A stability example

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Abstract

An example is presented of a pair of time-dependent ordinary differential equations for which an energy-based stability proof similar to the one for the Navier-Stokes equations can be made. Two discretizations are considered: nonlinear backward Euler and a linearized backward Euler with the nonlinear term computed partly at the previous time step. It is shown that the continuous and nonlinear backward Euler discretization are stable and also that a particular steady solution u_0 is Lyapunov stable, with the same dependence on problem data. In contrast, while the linearized backward Euler system is stable, u_0 is Lyapunov stable only for small time increment.

1 Introduction

The example presented below was generated as an aid in understanding the behavior of a common discretization of the Navier-Stokes equations. When the Navier-Stokes equations are discretized using the "Trapezoid Rule Linear Extrapolated" (Algorithm 9.2 in [2]) approach or the "linearized (or lagged) Backward Euler" approach (Scheme 5.1 in [1]), it is sometimes observed that solutions depart from the expected solution and move to one with substantially larger energy. This author has observed cases where simple Poiseulle flow at Reynolds numbers well below 100 can exhibit an excursion to an unexpected high energy transient state. It is conjectured that the excursion is a consequence of loss of Lyapunov stability of steady Poiseulle flow in the discrete case when the time increment is large, despite provable unconditional stability of the discrete system.

To see that it is possible for a system to be at once unconditionally stable and to have solutions that are not Lyapunov stable, we present an example of a system of ODEs along with a temporal discretization that has the following properties.

1. Both the continuous and discrete systems possess energy functionals that are bounded for all time, with the bound depending only on problem data.

2. For some data choices, the continuous system possesses (constant) solutions that are stable in the Lyapunov sense.

3. These solutions are also solutions of the system discretized with the usual (nonlinear) backward Euler method and are stable in the Lyapunov sense for the same problem data as the continuous case.

4. These solutions are also solutions of the system discretized with a linearized backward Euler method analogous to Scheme 5.1 in [1] or Algorithm 9.2 in [2]. In this case the solutions are only conditionally stable in the Lyapunov sense, depending on the time step as well as problem data.

2 The continuous system

Consider the following nonlinear system of ordinary differential equations.

$$\begin{aligned} \dot{u}_1 + c(u_1,u_2)u_2 + \nu u_1 &= f_1 \\ \dot{u}_2 - c(u_1,u_2)u_1 + \nu u_2 &= f_2 \end{aligned} \tag{1}$$

where $c(u_1, u_2)$ is a smooth (scalar) function of $\mathbf{u} = (u_1, u_2)$ and $\mathbf{f} = (f_1, f_2)$ is a smooth function of time, t. In this note, c will generally be taken to be an affine function of $\mathbf{u}$ and $\mathbf{f}$ a constant. In addition, ν will be assumed to be a positive constant and the system satisfies initial conditions $\mathbf{u}(0) = \mathbf{u}_0 = (v_1, v_2)$.

Equation (1) can be rewritten as

$$\begin{aligned} \dot{\mathbf{u}} + c(\mathbf{u})P\mathbf{u} + \nu\mathbf{u} &= \mathbf{f} \\ \mathbf{u}(0) &= \mathbf{u}_0 \end{aligned} \tag{2}$$

where

$$P\mathbf{u} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{u}.$$

Proposition 2.1. *Solutions* $\mathbf{u}$ *to (2) are bounded in time.*

Proof. Take the dot product of the first equation in (2) with $\mathbf{u}$ to get an energy equality

$$\frac{1}{2}\frac{d\|\mathbf{u}\|^2}{dt} + \nu\|\mathbf{u}\|^2 = \mathbf{f}\cdot\mathbf{u}.$$

Apply the Schwartz and Young's inequalities to get

$$\frac{1}{2}\frac{d\|\mathbf{u}\|^2}{dt} + \nu\|\mathbf{u}\|^2 \leq \frac{1}{2\nu}\|\mathbf{f}\|^2 + \frac{\nu}{2}\|\mathbf{u}\|^2,$$

so that

$$\frac{d\|\mathbf{u}\|^2}{dt} + \nu\|\mathbf{u}\|^2 \leq \frac{1}{\nu}\|\mathbf{f}\|^2.$$

Gronwall's inequality completes the proof. □

Next, we wish to discuss stability of solutions of (2). To this end, we turn to the notion of Lyapunov stability (see, for example, [3]).

Definition Suppose the constant $\mathbf{u}_0$ is a solution of (2). Then the solution $\mathbf{u}_0$ is termed "Lyapunov stable" if, for given $\epsilon > 0$, there exists a $\delta > 0$ so that if $\mathbf{y}_0$ satisfies $\|\mathbf{y}_0 - \mathbf{u}_0\| < \delta$, then a solution $\mathbf{y}(t)$ to (2) with initial condition $\mathbf{y}_0$ satisfies $\|\mathbf{y}(t) - \mathbf{u}_0\| < \epsilon$.

One way of determining Lyapunov stability of a constant solution $\mathbf{u}_0$ of (2) would be to linearize (2) about $\mathbf{u}_0$ and then check that the resulting linear system is stable. Suppose, then that $\mathbf{u}_0$ is a constant solution of (2) and that $\mathbf{y}$ is a solution with $\mathbf{y}(0) = \mathbf{u}_0$. Then we can write

$$\begin{aligned}\dot{\mathbf{y}} + c(\mathbf{y})P\mathbf{y} + \nu\mathbf{y} &= \mathbf{f},\\ c(\mathbf{u}_0)P\mathbf{u_0} + \nu\mathbf{u_0} &= \mathbf{f}.\end{aligned}$$

Subtracting, rearrainging, ignoring the higher order term $(c(\mathbf{y})-c(\mathbf{u}_0))P(\mathbf{y}-\mathbf{u}_0)$, and taking limits yields the following linearized ODE for $\mathbf{w} = \mathbf{y} - \mathbf{u}_0$.

$$\dot{\mathbf{w}} + (\nabla c_0 \cdot \mathbf{w})P\mathbf{u}_0 + c_0 P\mathbf{w} + \nu\mathbf{w} = 0, \tag{3}$$

where c_0 denotes $c_0(\mathbf{u}_0)$ and ∇c_0 denotes $\nabla c(\mathbf{u}_0)$.

Close examination of the term $(\nabla c_0 \cdot \mathbf{w})P\mathbf{u}_0$ reveals that it can be written as $P\mathbf{u}_0(\nabla c_0)^T\mathbf{w}$. To see why this is the case, write

$$(\nabla c_0 \cdot \mathbf{w})P\mathbf{u}_0 = P\mathbf{u}_0(\nabla c_0 \cdot \mathbf{w}) = P\mathbf{u}_0\left((\nabla c_0)^T\mathbf{w}\right)$$

because $(\nabla c_0 \cdot \mathbf{w})$ is a scalar. As a result, (3) can be written as

$$\dot{\mathbf{w}} + P\mathbf{u}_0(\nabla c_0)^T\mathbf{w} + c_0 P\mathbf{w} + \nu\mathbf{w} = 0, \tag{4}$$

and this equation is stable if the linear operator

$$A = \nu I + P\mathbf{u_0}(\nabla c_0)^T + c_0 P \tag{5}$$

has no eigenvalues with negative real part.

Remark If $c(\mathbf{u}) = u_2$, so that $c_0 = 0$ and $(\nabla c_0)^T = (0,1)$, then

$$A = \begin{pmatrix} \nu & (\mathbf{u_0})_2 \\ 0 & \nu - (\mathbf{u_0})_1 \end{pmatrix}$$

so that the solution $\mathbf{u}_0$ of (2) is Lyapunov stable so long as $(\mathbf{u}_0)_1 \leq \nu$.

3 Linearized backward Euler with lagged nonlinear term (LBE)

One possible discrete form of (2) is the LBE form

$$\begin{aligned} \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + c(\mathbf{u}^n)P\mathbf{u}^{n+1} + \nu\mathbf{u}^{n+1} &= \mathbf{f} \\ \mathbf{u}^0 &= \mathbf{u}_0 \end{aligned} \tag{6}$$

It turns out that LBE produces solutions that are bounded, just as in the continuous case.

Proposition 3.1. *Solutions to (6) are bounded independently of n, with the bound depending on f, ν and Δt.*

Proof. This proof follows one appearing in [1]. Take the dot product of (6) with $\mathbf{u}^{n+1}$, much as in the continuous case. The result is

$$\frac{1}{\Delta t}\|\mathbf{u}^{n+1}\|^2 - \frac{1}{\Delta t}\mathbf{u}^{n+1}\cdot\mathbf{u}^n + \nu\|\mathbf{u}^{n+1}\|^2 = \mathbf{f}\cdot\mathbf{u}^{n+1} \tag{7}$$

Using a well-known equality

$$\mathbf{u}^{n+1}\cdot\mathbf{u}^n = \frac{1}{2}\|\mathbf{u}^{n+1}\|^2 + \frac{1}{2}\|\mathbf{u}^n\|^2 - \frac{1}{2}\|(\mathbf{u}^{n+1}-\mathbf{u}^n)\|^2,$$

and the Cauchy-Schwarz and Young's inequalities, (7) yields

$$\frac{1}{2\Delta t}\|\mathbf{u}^{n+1}\|^2 - \frac{1}{2\Delta t}\|\mathbf{u}^n\|^2 + \frac{1}{2\Delta t}\|(\mathbf{u}^{n+1}-\mathbf{u}^n)\|^2 + \frac{1}{2}\nu\|\mathbf{u}^{n+1}\|^2 \leq \frac{1}{2\nu}\|\mathbf{f}\|^2$$

Dropping one term on the left yields the expression

$$(\frac{1}{\Delta t} + \nu)\|\mathbf{u}^{n+1}\|^2 - \frac{1}{\Delta t}\|\mathbf{u}^n\|^2 \leq \frac{1}{\nu}\|\mathbf{f}\|^2.$$

Solutions to this inequality are bounded by a version of discrete Gronwall inequality. It turns out that an elementary proof of boundedness for this simple inequality is available.

Simplifying slightly and denoting $\rho = \frac{1}{\Delta t}/(\frac{1}{\Delta t} + \nu) < 1$ yields

$$\|\mathbf{u}^{n+1}\|^2 \leq \rho \|\mathbf{u}^n\|^2 + \frac{1}{\nu^2}\|\mathbf{f}\|^2. \tag{8}$$

Notice that

$$\upsilon_{n+1} = \rho^n \upsilon_0 + \phi \sum_0^{n-1} \rho^k \leq \upsilon_0 + \frac{\phi}{1-\rho}$$

satisifies the recursion

$$\upsilon_{n+1} = \rho \upsilon_n + \phi, \tag{9}$$

so that taking $\phi = \|\mathbf{f}\|^2/\nu^2$ and $\upsilon_0 = 0$, along with a simple recursive argument comparing (8) with (9) shows that $\|\mathbf{u}^n\|^2 \leq \frac{\phi}{1-\rho}$. $\square$

Proposition 3.2. *Solutions to (6) can be Lyapunov stable or not, depending on* $\mathbf{f}$, ν *and* Δt*.*

Proof. The calculation leading up to (4) can be repeated for LBE to yield the following linearized discrete equation

$$\frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} + P\mathbf{u}_0(\nabla c_0)^T \mathbf{w}^n + c_0 P \mathbf{w}^{n+1} + \nu \mathbf{w}^{n+1} = 0. \tag{10}$$

As a consequence, LBE is stable iff the following matrix has eigenvalues not larger than one.

$$A = \left((\frac{1}{\Delta t} + \nu) I + c_0 P \right)^{-1} \left(\frac{1}{\Delta t} I - P\mathbf{u}_0 \nabla c_0^T \right) \tag{11}$$

$\square$

Remark If $c(u) = u_2$, so that $c_0 = 0$ and $(\nabla c_0)^T = (0, 1)$, then

$$A = \frac{1}{\frac{1}{\Delta t} + \nu} \begin{pmatrix} \frac{1}{\Delta t} & -(\mathbf{u}_0)_2 \\ 0 & \frac{1}{\Delta t} + (\mathbf{u}_0)_1 \end{pmatrix}.$$

One of the eigenvalues of A is $1/\Delta t/(1/\Delta t + \nu) < 1$, but the other is

$$\frac{\frac{1}{\Delta t} + (\mathbf{u}_0)_1}{\frac{1}{\Delta t} + \nu}$$

so that, for example, when $(\mathbf{u}_0)_1 = -2\nu$ and $\Delta t = 2/\nu$, then the smaller eigenvalue is -1 and (10) is stable but not asymptotically stable. Larger values of Δt result in (10) being unstable and smaller values result in (10) being asymptotically stable. All three of these cases satisfy the condition $(\mathbf{u}_0)_1 < \nu$ so that the solution $\mathbf{u}_0$ of the continuous system is Lyapunov stable for any choice of $(\mathbf{u}_0)_2$.

Remark If the (nonlinear) Backward Euler method (BE) were used,

$$\begin{aligned} \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + c(\mathbf{u}^{n+1})P\mathbf{u}^{n+1} + \nu\mathbf{u}^{n+1} &= \mathbf{f} \\ \mathbf{u}^0 &= \mathbf{u}_0 \end{aligned} \tag{12}$$

then the same linearization steps would yield

$$\frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} + P\mathbf{u}_0(\nabla c_0)^T\mathbf{w}^{n+1} + c_0 P\mathbf{w}^{n+1} + \nu\mathbf{w}^{n+1} = 0. \tag{13}$$

with the resulting matrix

$$A = \left((\frac{1}{\Delta t} + \nu)I + P\mathbf{u}_0\nabla c_0^T + c_0 P \right)^{-1} \left(\frac{1}{\Delta t} I \right)$$

If $c(u) = u_2$, so that $c_0 = 0$ and $(\nabla c_0)^T = (0, 1)$, then

$$A = \frac{1}{\Delta t} \begin{pmatrix} \frac{1}{\Delta t} + \nu & (\mathbf{u}_0)_2 \\ 0 & \frac{1}{\Delta t} + \nu - (\mathbf{u}_0)_1 \end{pmatrix}^{-1}.$$

One of the eigenvalues is smaller than 1 for all $\nu > 0$, and the other is smaller than 1 for $(\mathbf{u}_0)_1 < \nu$, exactly the condition for the continuous case.

References

[1] Temam, R., *Navier-Stokes equations: theory and numerical analysis*, originally published by North-Holland (New York, 1977), reprinted by AMS Chelsea Publishing (Providence, 2001).

[2] Layton, W., *Introduction to the numerical analysis of incompressible viscous flows*, SIAM (Philadelphia, 2008).

[3] Hartman, P., *Ordinary differential equations*, John Wiley & Sons (London, 1964).