NUMERICAL ANALYSIS OF MODULAR VMS METHODS WITH NONLINEAR EDDY VISCOSITY FOR THE NAIVER-STOKES EQUATIONS

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Abstract. This paper presents a stability, convergence and error analysis for two modular, projection-based variational multiscale (VMS) methods for the incompressible Naiver-Stokes equations. In VMS methods, the influence of the unresolved scales onto the resolved small scales is modeled by a Smagorinsky-type turbulent viscosity acting only on the marginally resolved scales. We analyze a method of inducing a VMS treatment of turbulence in an existing NSE discretization through an additional, uncoupled projection step. For two nonlinear eddy viscosity pparameterizations, we prove an error estimate for this approach. Numerical tests are given that confirm and illustrate the theoretical estimates. One method uses a fully nonlinear step inducing the VMS discretization. The second induces a nonlinear eddy viscosity model with a linear solve of much less cost.

Key words. Navier-Stokes equations; nonlinear eddy viscosity; projection-based VMS method; uncoupled approach; error estimate.

AMS subject classifications. 65M55, 65M70

1. Introduction. Variational multiscale (VMS) methods have proven (see Section 1.2 for its generation and some recent work) to be an important approach to the numerical simulation of turbulent flows. VMS methods are an efficient, clever and simple realization of the idea of introducing eddy viscosity locally in scale space only on the marginally resolved scales and tuned to add dissipation to mimic the loss of energy in the marginally resolved scales caused by breakdown of eddies to unresolved scales:

$$(\nu_T(\mathbf{u}^h)\mathbb{D}(I-P)\mathbf{u}^h, \mathbb{D}(I-P)\mathbf{v}^h), \tag{1.1}$$

where $\mathbb{D}(\mathbf{v}) = (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)/2$ is the velocity deformation tensor (symmetric part of the gradient), P is an elliptic projection onto the well-resolved velocities on a given mesh (so $(I-P)\mathbf{u}^h$ is the marginally resolved velocity scales).

The success of VMS methods lead naturally to the question of how to introduce them into legacy codes and other multi-physics codes so large as to discourage abandoning a method or a model that is already implemented to reprogram another one. In [28], this question was addressed: a VMS method can be induced into a black box (even laminar) flow simulation by adding a modular projection step, uncoupled from the (possibly black box) flow code. Although the numerical tests were quite general, the mathematical/numerical analysis in [28] in support of modular VMS methods was for constant eddy viscosity parametrizations $\nu_T(\cdot)$. In this report we continue the development of mathematical support for modular VMS methods in two ways. First we expand the analysis of [28] to include the fully nonlinear, eddy viscosity case of the (ideal) "small-small" Smagorinsky model for which

$$\nu_T(\mathbf{u}^h) = (C_s\delta)^2|\mathbb{D}(I-P)\mathbf{u}^h|. \tag{1.2}$$

We shall see that this ideal case has the strongest mathematical theory due to the strong monotonicity on the marginally resolved scales of (1.1) with (1.2). Unfortunately, the choice (1.2) also increases dramatically the cost of the modular Step 2 required. We therefore consider methods (i) whose realization is as close as possible to the ideal small-small Smagorinsky model,

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(ii) for which a complete and rigorous mathematical foundation can be given, and (iii) whose implementation is comparable in cost and complexity to the linear case of $\nu_T \equiv$constant. These issues lead to our second, related method for which (where e is any given element)

$$\nu_T(\mathbf{u}^h) = (C_s\delta)^2 Average_e(|\mathbb{D}(I-P)\mathbf{u}^h|), \tag{1.3}$$

where $Average_e(\phi) = \frac{1}{area(e)}\int_e \phi dx$. We shall see that because ν_T is now elementwise constant, there arises enormous simplification of the modular Step 2. Interestingly, we note that the restriction to elementwise constant eddy viscosities has also occurred in the works of Lube and Roehe [8] on full (or monolithic) VMS methods.

To introduce the idea, suppose the Navier-Stokes equations are written as

$$\frac{\partial \mathbf{u}}{\partial t} + N(\mathbf{u}) + \nu A\mathbf{u} = \mathbf{f}(t). \tag{1.4}$$

Let Π denote a postprocessing operator. The method we extend and then analyze, adds one uncoupled postprocessing step to a given method (we select the commonly used Crank-Nicolson time discretization for Step 1 for specificity): given $\mathbf{u}^n \cong \mathbf{u}(t^n)$, compute $\mathbf{u}^{n+1}$ by

Step 1: Compute $\mathbf{w}^{n+1}$ via:

$$\frac{\mathbf{w}^{n+1}-\mathbf{u}^n}{\triangle t} + N(\frac{\mathbf{w}^{n+1}+\mathbf{u}^n}{2}) + \nu A\frac{\mathbf{w}^{n+1}+\mathbf{u}^n}{2} = \mathbf{f}^{n+\frac{1}{2}}. \tag{1.5}$$

Step 2: Postprocess $\mathbf{w}^{n+1}$ to obtain $\mathbf{u}^{n+1}$:

$$\mathbf{u}^{n+1} = \Pi\mathbf{w}^{n+1}. \tag{1.6}$$

Both steps can be done by black box modules. Following Mathew et al [29], eliminating Step 2 gives:

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t} + N(\frac{\mathbf{w}^{n+1}+\mathbf{u}^n}{2}) + \nu A\frac{\mathbf{w}^{n+1}+\mathbf{u}^n}{2} + \frac{1}{\triangle t}(\mathbf{w}^{n+1} - \Pi\mathbf{w}^{n+1}) = \mathbf{f}^{n+\frac{1}{2}}, \tag{1.7}$$

where $\mathbf{f}^{n+\frac{1}{2}} = (\mathbf{f}^{n+1}+\mathbf{f}^n)/2$. We define the operator Π in Step 2, following [28] so that the extra term is exactly a nonlinear Smagorinsky model acting on small resolved scales.

$$\frac{1}{\triangle t}(\mathbf{w}^{n+1}-\mathbf{u}^{n+1}, \mathbf{v}_h) = (SmagorinskyModel, \mathbf{v}_h). \tag{1.8}$$

We consider herein two algorithmic realizations of (1.8). The first method analyzed is a full Smagorinsky model. Let P denote an L^2 projection onto a space of "well resolved" deformations, see Section 1.2 for a precise formulation in Definition 1.2.

Method 1. Let $\nu_T(\phi) = (C_s\delta)^2|[I-P]\mathbb{D}(\phi)|_F$, then

$$\frac{1}{\triangle t}(\mathbf{w}^{n+1}-\mathbf{u}^{n+1}, \mathbf{v}_h) = (\nu_T(\frac{\mathbf{w}^{n+1}+\mathbf{u}^{n+1}}{2})[I-P]\mathbb{D}(\frac{\mathbf{w}^{n+1}+\mathbf{u}^{n+1}}{2}), [I-P]\mathbb{D}(\mathbf{v}_h)), \tag{1.9}$$

where $C_s > 0$ is a Smagorinsky constant, $\delta > 0$ is the averaging radius, which is connected to the resolution of the finite element spaces involved in the VMS method (mesh size h of the fine scales or H of the large scales, see below) and $|\cdot|_F$ denotes the usual Frobenius norm of a tensor defined by for all $\mathbb{T} \in \mathbb{R}^{N\times N}$, $|\mathbb{T}|_F^2 = \sum_{i,j=1,N}(\mathbb{T}_{ij})^2$. Computationally, Step 2 reduces to the following nonlinear problem of each time step:

Given $\mathbf{w}^{n+1}$, solve the nonlinear system (1.9) for $\mathbf{u}^{n+1}$, subject to the constraint that $\nabla\cdot\mathbf{u}^{n+1} = \mathbf{0}$.

The difficulty with the modular, full or ideal Smagorinsky VMS method is exactly the cost of this nonlinear solve each time step. To reduce this cost we also give a full numerical analysis of the following Method 2 which is closely related and much less expensive.

If e denotes a typical triangle, define

$$A_e(\phi) = Average_e(\phi) = \frac{1}{|e|}\int_e \phi d\mathbf{x}.$$

Let $\nu_T(\cdot)$ denote the same turbulent viscosity parameterization, then lagging and averaging $\nu_T(\cdot)$ in Step 2 gives

Method 2. (See Algorithm 4.1, Section 4)

$$\begin{aligned}&\frac{1}{\triangle t}(\mathbf{w}^{n+1} - \mathbf{u}^{n+1} \ , \ \mathbf{v}_h)\\ &\qquad = (A_e(\nu_T(\frac{\mathbf{w}^n + \mathbf{u}^n}{2}))[I-P]\mathbb{D}(\frac{\mathbf{w}^{n+1} + \mathbf{u}^{n+1}}{2}), [I-P]\mathbb{D}(\mathbf{v}_h)).\end{aligned} \tag{1.10}$$

There are two ideas behind (1.10). The first and obvious one is that lagging $\nu_T(\cdot)$ reduces the computational problem of (1.10) to solving one (multiscale) linear equation per time step for $\mathbf{u}^{n+1}$. The second one is that with the projection operators employed, (1.10) reduces to: given $\mathbf{w}^{n+1}$ solve for $\mathbf{u}^{n+1}$ subject to $\nabla\cdot\mathbf{u}^{n+1} = 0$:

$$\begin{aligned}&(A_e(\nu_T(\cdot))[I-P]\mathbb{D}(\mathbf{u}^{n+1}), \mathbb{D}(\mathbf{v}_h)) + \frac{2}{\triangle t}(\mathbf{u}^{n+1}, \mathbf{v}_h)\\ &\qquad = \frac{2}{\triangle t}(\mathbf{w}^{n+1}, \mathbf{v}_h) - (A_e(\nu_T(\cdot)[I-P]\mathbb{D}(\mathbf{w}^{n+1}), \mathbb{D}(\mathbf{v}_h)).\end{aligned} \tag{1.11}$$

Note in particular the $\mathbb{D}(\mathbf{v}_h)$ replaces $[I-P]\mathbb{D}(\mathbf{v}_h)$. This change simplifies the computational work of (1.10) substantially.

For both methods we prove unconditional stability and delineate their energy balance (including induced model and numerical dissipation). We give a full convergence analysis of Method 1 in Theorem 3.1. This analysis uses the discrete Gronwall inequality at the last step and thus inherits the limitation introduced by its use (i.e. small time step restriction). These consequences have recently been thoroughly analyzed in [9] (for Step 1 without Step 2). Confirming numerical experiments are given in Section 5. For more numerical tests of the modular/partitioned VMS approach, see [28].

1.1. Previous Work. The VMS method is an active and rapidly developing approach to the simulation of turbulent flows; see the work of Hughes and his co-workers [5, 6, 10, 11] for its inception and recent developments. Mathematical study of it has taken several approaches, see [4, 21] for early work and [2, 3, 12, 13, 14, 15, 16, 17, 22] for some recent developments. The idea of imposing a VMS treatment of turbulence through an uncoupled Step 2 is from [28]. This work builds an work on time relaxation and filter based stabilization in [7, 19, 25, 26, 27, 29].

In the VMS considered in this paper, a "small-small" Smagorinsky eddy viscosity model is introduced acting only on the discrete resolved small scales (fluctuations). The main motivation behind the Smagorinsky model is the concept of energy cascade [1, 7], which suggests that the main role of the small scales is to extract kinetic energy out of the system. Many people use this model, [2, 10, 11, 13, 14, 15, 16, 17]. The VMS approach has gained a remarkable success in simulating the behavior of turbulence, so there is a natural need to introduce a VMS treatment of turbulence within legacy codes, in complex multi-physics applications and in other settings where reprogramming a new method from scratch in not palatable.

The idea of stabilization by a separate, modular step first appears with filtering in [26, 27], see also [25, 29]. This work also is connected to research on time relaxation stabilizations in numerical methods, continuum models and approximate models via (1.7).

This paper is organized into four sections. In the remainder of this section we establish the notations that will be used throughout the work and present a standard weak formulation of the Navier-Stokes equations. In the second section, the uncoupled projection-based VMS scheme is described and the stability of the method is provided. In the third section we present the error estimate for the algorithm. We also present the variant of the method and analyze its stability in section 4. The last section describes the implementation of two algorithms and presents the some numerical results to confirm the theoretical analysis.

1.2. Notations. Let Ω be an open, bounded region in $\mathbb{R}^d, d = 2$ *or* 3 with a Lipschitz continuous boundary. Throughout this paper, standard notations are used for Lebesgue space $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega), 1 < p < \infty, k = 0, 1, 2, \ldots$. The corresponding norms are denoted by $||\cdot||_{L^p}$ and $||\cdot||_{W^{k,p}}$, respectively. $H^k(\Omega)$ is used to represent the Sobolev space $W^{k,2}(\Omega)$, $|\cdot|_k$ and $||\cdot||_k$ denote the semi-norm and norm in $H^k(\Omega)$, respectively. Particularly, we will denote $H^0(\Omega)$ by $L^2(\Omega)$ and the standard L^2 inner product by $(\cdot,\cdot)$, L^2 norm by $||\cdot||$. The space $H^{-k}(\Omega)$ denotes the dual space of $H^k(\Omega)$. In addition, the vector spaces and vector functions will be indicated by boldface type letters. For the function $\mathbf{v}(x, t)$ defined on the entire time interval $(0, T)$, we define

$$||\mathbf{v}||_{\infty,k} := EssSup_{[0,T]}||\mathbf{v}(t,\cdot)||_k \text{ and } ||\mathbf{v}||_{m,k} := (\int_0^T ||\mathbf{v}(t,\cdot)||_k^m dt)^{1/m}.$$

Define the velocity space $\mathbf{X}$, the pressure space Q and the deformation space $\mathbf{L}$ as follows:

$$\begin{aligned}
\mathbf{X} :&= \mathbf{H}_0^1(\Omega) = \{\mathbf{v} : \mathbf{v} \in \mathbf{H}^1(\Omega), \mathbf{v} = 0 \text{ on } \partial\Omega\}, \\
Q :&= L_0^2(\Omega) := \{q \in L^2(\Omega), \int_\Omega q dx = 0\}, \\
\mathbf{L} :&= \{\mathbb{L} \in (L^2(\Omega))^{d\times d}, \mathbb{L} = \mathbb{L}^T\}.
\end{aligned}$$

We denote the dual space of $\mathbf{X}$ by $\mathbf{X}^*$, with the norm $||\cdot||_*$. The space of divergence free functions is given by

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : (\nabla\cdot\mathbf{v}, q) = 0, \ \forall\, q \in Q\}.$$

Let $\mathcal{T}_H$ denotes a coarse finite element mesh which is refined (once, twice, ...) to produce the finer mesh $\mathcal{T}_h$, so $h < H$. Let $(\mathbf{X}^h, Q^h)$ be a pair of conforming velocity-pressure finite element spaces satisfying the usual inf-sup condition (see Gunzburger [33]): there exists a constant β independent of h such that

$$\inf_{q^h\in Q^h} \sup_{\mathbf{v}^h\in\mathbf{X}^h} \frac{(q^h, \nabla\cdot\ \mathbf{v}^h)}{||q^h||||\nabla\mathbf{v}^h||} \geq \beta > 0. \tag{1.12}$$

Examples of such compatible spaces are the mini-element spaces [34], the Taylor-Hood spaces [33] and the continuous piecewise quadratics for the velocity space and discontinuous piecewise constants for the pressure space [35]. We assume that the spaces $\mathbf{X}^h$ and Q^h contain piecewise continuous polynomials of degree k and $k-1$, respectively, and suppose that the spaces $(\mathbf{X}^h, Q^h)$ satisfy the following approximation properties:

$$\begin{aligned}
\inf_{\mathbf{v}^h\in\mathbf{X}^h}\{||\mathbf{u}-\mathbf{v}^h|| + h||\nabla(\mathbf{u}-\mathbf{v}^h)||\} &\leq Ch^{k+1}|u|_{k+1}, \forall\ u \in \mathbf{H}^{k+1}(\Omega)\cap\mathbf{X}, \\
\inf_{q^h\in Q^h} ||p - q^h|| &\leq Ch^k|p|_k,\ \forall\ p \in H^k(\Omega)\cap Q.
\end{aligned}$$

Through the paper, C denotes a generic constant which is does not depend ν, h, H, δ, unless specified.

Furthermore, we introduce the discretely divergence free subspace of $\mathbf{X}^h$,

$$\mathbf{V}^h := \{\mathbf{v}^h \in \mathbf{X}^h : (\nabla \cdot \mathbf{v}^h, q^h) = 0, \ \forall\, q^h \in Q^h\}.$$

We shall use a space of "well resolved" velocity deformations. There are two natural ways to define this $\mathbf{L}^H$: (i) via a coarser mesh, and (ii) via a lower polynomial degree element on the same mesh. If $\mathbf{X}^h$ is a higher order finite element space on a given mesh, one approach is to define the large scale space using lower order finite elements on the same mesh. For low order elements, the only option is to define the large scale space $\mathbf{L}^H$ on a coarse mesh leading to a two-level discretization. In our numerical tests, we choose finite element spaces $\mathbf{L}^H \subset \mathbf{L}$ on the coarse finite element mesh $\mathcal{T}_H$. To present the method, we introduce the following definitions of projection operators.

Definition 1.1. *(L^2 projection) $P_{L^H} : \mathbf{L} \to \mathbf{L}^H$ is the L^2- orthogonal projection operator.*

We take a coarse mesh velocity space, denoted $\mathbf{X}^H$, and select

$$\mathbf{L}^H = \{\mathbb{D}(\mathbf{v}^h) : \forall\, \mathbf{v}^H \in \mathbf{X}^H, H > h\},$$

so that P_{L^H} satisfies

$$(P_{L^H}\mathbb{L}, \mathbb{L}^H) = (\mathbb{L}, \mathbb{L}^H), \forall\, \mathbb{L} \in \mathbf{L}, \mathbb{L}^H \in \mathbf{L}^H, \tag{1.13}$$

$$||(I - P_{L^H})\mathbb{L}|| \leq CH^k|\mathbb{L}|_k, \forall\, \mathbb{L} \in \mathbf{L} \cap \mathbf{H}^k(\Omega). \tag{1.14}$$

Definition 1.2. *(Elliptic projection). $P_H : \mathbf{X} \to \mathbf{X}^H$ is the projection operator satisfying*

$$(\mathbb{D}[\mathbf{w} - P_H(\mathbf{w})], \mathbb{D}(\mathbf{v}^H)) = 0, \quad \forall\, \mathbf{v}^H \in \mathbf{V}^H. \tag{1.15}$$

From [24], see also [22] and [13], we have the following result:

Lemma 1.3. *Let $\mathbf{v} \in \mathbf{X}$ and $\mathbf{L}^H = \mathbb{D}(\mathbf{X}^H)$, Then*

$$P_{L^H}(\mathbb{D}(\mathbf{v})) = \mathbb{D}(P_H\mathbf{v}) \ \texttt{and}\ (I - P_{L^H})\mathbb{D}(\mathbf{v}) = \mathbb{D}((I - P_H)\mathbf{v}), \ \forall \mathbf{v} \in \mathbf{X}. \tag{1.16}$$

We are interested in approximating the solution of the evolutionary Naiver-Stokes equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}) + \nabla p = \mathbf{f} \quad \texttt{in}\ (0, T] \times \Omega, \tag{1.17}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \texttt{in}\ [0, T] \times \Omega, \tag{1.18}$$

$$\mathbf{u} = 0 \quad \texttt{in}\ [0, T] \times \partial\Omega, \tag{1.19}$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0 \ \texttt{in}\ \Omega, \tag{1.20}$$

$$\int_\Omega p dx = 0 \quad \texttt{in}\ (0, T]. \tag{1.21}$$

Here, $\mathbb{D}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$ and $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ is the given body force, ν is the kinematic viscosity, which is inverse proportional to the Reynolds number R_e, $\mathbf{u}_0$ is the initial velocity field, and $[0, T]$ is a finite time interval.

The variational formulation of the Navier-Stokes equations (1.17)-(1.21): find $\mathbf{u} : [0, T] \to \mathbf{X}, p : (0, T] \to Q$ satisfying

$$(\mathbf{u}_t,\mathbf{v})+b_s(\mathbf{u},\mathbf{u},\mathbf{v})+\nu(\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))-(p,\nabla\cdot\mathbf{v})=(\mathbf{f},\mathbf{v}), \tag{1.22}$$

$$(q,\nabla\cdot\mathbf{u})=0, \tag{1.23}$$

for all $(\mathbf{v},q)\in(\mathbf{X},Q)$. Here

$$b_s(\mathbf{u},\mathbf{v},\mathbf{w})=\frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})-\frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})$$

is the skew-symmetric trilinear form of the convective term. It has the following properties:

$$b_s(\mathbf{u},\mathbf{v},\mathbf{w})=-b_s(\mathbf{u},\mathbf{w},\mathbf{v}), \tag{1.24}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}$ and consequently

$$b_s(\mathbf{u},\mathbf{v},\mathbf{v})=0,\quad\forall\ \mathbf{u},\mathbf{v}\in\mathbf{X}. \tag{1.25}$$

LEMMA 1.4. *Let $\Omega\subset\mathbb{R}^d$, then*

$$|b_s(\mathbf{u},\mathbf{v},\mathbf{w})|\le C||\nabla\mathbf{u}||||\nabla\mathbf{v}||||\nabla\mathbf{w}||, \tag{1.26}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}$.

LEMMA 1.5. *Let $\Omega\subset\mathbb{R}^3$, then*

$$|b_s(\mathbf{u},\mathbf{v},\mathbf{w})|\le C(\Omega)||\mathbf{u}||^{1/2}||\nabla\mathbf{u}||^{1/2}||\nabla\mathbf{v}||||\nabla\mathbf{w}||, \tag{1.27}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}$.

In the divergence-free space (1.22)-(1.23) can be reformulated as follows: find $\mathbf{u}:[0,T]\to\mathbf{V}$ satisfying

$$(\mathbf{u}_t,\mathbf{v})+b_s(\mathbf{u},\mathbf{u},\mathbf{v})+\nu(\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))=(\mathbf{f},\mathbf{v}), \tag{1.28}$$

for all $\mathbf{v}\in\mathbf{V}$.

2. Uncoupled projection-based VMS method with nonlinear eddy viscosity. In this section, we consider an uncoupled algorithm with nonlinear eddy viscosity for the finite element discretization of NSE (1.17)-(1.21). FEM in space discretization and Crank-Nicolson (CN) method in time discretization with an additional postprocessing step presented as follows.

Algorithm 2.1

Step 1: Given $\mathbf{u}_h^n$, compute $\mathbf{w}_h^{n+1}\in\mathbf{X}^h, p_h^{n+1}\in Q^h$ satisfying

$$\left\{\begin{array}{l}(\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^n}{\triangle t},\mathbf{v}_h)+b_s(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2},\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2},\mathbf{v}_h)\\ \qquad\qquad+\nu(\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}),\mathbb{D}(\mathbf{v}_h))-(p_h^{n+1},\nabla\cdot\mathbf{v}_h)=(\mathbf{f}^{n+\frac{1}{2}},\mathbf{v}_h),\\ (\nabla\cdot\mathbf{w}_h^{n+1},q_h)=0,\end{array}\right. \tag{2.1}$$

for all $(\mathbf{v}_h,q_h)\in(\mathbf{X}^h,Q^h)$.

Step 2: Given $\mathbf{w}_h^{n+1}$ solve the following to obtain $\mathbf{u}_h^{n+1}$:

$$\left\{\begin{array}{l}(\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^{n+1}}{\triangle t},\mathbf{v}_h)=(p_h^{n+1},\nabla\cdot\mathbf{v}_h)\\ \quad+((C_s\delta)^2|[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})|_F[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}),[I-P_{L^H}]\mathbb{D}(\mathbf{v}_h)),\\ (\nabla\cdot\mathbf{u}_h^{n+1},q_h)=0,\end{array}\right. \tag{2.2}$$

for all $(\mathbf{v}_h, q_h) \in (\mathbf{X}^h, Q^h)$, where P_{L^H} is a L^2-projection operator defined by (1.13).

Using the property of projection (1.16), one can rewrite Step 2 in the following way.
Restated Step 2: Given $\mathbf{w}_h^{n+1}$ solve the following to obtain $\mathbf{u}_h^{n+1}$:

$$(2.3)\quad \begin{cases} (\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^{n+1}}{\triangle t}, \mathbf{v}_h) = (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) \\ \quad +((C_s\delta)^2|\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})|_F\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}), \mathbb{D}([I-P_H]\mathbf{v}_h)), \\ (\nabla\cdot\mathbf{u}_h^{n+1}, q_h) = 0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in (\mathbf{X}^h, Q^h)$, where P_H is an elliptic projector defined by (1.15).

Before discussing the stability of the method, we recall some important analytical tools in the analysis of the Smagorinsky model, see [21].

Lemma 2.1. *(Strong monotonicity and local Lipschitz continuity) There is a constant $C > 0$ such that for all $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{1,3}(\Omega)$,*

$$(2.4)\quad (|\mathbb{D}(\mathbf{u})|_F\mathbb{D}(\mathbf{u}) - |\mathbb{D}(\mathbf{v})|_F\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{u}-\mathbf{v})) \geq C||\mathbb{D}(\mathbf{u}-\mathbf{v})||_{L^3}^3,$$

and the local Lipschitz continuity: there exists a constant $C > 0$ such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{W}^{1,3}(\Omega)$,

$$(2.5)\quad (|\mathbb{D}(\mathbf{u})|_F\mathbb{D}(\mathbf{u}) - |\mathbb{D}(\mathbf{v})|_F\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{w})) \leq CC_L||\mathbb{D}(\mathbf{u}-\mathbf{v})||_{L^3}||\mathbb{D}(\mathbf{w})||_{L^3},$$

where $C_L = max\{||\mathbb{D}(\mathbf{u})||_{L^3}, ||\mathbb{D}(\mathbf{v})||_{L^3}\}$.

The theory begins with a clear global energy balance. It is derived in proposition 2.3, and its proof utilizes the following lemma.

Lemma 2.2. *Let $C_s > 0, \delta > 0$ defined as above, then there holds*

$$(2.6)\quad ||\mathbf{w}_h^{n+1}||^2 = ||\mathbf{u}_h^{n+1}||^2 + 2\triangle tC(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||_{L^3}^3.$$

Proof. We choose $\mathbf{v}_h = \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}$ and $q_h = p_h^{n+1}$ in (2.3), by using the monotonicity (2.4) with $\mathbf{u} = \mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})$ and $\mathbf{v} = 0$, note that since $\mathbf{w}_h^{n+1} \in \mathbf{V}^h$, we get

$$\begin{aligned} &\frac{1}{2\triangle t}(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{u}_h^{n+1}||^2) \\ &= (C_s\delta)^2(|\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})|_F\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}), \mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})) \\ &\geq C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||_{L^3}^3. \end{aligned}$$

On the other hand, by using the local Lipschitz continuity (2.5) with $\mathbf{u} = \mathbf{w} = \mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})$ and $\mathbf{v} = 0$, we have

$$\begin{aligned} &\frac{1}{2\triangle t}(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{u}_h^{n+1}||^2) \\ &\quad = (C_s\delta)^2(|\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})|_F\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}), \mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})) \\ &\quad \leq C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||_{L^3}^3. \end{aligned}$$

Combining the above two inequalities results in (2.6). $\square$

Now, we prove the strong energy equality and associated strong, unconditional stability property of the method.

PROPOSITION 2.3. *Let $C_s > 0, \delta > 0$ defined as above, then the approximate velocity $\mathbf{u}_h^{n+1}$ given by the Algorithm 2.1 satisfies the energy equality*

$$\frac{1}{2}||\mathbf{u}_h^{l+1}||^2 + \triangle t\sum_{n=0}^{l}(C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||_{L^3}^3 + \nu||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}||^2) = \frac{1}{2}||\mathbf{u}_h^0||^2 + \triangle t\sum_{n=0}^{l}(f^{n+\frac{1}{2}}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}). \tag{2.7}$$

and the stability bound

$$\frac{1}{2}||\mathbf{u}_h^{l+1}||^2 + \triangle t\sum_{n=0}^{l}(C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||_{L^3}^3 + \frac{\nu}{2}||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}||^2) \le \frac{1}{2}||\mathbf{u}_h^0||^2 + \frac{\triangle t}{2\nu}\sum_{n=0}^{l}||\mathbf{f}^{n+\frac{1}{2}}||_*^2. \tag{2.8}$$

Proof. Setting $\mathbf{v}_h = \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}$ in (2.1) as the test function, this gives

$$\frac{1}{2\triangle t}(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{u}_h^n||^2) + \nu||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}||^2 = (f^{n+\frac{1}{2}}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}).$$

The application of Lemma 2.2 to this inequality gives

$$\frac{1}{2\triangle t}(||\mathbf{u}_h^{n+1}||^2 - ||\mathbf{u}_h^n||^2) + C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||_{L^3}^3 \quad + \nu||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}||^2 = (f^{n+\frac{1}{2}}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}),$$

summing over n establishes the energy equality. Using the Cauchy-Schwarz and Young's inequalities on the right-hand side, subsuming one term into the left-hand side gives

$$\frac{1}{2\triangle t}(||\mathbf{u}_h^{n+1}||^2 - ||\mathbf{u}_h^n||^2) + C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||_{L^3}^3 + \frac{\nu}{2}||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}||^2 \le \frac{1}{2\nu}||\mathbf{f}^{n+\frac{1}{2}}||_*^2.$$

Summing over n, the global stability estimate follows. $\square$

The method is thus stable. The viscous and numerical dissipation in the method are respectively

$$Viscous\ dissipation := \nu||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}||^2,$$
$$Numerical\ dissipation := C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||_{L^3}^3.$$

Another consequence of the above á priori bound is existence of a solution to the Step 2 problem. Uniqueness follows by monotonicity (which can also be used to prove existence) in a standard way.

Corollary 2.4. *Consider Step 2 in Algorithm 2.1. The solution $\mathbf{u}_h^{n+1}$ exists and is unique. If the discrete inf-sup condition (1.12) holds, then p_h^{n+1} exists and is unique.*

Furthermore, we also prove the stability for $\mathbf{w}_h^N$.

Proposition 2.5. *Under the assumption of proposition 2.3, the approximate velocity $\mathbf{w}_h^{n+1}$ given by the Algorithm 2.1 satisfies the energy inequality*

$$
\begin{aligned}
(2.9)\quad &\frac{1}{2}||\mathbf{w}_h^{l+1}||^2 + \triangle t\sum_{n=0}^{l-1} C(C_s\delta)^2||\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}||_{L^3}^3 + \frac{\triangle t\nu}{2}\sum_{n=0}^{l}||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n}}{2}||^2\\
&\leq ||\mathbf{u}_h^0||^2 + \frac{\triangle t}{2\nu}\sum_{n=0}^{l}||\mathbf{f}^{n+\frac{1}{2}}||_*^2.
\end{aligned}
$$

Proof. For $n=l$, a directly application of Lemma 2.2 gives

$$
||\mathbf{w}_h^{l+1}||^2 = ||\mathbf{u}_h^{l+1}||^2 + 2\triangle tC(C_s\delta)^2||\mathbb{D}[I-P_H]\frac{\mathbf{w}_h^{l+1}+\mathbf{u}_h^{l+1}}{2}||_{L^3}^3.
$$

Take it into the Proposition 2.3, we prove the claim. ◻

3. Error Estimate. In this section, we present a detailed error analysis for the approximation scheme. In order to establish the optimal asymptotic error estimates we assume the following regularity of the true solutions:

$$
(3.1)\quad \mathbf{u}\in L^\infty(0,T;W_4^{k+1}(\Omega))\cap H^1(0,T;H^{k+1}(\Omega))\cap H^3(0,T;L^2(\Omega))\cap W_4^2(0,T;H^1(\Omega)),
$$

$$
(3.2)\quad p\in L^\infty(0,T;H^k(\Omega)),\quad \mathbf{f}\in H^2(0,T;L^2(\Omega)).
$$

We denote $t^n = n\triangle t, n=0,1,2,\cdots,N_T$ and $T=N_T\triangle t$, we introduce the following discrete norms:

$$
|||v|||_{\infty,k} = \max_{0\leq n\leq N^T}||v^n||_k,\ \ |||v|||_{m,k} = (\triangle t\sum_{n=0}^{N_T}||v^n||_k^m)^{1/m},\ \ |||v_{1/2}|||_{m,k} = (\triangle t\sum_{n=0}^{N_T}||v^{n+1/2}||_k^m)^{1/m}.
$$

For notational convenience, we denote

$$
\tilde{\mathbf{w}}_h^{n+\frac{1}{2}} = \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}.
$$

To begin the analysis we rewrite Algorithm 2.1 in the following form: For $n=1,2,\cdots,N_T$, find $\mathbf{w}_h^{n+1},\mathbf{u}_h^{n+1}\in\mathbf{X}^h, p_h^{n+1}\in Q^h$ such that

$$
\begin{aligned}
(3.3)\quad &(\mathbf{w}_h^{n+1},\mathbf{v}_h) + \triangle tb_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}},\tilde{\mathbf{w}}_h^{n+\frac{1}{2}},\mathbf{v}_h) - \triangle t(p_h^{n+1},\nabla\cdot\mathbf{v}_h) + \triangle t\nu(\mathbb{D}\tilde{\mathbf{w}}_h^{n+1},\mathbb{D}\mathbf{v}_h)\\
&\qquad = (\mathbf{u}_h^n,\mathbf{v}_h) + \triangle t(\mathbf{f}^{n+\frac{1}{2}},\mathbf{v}_h),\\
(3.4)\quad &(\nabla\cdot\mathbf{w}_h^{n+1},q_h) = 0,\\
(3.5)\quad &\mathbf{u}_h^{n+1} = \Pi\mathbf{w}_h^{n+1},
\end{aligned}
$$

for all $(\mathbf{v}_h,q_h)\in(\mathbf{X}^h,Q^h)$.

We can also consider the following equivalent problem: For $n=1,2,\cdots,N_T$ find $\mathbf{w}_h^{n+1},\mathbf{u}_h^{n+1}\in\mathbf{V}^h$ such that

$$
\begin{aligned}
(3.6)\quad &(\ \mathbf{w}_h^{n+1},\mathbf{v}_h) + \triangle tb_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}},\tilde{\mathbf{w}}_h^{n+\frac{1}{2}},\mathbf{v}_h) + \triangle t\nu(\mathbb{D}(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}),\mathbb{D}(\mathbf{v}_h))\\
&\qquad = (\mathbf{u}_h^n,\mathbf{v}_h) + \triangle t(\mathbf{f}^{n+\frac{1}{2}},\mathbf{v}_h),\\
(3.7)\quad &\frac{1}{\triangle t}(\mathbf{w}_h^{n+1}-\mathbf{u}_h^{n+1},\mathbf{v}_h)\\
&= ((C_s\delta)^2|\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})|_F\mathbb{D}([I-P_H]\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}),\mathbb{D}([I-P_H]\mathbf{v}_h)),
\end{aligned}
$$

for all $\mathbf{v}_h \in \mathbf{V}_h$.

Let us start the error estimate at time $t^{n+\frac{1}{2}} = (n+\frac{1}{2})\triangle t$ with $\mathbf{u}$ given by (1.22)-(1.23), which satisfies

$$(3.8)\qquad \begin{aligned}(\mathbf{u}^{n+1}-\mathbf{u}^n,\mathbf{v}_h)+\triangle t\nu(\mathbb{D}(\mathbf{u}^{n+\frac{1}{2}}),\mathbb{D}(\mathbf{v}_h))+\triangle t b_s(\mathbf{u}^{n+\frac{1}{2}},\mathbf{u}^{n+\frac{1}{2}},\mathbf{v}_h)\\ -\triangle t(p(t^{n+\frac{1}{2}}),\nabla\cdot\mathbf{v}_h)=\triangle t(\mathbf{f}^{n+\frac{1}{2}},\mathbf{v}_h)+\triangle tR(\mathbf{u}^{n+1},\mathbf{v}_h),\end{aligned}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$, where $R(\mathbf{u}^{n+1},\mathbf{v}_h)$ represents the interpolating error, i.e.

$$(3.9)\ \begin{aligned}R(\mathbf{u}^{n+1},\mathbf{v}_h)=&(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t},\mathbf{v}_h)+\nu(\mathbb{D}(\mathbf{u}^{n+\frac{1}{2}})-\mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}})),\mathbb{D}(\mathbf{v}_h))\\ &+b_s(\mathbf{u}^{n+\frac{1}{2}},\mathbf{u}^{n+\frac{1}{2}},\mathbf{v}_h)-b_s(\mathbf{u}(t^{n+\frac{1}{2}}),\mathbf{u}(t^{n+\frac{1}{2}}),\mathbf{v}_h)+(\mathbf{f}(t^{n+\frac{1}{2}})-\mathbf{f}^{n+\frac{1}{2}},\mathbf{v}_h).\end{aligned}$$

We split the error into a model error ε_h according to (3.3)-(3.4), a model error e_h according to (3.5), and an approximation error Λ as

$$(3.10)\qquad \mathbf{u}^{n+1}-\mathbf{w}_h^{n+1}=(\mathbf{u}^{n+1}-I_h\mathbf{u}^{n+1})+(I_h\mathbf{u}^{n+1}-\mathbf{w}_h^{n+1})\triangleq\Lambda^{n+1}+\varepsilon_h^{n+1},$$

$$(3.11)\qquad \mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}=(\mathbf{u}^{n+1}-I_h\mathbf{u}^{n+1})+(I_h\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1})\triangleq\Lambda^{n+1}+e_h^{n+1},$$

where $I_h\mathbf{u}^{n+1} \in \mathbf{V}^h$ will be an interpolation of $\mathbf{u}^{n+1}$ in $\mathbf{V}^h$.

THEOREM 3.1. *For $\mathbf{u}, p$ and $\mathbf{f}$ given by (1.22)-(1.23) satisfying regularity assumptions (3.1)-(3.2), and $\mathbf{u}_h^n, \mathbf{w}_h^n$ given by Algorithm 2.1, then for $\triangle t$ sufficiently small, i.e., $\triangle t < (1+\nu^{-3}||\nabla u(t^{n+\frac{1}{2}})||^4)^{-1}$, we have*

$$(3.12)\qquad \begin{aligned}|||\mathbf{u}-\mathbf{u}_h|||_{\infty,0}^2+|||\mathbf{u}-\mathbf{w}_h|||_{\infty,0}^2\le E(\triangle t,h,H,\nu,\delta)+C(\Omega)h^{2k+2}|||\mathbf{u}|||_{\infty,k+1}^2,\\ \nu\triangle t\sum_{n=0}^{N-1}||\mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}})-\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2})||^2\ \le E(\triangle t,h,H,\nu,\delta)+C(\Omega)\nu\triangle t^4|||\nabla\mathbf{u}_{tt}|||_{2,0}^2\\ +C(\Omega)\nu h^{2k}|||\mathbf{u}|||_{2,k+1}^2,\ \forall\ 0\le N\le N_T,\end{aligned}$$

where

$$\begin{aligned}E(\triangle t,h,H,\nu,\delta)=&C(\Omega)\nu^{-1}(h^{2k+1}|||\mathbf{u}|||_{4,k+1}^4+h^{2k+1}|||\nabla\mathbf{u}|||_{4,0}^4+h^{2k}|||p|||_{2,k}^2)\\ &+C(\Omega)h^{2k+2}|||\mathbf{u}_t|||_{2,k+1}^2+C(\Omega)\nu^{-2}h^{2k}|||\mathbf{u}|||_{\infty,k+1}^2+C(\Omega)\nu h^{2k}|||\mathbf{u}|||_{2,k+1}^2\\ &+C(C_s\delta)^2h^{-\frac{d}{2}}H^{3k}|||\mathbf{u}|||_{3,k+1}^3+C(\Omega)\triangle t^4\nu^{-1}(|||\nabla\mathbf{u}|||_{4,0}^4+|||\nabla\mathbf{u}_{1/2}|||_{4,0}^4)\\ &+C(\Omega)\triangle t^4(|||\mathbf{u}_{ttt}|||_{2,0}^2+\nu|||\nabla\mathbf{u}_{tt}|||_{2,0}^2+\nu^{-1}|||\nabla\mathbf{u}_{tt}|||_{4,0}^4+|||\mathbf{f}_{tt}|||_{2,0}^2).\end{aligned}$$

Proof. First, Let us take the difference of the equation (3.6) from (3.8), and choose $\mathbf{v}_h = \frac{\varepsilon_h^{n+1}+e_h^n}{2} \in \mathbf{V}^h$ as the test function, we obtain

$$(3.13)\qquad \begin{aligned}&\frac{1}{2}(||\varepsilon_h^{n+1}||^2-||e_h^n||^2)+\triangle t\nu||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2\\ &\quad=-(\Lambda^{n+1}-\Lambda^n,\frac{\varepsilon_h^{n+1}+e_h^n}{2})-\triangle t\nu(\mathbb{D}(\Lambda^{n+\frac{1}{2}}),\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2})\\ &\quad-\triangle tb_s(\mathbf{u}^{n+\frac{1}{2}},\mathbf{u}^{n+\frac{1}{2}},\frac{\varepsilon_h^{n+1}+e_h^n}{2})+\triangle tb_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}},\tilde{\mathbf{w}}_h^{n+\frac{1}{2}},\frac{\varepsilon_h^{n+1}+e_h^n}{2})\\ &\quad+\triangle t(p(t^{n+\frac{1}{2}})-q_h,\nabla\cdot\frac{\varepsilon_h^{n+1}+e_h^n}{2})+\triangle tR(\mathbf{u}^{n+1},\frac{\varepsilon_h^{n+1}+e_h^n}{2}).\end{aligned}$$

Note that since $\frac{\varepsilon_h^{n+1}+e_h^n}{2} \in \mathbf{V}^h$, $(q_h, \nabla \cdot \frac{\varepsilon_h^{n+1}+e_h^n}{2}) = 0$, and we can write

$$(p(t^{n+\frac{1}{2}}), \nabla \cdot \frac{\varepsilon_h^{n+1}+e_h^n}{2}) = (p(t^{n+\frac{1}{2}}) - q_h, \nabla \cdot \frac{\varepsilon_h^{n+1}+e_h^n}{2}),$$

for all $q^h \in Q^h$.

We want to bound the terms on the right-hand side of (3.13). Consider first the convection term in (3.13), adding and subtracting the term $b_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2})$, taking (1.25) into account, then the trilinear terms can be rewritten as follows:

$$\begin{aligned}
&b_s(\mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}) - b_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2})\\
&= b_s(\frac{\mathbf{u}^{n+1}-\mathbf{w}^{n+1}+\mathbf{u}^n-\mathbf{u}_h^n}{2}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}) + b_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \frac{\mathbf{u}^{n+1}-\mathbf{w}_h^{n+1}+\mathbf{u}^n-\mathbf{u}_h^n}{2}, \frac{\varepsilon_h^{n+1}+e_h^n}{2})\\
&= b_s(\Lambda^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}) + b_s(\frac{\varepsilon_h^{n+1}+e_h^n}{2}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}) + b_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \Lambda^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}).
\end{aligned}$$

In order to bound the first two nonlinear terms on the right-hand side of the last equation, we use Lemma 1.5, Young's and Korn's inequalities:

$$\begin{aligned}
b_s(\Lambda^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}) &\le C(\Omega)||\Lambda^{n+\frac{1}{2}}||^{\frac{1}{2}}||\nabla\Lambda^{n+\frac{1}{2}}||^{\frac{1}{2}}||\nabla\mathbf{u}^{n+\frac{1}{2}}||||\nabla\frac{\varepsilon_h^{n+1}+e_h^n}{2}||\\
&\le \frac{\nu}{10}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2 + \frac{C(\Omega)}{\nu}||\Lambda^{n+\frac{1}{2}}||||\nabla\Lambda^{n+\frac{1}{2}}||||\nabla\mathbf{u}^{n+\frac{1}{2}}||^2, \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
b_s(\frac{\varepsilon_h^{n+1}+e_h^n}{2}, \mathbf{u}^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}) &\le C(\Omega)||\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^{\frac{1}{2}}||\nabla\mathbf{u}^{n+\frac{1}{2}}||||\nabla\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^{\frac{3}{2}}\\
&\le \frac{\nu}{10}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2 + \frac{C(\Omega)}{\nu^3}||\nabla\mathbf{u}^{n+\frac{1}{2}}||^4(||\varepsilon_h^{n+1}||^2 + ||e_h^n||^2). \qquad (3.15)
\end{aligned}$$

The last nonlinear term is bounded by Lemma 1.4, Korn's and Young's inequalities:

$$\begin{aligned}
b_s(\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}, \Lambda^{n+\frac{1}{2}}, \frac{\varepsilon_h^{n+1}+e_h^n}{2}) &\le C||\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}||||\nabla\Lambda^{n+\frac{1}{2}}||||\nabla\frac{\varepsilon_h^{n+1}+e_h^n}{2}||\\
&\le \frac{\nu}{10}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2 + \frac{C}{\nu}||\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}||^2||\nabla\Lambda^{n+\frac{1}{2}}||^2. \qquad (3.16)
\end{aligned}$$

The remaining terms in (3.13) are estimated by Cauchy-Schwarz, Young's and Minkowski's inequalities as follows:

$$\begin{aligned}
(\Lambda^{n+1}-\Lambda^n, \frac{\varepsilon_h^{n+1}+e_h^n}{2}) &= \triangle t(\frac{\Lambda^{n+1}-\Lambda^n}{\triangle t}, \frac{\varepsilon_h^{n+1}+e_h^n}{2})\\
&\le \frac{\triangle t}{2}||\frac{\Lambda^{n+1}-\Lambda^n}{\triangle t}||^2 + \frac{\triangle t}{2}||\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2\\
&= \frac{\triangle t}{2}\int_\Omega(\frac{1}{\triangle t}\int_{t^n}^{t^{n+1}}\Lambda_t dt)^2 d\Omega + \frac{\triangle t}{2}||\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2\\
&\le \frac{\triangle t}{2}\int_\Omega(\frac{1}{\triangle t}\int_{t^n}^{t^{n+1}}|\Lambda_t|^2 dt) d\Omega + \frac{\triangle t}{2}||\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2\\
&\le \frac{1}{2}\int_{t^n}^{t^{n+1}}||\Lambda_t||^2 dt + \frac{\triangle t}{4}(||\varepsilon_h^{n+1}||^2 + ||e_h^n||^2), \qquad (3.17)
\end{aligned}$$

$$\triangle t\nu(\mathbb{D}(\Lambda^{n+\frac{1}{2}}),\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2})\le\frac{\nu\triangle t}{10}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+\frac{5\triangle t\nu}{2}||\nabla\Lambda^{n+\frac{1}{2}}||^2, \tag{3.18}$$

$$\begin{aligned}(p(t^{n+\frac{1}{2}})-q_h,\nabla\cdot\frac{\varepsilon_h^{n+1}+e_h^n}{2}) &\le ||p(t^{n+\frac{1}{2}})-q_h||||\nabla\cdot\frac{\varepsilon_h^{n+1}+e_h^n}{2}||\\ &\le\frac{\nu}{10}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+\frac{C}{\nu}||p(t^{n+\frac{1}{2}})-q_h||^2.\end{aligned} \tag{3.19}$$

For the last term in (3.13), we present its estimate in the following lemma.

Lemma 3.2. *There holds*

$$\begin{aligned}\triangle tR(\mathbf{u}^{n+1},\frac{\varepsilon_h^{n+1}+e_h^n}{2}) &\le\frac{\triangle t}{2}(||\varepsilon_h^{n+1}||^2+||e_h^n||^2)+\frac{\nu\triangle t}{4}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2\\ &+\frac{C(\Omega)\triangle t^5}{\nu}(||\nabla\mathbf{u}^{n+\frac{1}{2}}||^4+||\nabla u(t^{n+\frac{1}{2}})||^4)\\ &+C(\Omega)\triangle t^4\int_{t^n}^{t^{n+1}}(||\mathbf{u}_{ttt}||^2+\nu||\nabla\mathbf{u}_{tt}||^2+\frac{1}{\nu}||\nabla u_{tt}||^4+||\mathbf{f}_{tt}||^2)dt.\end{aligned} \tag{3.20}$$

Proof. In the same way, we estimate every term in the definition (3.9) of $R(\cdot,\cdot)$. The application of the Cauchy-Schwarz inequality, Young's inequality and Minkowski's inequality shows

$$\begin{aligned}(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t},\frac{\varepsilon_h^{n+1}+e_h^n}{2}) &\le\frac{1}{2}||\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+\frac{1}{2}||\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t}||^2\\ &\le\frac{1}{4}||\varepsilon_h^{n+1}||^2+\frac{1}{4}||e_h^{n+1}||^2+\frac{\triangle t^3}{1152}\int_{t^n}^{t^{n+1}}||u_{ttt}||^2dt.\end{aligned} \tag{3.21}$$

Similarly,

$$\begin{aligned}\nu(\mathbb{D}(\mathbf{u}^{n+\frac{1}{2}})-\mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}})),\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}) &\le\frac{\nu}{8}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+2\nu||\nabla\mathbf{u}^{n+\frac{1}{2}}-\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^2\\ &\le\frac{\nu}{8}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+\frac{\nu\triangle t^3}{8}\int_{t^n}^{t^{n+1}}||\nabla\mathbf{u}_{tt}||^2dt,\end{aligned} \tag{3.22}$$

and

$$(\mathbf{f}(t^{n+\frac{1}{2}})-\mathbf{f}^{n+\frac{1}{2}},\frac{\varepsilon_h^{n+1}+e_h^n}{2})\le\frac{1}{4}||\varepsilon_h^{n+1}||^2+\frac{1}{4}||e_h^{n+1}||^2+\frac{\triangle t^3}{32}\int_{t^n}^{t^{n+1}}||\mathbf{f}_{tt}||^2dt. \tag{3.23}$$

By adding and subtracting the term $b_s(\mathbf{u}(t^{n+\frac{1}{2}}),\mathbf{u}^{n+\frac{1}{2}},\frac{\varepsilon_h^{n+1}+e_h^n}{2})$, using Lemma 1.4, Young's and Minkowski's inequalities, we can estimate the trilinear terms as follows:

$$\begin{aligned}&b_s(\mathbf{u}^{n+\frac{1}{2}},\mathbf{u}^{n+\frac{1}{2}},\frac{\varepsilon_h^{n+1}+e_h^n}{2})-b_s(\mathbf{u}(t^{n+\frac{1}{2}}),\mathbf{u}(t^{n+\frac{1}{2}}),\frac{\varepsilon_h^{n+1}+e_h^n}{2})\\ &= b_s(\mathbf{u}^{n+\frac{1}{2}}-\mathbf{u}(t^{n+\frac{1}{2}}),\mathbf{u}^{n+\frac{1}{2}},\frac{\varepsilon_h^{n+1}+e_h^n}{2})+b_s(u(t^{n+\frac{1}{2}},\mathbf{u}^{n+\frac{1}{2}}-u(t^{n+\frac{1}{2}}),\frac{\varepsilon_h^{n+1}+e_h^n}{2})\\ &\le C(\Omega)||\nabla(\mathbf{u}^{n+\frac{1}{2}}-\mathbf{u}(t^{n+\frac{1}{2}}))||||\nabla\frac{\varepsilon_h^{n+1}+e_h^n}{2}||(||\nabla\mathbf{u}^{n+\frac{1}{2}}||+||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||)\\ &\le\frac{\nu}{8}||\nabla\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+\frac{2C(\Omega)}{\nu}(||\nabla\mathbf{u}^{n+\frac{1}{2}}||^2+||\nabla\mathbf{u}(t^{n+\frac{1}{2}}))||^2)||\frac{\triangle t^2}{4}\nabla\mathbf{u}_{tt}(t^{n+\frac{1}{2}})||^2\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\nu}{8}||\nabla\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+\frac{\triangle t^3 C(\Omega)}{8\nu}(||\nabla\mathbf{u}^{n+\frac12}||^2+||\nabla\mathbf{u}(t^{n+\frac12}))||^2)(\int_{t^n}^{t^{n+1}}||\nabla\mathbf{u}_{tt}||^2dt)\\
&\leq \frac{\nu}{8}||\nabla\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+\frac{\triangle t^3 C(\Omega)}{8\nu}\int_{t^n}^{t^{n+1}}||\nabla\mathbf{u}_{tt}||^2(||\nabla\mathbf{u}^{n+\frac12}||^2+||\nabla\mathbf{u}(t^{n+\frac12}))||^2)dt\\
&\leq \frac{\nu}{8}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+\frac{\triangle t^3 C(\Omega)}{8\nu}(\int_{t^n}^{t^{n+1}}||\nabla\mathbf{u}_{tt}||^4dt+\triangle t(||\nabla\mathbf{u}^{n+\frac12}||^4+||\nabla\mathbf{u}(t^{n+\frac12}))||^4). \qquad (3.24)
\end{aligned}$$

Finally, Combining all estimates (3.21)-(3.24) and incorporating the similar terms gives the Lemma. □

Subsequently, combining Lemma 3.2 with (3.17)-(3.19) gives

$$\begin{aligned}
\frac12(||\varepsilon_h^{n+1}||^2-||e_h^n||^2)+\frac{\triangle t\nu}{4}||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2 &\leq C(\Omega)\triangle t(1+\nu^{-3}||\nabla\mathbf{u}(t^{n+\frac12})||^4)(||\varepsilon_h^{n+1}||^2+||e_h^n||^2)\\
&+C(\Omega)\nu\triangle t||\nabla\Lambda^{n+\frac12}||^2+\frac{C(\Omega)\triangle t}{\nu}||\nabla\tilde{\mathbf{w}}_h^{n+\frac12}||^2||\nabla\Lambda^{n+\frac12}||^2\\
&+\frac{C(\Omega)\triangle t}{\nu}||\Lambda^{n+\frac12}||||\nabla\Lambda^{n+\frac12}||||\nabla\mathbf{u}^{n+\frac12}||^2+C(\Omega)\int_{t^n}^{t^{n+1}}||\Lambda_t||^2dt\\
&+\frac{C(\Omega)\triangle t}{\nu}||p(t^{n+\frac12})-q_h||^2+\frac{C(\Omega)\triangle t^5}{\nu}(||\nabla\mathbf{u}^{n+\frac12}||^4+||\nabla u(t^{n+\frac12})||^4)\\
&+C(\Omega)\triangle t^4\int_{t^n}^{t^{n+1}}(||\mathbf{u}_{ttt}||^2+\nu||\nabla\mathbf{u}_{tt}||^2+\frac{1}{\nu}||\nabla u_{tt}||^4+||\mathbf{f}_{tt}||^2)dt. \qquad (3.25)
\end{aligned}$$

In order to estimate the error of e_h^n, we need to formulate the relationship between ε_h^n and e_h^n in the next step.

Since $\mathbf{u}_h^{n+1}$ and $\mathbf{w}_h^{n+1}$ are connected through the variational multiscale equation, therefore, we take $\mathbf{v}_h=\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2}$ in (2.3), and note that $\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}=I_h\mathbf{u}^{n+1}-\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2}$, with $I_h\mathbf{u}^{n+1}=\mathbf{u}^{n+1}-\Lambda^{n+1}$. For notational simplicity, we denote $\alpha=[I-P_H]I_h\mathbf{u}^{n+1}$, $\beta=[I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2}$, then by using the monotonicity (2.4) and Lipschitz continuity (2.5) as well as the Young's inequality with exponents 3 and 3/2 , we get

$$\begin{aligned}
\frac{1}{2\triangle t}(||e_h^{n+1}||^2-||\varepsilon_h^{n+1}||^2) &= (C_s\delta)^2(|\mathbb{D}(\alpha-\beta)|_F\mathbb{D}(\alpha-\beta),\mathbb{D}(\beta))\\
&= -(C_s\delta)^2(|\mathbb{D}(\beta-\alpha)|_F\mathbb{D}(\beta-\alpha)-|\mathbb{D}(-\alpha)|_F\mathbb{D}(-\alpha),\mathbb{D}(\beta))+(C_s\delta)^2(|\mathbb{D}(\alpha)|_F\mathbb{D}(\alpha),\mathbb{D}(\beta))\\
&\leq -C(C_s\delta)^2||\mathbb{D}(\beta)||_{L^3}^3+C(C_s\delta)^2||\mathbb{D}(\alpha)||_{L^3}^2||\mathbb{D}(\beta)||_{L^3}\\
&\leq -C(C_s\delta)^2||\mathbb{D}(\beta)||_{L^3}^3+\frac{C(C_s\delta)^2}{2}||\mathbb{D}(\beta)||_{L^3}^3+\frac{16C(C_s\delta)^2}{27}||\mathbb{D}(\alpha)||_{L^3}^3\\
&= -\frac{C(C_s\delta)^2}{2}||\mathbb{D}(\beta)||_{L^3}^3+\frac{16C(C_s\delta)^2}{27}||\mathbb{D}(\alpha)||_{L^3}^3,
\end{aligned}$$

which means

$$\begin{aligned}
\frac12||\varepsilon_h^{n+1}||^2 \geq \frac12||e_h^{n+1}||^2+\frac{C(C_s\delta)^2\triangle t}{2}||\mathbb{D}[I-P_H](\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})||_{L^3}^3\\
-\frac{16C(C_s\delta)^2}{27}\triangle t||\mathbb{D}([I-P_H]I_h\mathbf{u}^{n+1})||_{L^3}^3. \qquad (3.26)
\end{aligned}$$

On the other hand, note the fact $\beta=\alpha-\beta-(\alpha-2\beta)$, by repeated application of monotonicity (2.4) and Lipschitz continuity (2.5), as well as Young's inequality with exponents 3 and

3/2 and Minkowski's inequality gives

$$
\begin{aligned}
\frac{1}{2\triangle t}(||e_h^{n+1}||^2 - ||\varepsilon_h^{n+1}||^2) &= (C_s\delta)^2(|\mathbb{D}(\alpha-\beta)|_F\mathbb{D}(\alpha-\beta),\mathbb{D}(\beta)) \\
&= (C_s\delta)^2(|\mathbb{D}(\alpha-\beta)|_F\mathbb{D}(\alpha-\beta),\mathbb{D}(\alpha-\beta)) - (C_s\delta)^2(|\mathbb{D}(\alpha-\beta)|_F\mathbb{D}(\alpha-\beta),\mathbb{D}(\alpha-2\beta)) \\
&\geq C(C_s\delta)^2||\mathbb{D}(\alpha-\beta)||_{L^3}^3 - C(C_s\delta)^2||\mathbb{D}(\alpha-\beta)||_{L^3}^2||\mathbb{D}(\alpha-2\beta)||_{L^3} \\
&\geq C(C_s\delta)^2||\mathbb{D}(\alpha-\beta)||_{L^3}^3 - C(C_s\delta)^2||\mathbb{D}(\alpha-\beta)||_{L^3}^3 - \frac{4C(C_s\delta)^2}{27}||\mathbb{D}(\alpha-2\beta)||_{L^3}^3 \\
&\geq -\frac{4C(C_s\delta)^2}{27}(||\mathbb{D}(\alpha)||_{L^3} + 2||\mathbb{D}(\beta)||_{L^3})^3 \\
&\geq -\frac{16C(C_s\delta)^2}{27}||\mathbb{D}(\alpha)||_{L^3}^3 - \frac{128C(C_s\delta)^2}{27}||\mathbb{D}(\beta)||_{L^3}^3,
\end{aligned}
$$

here we use the fact $(a+b)^3 \leq 4(a^3+b^3)$, thus the above inequality implies

$$
\begin{aligned}
||\varepsilon_h^{n+1}||^2 \leq ||e_h^{n+1}||^2 &+ \frac{32\triangle tC(C_s\delta)^2}{27}||\mathbb{D}([I-P_H]I_h\mathbf{u}^{n+1})||_{L^3}^3 \\
&+ \frac{256\triangle tC(C_s\delta)^2}{27}||\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})||_{L^3}^3.
\end{aligned} \tag{3.27}
$$

Substitute (3.26)-(3.27) into (3.25) and assume $||e_h^0|| = 0, i.e., u^0 \in \mathbf{X}_h$, we obtain

$$
\begin{aligned}
&\frac{1}{2}(||e_h^{n+1}||^2 - ||e_h^n||^2) + \frac{\triangle t}{4}(\nu||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2 + 2C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})||_{L^3}^3) \\
&\quad\leq C(\Omega)\triangle t(1+\nu^{-3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4)(||e_h^{n+1}||^2 + ||e_h^n||^2) + C(\Omega)\nu\triangle t||\nabla\Lambda^{n+\frac{1}{2}}||^2 \\
&\quad+ C(\Omega)\triangle t^2(1+\nu^{-3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4)C(C_s\delta)^2||\mathbb{D}([I-P_H](\mathbf{u}^{n+1}-\Lambda^{n+1}))||_{L^3}^3 \\
&\quad+ C(\Omega)\triangle t^2(1+\nu^{-3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4)C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})||_{L^3}^3 \\
&\quad+ C(\Omega)\triangle tC(C_s\delta)^2||\mathbb{D}([I-P_H](\mathbf{u}^{n+1}-\Lambda^{n+1}))||_{L^3}^3 + \frac{C(\Omega)\triangle t}{\nu}||\nabla\tilde{\mathbf{w}}_h^{n+\frac{1}{2}}||^2||\nabla\Lambda^{n+\frac{1}{2}}||^2 \\
&\quad+ \frac{C(\Omega)\triangle t}{\nu}||\Lambda^{n+\frac{1}{2}}||||\nabla\Lambda^{n+\frac{1}{2}}||||\nabla\mathbf{u}^{n+\frac{1}{2}}||^2 + \frac{C(\Omega)\triangle t}{\nu}||p(t^{n+\frac{1}{2}}) - q_h||^2 \\
&\quad+ \frac{C(\Omega)\triangle t^5}{\nu}(||\nabla\mathbf{u}^{n+\frac{1}{2}}||^4 + ||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4) + C(\Omega)\int_{t^n}^{t^{n+1}}||\Lambda_t||^2dt \\
&\quad+ C(\Omega)\triangle t^4\int_{t^n}^{t^{n+1}}(||\mathbf{u}_{ttt}||^2 + \nu||\nabla\mathbf{u}_{tt}||^2 + \frac{1}{\nu}||\nabla\mathbf{u}_{tt}||^4 + ||\mathbf{f}_{tt}||^2)dt.
\end{aligned} \tag{3.28}
$$

We restrict $\triangle t$ to be small enough that $\triangle t < C(\Omega)(1+\nu^{-3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4)^{-1}$, then we can "absorb" the terms stemming from the VMS method on the right-hand side into the last term on the left-hand side, then summing (3.28) up from $n=0$ to $n=N-1$ and using Minkowski's inequality results in

$$
\begin{aligned}
&\frac{1}{2}||e_h^N||^2 + \frac{\triangle t}{4}\sum_{n=0}^{N-1}(\nu||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2 + C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})||_{L^3}^3) \\
&\quad\leq C(\Omega)\triangle t\sum_{n=0}^{N-1}(1+\nu^{-3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4)||e_h^{n+1}||^2 \\
&\quad+ C(\Omega)\nu\triangle t\sum_{n=0}^{N-1}||\nabla\Lambda^{n+\frac{1}{2}}||^2 + C(\Omega)\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}||\Lambda_t||^2dt
\end{aligned}
$$

$$
\begin{aligned}
&+ C(\Omega)\triangle t \sum_{n=0}^{N-1} [((C_s\delta)^2 ||\mathbb{D}[I-P_H]\Lambda^{n+1}||_{L^3}^3 + (C_s\delta)^2 ||\mathbb{D}[I-P_H]\mathbf{u}^{n+1}||_{L^3}^3]) \\
&+ \frac{C(\Omega)\triangle t}{\nu} \sum_{n=0}^{N-1} ||\nabla \tilde{\mathbf{w}}_h^{n+\frac{1}{2}}||^2 ||\nabla \Lambda^{n+\frac{1}{2}}||^2 + \frac{C(\Omega)\triangle t}{\nu} \sum_{n=0}^{N-1} ||\Lambda^{n+\frac{1}{2}}|| ||\nabla \Lambda^{n+\frac{1}{2}}|| ||\nabla \mathbf{u}^{n+\frac{1}{2}}||^2 \\
&+ \frac{C(\Omega)\triangle t}{\nu} \sum_{n=0}^{N-1} ||p(t^{n+\frac{1}{2}}) - q_h||^2 + \frac{C(\Omega)\triangle t^5}{\nu} \sum_{n=0}^{N-1} (||\nabla \mathbf{u}^{n+\frac{1}{2}}||^4 + ||\nabla u(t^{n+\frac{1}{2}})||^4) \\
&+ C(\Omega)\triangle t^4 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (||\mathbf{u}_{ttt}||^2 + \nu||\nabla \mathbf{u}_{tt}||^2 + \frac{1}{\nu}||\nabla u_{tt}||^4 + ||\mathbf{f}_{tt}||^2) dt. \qquad (3.29)
\end{aligned}
$$

The terms on the right-hand side of (3.29) can be further simplified as follows

$$
C(\Omega)\nu\triangle t \sum_{n=0}^{N-1} ||\nabla \Lambda^{n+\frac{1}{2}}||^2 \le C(\Omega)\nu\triangle t \sum_{n=0}^{N-1} h^{2k} |\mathbf{u}|_{k+1}^2 \le C(\Omega)\nu h^{2k} |||\mathbf{u}|||_{2,k+1}^2.
$$

By using the boundedness of $\nu\triangle t \sum_{n=0}^{N-1} ||\mathbb{D}\tilde{\mathbf{w}}_h^{N+\frac{1}{2}}||^2$(Proposition 2.3) and Korn's inequality, we have

$$
\begin{aligned}
C(\Omega)\triangle t\nu^{-1} \sum_{n=0}^{N-1} ||\nabla \tilde{\mathbf{w}}_h^{n+\frac{1}{2}}||^2 ||\nabla \Lambda^{n+\frac{1}{2}}||^2 &\le \frac{C(\Omega)}{2}\triangle t\nu^{-1} \sum_{n=0}^{N-1} ||\nabla \tilde{\mathbf{w}}_h^{n+\frac{1}{2}}||^2 (||\nabla \Lambda^{n+1}||^2 + ||\nabla \Lambda^n||^2) \\
&\le \frac{C(\Omega)}{2}\triangle t\nu^{-1} \sum_{n=0}^{N-1} ||\nabla \tilde{\mathbf{w}}_h^{n+\frac{1}{2}}||^2 h^{2k} (|\mathbf{u}^{n+1}|_{k+1}^2 + |\mathbf{u}^n|_{k+1}^2) \\
&\le C(\Omega)\nu^{-2} h^{2k} |||\mathbf{u}|||_{\infty,k+1}^2.
\end{aligned}
$$

For the next term, using Young's inequality gives

$$
\begin{aligned}
&C(\Omega) \triangle t\nu^{-1} \sum_{n=0}^{N-1} ||\Lambda^{n+\frac{1}{2}}|| ||\nabla \Lambda^{n+\frac{1}{2}}|| ||\nabla \mathbf{u}^{n+\frac{1}{2}}||^2 \\
&\quad \le C(\Omega)\triangle t\nu^{-1} \sum_{n=0}^{N-1} (||\Lambda^{n+1}|| ||\nabla \Lambda^{n+1}|| + ||\Lambda^n|| ||\nabla \Lambda^n|| + ||\Lambda^n|| ||\nabla \Lambda^{n+1}|| + ||\Lambda^{n+1}|| ||\nabla \Lambda^n||) ||\nabla \mathbf{u}^{n+\frac{1}{2}}||^2 \\
&\quad \le C(\Omega)\nu^{-1} h^{2k+1} (\triangle t \sum_{n=0}^{N-1} (|\mathbf{u}^{n+1}|_{k+1}^2 + |\mathbf{u}^n|_{k+1}^2 + |\mathbf{u}^{n+1}|_{k+1} |\mathbf{u}^n|_{k+1}) ||\nabla \mathbf{u}^{n+\frac{1}{2}}||^2) \\
&\quad \le C(\Omega)\nu^{-1} h^{2k+1} (\triangle t \sum_{n=0}^{N-1} |\mathbf{u}^{n+1}|_{k+1}^4 + \triangle t \sum_{n=0}^{N-1} ||\nabla \mathbf{u}^{n+1}||^4) \\
&\quad \le C(\Omega)\nu^{-1} h^{2k+1} (|||\mathbf{u}|||_{4,k+1}^4 + |||\nabla \mathbf{u}|||_{4,0}^4),
\end{aligned}
$$

as well as

$$
C(\Omega)\triangle t\nu^{-1} \sum_{n=0}^{N-1} ||p(t^{n+\frac{1}{2}}) - q_h||^2 \le C(\Omega)\triangle t\nu^{-1} \sum_{n=0}^{N-1} h^{2k} |p(t^{n+\frac{1}{2}})|_k^2 \le C(\Omega)\nu^{-1} h^{2k} |||p|||_{2,k}^2,
$$

and

$$
C(\Omega) \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} ||\Lambda_t||^2 dt \le C(\Omega) \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} h^{2k+2} |\mathbf{u}_t|_{k+1}^2 dt \le C(\Omega) h^{2k+2} |||\mathbf{u}_t|||_{2,k+1}^2.
$$

Using Korn's inequality and the property of projection P_{L^H} (1.13), we get

$$\begin{aligned}C\triangle t\sum_{n=0}^{N-1}((C_s\delta)^2||\mathbb{D}[I-P_H]\Lambda^{n+1}||_{L^3}^3 &\leq C\triangle t(C_s\delta)^2h^{-\frac{d}{2}}\sum_{n=0}^{N-1}||[I-P_{L^H}]\nabla\Lambda^{n+1}||^3\\ &\leq C(C_s\delta)^2h^{-\frac{d}{2}}H^{3k}\triangle t\sum_{n=0}^{N-1}|\mathbf{u}^{n+1}|_{k+1}^3\\ &\leq C(C_s\delta)^2h^{-\frac{d}{2}}H^{3k}|||\mathbf{u}|||_{3,k+1}^3,\end{aligned}$$

and

$$C\triangle t(C_s\delta)^2\sum_{n=0}^{N-1}||\mathbb{D}[I-P_H]\mathbf{u}^{n+1}||_{L^3}^3 \leq C(C_s\delta)^2h^{-\frac{d}{2}}H^{3k}|||\mathbf{u}|||_{3,k+1}^3.$$

Combining above seven inequalities, equation (3.29) reduces to

$$\begin{aligned}&\frac{1}{2}||e_h^N||^2+\frac{\triangle t}{4}\sum_{n=0}^{N-1}(\nu||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})||_{L^3}^3)\\ &\quad\leq C\,(\Omega)\triangle t\sum_{n=0}^{N-1}(1+\nu^{-3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4)||e_h^{n+1}||^2\\ &\qquad+C(\Omega)\nu^{-1}(h^{2k+1}|||\mathbf{u}|||_{4,k+1}^4+h^{2k+1}|||\nabla\mathbf{u}|||_{4,0}^4+h^{2s+2}|||p|||_{2,s+1}^2)\\ &\qquad+C(\Omega)h^{2k+2}|||\mathbf{u}_t|||_{2,k+1}^2+C(\Omega)\nu^{-2}h^{2k}|||\mathbf{u}|||_{\infty,k+1}^2+C(\Omega)\nu h^{2k}|||\mathbf{u}|||_{2,k+1}^2\\ &\qquad+C(C_s\delta)^2h^{-\frac{d}{2}}H^{3k}|||\mathbf{u}|||_{3,k+1}^3+C(\Omega)\triangle t^4\nu^{-1}(|||\nabla\mathbf{u}|||_{4,0}^4+|||\nabla\mathbf{u}_{1/2}|||_{4,0}^4)\\ &\qquad+C(\Omega)\triangle t^4(|||\mathbf{u}_{ttt}|||_{2,0}^2+\nu|||\nabla\mathbf{u}_{tt}|||_{2,0}^2+\nu^{-1}|||\nabla\mathbf{u}_{tt}|||_{4,0}^4+|||\mathbf{f}_{tt}|||_{2,0}^2).\end{aligned}\tag{3.30}$$

Hence, with $\triangle t$ sufficiently small, i.e. $\triangle t < C(\Omega)(1+\nu^{-3}||\nabla\mathbf{u}(t^{n+\frac{1}{2}})||^4)^{-1}$, from the discrete Gronwall's inequality, we have

$$\begin{aligned}&\frac{1}{2}||e_h^N||^2+\frac{\triangle t}{4}\sum_{n=0}^{N-1}(\nu||\mathbb{D}\frac{\varepsilon_h^{n+1}+e_h^n}{2}||^2+C(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\varepsilon_h^{n+1}+e_h^{n+1}}{2})||_{L^3}^3)\\ &\quad\leq C\,(\Omega)\nu^{-1}(h^{2k+1}|||\mathbf{u}|||_{4,k+1}^4+h^{2k+1}|||\nabla\mathbf{u}|||_{4,0}^4+h^{2s+2}|||p|||_{2,s+1}^2)\\ &\qquad+C(\Omega)h^{2k+2}|||\mathbf{u}_t|||_{2,k+1}^2+C(\Omega)\nu^{-2}h^{2k}|||\mathbf{u}|||_{\infty,k+1}^2+C(\Omega)\nu h^{2k}|||\mathbf{u}|||_{2,k+1}^2\\ &\qquad+C(C_s\delta)^2h^{-\frac{d}{2}}H^{3k}|||\mathbf{u}|||_{3,k+1}^3+C(\Omega)\triangle t^4\nu^{-1}(|||\nabla\mathbf{u}|||_{4,0}^4+|||\nabla\mathbf{u}_{1/2}|||_{4,0}^4)\\ &\qquad+C(\Omega)\triangle t^4(|||\mathbf{u}_{ttt}|||_{2,0}^2+\nu|||\nabla\mathbf{u}_{tt}|||_{2,0}^2+\nu^{-1}|||\nabla\mathbf{u}_{tt}|||_{4,0}^4+|||\mathbf{f}_{tt}|||_{2,0}^2).\end{aligned}\tag{3.31}$$

The estimate given in theorem 3.1 for $||\mathbf{u}^N-\mathbf{u}_h^N||^2$ then follows from the Minkowski's inequality and (3.31). The estimate for $||\mathbf{u}-\mathbf{w}_h||^2$ follows from (3.27) and Minkowski's inequality,

$$\begin{aligned}||\mathbf{u}^N-\mathbf{w}_h^N||^2 = ||\ \Lambda^N+\varepsilon_h^N||^2 &\leq 2||\Lambda^N||^2+2||\varepsilon_h^N||^2\\ &\leq 2||\Lambda^N||^2+2||e_h^N||^2+2\triangle tC(C_s\delta)^2||\mathbb{D}([I-P_H]I_h\mathbf{u}^N)||_{L^3}^3\\ &\quad+2\triangle tC(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\varepsilon_h^N+e_h^N}{2})||_{L^3}^3\\ &\leq Ch^{2k+2}|\mathbf{u}^{n+1}|_{k+1}^2+C\triangle t(C_s\delta)^2h^{-\frac{d}{2}}H^{3k}|\mathbf{u}|_{k+1}^3\\ &\quad+||e_h^N||^2+\triangle tC(C_s\delta)^2||\mathbb{D}([I-P_H]\frac{\varepsilon_h^N+e_h^N}{2})||_{L^3}^3.\end{aligned}\tag{3.32}$$

To obtain theorem 3.1, we also use

$$||\mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}}) - \frac{\mathbf{w}_h^{n+1} + \mathbf{u}_h^n}{2})||^2 \le ||\mathbb{D}(\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}(t^{n+\frac{1}{2}}))||^2 + ||\nabla\Lambda^{n+\frac{1}{2}}||^2 + ||\mathbb{D}\frac{\varepsilon_h^{n+1} + e_h^n}{2}||^2)$$

$$\le \frac{\triangle t^3}{48}\int_{t^n}^{t^{n+1}} ||\nabla \mathbf{u}_{tt}||^2 dt + Ch^{2k}|\mathbf{u}^{n+1}|_{k+1}^2 + Ch^{2k}|\mathbf{u}^n|_{k+1}^2 + ||\mathbb{D}\frac{\varepsilon_h^{n+1} + e_h^n}{2}||^2. \tag{3.33}$$

Finally, combining (3.32)-(3.33) with (3.31) yields the result of the theorem. □

For the case of Taylor-Hood approximating elements, i.e., $k = 2$, we have the following estimate.

COROLLARY 3.3. *Under the assumptions of Theorem 3.1, with $\triangle t = Ch, \delta = Ch, h = H^2$ and $(\mathbf{X}^h, Q^h)$ given by the Taylor-Hood approximation elements, we have*

$$|||\mathbf{u} - \mathbf{u}_h|||_{\infty,0} + |||\mathbf{u} - \mathbf{w}_h|||_{\infty,0} + (\triangle t\nu \sum_{n=0}^{N-1} ||\mathbb{D}(\mathbf{u}(t^{n+\frac{1}{2}}) - \frac{w_h^{n+1} + u_h^n}{2})||^2)^{1/2} \le C(\triangle t^2 + h^2). \tag{3.34}$$

4. A variant with reduced complexity. The difficulty with the modular, full or ideal Smagorinsky VMS method is exactly the cost of this nonlinear solve each time step. To reduce this cost we present a variant on Algorithm 2.1 which is closely related and much less expensive.

Recalling (2.2), one difficulty with this computation is the nonlinearity of the eddy viscosity, the other is the coupling of fine mesh elements and coarse mesh elements which caused by the projection, especially in the factor $(I - P_{L^H})\mathbb{D}(\mathbf{v})$. Thus the first and obvious treatment is lagging $\nu_T(\cdot)$ to reduce the complexity to solving a (multiscale) linear equation per time step for $\mathbf{u}^{n+1}$, i.e., we replace the coefficient of the viscosity eddy $|[I - P_{L^H}]\mathbb{D}\frac{\mathbf{w}^{n+1}+\mathbf{u}^{n+1}}{2})|_F$ by $|[I - P_{L^H}]\mathbb{D}\frac{\mathbf{w}^n+\mathbf{u}^n}{2})|_F$. Secondly, we redefine the coefficient of the viscosity eddy $|[I - P_{L^H}]\mathbb{D}\frac{\mathbf{w}^n+\mathbf{u}^n}{2})|_F$ element by element in the following way:

$$A_e := Average_e(|[I - P_{L^H}]\mathbb{D}\frac{\mathbf{w}^n + \mathbf{u}^n}{2})|_F) = \frac{1}{|e|}\int_e |[I - P_{L^H}]\mathbb{D}\frac{\mathbf{w}^n + \mathbf{u}^n}{2})|_F dx,$$

where e represents the element, $|e|$ represents the area of element e. This means we take the average of $|[I - P_{L^H}]\mathbb{D}\frac{\mathbf{w}^n+\mathbf{u}^n}{2})|_F$ on each element as the coefficient of the viscosity eddy. Obviously, now the coefficient A_e is a piecewise constant, so it can be commuted with the operator $[I - P_{L^H}]\mathbb{D}$. Thanks to the orthogonality of projection P_{L^H}, we can simplify Step 2 in Algorithm 2.1 as follow:

$$(\frac{\mathbf{w}_h^{n+1} - \mathbf{u}_h^{n+1}}{\triangle t}, \mathbf{v}_h) = (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) + (C_s\delta)^2 A_e([I - P_{L^H}]\mathbb{D}\frac{\mathbf{w}_h^{n+1} + \mathbf{u}_h^{n+1}}{2}, \mathbb{D}(\mathbf{v}_h)).$$

Note in particular that $\mathbb{D}(\mathbf{v}_h)$ replaces $[I - P]\mathbb{D}(\mathbf{v}_h)$. This change simplifies the computational work of Step 2 substantially. Rearranging terms we can rewrite last equation as follows: given $\mathbf{w}_h^{n+1}$ to solve for $\mathbf{u}_h^{n+1}$ by the following equation:

$$(C_s\delta)^2 A_e([I - P_{L^H}]\mathbb{D}(\mathbf{u}_h^{n+1}), \mathbb{D}(\mathbf{v}_h)) + \frac{2}{\triangle t}(\mathbf{u}_h^{n+1}, \mathbf{v}_h) + 2(p_h^{n+1}, \nabla\cdot\mathbf{v}_h)$$
$$= \frac{2}{\triangle t}(\mathbf{w}_h^{n+1}, \mathbf{v}_h) - (C_s\delta)^2 A_e([I - P_{L^H}]\mathbb{D}(\mathbf{w}_h^{n+1}), \mathbb{D}(\mathbf{v}_h)), \tag{4.1}$$

which is much easier to implement than full nonlinear Smagorinsky model. Next, we present this variant on the method.

Algorithm 4.1

Step 1: Given $\mathbf{u}_h^n$, compute $\mathbf{w}_h^{n+1} \in \mathbf{X}^h, p_h^{n+1} \in Q^h$ satisfying

$$\text{(4.2)} \quad \begin{cases} (\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^n}{\triangle t}, \mathbf{v}_h) + b_s(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}, \mathbf{v}_h) \\ \qquad +\nu(\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}), \mathbb{D}(\mathbf{v}_h)) - (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) = (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{v}_h), \\ (\nabla\cdot\mathbf{w}_h^{n+1}, q_h) = 0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in (\mathbf{X}^h, Q^h)$,

Step 2: Given $\mathbf{w}_h^{n+1}$ solve the following to obtain $\mathbf{u}_h^{n+1}$:

$$\text{(4.3)} \quad \begin{cases} (\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^{n+1}}{\triangle t}, \mathbf{v}_h) = (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) + (C_s\delta)^2 A_e([I-P_{L^H}]\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}, \mathbb{D}(\mathbf{v}_h)), \\ (\nabla\cdot\mathbf{u}_h^{n+1}, q_h) = 0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in (\mathbf{X}^h, Q^h)$, where A_e is defined above.

Now, we prove a strong energy equality and associated strong, unconditional stability property for this variant.

THEOREM 4.1. *The approximate velocity $\mathbf{u}_h^{n+1}$ given by the Algorithm 4.1 satisfies the energy equality*

$$\text{(4.4)} \quad \frac{1}{2}||\mathbf{u}_h^{l+1}||^2 + \triangle t\sum_{n=0}^{l}(C_s\delta)^2 A_e||[I-P_{L^H}]\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}||^2 + \triangle t\sum_{n=0}^{l}\nu||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}||^2 = \frac{1}{2}||\mathbf{u}_h^0||^2 + \triangle t\sum_{n=0}^{l}(\mathbf{f}^{n+\frac{1}{2}}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}),$$

and the stability bound

$$\text{(4.5)} \quad \frac{1}{2}||\mathbf{u}_h^{l+1}||^2 + \triangle t\sum_{n=0}^{l}(C_s\delta)^2 A_e||[I-P_{L^H}]\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}||^2 + \triangle t\sum_{n=0}^{l}\frac{\nu}{2}||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}||^2 \le \frac{1}{2}||\mathbf{u}_h^0||^2 + \frac{\triangle t}{2\nu}\sum_{n=0}^{l}||\mathbf{f}^{n+\frac{1}{2}}||_*^2,$$

where $A_e = \frac{1}{|e|}\int_e |[I-P_{L^H}]\mathbb{D}\frac{\mathbf{w}^n+\mathbf{u}^n}{2})|_F dx$.

Proof. First, from the orthogonality of the projection P_{L^H}, we can rewrite the first equation in (4.3) as follows:

$$(\frac{\mathbf{w}_h^{n+1}-\mathbf{u}_h^{n+1}}{\triangle t}, \mathbf{v}_h) = (p_h^{n+1}, \nabla\cdot\mathbf{v}_h) + (C_s\delta)^2 A_e([I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}), [I-P_{L^H}]\mathbb{D}(\mathbf{v}_h)).$$

Set $\mathbf{v}_h = \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2}$ in the above equation, this gives

$$\frac{1}{2\triangle t}||\mathbf{w}_h^{n+1}||^2 = \frac{1}{2\triangle t}||\mathbf{u}_h^{n+1}||^2 + (C_s\delta)^2 A_e||[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||^2.$$

Set $\mathbf{v}_h = \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}$ in (4.2), by using the Cauchy-Schwarz and Young's inequalities, we get

$$\frac{1}{2\triangle t}(||\mathbf{w}_h^{n+1}||^2 - ||\mathbf{u}_h^n||^2) + \nu||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}||^2 = (\mathbf{f}^{n+\frac{1}{2}}, \frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}).$$

Combining the above two equations gives

$$\frac{1}{2\triangle t}(||\mathbf{u}_h^{n+1}||^2-||\mathbf{u}_h^n||^2)+(C_s\delta)^2A_e||[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||^2 \quad +\nu||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}||^2 = (\mathbf{f}^{n+\frac{1}{2}},\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}).$$

Summing this establishes the energy equality. Using the Cauchy-Schwarz inequality and Young's inequality on the right-hand side, subsuming one term into the left-hand side gives

$$\frac{1}{2}(||\mathbf{u}_h^{n+1}||^2-||\mathbf{u}_h^n||^2)+\triangle t(C_s\delta)^2A_e||[I-P_{L^H}]\mathbb{D}(\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^{n+1}}{2})||^2 + \frac{\triangle t\nu}{2}||\mathbb{D}\frac{\mathbf{w}_h^{n+1}+\mathbf{u}_h^n}{2}||^2 \le \frac{\triangle t\nu}{2}\sum_{n=0}^{N-1}||\mathbf{f}^{n+\frac{1}{2}}||_*^2.$$

Summing over the index n, the global stability estimate follows. □

5. Numerical results. In all experiments, the algorithms are implemented using public domain finite element software Freefem++ [30].

5.1. Convergence study. Let Ω be the unit square in $\mathbb{R}^2$. The uniform mesh is obtained by dividing Ω into squares and then drawing a diagonal in each square in the same direction. The Taylor-Hood element are chosen for the velocity-pressure finite element space $(\mathbf{X}^h, Q^h)$, the large scale space $\mathbb{L}^H$ is using the piecewise constant space on the same grid.

Then, choose the true solution $(\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2),\, p)$ as follows:

$$\begin{aligned}\mathbf{u}_1 &= -cos(\pi x)sin(\pi y)exp(-2\pi^2 t/Re),\\ \mathbf{u}_2 &= sin(\pi x)cos(\pi y)exp(-2\pi^2 t/Re),\\ p &= -0.25(cos(2\pi x)+cos(2\pi y))exp(-4\pi^2 t/Re),\end{aligned}$$

which is the Green-Taylor vortex. It was used as a numerical test in Chorin [18], Tafti [31] and John and Layton [32] among many others.

First, we compare Algorithm 2.1 and Algorithm 4.1 of uncoupled VMS method with the classical or monolithic VMS method. We choose $C_S=0.1,\ \delta=0.1h$. In Table 5.1, we display the errors of the classical VMS method for $\mathbf{u}_h$ and p_h, while Table 5.2 and 5.3 give the results of both Algorithm 2.1 and Algorithm 4.1 for $\mathbf{w}_h$, $\mathbf{u}_h$ and p_h. Here we introduce the following symbols

$$e_w=\mathbf{u}-\mathbf{w}_h,\quad e_u=\mathbf{u}-\mathbf{u}_h,\quad e_p=p-p_h.$$

TABLE 5.1
Errors of convergence using classical VMS, Re=1000

$\frac{h}{\Delta t}$	$\|\|e_u\|\|_{L^2(0,T;L^2)}$	$\|\|e_u\|\|_{L^2(0,T;H^1)}$	$\|\|e_p\|\|_{L^2(0,T;L^2)}$
$\frac{0.1}{0.05}$	0.0113960	0.6584710	0.00542664
$\frac{0.05}{0.025}$	0.0009871	0.1329400	0.00095956
$\frac{0.025}{0.0125}$	6.42669e-5	0.019094	0.00022907

From these tables, we notice that under the same magnitude of h and Δt, all three algorithms obtain the similar accurate, which means that uncoupled VMS method is comparable accurate to the one-step, classical VMS method. As we mentioned in the introduction, Algorithm 4.1 is easier to implement, but the results confirm that it works as well as Algorithm 2.1.

Table 5.2
Errors of convergence using Algorithm 2.1 of uncoupled VMS, Re=1000

$\frac{h}{\Delta t}$	$\|\|e_w\|\|_{L^2(0,T;L^2)}$	$\|\|e_u\|\|_{L^2(0,T;L^2)}$	$\|\|e_w\|\|_{L^2(0,T;H^1)}$	$\|\|e_u\|\|_{L^2(0,T;H^1)}$	$\|\|e_p\|\|_{L^2(0,T;L^2)}$
$\frac{0.1}{0.05}$	0.0114657	0.0114564	0.663174	0.662166	0.0054379
$\frac{0.05}{0.025}$	0.0009905	0.0009904	0.133443	0.133411	0.0009597
$\frac{0.025}{0.0125}$	6.43394e-5	6.43362e-5	0.019117	0.019116	0.0002291

Table 5.3
Errors of convergence using Algorithm 4.1 of uncoupled VMS, Re=1000

$\frac{h}{\Delta t}$	$\|\|e_w\|\|_{L^2(0,T;L^2)}$	$\|\|e_u\|\|_{L^2(0,T;L^2)}$	$\|\|e_w\|\|_{L^2(0,T;H^1)}$	$\|\|e_u\|\|_{L^2(0,T;H^1)}$	$\|\|e_p\|\|_{L^2(0,T;L^2)}$
$\frac{0.1}{0.05}$	0.01147343	0.0114676	0.6637040	0.663071	0.0054390
$\frac{0.05}{0.025}$	0.00099064	0.0009905	0.1333457	0.133438	0.0009598
$\frac{0.025}{0.0125}$	6.43402e-5	6.43386e-5	0.0191171	0.019117	0.0002291

5.2. Flow around a cylinder. The second example is the 'flow around a cylinder' which is a popular benchmark problem for testing numerical schemes. This is a well known benchmark problem taken from Shafer and Turek [36] and John [37]. The domain with meshes is presented in Figure 5.1.

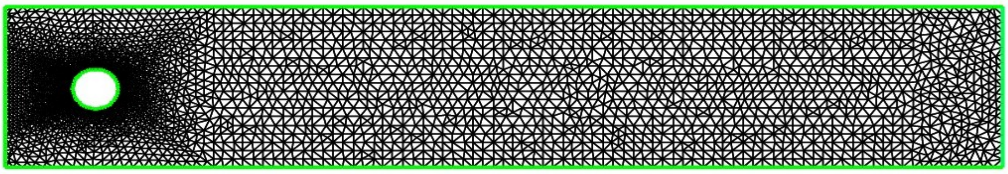

Fig. 5.1. *The triangulation of the computational domain for uncoupled VMS method.*

The time-dependent inflow profile is

$$\mathbf{u}_1(0,y,t) = \mathbf{u}_1(2.2,y,t) = \frac{6}{0.41^2} sin(\frac{\pi t}{8}) y(0.41-y),$$
$$\mathbf{u}_2(0,y,t) = \mathbf{u}_2(2.2,y,t) = 0.$$

No-slip conditions are prescribed at the other boundaries. Computations are performed for the Reynolds number corresponding to $\nu = 10^{-3}$, and the external force $f = 0$. A mesh with 7510 triangles is used, and $C_S = 0.1,\ \delta = 0.1h,\ h = \max_{T\in\tau_h}\{\text{diam}(T)\}$.

The development of the flows by both uncoupled VMS algorithms are depicted in Figure 5.2, 5.3, respectively. From these figures, we notice that from $t = 2$ to $t = 4$, along with the flow increasing, two vortices start to develop behind the cylinder. Then, the vortices separate from the cylinder between $t = 4$ and $t = 5$, and a vortex street develops, and they continue to be visible through the final time $t = 8$, which agrees with the results of [19, 36, 37].

The evolutions of $c_{d,max},\ c_{l,max}$ and Δp with $\Delta t = 0.0025$ for Algorithm 2.1 and 4.1 are presented in Figure 5.4 and 5.5, respectively. The values for the maximal drag $c_{d,max}$, maximal lift $c_{l,max}$ and $\Delta p(8s)$ (here $\Delta p(t) = p(t; 0.15, 0.2) - p(t; 0.25, 0.2)$) with different time step size Δt for Algorithm 2.1 and Algorithm 4.1 are presented in Table 5.4 and 5.5, respectively. The

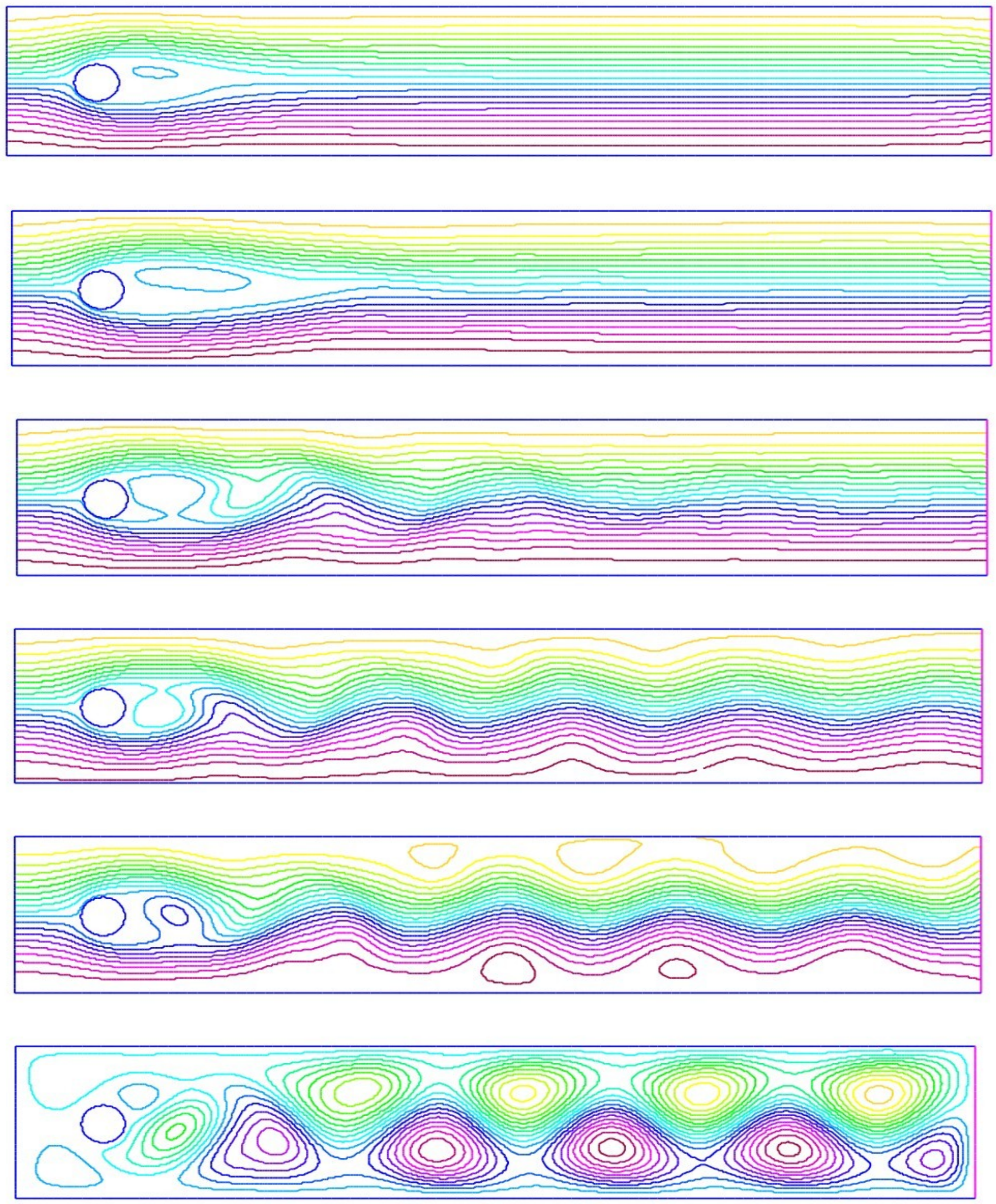

FIG. 5.2. *The streamline at $t = 2, 4, 5, 6, 7, 8$ by Algorithm 2.1 of uncoupled VMS method with* $\delta t = 0.0025$.

following reference intervals are given in [36],

$$c_{d,max}^{ref} \in [2.93, 297], \quad c_{l,max}^{ref} \in [047, 049], \quad \Delta p(8s)^{ref} \in [-0.115, -0.105].$$

The computation results in both tables show that when the time step size decreases, all coefficients approach the reference results, which mean that both uncoupled VMS method,

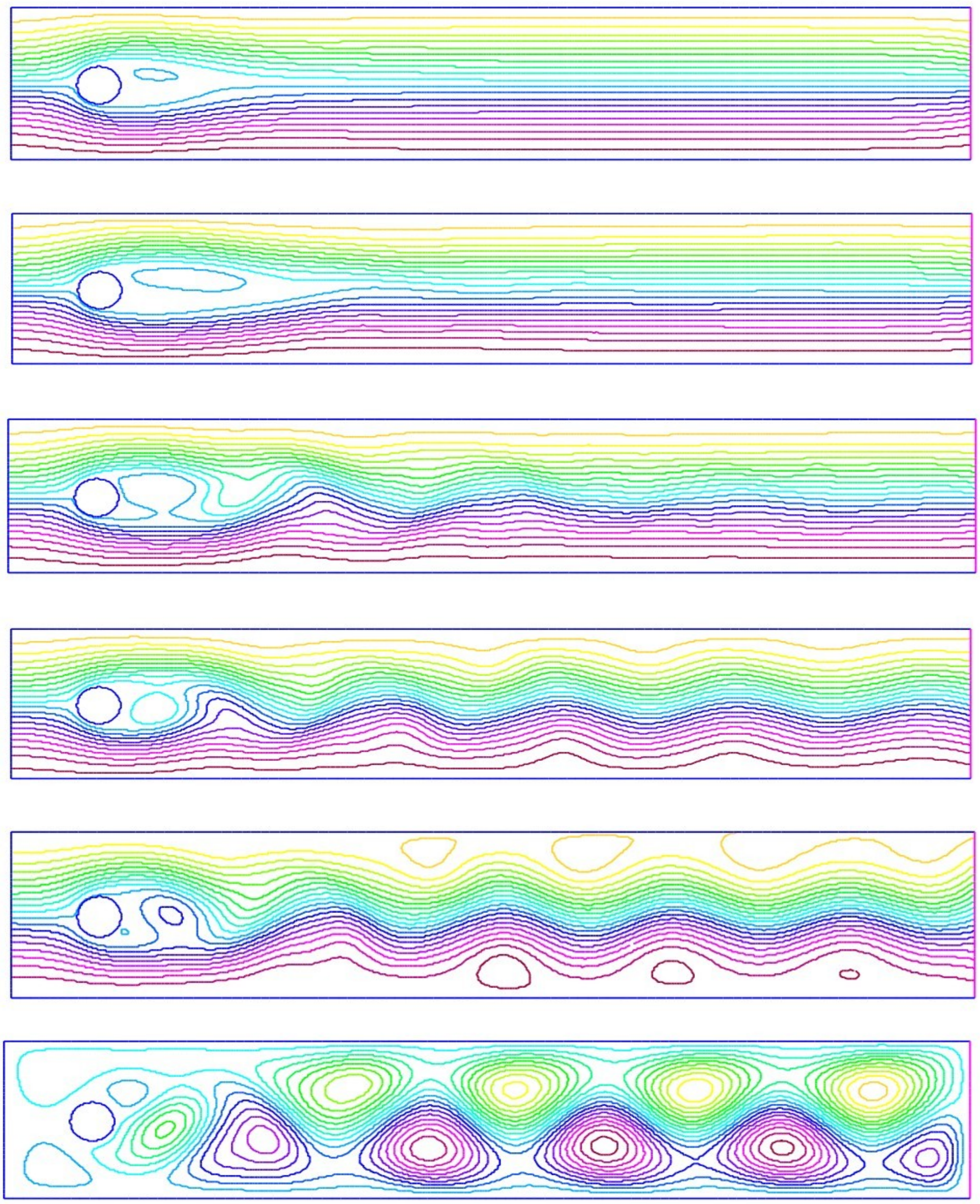

FIG. 5.3. *The streamline at $t = 2,\ 4,\ 5,\ 6,\ 7,\ 8$ by Algorithm 4.1 of uncoupled VMS method with* $\delta t = 0.0025$.

Algorithms 2.1 and 4.1 are efficient and practical, moreover, Algorithm 4.1 is a little more accurate than Algorithm 2.1.

6. Conclusions. In this paper, we have analyzed two modular, uncoupled variational multiscale methods focusing on analysis specifically on the case of nonlinear eddy viscosity for the Navier-Stokes equations. We separated the VMS treatment as a separate step, which means

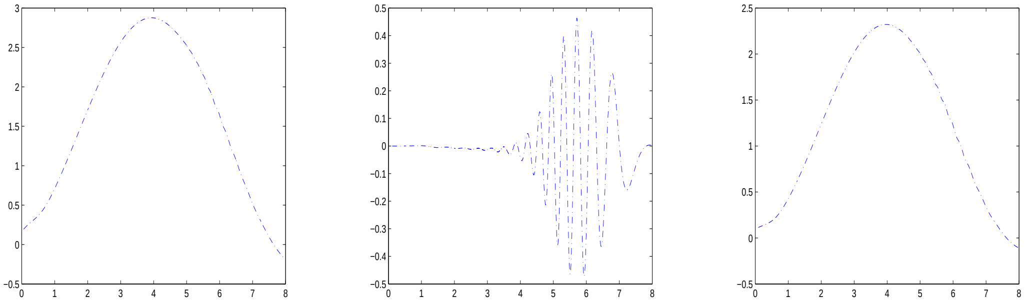

FIG. 5.4. *The evolutions of $c_{d,max}$, $c_{l,max}$ and Δp by Algorithm 2.1 of uncoupled VMS method with* $\delta t = 0.0025$.

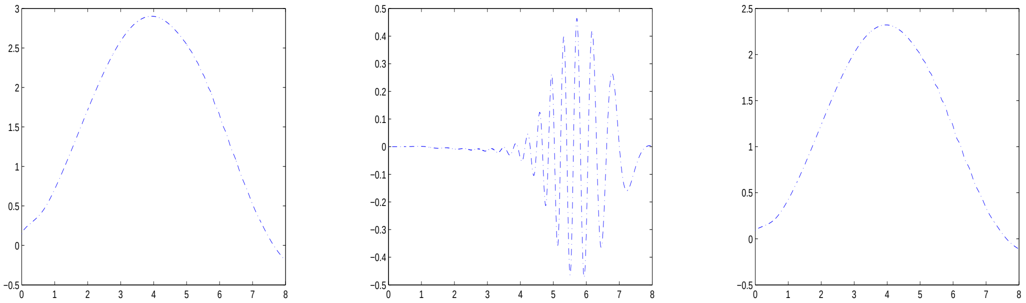

FIG. 5.5. *The evolutions of $c_{d,max}$, $c_{l,max}$ and Δp by Algorithm 4.1 of uncoupled VMS method with* $\delta t = 0.0025$.

TABLE 5.4
Results maximal drag $c_{d,max}$, maximal lift $c_{l,max}$ and $\Delta p(8s)$ for different time step size by Algorithm 2.1

Δt	$t(c_{d,max})$	$c_{d,max}$	$t(c_{d,max})$	$c_{d,max}$	$\Delta p(8s)$
0.025	3.9	2.39891	5.8	0.424975	-0.098419
0.01	3.91	2.70969	5.73	0.454301	-0.109673
0.005	3.92	2.82000	5.715	0.460827	-0.111061
0.0025	3.92	2.87696	5.7125	0.463704	-0.111484

TABLE 5.5
Results maximal drag $c_{d,max}$, maximal lift $c_{l,max}$ and $\Delta p(8s)$ for different time step size by Algorithm 4.1

Δt	$t(c_{d,max})$	$c_{d,max}$	$t(c_{d,max})$	$c_{d,max}$	$\Delta p(8s)$
0.025	3.9	2.59824	5.8	0.430873	-0.097808
0.01	3.91	2.79716	5.73	0.456948	-0.109780
0.005	3.93	2.86682	5.715	0.462216	-0.111104
0.0025	3.93	2.90233	5.7125	0.464294	-0.111503

one can utilize legacy codes to deal with the NSE in Step 1. We proved stability and performed an error analysis of the method. Numerical tests were given that confirm and illustrate the theoretical results as well.

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