ON THE QUASISTATIC APPROXIMATION IN THE STOKES-DARCY MODEL OF GROUNDWATER-SURFACE WATER FLOWS

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Abstract

We study the validity of the quasistatic approximation in the fully evolutionary Stokes-Darcy problem for the coupling of groundwater and surface water flows, as well as the dependence of the problem on the specific storage parameter. In the coupled equations that describe the groundwater and surface water flows for an incompressible fluid, the specific storage, S_0, represents the volume of water that a fully saturated porous medium will expel (or absorb) per unit volume per unit change in hydraulic head. In confined aquifers, S_0 takes values ranging from 10^{-6} or smaller to 10^{-2}. In this work we analyze the validity of the previously studied quasistatic approximation (setting $S_0 = 0$ in the Stokes-Darcy equations) by proving that the weak solution of the fully evolutionary Stokes-Darcy problem approaches the weak solution of the quasistatic problem as $S_0 \to 0$. We also estimate the rate of convergence.

Keywords: Stokes-Darcy, Groundwater flow, Surface water flow, Specific Storage.

1. Introduction

In the time-dependent Stokes-Darcy problem that models the coupling of groundwater with surface water flows, given in (1.1), the term $S_0 \frac{\partial \phi}{\partial t}$, where ϕ is the hydraulic head and S_0 is the specific storage, arises because aquifers are poroelastic media. This means that the space between the pores responds to changes in the pressure of the water occupying the pores. See e.g., Bear [1] for a clear derivation of (1.1) from basic conservation laws. There has been considerable study of the poroelastic effect and its many consequences, see e.g., Biot [2] and Wang [3]. In confined aquifers the values of S_0 range from 10^{-6} or smaller for rock to 10^{-2} for plastic clay, see Domenico [4], while in unconfined aquifers, S_0 can be larger[1]. In Table 1 we give a few representative values for S_0 in confined aquifers, see Anderson [5], Batu [6], Domenico and Mifflin [7], and Johnson [8]. The quasistatic approximation is obtained by setting $S_0 = 0$ in the Stokes-Darcy problem. It is thus equivalent to an inelastic assumption on the aquifer and is used in e.g., Cesmelioglu and Riviére [9], and Badea, Discacciati, and Quarteroni [10]. In this report we justify the validity of this quasistatic approximation. We prove in Theorems 4.1 and 4.2 that the solution of the fully evolutionary Stokes-Darcy problem, (u, ϕ), where u is the velocity in the fluid region and ϕ is the hydraulic head in the porous media region, converges to the solution of the quasistatic Stokes-Darcy model, denoted (u^{QS}, ϕ^{QS}), as S_0 converges to zero, under mild assumptions on the initial data and body forces, and the shape of the fluid and porous media domains. We obtain one half or first order convergence depending on the assumptions.

Table 1: Specific Storage (S_0) values for different materials

Material	**Specific Storage** S_0 (m^{-1})
plastic clay	$2.0 \times 10^{-2} - 2.6 \times 10^{-3}$
stiff clay	$2.6 \times 10^{-3} - 1.3 \times 10^{-3}$
medium hard clay	$1.3 \times 10^{-3} - 9.2 \times 10^{-4}$
loose sand	$1.0 \times 10^{-3} - 4.9 \times 10^{-4}$
dense sand	$2.0 \times 10^{-4} - 1.3 \times 10^{-4}$
dense sandy gravel	$1.0 \times 10^{-4} - 4.9 \times 10^{-5}$
rock, fissured jointed	$6.9 \times 10^{-5} - 3.3 \times 10^{-6}$
rock, sound	less than 3.3×10^{-6}

Let the fluid and porous media domains be denoted by Ω_f and Ω_p respectively, $\Omega_{f/p} \subset \mathbb{R}^d$, $d = 2$ or 3, and assume they lie across an interface, I, from each other as shown in Figure 1. Both domains are assumed to be bounded and regular. Before introducing the Stokes-Darcy problem and its quasistatic approximation, we list below the variables and parameters of the problem:

$$
\begin{aligned}
u = u(x,t) = (u_1(x,t), \ldots, u_d(x,t)) = &\ \text{fluid velocity in } \Omega_f, \\
\phi = \phi(x,t) = &\ \text{hydraulic head in } \Omega_p,\ x = (x_1, \ldots, x_d) \in \Omega_{f/p}, \\
f_f, f_p = &\ \text{body forces in fluid region and sources or sinks in porous region}, \\
\mathcal{K} = &\ \text{hydraulic conductivity tensor (symmetric positive definite)}, \\
g = &\ \text{gravitational acceleration constant}, \\
\nu = &\ \text{kinematic viscosity of fluid}, \\
\hat{n}_{f/p} = &\ \text{unit outward pointing normal on } \Omega_{f/p},
\end{aligned}
$$

[1]A confined aquifer is one bounded above and below by impervious formations. In a well penetrating such an aquifer, the water level will rise above the base of the confining formation. An unconfined aquifer is one with a water table serving as its upper boundary, see Bear [1].

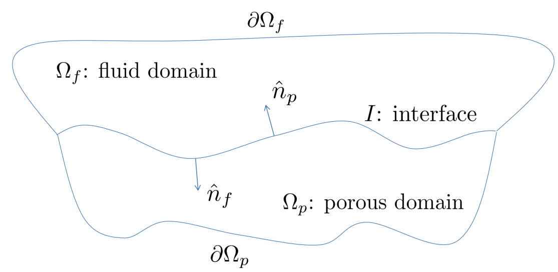

Figure 1: Fluid and porous media domains

The specific storage[2], S_0, represents the volume of water that a portion of a fully saturated porous medium releases from storage, per unit volume, per unit change in hydraulic head, see Freeze and Cherry [12], and Hantush [13]. All material and fluid parameters above are positive. Moreover,

$$0 < k_{min} \leq \lambda \leq k_{max},$$

where $\lambda \in \lambda(\mathcal{K})$, and $\lambda(\mathcal{K})$ is the spectrum of the hydraulic conductivity tensor, $\mathcal{K}$. The fluid velocity $u = u(x,t)$, defined on Ω_f, and porous media hydraulic head $\phi = \phi(x,t)$, defined on Ω_p, satisfy

$$\begin{aligned} u_t - \nu\Delta u + \nabla p &= f_f(x,t), \nabla \cdot u = 0, \text{ in } \Omega_f, \\ S_0\phi_t - \nabla \cdot (\mathcal{K}\nabla\phi) &= f_p(x,t), \text{ in } \Omega_p, \\ u(x,0) &= u_0, \text{ in } \Omega_f, \ \phi(x,0) = \phi_0, \text{ in } \Omega_p, \\ u(x,t) &= 0, \text{ in } \partial\Omega_f \backslash I, \text{ and } \phi(x,t) = 0, \text{ in } \partial\Omega_p \backslash I, \\ &+ \text{coupling conditions across } I, \end{aligned} \tag{1.1}$$

where the pressure in the fluid domain, p, as well as f_p are rescaled by the fluid density. We assume Dirichlet boundary conditions on the exterior boundaries (not including the interface I); our analysis extends to other boundary conditions as well.

The coupling conditions are conservation of mass across the interface

$$u \cdot \hat{n}_f - \mathcal{K}\nabla\phi \cdot \hat{n}_p = 0, \text{ on } I,$$

and balance of forces across the interface

$$p - \nu\hat{n}_f \cdot \nabla u \cdot \hat{n}_f = g\phi, \text{ on } I.$$

The last condition is a condition on the tangential velocity on I. Let $\hat{\tau}_i$, $i =$

[2]The specific storage is defined as $S_0 = \frac{S}{b}$, where S the storativity coefficient (dimensionless) and b the height of the aquifer. For more information see Watson and Burnett [11].

$1, \dots, d-1$, denote an orthonormal basis of tangent vectors on I, $d = 2$ or 3. We use the Beavers-Joseph-Saffman-Jones condition, see Joseph [14] and Saffman [15]:

$$-\nu\hat{\tau}_i \cdot \nabla u \cdot \hat{n}_f = \frac{\alpha}{\sqrt{\hat{\tau}_i \cdot \mathcal{K} \cdot \hat{\tau}_i}} u \cdot \hat{\tau}_i, \text{ for } i = 1, \dots, d-1, \text{ on } I, \tag{1.2}$$

which is a simplification of the original and more physically realistic Beavers-Joseph condition, see Beavers and Joseph [16]. The latter states that the tangential component of the normal stress of the flow in the conduit at the interface is proportional to the tangential velocity in the conduit at the interface. In (1.2), $\alpha > 0$ is a dimensionless, experimentally determined constant. For more information on this condition see e.g., Mikeliĉ and Jäger [17], and Payne and Straughan [18].

One common model used in e.g., Cesmelioglu and Riviére [9], and Badea, Discacciati, and Quarteroni [10], is based on the assumption that the porous media pressure adjusts instantly to changes in the fluid velocity, in other words, the term $S_0\phi_t$ is dropped in the Stokes-Darcy equations. This leads to replacing (1.1) by the quasistatic approximation:

$$\begin{aligned} u_t^{QS} - \nu\Delta u^{QS} + \nabla p^{QS} = \mathbf{f}_f(x,t), \nabla \cdot u^{QS} = 0, \text{ in } \Omega_f, \\ -\nabla \cdot \left(\mathcal{K}\nabla\phi^{QS}\right) = f_p(x,t), \text{ in } \Omega_p, \end{aligned} \tag{1.3}$$

with the same interface coupling and boundary conditions for u^{QS}, ϕ^{QS}, and initial condition for u^{QS}. We consider herein the mathematical foundation for this simplification. Problems of the type $\epsilon u_t + Au = 0$, ϵ small, are treated in Lions [19]. However, problem (1.1), with S_0 small, does not fit within the general theory in [19].

The rest of this paper is organized as follows. Section 2 presents the variational formulation for both the Stokes-Darcy and the quasistatic problems. In Section 3 we obtain à priori bounds on the velocity and hydraulic head for both problems. In Section 4 we justify that

$$u \to u^{QS} \text{ and } \phi \to \phi^{QS}, \quad \text{as } S_0 \to 0,$$

through Theorems 4.1 and 4.2. We prove one half and first order convergence in S_0 of the Stokes-Darcy solution to the quasistatic solution, under assumptions on the initial data and body forces. Convergence of the pressure is then standard. This analysis justifies the inelastic or quasistatic approximation provided

$$S_0 << k_{min}.$$

Finally, we present conclusions in Section 5.

2. Variational formulation

We denote the L^2 norms on $\Omega_{f/p}$ by $\|\cdot\|_{f/p}$ respectively, and the L^2 norm on the interface, I, by $\|\cdot\|_I$; the corresponding inner products on $\Omega_{f/p}$ are denoted by $(\cdot,\cdot)_{f/p}$. Moreover, the H^1 norm on $\Omega_{f/p}$ is denoted by $\|\cdot\|_{1,f/p}$. Define the spaces

$$\begin{aligned}
X_f &:= \left\{ v \in (H^1(\Omega_f))^d : v = 0 \text{ on } \partial\Omega_f \backslash I \right\},\\
X_p &:= \left\{ \psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p \backslash I \right\},\\
Q &:= L_0^2(\Omega_f),\\
V_f &:= \{ v \in X_f : (\nabla \cdot v, q)_f = 0 \text{ for all } q \in Q \}.
\end{aligned}$$

Define the norms on the dual spaces X_f^* and X_p^* by

$$\|f\|_{-1,f/p} = \underset{0\neq v\in X_{f/p}}{sup} \frac{(f,v)_{f/p}}{\|\nabla v\|_{f/p}}.$$

In what follows, we will use the basic estimates

$$\|u\|_{L^2(\partial\Omega_f)} \leq C(\Omega_f)\sqrt{\|u\|_f \|\nabla u\|_f} \tag{2.1}$$

$$\|\phi\|_{L^2(\partial\Omega_p)} \leq C(\Omega_p)\sqrt{\|\phi\|_p \|\nabla \phi\|_p}, \tag{2.2}$$

where by a scaling argument, $C(\Omega_{f/p}) = O(\sqrt{L_{f/p}}), L_{f/p} = diameter(\Omega_{f/p})$. Define the bilinear forms

$$\begin{aligned}
a_f(u,v) &= (\nu\nabla u, \nabla v)_f + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i}}(u\cdot\hat{\tau}_i)(v\cdot\hat{\tau}_i)\; ds,\\
a_p(\phi,\psi) &= (\mathcal{K}\nabla\phi, \nabla\psi)_p\,,\\
c_I(u,\phi) &= g\int_I \phi u\cdot\hat{n}_f\; ds.
\end{aligned}$$

Lemma 2.1. *The bilinear forms $a_f(\cdot,\cdot), a_p(\cdot,\cdot)$ satisfy*

$$a_f(u,v) \leq \max\{\nu+1, \frac{C(\Omega_f)\alpha}{2\sqrt{k_{min}}}\}\|u\|_{1,f}\|v\|_{1,f}, \tag{2.3}$$

$$a_f(u,u) \geq \nu\|\nabla u\|_f^2 + \frac{\alpha}{\sqrt{k_{max}}}\sum_{i=1}^{d-1}\int_I (u\cdot\hat{\tau}_i)^2\; d\sigma =: \nu\|\nabla u\|_f^2 + \frac{\alpha}{\sqrt{k_{max}}}\|u\cdot\hat{\tau}\|_I^2, \tag{2.4}$$

$$a_p(\phi,\psi) \leq k_{max}\|\nabla\phi\|_p\|\nabla\psi\|_p, \tag{2.5}$$

$$a_p(\phi,\phi) \geq k_{min}\|\nabla\phi\|_p^2, \tag{2.6}$$

for all $u, v \in X_f$ and all $\phi, \psi \in X_p$.

Proof. Let $\phi,\psi \in X_p$. Since $\mathcal{K}$ is positive definite, and $0 < k_{min} \leq \lambda(\mathcal{K}) \leq k_{max}$, (2.5) and (2.6) are straightforward. For $u, v \in X_f$, and using $\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i \geq k_{min}$,

$\forall i$, the Cauchy-Schwarz inequality, and the trace inequality (2.1) we have

$$a_f(u,v) \le \nu \|\nabla u\|_f \|\nabla v\|_f + \frac{C(\Omega_f)\alpha}{\sqrt{k_{min}}} \sqrt{\|u\|_f \|\nabla u\|_f} \sqrt{\|v\|_f \|\nabla v\|_f}.$$

Applying the arithmetic-geometric mean inequality we obtain (2.3). Finally, using $\hat{\tau}_i \cdot \mathcal{K} \cdot \hat{\tau}_i \le k_{max}$, $\forall i$, we get

$$a_f(u,u) \ge \nu \|\nabla u\|_f^2 + \frac{\alpha}{\sqrt{k_{max}}} \sum_{i=1}^{d-1} \int_I (u \cdot \hat{\tau}_i)^2 \, d\sigma =: \nu \|\nabla u\|_f^2 + \frac{\alpha}{\sqrt{k_{max}}} \|u \cdot \hat{\tau}\|_I^2.$$

□

The key quantity in the analysis is the interface coupling term, $c_I(\cdot,\cdot)$.

Lemma 2.2. *The bilinear form $c_I(\cdot,\cdot)$ satisfies*

$$|c_I(u,\phi)| \le \frac{gC(\Omega_f,\Omega_p)}{2} \|u\|_{1,f} \|\phi\|_{1,p} \tag{2.7}$$

for all $u, \phi \in X_f, X_p$.

Proof. (2.7) follows by applying the Cauchy-Schwarz inequality and the trace inequalities (2.1)-(2.2). □

The variational formulation of the Stokes-Darcy problem then is to find $u : [0,\infty) \to V_f$, $\phi : [0,\infty) \to X_p$ satisfying

$$(u_t, v)_f + a_f(u,v) + c_I(v,\phi) = (f_f, v)_f, \tag{2.8}$$

$$gS_0(\phi_t, \psi)_p + ga_p(\phi,\psi) - c_I(u,\psi) = g(f_p,\psi)_p, \tag{2.9}$$

$\forall v \in V_f$, $\forall \psi \in X_p$, where $u(x,0) = u_0(x)$, $\phi(x,0) = \phi_0(x)$ are given. Existence and uniqueness of a solution (u,ϕ) to problem (2.8)-(2.9) follow by the Hille - Yosida theorem, see Brézis [20].

The variational formulation of the quasistatic approximation is obtained by setting $S_0 = 0$ in (2.8)-(2.9):

Find $u^{QS} : [0,\infty) \to V_f$, $\phi^{QS} : [0,\infty) \to X_p$ satisfying

$$(u_t^{QS}, v)_f + a_f(u^{QS}, v) + c_I(v, \phi^{QS}) = (f_f, v)_f, \tag{2.10}$$

$$ga_p(\phi^{QS}, \psi) - c_I(u^{QS}, \psi) = g(f_p, \psi)_p, \tag{2.11}$$

$\forall v \in V_f$, $\forall \psi \in X_p$, where $u^{QS}(x,0) = u_0(x)$ is given. $\phi^{QS}(x,0)$ is defined through (2.11), by solving

$$ga_p(\phi^{QS}(x,0), \psi(x)) = c_I(u_0(x), \psi(x)) + g(f_p(x,0), \psi(x)), \forall \psi \in X_p,$$

for the unknown $\phi^{QS}(x,0)$.

3. À priori estimates

The difference between variational formulations (2.8)-(2.9) and (2.10)-(2.11) is the term $gS_0(\phi_t, \psi)_p$. Thus, convergence will hinge on à priori bounds on the hydraulic head ϕ.

$C^* = C^*(u_0, \phi_0, f_{f/p}, g, \Omega_p)$ and $C^{**} = C^{**}(u_0, \phi_0, f_{f/p}, \nu, g, k_{min}, k_{max})$ will denote finite constants, independent of S_0, and

$$L^2(0,T;X) = \{v : [0,T] \to X : \int_0^T \|v(t)\|_X^2 \, dt < \infty\},$$
$$L^\infty(0,T;X) = \{v : [0,T] \to X : \sup_{[0,T]} \{\|v(t)\|_X\} < \infty\}.$$

Theorem 3.1 Part 1, gives à priori bounds for the velocity and hydraulic head for both problems. The second part gives bounds for the time derivatives of the same quantities for both problems.

Theorem 3.1. *1. In the variational formulations (2.8)-(2.9) and (2.10)-(2.11) assume the initial data and body forces satisfy*

$$u_0 \in L^2(\Omega_f), f_f \in L^2(0,T;H^{-1}(\Omega_f)), f_p \in L^2(0,T;H^{-1}(\Omega_p)).$$

Then for u^{QS} given by (2.10)-(2.11)

$$\begin{aligned} &u^{QS} \in L^\infty(0,T;L^2(\Omega_f)), \nabla u^{QS} \in L^2(0,T;L^2(\Omega_f)), \\ &u^{QS} \cdot \hat{\tau} \in L^2(0,T;L^2(I)), \nabla \phi^{QS} \in L^2(0,T;L^2(\Omega_p)). \end{aligned} \tag{3.1}$$

If in addition $\phi_0 \in L^2(\Omega_p)$, then for u, ϕ given by (2.8)-(2.9)

$$\begin{aligned} &u \in L^\infty(0,T;L^2(\Omega_f)), \sqrt{S_0}\phi \in L^\infty(0,T;L^2(\Omega_p)), \\ &\nabla u \in L^2(0,T;L^2(\Omega_f)), u \cdot \hat{\tau} \in L^2(0,T;L^2(I)), \nabla \phi \in L^2(0,T;L^2(\Omega_p)). \end{aligned} \tag{3.2}$$

2. Assume that the body forces satisfy

$$f_{f,t} \in L^2(0,T;H^{-1}(\Omega_f)), f_{p,t} \in L^2(0,T;H^{-1}(\Omega_p)),$$

where $f_{f/p,t}$ denotes differentiation with respect to time. If the initial data for (2.8)-(2.9) satisfy $u_t(0) \in L^2(\Omega_f)$, $\phi_t(0) \in L^2(\Omega_p)$, then

$$\begin{aligned} &u_t \in L^\infty(0,T;L^2(\Omega_f)), \sqrt{S_0}\phi_t \in L^\infty(0,T;L^2(\Omega_p)), \\ &\nabla u_t \in L^2(0,T;L^2(\Omega_f)), u_t \cdot \hat{\tau} \in L^2(0,T;L^2(I)), \nabla \phi_t \in L^2(0,T;L^2(\Omega_p)). \end{aligned} \tag{3.3}$$

If the initial data for (2.10)-(2.11) satisfy $u_t^{QS}(0) \in L^2(\Omega_f)$, then

$$\begin{aligned} &u_t^{QS} \in L^\infty(0,T;L^2(\Omega_f)), \nabla u_t^{QS} \in L^2(0,T;L^2(\Omega_f)), \\ &u_t^{QS} \cdot \hat{\tau} \in L^2(0,T;L^2(I)), \nabla \phi_t^{QS} \in L^2(0,T;L^2(\Omega_p)). \end{aligned} \tag{3.4}$$

Proof. The result follows from Propositions 3.1-3.4 below. □

Proposition 3.1 is the first energy estimate for the Stokes-Darcy weak formulation (2.8)-(2.9):

Proposition 3.1. *Consider the fully evolutionary Stokes-Darcy problem (2.8)-(2.9). Assume the initial data and body forces satisfy*

$$u_0 \in L^2(\Omega_f), \phi_0 \in L^2(\Omega_p), f_f \in L^2(0,T;H^{-1}(\Omega_f)), f_p \in L^2(0,T;H^{-1}(\Omega_p)).$$

We have

$$\begin{aligned}\sup_{[0,T]} \left\{\|u(t)\|_f^2 + gS_0\|\phi(t)\|_p^2\right\} + \int_0^T \left\{\nu\|\nabla u(t)\|_f^2 + \frac{2\alpha}{\sqrt{k_{max}}}\|u\cdot\hat{\tau}\|_I^2 + gk_{min}\|\nabla\phi\|_p^2\right\} dt \\ \leq \|u_0\|_f^2 + gS_0\|\phi_0\|_p^2 + \int_0^T \left\{\frac{1}{\nu}\|f_f\|_{-1,f}^2 + \frac{g}{k_{min}}\|f_p\|_{-1,p}^2\right\} dt \leq C^{**}.\end{aligned} \tag{3.5}$$

Proof. Fix $t > 0$. Set $v = u(t)$, $\psi = \phi(t)$ in (2.8)-(2.9) and add. Note that the two coupling terms exactly cancel. The remainder follows by standard manipulations. Using coercivity of the bilinear forms and Young's inequality we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{\|u(t)\|_f^2 + gS_0\|\phi(t)\|_p^2\right\} + \nu\|\nabla u(t)\|_f^2 + \frac{\alpha}{\sqrt{k_{max}}}\|u\cdot\hat{\tau}\|_I^2 + gk_{min}\|\nabla\phi(t)\|_p^2 \\ \leq \|f_f\|_{-1,f}\|\nabla u\|_f + g\|f_p\|_{-1,p}\|\nabla\phi\|_p \leq \frac{\nu}{2}\|\nabla u\|_f^2 + \frac{1}{2\nu}\|f_f\|_{-1,f}^2 \\ + \frac{gk_{min}}{2}\|\nabla\phi\|_p^2 + \frac{g}{2k_{min}}\|f_p\|_{-1,p}^2.\end{aligned}$$

Rearranging and integrating over $[0,t]$ for any t in $(0,T]$ and $T < \infty$, yields

$$\begin{aligned}\|u(t)\|_f^2 + gS_0\|\phi(t)\|_p^2 + \int_0^t \left\{\nu\|\nabla u(s)\|_f^2 + \frac{2\alpha}{\sqrt{k_{max}}}\|u\cdot\hat{\tau}\|_I^2 + gk_{min}\|\nabla\phi(s)\|_p^2\right\} ds \\ \leq \|u(0)\|_f^2 + gS_0\|\phi(0)\|_p^2 + \int_0^t \left\{\frac{1}{\nu}\|f_f\|_{-1,f}^2 + \frac{g}{k_{min}}\|f_p\|_{-1,p}^2\right\} ds.\end{aligned}$$

Finally, the result follows by taking supremum over $[0,T]$. □

The next proposition gives the corresponding energy estimate for the quasistatic weak formulation (2.10)-(2.11).

Proposition 3.2. *Consider the quasistatic weak formulation (2.10)-(2.11). Assume the initial data and body forces satisfy*

$$u_0 \in L^2(\Omega_f), f_f \in L^2(0,T;H^{-1}(\Omega_f)), f_p \in L^2(0,T;H^{-1}(\Omega_p)).$$

We have

$$\underset{[0,T]}{sup}\|u^{QS}\|_f^2 + \int_0^T \left\{\nu\|\nabla u^{QS}(t)\|_f^2 + \frac{2\alpha}{k_{max}}\|u^{QS}\cdot\hat{\tau}\|_I^2 + gk_{min}\|\nabla\phi^{QS}(t)\|_p^2\right\} dt$$
$$\leq \|u_0\|_f^2 + \int_0^T \left\{\frac{1}{\nu}\|f_f\|_{-1,f}^2 + \frac{g}{k_{min}}\|f_p\|_{-1,p}^2\right\} dt \leq C^{**}.$$

Proof. Fix $t > 0$. In (2.10)-(2.11) pick $v = u^{QS}(t), \psi = \phi^{QS}(t)$ and add. The coupling terms cancel and the result follows by manipulations similar to the ones in the proof of Proposition 3.1. □

Propositions 3.3 and 3.4 provide à priori bounds for the time derivatives of u and ϕ in the Stokes-Darcy and the quasistatic Stokes-Darcy problems, respectively.

Proposition 3.3. *Consider the fully evolutionary Stokes-Darcy problem (2.8)-(2.9). Assume the initial data and body forces satisfy*

$$u_t(0) \in L^2(\Omega_f), \phi_t(0) \in L^2(\Omega_p),$$
$$f_{f,t} \in L^2(0,T;H^{-1}(\Omega_f)), f_{p,t} \in L^2(0,T;H^{-1}(\Omega_p)).$$

Then

$$\sup_{[0,T]}\left\{\|u_t\|_f^2 + gS_0\|\phi_t\|_p^2\right\} + \int_0^T \left\{\nu\|\nabla u_t\|_f^2 + \frac{2\alpha}{\sqrt{k_{max}}}\|u_t\cdot\hat{\tau}\|_I^2 + gk_{min}\|\nabla\phi_t\|_p^2\right\} dt$$
$$\leq \|u_t(0)\|_f^2 + gS_0\|\phi_t(0)\|_p^2 + \int_0^T \left\{\frac{1}{\nu}\|f_{f,t}\|_{-1,f}^2 + \frac{g}{k_{min}}\|f_{p,t}\|_{-1,p}^2\right\} dt \leq C^{**}.$$

Proof. Start with the weak formulation (2.8)-(2.9), and take the derivative with respect to time

$$(u_{tt}, v)_f + a_f(u_t, v) + c_I(v, \phi_t) = (f_{f,t}, v)_f, \tag{3.6}$$
$$gS_0(\phi_{tt}, \psi)_p + ga_p(\phi_t, \psi) - c_I(u_t, \psi) = g(f_{p,t}, \psi)_p. \tag{3.7}$$

Fix $t > 0$. In (3.6)-(3.7) pick $v = u_t(t), \psi = \phi_t(t)$, and add. The coupling terms will cancel and the rest of the proof is similar to the proof of Proposition 3.1. □

Proposition 3.4. *Consider the quasistatic weak formulation (2.10)-(2.11). If*

$$u_t^{QS}(0) \in L^2(\Omega_f), f_{f,t} \in L^2(0,T;H^{-1}(\Omega_f)), f_{p,t} \in L^2(0,T;H^{-1}(\Omega_p))$$

then

$$\sup_{[0,T]}\|u_t^{QS}(t)\|_f^2+\int_0^T\left\{\nu\|\nabla u_t^{QS}\|_f^2+\frac{2\alpha}{\sqrt{k_{max}}}\|u_t^{QS}\cdot\hat{\tau}\|_I^2+gk_{min}\|\nabla\phi_t^{QS}\|_p^2\right\}\,dt$$
$$\leq\|u_t^{QS}(0)\|_f^2+\int_0^T\left\{\frac{1}{\nu}\|f_{f,t}\|_{-1,f}^2+\frac{g}{k_{min}}\|f_{p,t}\|_{-1,p}^2\right\}\,dt\leq C^{**}.$$

Proof. Starting with the weak formulation (2.10)-(2.11), take the derivative with respect to time to get

$$(u_{tt}^{QS},v)_f+a_f(u_t^{QS},v)+c_I(v,\phi_t^{QS})=(f_{f,t},v)_f, \tag{3.8}$$
$$ga_p(\phi_t^{QS},\psi)-c_I(u_t^{QS},\psi)=g(f_{p,t},\psi)_p. \tag{3.9}$$

Fix $t>0$. Pick $v=u_t^{QS}(t),\psi=\phi_t^{QS}(t)$ in (3.8)-(3.9) and add so that the coupling terms cancel. The remainder of the proof is similar to the one of Proposition 3.1. □

In the subsection that follows we obtain á priori bounds assuming less regularity on the body forces, but introducing a constraint on the domains Ω_f,Ω_p.

3.1. À priori estimates using less regular body forces

By assuming less regularity on the body forces we obtain á priori bounds on the velocity and hydraulic head. In this case, however, we restrict the domains Ω_f and Ω_p by assuming that there exists a C^1-diffeomorphism from Ω_f to Ω_p. We begin with a Lemma that gives a bound on the interface term, that is essential in the analysis of the less regular case.

Lemma 3.1. *Assume there exists a $\mathcal{C}^1-$diffeomorphism $F:\Omega_f\to\Omega_p$. Then there exists a constant C such that*

$$|c_I(u,\phi)|\leq gC\|u\|_{H(div,f)}\|\phi\|_{1,p}, \tag{3.10}$$

where $\|v\|_{H(div,f)}^2=\|v\|_f^2+\|\nabla\cdot v\|_f^2$.

Proof. Define $\tilde{\phi}:\Omega_f\to\Omega_p$ by

$$\tilde{\phi}(x)=\begin{cases}(\phi\circ F)(x) & ,x\in\Omega_f\\ \phi(x) & ,x\in I.\end{cases}$$

Since F is a $\mathcal{C}^1-$diffeomorphism, there exist constants C_1,C_2 such that

$$\frac{1}{\sqrt{|det(F')|}}\leq C_1,\ \text{in }\Omega_f, \tag{3.11}$$
$$|F'|_{Hilb}\leq C_2,\ \text{in }\Omega_f, \tag{3.12}$$

where F' is the Jacobian matrix of F, and $|\cdot|_{Hilb}$ denotes the Hilbert norm. We have

$$\int_I \phi u \cdot \hat{n}_f \; ds = \int_I \tilde{\phi} u \cdot \hat{n}_f \; ds = \int_{\partial\Omega_f} \tilde{\phi} u \cdot \hat{n}_f \; ds = \int_{\Omega_f} \nabla \cdot \left(\tilde{\phi} u\right) \; ds$$
$$= \int_{\Omega_f} \tilde{\phi} \nabla \cdot u \; dx + \int_{\Omega_f} \nabla \tilde{\phi} \cdot u \; dx,$$

by the divergence theorem. Thus we obtain

$$|c_I(u,\phi)| \le g \left| \int_{\Omega_f} \tilde{\phi} \nabla \cdot u \; dx \right| + g \left| \int_{\Omega_f} \nabla \tilde{\phi} \cdot u \; dx \right|$$
$$\le g \|u\|_{H(div;f)} \|\tilde{\phi}\|_{1,f}, \tag{3.13}$$

by the Cauchy-Schwarz inequality. The change of variables theorem yields

$$\begin{aligned}
\|\tilde{\phi}\|_{1,f} &= \left(\int_{\Omega_f} \left(|\phi \circ F|^2 + |\nabla(\phi \circ F)|^2 \right) \frac{|det(F')|}{|det(F')|} \; dx \right)^{\frac{1}{2}} \\
&\le \left(\int_{\Omega_f} \left(|\phi \circ F|^2 + |\nabla(\phi \circ F)|^2 \right) |det(F')| C_1^2 \; dx \right)^{\frac{1}{2}} \\
&= C_1 \left(\int_{\Omega_p} \left(|\phi|^2 + |\nabla_x \phi|^2 \right) \; d\eta \right)^{\frac{1}{2}} \\
&\le C_1 \left(\int_{\Omega_p} \left(|\phi|^2 + |F'|_{Hilb}^2 |\nabla_\eta \phi|^2 \right) \; d\eta \right)^{\frac{1}{2}} \\
&\le C_1 \max\{1, C_2\} \left(\int_{\Omega_p} \left(|\phi|^2 + \nabla_\eta \phi|^2 \right) \; d\eta \right)^{\frac{1}{2}} = C \|\phi\|_{1,p},
\end{aligned} \tag{3.14}$$

where $C := C_1 \max\{1, C_2\}$, $\nabla_x = \nabla_{(x_1,\ldots,x_d)}, x \in \Omega_f$, denotes the gradient operator in Ω_f, and $\nabla_\eta = \nabla_{(\eta_1,\ldots,\eta_d)}, \eta \in \Omega_p$, denotes the gradient operator in Ω_p. The inequality now follows by combining (3.13) and (3.14). □

It is worth noting that in the special case when the interface is flat and Ω_f, Ω_p are any domains (Figure 2), inequality (3.10) holds with constant $C = 1$.

Lemma 3.2. *If the interface I is flat (i.e., I is the line $x_2 = 0$ in 2d or the plane $x_3 = 0$ in 3d), then*

$$|c_I(u,\phi)| \le g \|u\|_{H(div,f)} \|\phi\|_{1,p}.$$

Proof. Embed the two domains in equal sized boxes as shown in Figure 2. Extend u by zero on B and ϕ by zero on $-B$, and denote the extended functions

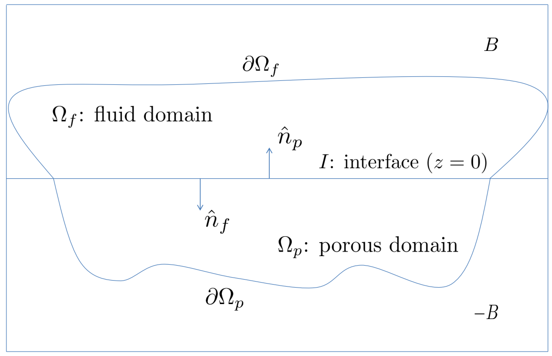

Figure 2: Fluid and porous media domains with flat interface

by u and ϕ respectively. Let $F(x_1, x_2, \ldots, x_{d-1}, x_d) = (x_1, x_2, \ldots, x_{d-1}, -x_d)$, $(x_1, \ldots, x_d) \in B$. Then $|det(F')| = 1$. Define $\tilde{\phi} : B \to \mathbb{R}$ by

$$\tilde{\phi}(x_1, \ldots, x_d) = \begin{cases} (\phi \circ F)(x_1, \ldots, x_d) = \phi(x_1, \ldots, -x_d) & , \text{ on } B \\ \phi(x_1, \ldots, x_d) & , \text{ on } I. \end{cases}$$

Using the same steps as in the proof of (3.10), with B replacing Ω_f, we obtain

$$\begin{aligned} \int_I \phi u \cdot \hat{n}_f \, ds = \int_I \tilde{\phi} u \cdot \hat{n}_f \, ds &= \int_{\partial B} \tilde{\phi} u \cdot \hat{n}_f \, ds \\ &= \int_B \nabla \cdot (\tilde{\phi} u) \, dx = \int_B \nabla \tilde{\phi} \cdot u \, dx + \int_B \tilde{\phi} \nabla \cdot u \, dx \\ &\le \|u\|_{H(div,f)} \left(\int_B (|\tilde{\phi}|^2 + |\nabla_x \tilde{\phi}|^2) \, dx \right)^{\frac{1}{2}} \\ &= \|u\|_{H(div,f)} \left(\int_{-B} (|\phi|^2 + |\nabla_x \phi|^2) \, d\eta \right)^{\frac{1}{2}} \\ &= \|u\|_{H(div,f)} \left(\int_{-B} (|\phi|^2 + |\nabla_\eta \phi|^2) \, d\eta \right)^{\frac{1}{2}} \\ &= \|u\|_{H(div,f)} \|\phi\|_{1,p}, \end{aligned}$$

where, as in the general proof of the inequality, $\nabla_{x/\eta}$ denotes the gradient operator in $\Omega_{f/p}$ respectively. Thus, the inequality follows. □

In our problem, $\nabla \cdot u = 0$ in Ω_f, in which case the proof of (3.10) yields the following inequality instead

$$|c_I(u, \phi)| \le gC\|u\|_f \|\nabla \phi\|_p. \tag{3.15}$$

The constant C in (3.15) is given instead by $C := C_1C_2$, where C_1, C_2 are the constants in (3.11)-(3.12).

Theorem 3.2. *Assume the initial data and body forces satisfy*

$$u_0, \nabla u(0) \in L^2(\Omega_f), u \cdot \hat{\tau}_i|_{t=0} \in L^2(I), i = 1, \dots, d-1, \nabla\phi(0) \in L^2(\Omega_p),$$

$$f_f \in L^2(0,T; L^2(\Omega_f)), f_p \in L^2(0,T; L^2(\Omega_p)),$$

and that the domains Ω_f, Ω_p are such that (3.15) holds. We have

$$\frac{1}{2}\int_0^T \|u_t\|_f^2 \, dt + gS_0 \int_0^T \|\phi_t\|_p^2 \, dt + \sup_{[0,T]} \left\{ \nu\|\nabla u\|_f^2 + \frac{\alpha}{\sqrt{k_{max}}}\|u \cdot \hat{\tau}\|_I^2 \right.$$
$$\left. + \frac{gk_{min}}{2}\|\nabla\phi\|_p^2 \right\}(t) \le \int_0^T \|f_f\|_f^2 \, dt + \frac{g}{S_0}\int_0^T \|f_p\|_p^2 \, dt + 8(gC)^2 \int_0^T \|\nabla\phi\|_p^2 \, dt$$
$$+ \sup_{[0,T]}\left\{ \frac{gC^2}{2k_{min}}\|u\|_f^2 \right\}(t) + \left\{ \nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i \cdot \mathcal{K} \cdot \hat{\tau}_i}}(u \cdot \hat{\tau}_i)^2 \, ds \right.$$
$$\left. + g(\mathcal{K}\nabla\phi, \nabla\phi)_p - 2c_I(u,\phi) \right\}(0) \le C^{**}.$$

Then, specifically

$$u_t \in L^2(0,T; L^2(\Omega_f)), \sqrt{S_0}\phi_t \in L^2(0,T; L^2(\Omega_p)),$$

$$\nabla u \in L^\infty(0,T; L^2(\Omega_f)), u \cdot \hat{\tau} \in L^\infty(0,T; L^2(I)), \nabla\phi \in L^\infty(0,T; L^2(\Omega_p)).$$

Proof. Fix $t > 0$ and set $v = u_t(t), \psi = \phi_t(t)$ in (2.8)-(2.9) and add:

$$(u_t, u_t)_f + gS_0(\phi_t, \phi_t)_p + a_f(u, u_t) + ga_p(\phi, \phi_t) + c_I(u_t, \phi) - c_I(u, \phi_t)$$
$$= (f_f, u_t)_f + g(f_p, \phi_t)_p.$$

Thus,

$$\|u_t(t)\|_f^2 + gS_0\|\phi_t(t)\|_p^2 + \nu(\nabla u, \nabla u_t)_f + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i \cdot \mathcal{K} \cdot \hat{\tau}_i}}(u \cdot \hat{\tau}_i)(u_t \cdot \hat{\tau}_i) \, ds$$
$$+ g(\mathcal{K}\nabla\phi, \nabla\phi_t)_p + c_I(u_t, \phi) - c_I(u, \phi_t) = (f_f, u_t)_f + g(f_p, \phi_t)_p.$$

Using the Cauchy-Schwarz and Young's inequalities we obtain

$$\|u_t\|_f^2 + gS_0\|\phi_t\|_p^2 + \frac{1}{2}\frac{d}{dt}\left\{ \nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i \cdot \mathcal{K} \cdot \hat{\tau}_i}}(u \cdot \hat{\tau}_i)^2 \, ds \right.$$
$$\left. + g(\mathcal{K}\nabla\phi, \nabla\phi)_p \right\} + \{c_I(u_t, \phi) - c_I(u, \phi_t)\}$$
$$\le \|f_f\|_f\|u_t\|_f + g\|f_p\|_p\|\phi_t\|_p \le \frac{1}{2}\|u_t\|_f^2 + \frac{1}{2}\|f_f\|_f^2 + \frac{gS_0}{2}\|\phi_t\|_p^2 + \frac{g}{2S_0}\|f_p\|_p^2.$$

Rearranging then gives

$$\|u_t\|_f^2 + gS_0\|\phi_t\|_p^2 + \frac{d}{dt}\left\{\nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i}}(u\cdot\hat{\tau}_i)^2\ ds\right.$$
$$\left. + g(\mathcal{K}\nabla\phi,\nabla\phi)_p\right\} + 2\left\{c_I(u_t,\phi) - c_I(u,\phi_t)\right\} \le \|f_f\|_f^2 + \frac{g}{S_0}\|f_p\|_p^2.$$

Using

$$c_I(u_t,\phi) - c_I(u,\phi_t) = -\frac{d}{dt}c_I(u,\phi) + 2c_I(u_t,\phi)$$

we get

$$\|u_t\|_f^2 + gS_0\|\phi_t\|_p^2 + \frac{d}{dt}\left\{\nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i}}(u\cdot\hat{\tau}_i)^2\ ds\right.$$
$$\left. + g(\mathcal{K}\nabla\phi,\nabla\phi)_p - 2c_I(u,\phi)\right\} \le \|f_f\|_f^2 + \frac{g}{S_0}\|f_p\|_p^2 - 4c_I(u_t,\phi). \tag{3.16}$$

By (3.15) and Young's inequality we obtain

$$-2c_I(u,\phi) \ge -gC\|u\|_f\|\nabla\phi\|_p \ge -\frac{g}{2k_{min}}C^2\|u\|_f^2 - \frac{gk_{min}}{2}\|\nabla\phi\|_p^2. \tag{3.17}$$

Moreover, since $\nabla\cdot u = 0$ in Ω_f implies that $\nabla\cdot u_t = 0$ in Ω_f, we also get

$$-4c_I(u_t,\phi) \le 4gC\|u_t\|_f\|\nabla\phi\|_p \le \frac{1}{2}\|u_t\|_f^2 + 8(gC)^2\|\nabla\phi\|_p^2. \tag{3.18}$$

Using (3.18) in (3.16) and rearranging terms gives

$$\frac{1}{2}\|u_t\|_f^2 + gS_0\|\phi_t\|_p^2 + \frac{d}{dt}\left\{\nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i}}(u\cdot\hat{\tau}_i)^2\ ds\right.$$
$$\left. + g(\mathcal{K}\nabla\phi,\nabla\phi)_p - 2c_I(u,\phi)\right\} \le \|f_f\|_f^2 + \frac{g}{S_0}\|f_p\|_p^2 + 8(gC)^2\|\nabla\phi\|_p^2.$$

Integrating over $(0,t]$, $\forall t \leq T$, then gives

$$\begin{aligned}
&\frac{1}{2}\int_0^t \|u_t\|_f^2 \ dt + gS_0\int_0^t \|\phi_t\|_p^2 \ dt + \Bigg\{\nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i}}(u\cdot\hat{\tau}_i)^2 \ ds \\
&\qquad + g(\mathcal{K}\nabla\phi,\nabla\phi)_p - 2c_I(u,\phi)\Bigg\}(t) \leq \int_0^t \|f_f\|_f^2 \ dt + \frac{g}{S_0}\int_0^t \|f_p\|_p^2 \ dt \\
&\qquad + 8(gC)^2\int_0^t \|\nabla\phi\|_p^2 \ dt + \Bigg\{\nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i}}(u\cdot\hat{\tau}_i)^2 \ ds \\
&\qquad + g(\mathcal{K}\nabla\phi,\nabla\phi)_p - 2c_I(u,\phi)\Bigg\}(0).
\end{aligned} \tag{3.19}$$

Using (3.17), we estimate

$$\begin{aligned}
&\Bigg\{\nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i}}(u\cdot\hat{\tau}_i)^2 \ ds + g(\mathcal{K}\nabla\phi,\nabla\phi)_p - 2c_I(u,\phi)\Bigg\}(t) \\
&\geq \Bigg\{\nu\|\nabla u\|_f^2 + \frac{\alpha}{\sqrt{k_{max}}}\|u\cdot\hat{\tau}_i\|_I^2 + gk_{min}\|\nabla\phi\|_p^2 - \frac{gC^2}{2k_{min}}\|u\|_f^2 - \frac{gk_{min}}{2}\|\nabla\phi\|_p^2\Bigg\}(t).
\end{aligned} \tag{3.20}$$

With (3.20) and after rearranging, (3.19) becomes

$$\begin{aligned}
&\frac{1}{2}\int_0^t \|u_t\|_f^2 \ dt + gS_0\int_0^t \|\phi_t\|_p^2 \ dt + \Bigg\{\nu\|\nabla u\|_f^2 + \frac{\alpha}{\sqrt{k_{max}}}\|u\cdot\hat{\tau}\|_I^2 + \frac{gk_{min}}{2}\|\nabla\phi\|_p^2\Bigg\}(t) \\
&\qquad \leq \int_0^t \|f_f\|_f^2 \ dt + \frac{g}{S_0}\int_0^t \|f_p\|_p^2 \ dt + 8(gC)^2\int_0^t \|\nabla\phi\|_p^2 \ dt + \frac{gC^2}{2k_{min}}\|u\|_f^2 \\
&\qquad + \Bigg\{\nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i}}(u\cdot\hat{\tau}_i)^2 \ ds + g(\mathcal{K}\nabla\phi,\nabla\phi)_p - 2c_I(u,\phi)\Bigg\}(0),
\end{aligned}$$

$\forall t \leq T$. Taking *supremum* over $[0,T]$ yields

$$\begin{aligned}
&\frac{1}{2}\int_0^T \|u_t\|_f^2 \ dt + gS_0\int_0^T \|\phi_t\|_p^2 \ dt + \sup_{[0,T]}\Bigg\{\nu\|\nabla u\|_f^2 + \frac{\alpha}{\sqrt{k_{max}}}\|u\cdot\hat{\tau}\|_I^2 \\
&\qquad + \frac{gk_{min}}{2}\|\nabla\phi\|_p^2\Bigg\}(t) \leq \int_0^T \|f_f\|_f^2 \ dt + \frac{g}{S_0}\int_0^T \|f_p\|_p^2 \ dt \\
&\qquad + 8(gC)^2\int_0^T \|\nabla\phi\|_p^2 \ dt + \frac{gC^2}{2k_{min}}\sup_{[0,T]}\|u\|_f^2(t) \\
&+ \Bigg\{\nu\|\nabla u\|_f^2 + \sum_{i=1}^{d-1}\int_I \frac{\alpha}{\sqrt{\hat{\tau}_i\cdot\mathcal{K}\cdot\hat{\tau}_i}}(u\cdot\hat{\tau}_i)^2 \ ds + g(\mathcal{K}\nabla\phi,\nabla\phi)_p - 2c_I(u,\phi)\Bigg\}(0).
\end{aligned}$$

Since $\|\cdot\|_{-1,f/p} \le C\|\cdot\|_{f/p}$, by (3.2) of Theorem 3.1, we have $u \in L^\infty(0,T;L^2(\Omega_f))$ and $\nabla\phi \in L^2(0,T;L^2(\Omega_p))$. Thus, the claim of the theorem follows. $\square$

4. Convergence to the quasistatic solution

In this section we prove that the Stokes-Darcy solution, (u,ϕ), given by (2.8)-(2.9), converges to the quasistatic solution, (u^{QS},ϕ^{QS}), given by (2.10)-(2.11), as S_0 approaches zero. We use the à priori estimates from the previous section to obtain estimates for the errors in the velocity and hydraulic head between the two problems. For the less regular body forces case we obtain one half order convergence in S_0. For the more regular case, we obtain first order convergence.

Let

$$e_u(x,t) := u(x,t) - u^{QS}(x,t),$$
$$e_\phi(x,t) := \phi(x,t) - \phi^{QS}(x,t),$$

denote the errors in u and ϕ respectively. Then we have $e_u(x,0) = 0$ and $e_\phi = \phi_0(x) - \phi^{QS}(x,0)$. Subtracting (2.10) from (2.8) and (2.11) from (2.9) we find that the errors satisfy the quasistatic weak formulation (2.10)-(2.11):

$$(e_{ut},v)_f + a_f(e_u,v) + c_I(v,e_\phi) = 0, \tag{4.1}$$
$$ga_p(e_\phi,\psi) - c_I(e_u,\psi) = -gS_0(\phi_t,\psi)_p. \tag{4.2}$$

This can also be written in the form of the Stokes-Darcy weak formulation (2.8)-(2.9):

$$(e_{ut},v)_f + a_f(e_u,v) + c_I(v,e_\phi) = 0, \tag{4.3}$$
$$gS_0(e_{\phi_t},\psi)_p + ga_p(e_\phi,\psi) - c_I(e_u,\psi) = -gS_0(\phi_t^{QS},\psi)_p. \tag{4.4}$$

Theorem 4.1 gives a result of first order convergence of the solution (u,ϕ) to the quasistatic solution (u^{QS},ϕ^{QS}), as S_0 converges to zero.

Theorem 4.1. *Consider the weak formulation (4.3)-(4.4). Assume the initial data and body forces satisfy*

$$u_t^{QS}(0) \in L^2(\Omega_f), \|\phi_t(0)\|_{-1,p} < \infty,$$

$$f_{f,t} \in L^2(0,T;H^{-1}(\Omega_f)), f_{p,t} \in L^2(0,T;H^{-1}(\Omega_p)).$$

Then

$$\sup_{[0,T]}\left\{\|e_u(t)\|_f^2 + gS_0\|e_\phi(t)\|_p^2\right\} + \int_0^T \left\{\nu\|\nabla e_u(t)\|_f^2 + \frac{2\alpha}{\sqrt{k_{max}}}\|e_u\cdot\hat{\tau}\|_I^2 + gk_{min}\|\nabla e_\phi(t)\|_p^2\right\} dt \le gS_0\|\phi_0 - \phi^{QS}(0)\|_p^2 + \frac{S_0^2}{k_{min}}C^* \le \frac{C^*}{k_{min}}(\frac{S_0}{k_{min}} + 1)S_0^2.$$

Proof. We apply the energy estimate obtained in Proposition 3.1 to the weak formulation (4.3)-(4.4) for the error, with $f_f \equiv 0$, $f_p = -S_0\phi_t^{QS}$, e_u replacing u and e_ϕ replacing ϕ. Using the Poincarè-Friedrichs inequality we have

$$\|\phi_t^{QS}\|_{-1,p}^2 \leq C(\Omega_p)\|\nabla\phi_t^{QS}\|_p^2.$$

By (3.4) of Theorem 3.1 we have $\nabla\phi_t^{QS} \in L^2(0,T;L^2(\Omega_p))$. The above inequality then implies that $\phi_t^{QS} \in L^2(0,T;H^{-1}(\Omega_p))$. Therefore,

$$\sup_{[0,T]}\left\{\|e_u(t)\|_f^2 + gS_0\|e_\phi(t)\|_p^2\right\} + \int_0^T \left(\nu\|\nabla e_u(t)\|_f^2 + \frac{2\alpha\|e_u\cdot\hat{\tau}\|_I^2}{\sqrt{k_{max}}} + gk_{min}\|\nabla e_\phi(t)\|_p^2\right)\,dt$$
$$\leq gS_0\|\phi_0 - \phi^{QS}(0)\|_p^2 + \frac{gS_0^2}{k_{min}}\int_0^T \|\phi_t^{QS}\|_{-1,p}^2\,dt \leq gS_0\|\phi_0 - \phi^{QS}(0)\|_p^2 + \frac{C^*}{k_{min}}S_0^2, \tag{4.5}$$

which proves the first part of the theorem. For the last inequality, set $t = 0$ in (2.9) and (2.11) and subtract the second from the first equation to obtain

$$gS_0(\phi_t(0),\psi)_p + ga_p(\phi_0 - \phi^{QS}(0),\psi) = 0, \forall\psi \in X_p, \tag{4.6}$$

where we used that $u^{QS}(x,0) = u_0(x)$. Take $\psi = \phi_0 - \phi^{QS}(0)$ in (4.6) to get

$$ga_p(\phi_0 - \phi^{QS}(0), \phi_0 - \phi^{QS}(0)) = gS_0(\phi_t(0), \phi^{QS}(0) - \phi_0)_p.$$

Using coercivity of the bilinear form $a_p(\cdot,\cdot)$ and the definition of the $\|\cdot\|_{-1}$ norm we have

$$k_{min}\|\nabla(\phi_0 - \phi^{QS}(0))\|_p^2 \leq S_0\|\phi_t(0)\|_{-1,p}\|\nabla(\phi_0 - \phi^{QS}(0))\|_p,$$

so that

$$\|\nabla(\phi_0 - \phi^{QS}(0))\|_p \leq \frac{S_0}{k_{min}}\|\phi_t(0)\|_{-1,p}. \tag{4.7}$$

Finally, using the Poincarè-Friedrichs inequality on the left hand side of (4.7) yields

$$\|\phi_0 - \phi^{QS}(0)\|_p \leq \frac{C(\Omega_p)S_0}{k_{min}}\|\phi_t(0)\|_{-1,p}. \tag{4.8}$$

The last inequality of the theorem now follows by combining inequalities (4.5) and (4.8). □

In Theorem 4.2 we assume less regularity on the body forces and prove one half order convergence of the Stokes-Darcy solution to the quasistatic solution as $S_0 \to 0$.

Theorem 4.2. *Consider the weak formulation (4.1)-(4.2). Assume the initial data and body forces satisfy*

$$u_0, \nabla u(0) \in L^2(\Omega_f), u\cdot\hat{\tau}_i|_{t=0} \in L^2(I), i = 1,\ldots,d-1, \phi_0 \in L^2(\Omega_p),$$

$$f_f \in L^2(0,T;L^2(\Omega_f)), f_p \in L^2(0,T;L^2(\Omega_p)),$$

and that the domains Ω_f and Ω_p are such that inequality (3.15) holds. We have

$$\sup_{[0,T]} \|e_u\|_f^2 + \int_0^T \left\{ \nu \|\nabla e_u\|_f^2 + \frac{2\alpha}{k_{max}} \|e_u \cdot \hat{\tau}\|_I^2 + g k_{min} \|\nabla e_\phi\|_p^2 \right\} \, dt$$
$$\leq \frac{g S_0}{k_{min}} \int_0^T \left(\sqrt{S_0} \|\phi_t\|_p \right)^2 \, dt \leq \frac{C^*}{k_{min}} S_0.$$

Proof. We will use the energy estimate obtained in Proposition 3.2 and apply it to the weak formulation (4.1)-(4.2) for the error, with $f_f \equiv 0$, $f_p = -S_0 \phi_t$, e_u in place of u^{QS} and e_ϕ in place of ϕ^{QS}. By Theorem 3.2 we have in addition that $\sqrt{S_0} \phi_t \in L^2(0,T;L^2(\Omega_p))$. Thus, we conclude

$$\sup_{[0,T]} \|e_u\|_f^2 + \int_0^T \left\{ \nu \|\nabla e_u\|_f^2 + \frac{2\alpha}{k_{max}} \|e_u \cdot \hat{\tau}\|_I^2 + g k_{min} \|\nabla e_\phi\|_p^2 \right\} \, dt$$
$$\leq \frac{g S_0}{k_{min}} \int_0^T \left(\sqrt{S_0} \|\phi_t\|_p \right)^2 \, dt \leq \frac{C^*}{k_{min}} S_0.$$

□

Theorem 4.2 is important because it proves convergence of the Stokes-Darcy solution to the quasistatic solution as S_0 converges to zero assuming less regular body forces. Notice that the assumption on the body forces in Theorem 4.1 is that the time derivatives of the body forces in $\Omega_{f,p}$ belong to $L^2(0,T;H^{-1}(\Omega_{f,p}))$ respectively, while the requirement in Theorem 4.2 is that $f_{f/p} \in L^2(0,T;L^2(\Omega_{f/p}))$. Less regular body forces occur, for instance, in settings involving wells. One half order convergence is obtained in Theorem 4.2 by making a few more assumptions on the initial data for the Stokes-Darcy solution (u, ϕ) given by (2.8)-(2.9) and assuming in addition that there exists a C^1-diffeomorphism between the domains Ω_f and Ω_p.

Remark 4.1. *From the results in Theorems 4.1 and 4.2 it is clear that dropping the term $S_0 \phi_t$ from the fully evolutionary Stokes-Darcy equations, if S_0 is small, is justified provided $S_0 << k_{min}$.*

5. Conclusions

The solution of the fully evolutionary Stokes-Darcy problem converges to the quasistatic solution, as S_0 approaches zero, under mild assumptions on the initial data and body forces. First order convergence is obtained. Provided that S_0 is small and $S_0 << k_{min}$, the term $S_0 \phi_t$ can be dropped from the Stokes-Darcy equations.

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