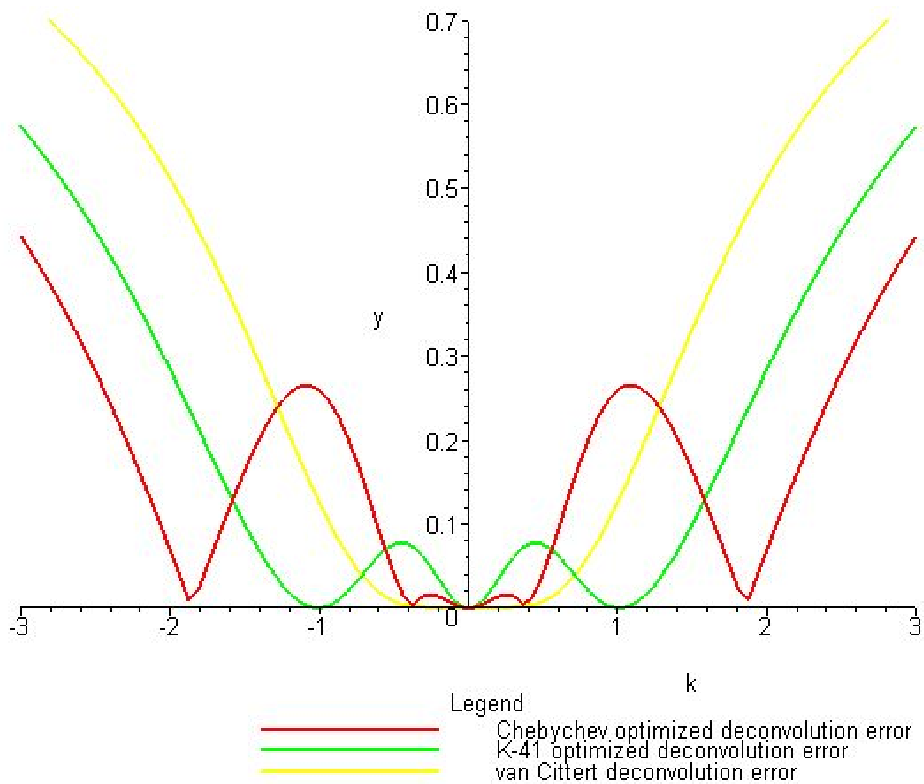


Figure 2: Deconvolution Error (N=2)

## 5 An Illustration

To begin, we test the deconvolution error when both filtering and deconvolution are done discretely using a finite element approximation of the Laplace operator in (1.2). The computations were performed with the software FreeFem++, see [8]. We choose $\mathbf{u} = (\sin(ky), \sin(kx))$ a known, divergence free velocity and calculate

$$\left[\frac{1}{|\Omega|}\int_\Omega |\mathbf{u} - D_N\overline{\mathbf{u}}|^2 d\mathbf{x}\right]^{\frac{1}{2}},$$

where $\Omega = (0, 2\pi)^2$. P2 elements were used in the discretization, i.e. the velocity is approximated by continuous piecewise quadratics. For each value of N deconvolution involves the solution of N+1 discrete Poisson problems. We solve the resulting linear system with GMRES. For these calculations, we consider the meshwidth h = 1/10, 1/20, 1/30, 1/40 and N=1, 2, 3. We fix $\delta = 0.1$ and $k = 1$ and 8. The case $k = 1$ is very smooth and the theory predicts regular van Cittert to be more accurate. The case $k = 8$ oscillates faster and the theory predicts both Accelerated van Cittert to be more competitive.

Comparing tables 5, 6, and 7 we see that for a very smooth $\mathbf{u}$ (the case k=1) unoptimized van Cittert ($\omega_i = 1$) is indeed more accurate, as expected. In this case, Chebychev optimized is superior to K-41 optimized. This result is not expected since the Chebychev is for a general $L^2$ field while K-41 optimized is for fields that are slightly more regular. Next we consider

| h | $\|(I - D_1G)\mathbf{u}\|$ | $\|(I - D_2G)\mathbf{u}\|$ | $\|(I - D_3G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.000385707 | 0.000157297 | 0.000067544 |
| 1/20 | 0.000454801 | 0.000238021 | 0.000129238 |
| 1/30 | 0.000469269 | 0.000265614 | 0.000153671 |
| 1/40 | 0.000471633 | 0.000273757 | 0.000163838 |

Table 4: Unoptimized Deconvolution Error: k = 1, $\delta = 0.1$

| h | $\|(I - D_1G)\mathbf{u}\|$ | $\|(I - D_2G)\mathbf{u}\|$ | $\|(I - D_3G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.0102168 | 0.00862918 | 0.00590472 |
| 1/20 | 0.0102158 | 0.00862688 | 0.00590488 |
| 1/30 | 0.0102158 | 0.00862691 | 0.00590513 |
| 1/40 | 0.0102158 | 0.00862699 | 0.00590512 |

Table 5: K-41 Optimized Deconvolution Error: k = 1, $\delta = 0.1$

the case a velocity field which is highly oscillatory with respect to the chosen filter radius, $k = 8$ and $\delta = 0.1$. We see in tables 8, 9, 10 that, for rougher velocity fields, both optimized van Cittert are superior to unoptimized van Cittert, in accord with the predictions of the theory.

| h | $\|(I-D_1G)\mathbf{u}\|$ | $\|(I-D_2G)\mathbf{u}\|$ | $\|(I-D_3G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.00757131 | 0.00426287 | 0.00190707 |
| 1/20 | 0.00757049 | 0.0042709 | 0.00191 |
| 1/30 | 0.00757075 | 0.00427072 | 0.00191063 |
| 1/40 | 0.00757091 | 0.0042705 | 0.00191061 |

Table 6: Chebychev Optimized Deconvolution Error: k = 1, $\delta = 0.1$

| h | $\|(I-D_1G)\mathbf{u}\|$ | $\|(I-D_2G)\mathbf{u}\|$ | $\|(I-D_3G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.166844 | 0.0797244 | 0.040896 |
| 1/20 | 0.154312 | 0.0200043 | 0.028794 |
| 1/30 | 0.152509 | 0.0191031 | 0.027441 |
| 1/40 | 0.15215 | 0.0186714 | 0.027323 |

Table 7: Unoptimize Deconvolution Error: k = 8, $\delta = 0.1$

| h | $\|(I-D_1G)\mathbf{u}\|$ | $\|(I-D_2G)\mathbf{u}\|$ | $\|(I-D_3G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.026368 | 0.027578 | 0.0372458 |
| 1/20 | 0.095817 | 0.028730 | 0.0200043 |
| 1/30 | 0.102346 | 0.027441 | 0.0191031 |
| 1/40 | 0.103715 | 0.027323 | 0.0186714 |

Table 8: K-41 Optimized Deconvolution Error: k = 8, $\delta = 0.1$

| h | $\|(I-D_1G)\mathbf{u}\|$ | $\|(I-D_2G)\mathbf{u}\|$ | $\|(I-D_3G)\mathbf{u}\|$ |
|---|---|---|---|
| 1/10 | 0.0359071 | 0.170887 | 0.0331716 |
| 1/20 | 0.0357505 | 0.195504 | 0.0383042 |
| 1/30 | 0.0398032 | 0.195824 | 0.0435825 |
| 1/40 | 0.0408305 | 0.195785 | 0.0446767 |

Table 9: Chebychev Optimized Deconvolution Error: k = 8, $\delta = 0.1$

# 6 Conclusions

For a LES with deconvolution model to be feasible, the model's consistency error must be small for large radii $\delta$, which are large with respect to the problems inherent length scales (which correspond to computationally attainable meshwidths). Thus, selection of parameters to minimize model's consistency error increases the problems for which LES is feasible and increases the reduction in computational effort obtainable when using LES.

The use of optimal parameters requires no extra computational effort. Two main results of this work are:

(i) the values of the optimal parameters (in Section 4) and

(ii) the reduction in the model consistency error that results in their use is is at least

50%, i.e ( with $F_N$ given by (3.9) )

$$\text{accuracy increase ratio} := \frac{\min_{\omega_0,\omega_1,\dots,\omega_{N-1}} F_N(\omega_0,\omega_1,\cdots,\omega_{N-1})}{F_N(1,1,\dots,1)} \ll \frac{e^{-1.64N}}{\frac{3}{2}+\frac{1}{4N+\frac{10}{3}}}.$$

This is also reflected by Table 3. Interestingly, in all cases (N= 1,2,3,4,5) the Chebychev optimized parameters resulted in comparable or better errors to K-41 optimization. Since Chebychev optimization give parameter values good for all flow fields and the latter only for special ones, this suggests that Chebychev optimized deconvolution is to be strongly preferred. It is important to note that the relative increase in accuracy obtained using optimal parameters itself increases with the order of the model.

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