A DEFECT CORRECTION METHOD FOR THE TIME-DEPENDENT NAVIER-STOKES EQUATIONS

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Abstract. A method for solving the time dependent Navier-Stokes equations, aiming at higher Reynolds' number, is presented. The direct numerical simulation of flows with high Reynolds' number is computationally expensive. The method presented is unconditionally stable, computationally cheap and gives an accurate approximation to the quantities sought. In the defect step, the artificial viscosity parameter is added to the inverse Reynolds number as a stability factor, and the system is antidiffused in the correction step. Stability of the method is proven, and the error estimations for velocity and pressure are derived for the one- and two-step defect-correction methods. The spacial error is $O(h)$ for the one-step defect-correction method, and $O(h^2)$ for the two-step method, where h is the diameter of the mesh. The method is compared to an alternative approach, and both methods are applied to a singularly perturbed convection-diffusion problem. The numerical results are given, which demonstrate the advantage (stability, no oscillations) of the method presented.

Key words. Navier-Stokes, defect-correction, time-dependent, artificial viscosity, residual, antidiffuse, Reynolds number.

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1. Introduction. In the numerical solution of higher Reynolds number flow problems one of the most commonly reported results is that "the method failed". Often "failure" means that the iterative method used to solve the linear and/or nonlinear system for the approximate solution at the new time level failed to converge within the time constraints of the problem or the resulting approximation had poor solution quality. The first type of failure can usually be overcome easily by using an upwind or artificial viscosity (AV) discretization at the expense of decreasing dramatically the accuracy of the method and possibly even altering the predictions of the simulation at the qualitative, O(1) level, therefore increasing the likelihood of the second type of failure.

One interesting approach to attaining (by a convergent method) an approximate solution of desired accuracy is the defect correction method (DCM). Briefly, let a k^{th} order accurate discretization of the *equilibrium* Navier-Stokes equations (NSE) be written as

$$NSE^h(u^h) = f, \tag{1.1}$$

The DCM computes $u^h_1, ..., u^h_k$ as

$$\begin{aligned} -\alpha h \Delta^h u^h_1 + NSE^h(u^h_1) &= f, \\ -\alpha h \Delta^h u^h_l + NSE^h(u^h_l) &= f - \alpha h \Delta^h u^h_{l-1}, \text{ for } l = 2, ..., k, \end{aligned} \tag{1.2}$$

where the velocity approximations u^h_i are sought in the finite element space of piecewise polynomials of degree k.

It has been proven under quite general conditions (see, e.g., [LLP02]) that for the intermediate approximations of the equilibrium NSE

$$\|u_{NSE} - u^h_l\|_{energy-norm} = O(h^k + h\|u_{NSE} - u^h_{l-1}\|_{energy-norm}) = O(h^k + h^l),$$

and thus, after $l = k$ steps,

$$\|u - u^h_k\|_{energy-norm} = O(h^k).$$

Note that (1.2) requires solving an AV approximation k times which is often cheaper and more reliable than solving (1.1) once.

In problems with high Reynolds number we may expect turbulence. In that case the DCM needs to be combined with appropriate turbulence models. These models tend to introduce extra nonlinearities (due to the closure of the model); it might be possible to incorporate them into the residual on the right-hand side, as was done in the quasistatic case by Ervin, Layton, Maubach [ELM00].

There has been an extensive study and development of this approach for equilibrium flow problems, see e.g. Hemker[Hem82], Koren[K91], Heinrichs[Hei96], Layton, Lee, Peterson[LLP02], Ervin, Lee[EL06], and subsection 1.1 for a review of this work.

For many years, it has been widely believed that (1.2) can be directly imported into implicit time discretizations of flow problems in the obvious way: discretize in time, given $u^h(t_{OLD})$, the quasistatic flow problem for $u^h(t_{NEW})$ is solved by applying (1.2) directly, resulting in

$$\begin{aligned} -\alpha h \Delta^h u^h_1(t_{NEW}) + B(u^h_1(t_{NEW}), u^h_k(t_{OLD})) + NSE^h(u^h_1(t_{NEW})) &= f, \\ -\alpha h \Delta^h u^h_l(t_{NEW}) + B(u^h_l(t_{NEW}), u^h_k(t_{OLD})) + NSE^h(u^h_l(t_{NEW})) & \\ = f - \alpha h \Delta^h u^h_{l-1}(t_{NEW}), \text{ for } l = 2, ..., k,& \end{aligned} \tag{1.3}$$

where B is a time stepping operator (e.g., Backward Euler), and k is the degree of piecewise polynomials in the finite element space.

Unfortunately, this natural idea doesn't seem to be even stable (see Section 7).

On the other hand, there is a parallel development of DCM's, for initial value problems in which no spacial stabilization (such as $-\alpha h\Delta^h$ in (1.2)) is used, but DCM is used to increase the accuracy of the time discretization. This work contains no reports of instabilities: see, e.g., Heywood, Rannacher[HR90], Hemker, Shishkin[HSS], Lallemand, Koren[LK93], Minion[M04]. Yet, in spite of this parallel development and after 30+ years of studies of (1.2), there has yet to be a successful extension of (1.2) to time dependent flow problems.

This report will present this extension of (1.2) to the time dependent problem. We notice that the obvious extension, described above, is in fact unstable, see Section 7 . We give a small but critically important modification of the above natural extension to time dependent problems, that we prove to be unconditionally stable (Theorem 3.3) and convergent (Theorem 4.3). We complement the stability proof of the modified DCM by a complete error analysis, which confirms the expected error in the resulting method: $\|u(t_n) - u_l^h(t_n)\|_{energy-norm} = O(\Delta t^a + h^k + h^l), l = 1, ..., k$, where a is the order of accuracy of the (implicit) time stepping employed.

The error analysis is necessarily technical. To keep the details under some control, we study the backward Euler time discretization (It will be clear from our analysis that extension to more accurate time discretizations requires no new ideas and only more pages).

In subsection 1.1 we review important previous work on DCM in space and DCM in time. Section 2 begins with (the inevitable) notation and preliminaries. Section 3, the heart of the report, gives the stability proof. The error analysis is given in Sections 4, 5.2 and Section 7 gives a numerical illustration.

Consider the time dependent, incompressible Navier-Stokes equations

$$\begin{aligned} \frac{\partial u}{\partial t} - Re^{-1}\Delta u + u\cdot\nabla u + \nabla p = f, &\text{ for } x\in\Omega, 0<t\le T, \\ \nabla\cdot u = 0, x\in\Omega, 0<t\le T,& \\ u(x,0) = u_0(x), x\in\Omega,& \\ u|_{\partial\Omega} = 0, \text{ for } 0<t\le T,& \\ \text{where } \Omega\subset\mathbb{R}^d, d=2,3.& \end{aligned} \tag{1.4}$$

Before proceeding with the analysis we shall present carefully next the precise extension of (1.2) to the time dependent NSE that we study.

Let $X^h\subset X, Q^h\subset Q$ be finite-dimensional finite element spaces. Denote the finite-element discretization of the Navier-Stokes operator by

$$N_{Re}^h(u,p) \equiv \frac{\partial u}{\partial t} - Re^{-1}\Delta^h u + (u\cdot\nabla^h)u + \nabla^h p.$$

Adding an artificial viscosity parameter to the inverse Reynolds number leads to the modified Navier-Stokes operator

$$N_{\tilde{Re}}^h(u,p) \equiv \frac{\partial u}{\partial t} - (h + Re^{-1})\Delta^h u + (u\cdot\nabla^h)u + \nabla^h p.$$

The method proceeds as follows: first we compute the AV approximation $(u_1, p_1) \in (X^h, Q^h)$ via

$$N_{\tilde{Re}}^h(u_1,p_1) = f.$$

The accuracy of the approximation is then increased by the correction step: compute $(u_2, p_2) \in (X^h, Q^h)$, satisfying

$$N^h_{\tilde{Re}}(u_2, p_2) - N^h_{\tilde{Re}}(u_1, p_1) = f - N^h_{Re}(u_1, p_1).$$

The Backward Euler time discretization, combined with the two-step defect correction method in space leads to the following system of equations for $(u_1^{h,n+1}, p_1^{h,n+1})$, $(u_2^{h,n+1}, p_2^{h,n+1}) \in (X^h, Q^h), \forall v^h \in X^h$ at $t = t_{n+1}$, $n \geq 0$, with $k := \Delta t = t_{i+1} - t_i$

$$\begin{aligned}
&\Big(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, v^h\Big) + (h + Re^{-1})(\nabla u_1^{h,n+1}, \nabla v^h) + b^*(u_1^{h,n+1}, u_1^{h,n+1}, v^h) \quad (1.5)\\
&\qquad\qquad - (p_1^{h,n+1}, \nabla \cdot v^h) = (f(t_{n+1}), v^h),\\
&\Big(\frac{u_2^{h,n+1} - u_2^{h,n}}{k}, v^h\Big) + (h + Re^{-1})(\nabla u_2^{h,n+1}, \nabla v^h) + b^*(u_2^{h,n+1}, u_2^{h,n+1}, v^h)\\
&\qquad\qquad - (p_2^{h,n+1}, \nabla \cdot v^h) = (f(t_{n+1}), v^h) + h(\nabla u_1^{h,n+1}, \nabla v^h),
\end{aligned}$$

where $b^*(\cdot, \cdot, \cdot)$ is the explicitely skew-symmetrized trilinear form, defined below.

The initial value approximations are taken to be $u_1^{h,0} = u_2^{h,0} = u_0^s$, where u_0^s is the modified Stokes projection of u_0 onto the space V^h of discretely divergence-free functions (this projection and this space are defined in section 2). The stability and error estimate for the modified Stokes projection are proven in the sections 3 and 4.

1.1. Previous results. Many iterative methods can be written as a Defect Correction method, see e. g. Bohmer, Hemker, Stetter [BHS]. In the DCM we consider, no iterates occur; a small number of updates are calculated to increase the accuracy of the velocity and pressure approximations. Thus it is most similar to DCM's which are close to Richardson extrapolation (see, for example, Mathews, Fink [MF04]). In the late 1970's Hemker (Bohmer, Stetter, Heinrichs and others) discovered that DCM, properly interpreted, is good also for nearly singular problems. Examples for which this has been successful include equilibrium Euler equations (Koren, Lallemand [LK93]), high Reynolds number problems (Layton, Lee, Peterson [LLP02]), viscoelastic problems (Ervin, Lee [EL06]).

There has also been interesting work on Spectral Deferred Correction (SDC) for IVP's (e.g., Minion [M04], Bourlioux, Layton, Minion [BLM03], Kress, Gustafsson [KG02], Dutt, Greengard, Rokhlin [DGR00]). With the exception of the SDC methods for time stepping, the majority of the results has been obtained for the equilibrium problems - an odd fact, since, e.g., for the Euler equations the time-dependent problem is natural. For example, it has not been known apparently if the natural idea of time stepping combined with the DCM in space for the associated quasi-equilibrium problem is stable.

2. Mathematical Preliminaries and Notations. Throughout this paper the norm $\|\cdot\|$ will denote the usual $L^2(\Omega)$-norm of scalars, vectors and tensors, induced by the usual L^2 inner-product, denoted by $(\cdot, \cdot)$. The space that velocity (at time t) belongs to, is

$$X = H_0^1(\Omega)^d = \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{d \times d} \text{ and } v = 0 \text{ on } \partial\Omega\}.$$

with the norm $\|v\|_X = \|\nabla v\|$. The space dual to X, is equipped with the norm

$$\|f\|_{-1} = \sup_{v \in X} \frac{(f, v)}{\|\nabla v\|}.$$

The pressure (at time t) is sought in the space

$$Q = L_0^2(\Omega) = \{q : q \in L^2(\Omega), \int_\Omega q(x)dx = 0\}.$$

Also introduce the space of weakly divergence-free functions

$$X \supset V = \{v \in X : (\nabla \cdot v, q) = 0, \forall q \in Q\}.$$

For measurable $v : [0, T] \to X$, we define

$$\|v\|_{L^p(0,T;X)} = (\int_0^T \|v(t)\|_X^p dt)^{\frac{1}{p}},\ 1 \le p < \infty,$$

and

$$\|v\|_{L^\infty(0,T;X)} = ess \sup_{0\le t\le T} \|v(t)\|_X.$$

Define the trilinear form on $X \times X \times X$

$$b(u, v, w) = \int_\Omega u \cdot \nabla v \cdot w dx.$$

The following lemma is also necessary for the analysis

LEMMA 2.1. *There exist finite constants $M = M(d)$ and $N = N(d)$ s.t. $M \ge N$ and*

$$M = \sup_{u,v,w\in X} \frac{b(u, v, w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|} < \infty\ ,\ N = \sup_{u,v,w\in V} \frac{b(u, v, w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|} < \infty.$$

The proof can be found, for example, in [GR79]. The corresponding constants M^h and N^h are defined by replacing X by the finite element space $X^h \subset X$ and V by $V^h \subset X$, which will be defined below. Note that $M \ge \max(M^h, N, N^h)$ and that as $h \to 0$, $N^h \to N$ and $M^h \to M$ (see [GR79]).

Throughout the paper, we shall assume that the velocity-pressure finite element spaces $X^h \subset X$ and $Q^h \subset Q$ are conforming, have typical approximation properties of finite element spaces commonly in use, and satisfy the discrete inf-sup, or LBB^h, condition

$$\inf_{q^h\in Q^h} \sup_{v^h\in X^h} \frac{(q^h, \nabla \cdot v^h)}{\|\nabla v^h\|\|q^h\|} \ge \beta^h > 0, \tag{2.1}$$

where β^h is bounded away from zero uniformly in h. Examples of such spaces can be found in [GR79]. We shall consider $X^h \subset X$, $Q^h \subset Q$ to be spaces of continuous piecewise polynomials of degree m and $m - 1$, respectively, with $m \ge 2$. The case of $m = 1$ is not considered, because the optimal error estimate (of the order h) is obtained after the first step of the method - and therefore the DCM in this case is reduced to the artificial viscosity approach.

The space of discretely divergence-free functions is defined as follows

$$V^h = \{v^h \in X^h : (q^h, \nabla \cdot v^h) = 0, \forall q^h \in Q^h\}.$$

In the analysis we use the properties of the following Modified Stokes Projection

Definition 2.2 (Modified Stokes Projection). *Define the Stokes projection operator P_S: $(X,Q) \to (X^h,Q^h)$, $P_S(u,p) = (\tilde{u},\tilde{p})$, satisfying*

$$(h + Re^{-1})(\nabla(u-\tilde{u}),\nabla v^h) - (p-\tilde{p},\nabla\cdot v^h) = 0, \qquad (2.2)$$
$$(\nabla\cdot(u-\tilde{u}),q^h) = 0,$$

for any $v^h \in V^h, q^h \in Q^h$.

In (V^h,Q^h) this formulation reads: given $(u,p)\in(X,Q)$, find $\tilde{u}\in V^h$ satisfying

$$(h + Re^{-1})(\nabla(u-\tilde{u}),\nabla v^h) - (p-q^h,\nabla\cdot v^h) = 0, \qquad (2.3)$$

for any $v^h \in V^h, q^h \in Q^h$.

Define the explicitly skew-symmetrized trilinear form

$$b^*(u,v,w) := \frac{1}{2}(u\cdot\nabla v,w) - \frac{1}{2}(u\cdot\nabla w,v).$$

The following estimate is easy to prove (see, e.g., [GR79]): there exists a constant $C = C(\Omega)$ such that

$$|b^*(u,v,w)| \le C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\|. \qquad (2.4)$$

The proofs will require the sharper bound on the nonlinearity. This upper bound is improvable in $\mathbb{R}^2$.

Lemma 2.3 (The sharper bound on the nonlinear term). *Let $\Omega\subset\mathbb{R}^d$, $d = 2,3$. For all $u,v,w\in X$*

$$|b^*(u,v,w)| \le C(\Omega)\sqrt{\|u\|\|\nabla u\|}\|\nabla v\|\|\nabla w\|.$$

Proof. See [GR79]. □

We will also need the following inequalities: for any $u \in V$

$$\inf_{v\in V^h}\|\nabla(u-v)\| \le C(\Omega)\inf_{v\in X^h}\|\nabla(u-v)\|, \qquad (2.5)$$

$$\inf_{v\in V^h}\|u-v\| \le C(\Omega)\inf_{v\in X^h}\|\nabla(u-v)\|. \qquad (2.6)$$

The proof of (2.5) can be found, e.g., in [GR79], and (2.6) follows from the Poincare-Friedrich's inequality and (2.5).

Define also the number of time steps $N := \frac{T}{k}$.

We conclude the preliminaries by formulating the discrete Gronwall's lemma, see, e.g. [HR90]

Lemma 2.4. *Let k, B, and $a_\mu, b_\mu, c_\mu, \gamma_\mu$, for integers $\mu \ge 0$, be nonnegative numbers such that:*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \le k\sum_{\mu=0}^{n}\gamma_\mu a_\mu + k\sum_{\mu=0}^{n} c_\mu + B \text{ for } n\ge 0.$$

Suppose that $k\gamma_\mu < 1$ for all μ, and set $\sigma_\mu = (1-k\gamma_\mu)^{-1}$. Then

$$a_n + k\sum_{\mu=0}^{n} b_\mu \le e^{k\sum_{\mu=0}^{n}\sigma_\mu\gamma_\mu}\cdot[k\sum_{\mu=0}^{n} c_\mu + B].$$

3. Stability of the Velocity. In this section we prove the unconditional stability of the discrete artificial viscosity approximation u_1^h and use this result to prove stability of the higher order approximation u_2^h. Over $0 \leq t \leq T < \infty$ the approximations u_1^h and u_2^h are bounded uniformly in Re.

Hence, the formulation (1.5) gives the unconditionally stable extension of the defect correction method to the time-dependent Navier-Stokes equations. We start by proving stability of the modified Stokes Projection, that we use as the approximation $\tilde{u}^0$ to the initial velocity u_0.

PROPOSITION 3.1 (Stability of the Stokes projection). *Let u, $\tilde{u}$ satisfy (2.3). The following bound holds*

$$(h + Re^{-1})\|\nabla \tilde{u}\|^2 \leq 2(h + Re^{-1})\|\nabla u\|^2 \\ +2d(h + Re^{-1})^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2, \tag{3.1}$$

where d is the dimension, $d = 2, 3$.

Proof. Take $v^h = \tilde{u} \in V^h$ in (2.3). This gives

$$(h + Re^{-1})\|\nabla \tilde{u}\|^2 = (h + Re^{-1})(\nabla u, \nabla \tilde{u}) \\ -(p - q^h, \nabla \cdot \tilde{u}). \tag{3.2}$$

Using the Cauchy-Schwarz and Young's inequalities, we obtain

$$(h + Re^{-1})\|\nabla \tilde{u}\|^2 \leq (h + Re^{-1})\|\nabla u\|^2 + \frac{h + Re^{-1}}{4}\|\nabla \tilde{u}\|^2 \\ +d(h + Re^{-1})^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2 + \frac{h + Re^{-1}}{4d}\|\nabla \cdot \tilde{u}\|^2. \tag{3.3}$$

Using the inequality $\|\nabla \cdot \tilde{u}\|^2 \leq d\|\nabla \tilde{u}\|^2$ and combining the like terms concludes the proof. $\square$

Now we prove the main results of this section - stability of the AV approximation u_1^h and the Correction Step approximation u_2^h.

LEMMA 3.2 (Stability of the AV approximation). *Let u_1^h satisfy the first equation of (1.5). Let $f \in L^2(0, T; H^{-1}(\Omega))$. Then for $n = 0, ..., N - 1$*

$$\|u_1^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \leq \|u_0^s\|^2 \\ + \frac{1}{h + Re^{-1}} k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2.$$

Also, if $f \in L^2(0, T; L^2(\Omega))$ and the time constraint T is finite, then there exists a constant $C = C(T)$ such that

$$\|u_1^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \\ \leq C(\|u_0^s\|^2 + k\Sigma_{i=1}^{n+1}\|f(t_i)\|^2). \tag{3.4}$$

Proof. Let $v^h = u_1^{h,n+1} \in V^h$ in the first equation of (1.5). Since $b^*(u, v, v) = 0$, we obtain

$$\frac{\|u_1^{h,n+1}\|^2 - (u_1^{h,n}, u_1^{h,n+1})}{k} + (h + Re^{-1})\|\nabla u_1^{h,n+1}\|^2 - (p_1^{h,n+1}, \nabla \cdot u_1^{h,n+1}) \\ = (f(t_{n+1}), u_1^{h,n+1}).$$

Since $p_1^{h,n+1} \in Q^h$ and $u_1^{h,n+1} \in V^h$ it follows that $(p_1^{h,n+1}, \nabla \cdot u_1^{h,n+1}) = 0$. Applying Cauchy-Schwartz and Young's inequalities gives

$$\frac{\|u_1^{h,n+1}\|^2 - \|u_1^{h,n}\|^2}{2k} + (h + Re^{-1})\|\nabla u_1^{h,n+1}\|^2 \leq (f(t_{n+1}), u_1^{h,n+1}). \tag{3.5}$$

The definition of the dual norm and the Young's inequality, applied to the inner-product on the right-hand side, lead to

$$\begin{aligned}(f^{n+1}, u_1^{h,n+1}) &\leq \|f^{n+1}\|_{-1}\|\nabla u_1^{h,n+1}\| \\ \leq \frac{h + Re^{-1}}{2}\|\nabla u_1^{h,n+1}\|^2 &+ \frac{1}{2(h + Re^{-1})}\|f(t_{n+1})\|_{-1}^2.\end{aligned} \tag{3.6}$$

We obtain

$$\frac{\|u_1^{h,n+1}\|^2 - \|u_1^{h,n}\|^2}{2k} + \frac{h + Re^{-1}}{2}\|\nabla u_1^{h,n+1}\|^2 \leq \frac{1}{2(h + Re^{-1})}\|f(t_{n+1})\|_{-1}^2. \tag{3.7}$$

Summing (3.7) over all time levels and multiplying by $2k$ gives

$$\begin{aligned}\|u_1^{h,n+1}\|^2 + (h + Re^{-1})k\Sigma_{i=1}^{n+1}\|\nabla u_1^{h,i}\|^2 &\leq \|u_0^s\|^2 \\ &+ \frac{1}{h + Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2.\end{aligned} \tag{3.8}$$

This proves the first part of Lemma.

Consider (3.5). Apply the Cauchy-Schwarz and Young's inequalities to the right-hand side. Different choice of constants in the Young's inequality gives

$$(f(t_{n+1}), u_1^{h,n+1}) \leq \|f(t_{n+1})\|\|u_1^{h,n+1}\| \leq \frac{1}{2}\|u_1^{h,n+1}\|^2 + \frac{1}{2}\|f(t_{n+1})\|^2 \tag{3.9}$$

and

$$(f(t_{n+1}), u_1^{h,n+1}) \leq \|f(t_{n+1})\|\|u_1^{h,n+1}\| \leq \frac{1}{4k}\|u_1^{h,n+1}\|^2 + k\|f(t_{n+1})\|^2. \tag{3.10}$$

Sum (3.5) over all time levels, using (3.9) at the time levels $t_0, t_1, ..., t_n$ and (3.10) at $t = t_{n+1}$. We obtain

$$\begin{aligned}&\frac{\|u_1^{h,n+1}\|^2 - \|u_0^s\|^2}{2k} + \Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \\ \leq \frac{1}{4k}\|u_1^{h,n+1}\|^2 &+ \frac{1}{2}\Sigma_{i=1}^{n}\|u_1^{h,i}\|^2 + k\|f(t_{n+1})\|^2 + \frac{1}{2}\Sigma_{i=1}^{n}\|f(t_i)\|^2.\end{aligned} \tag{3.11}$$

Multiply by $4k$ and simplify to obtain

$$\begin{aligned}&\|u_1^{h,n+1}\|^2 + 4k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \\ \leq 2\|u_0^s\|^2 + 4k^2\|f(t_{n+1})\|^2 &+ 2k\Sigma_{i=1}^{n}\|f(t_i)\|^2 + 2k\Sigma_{i=1}^{n}\|u_1^{h,i}\|^2.\end{aligned} \tag{3.12}$$

For the finite time constraint T, the discrete Gronwall's lemma yields

$$\begin{aligned}&\|u_1^{h,n+1}\|^2 + 4k\Sigma_{i=1}^{n+1}(h + Re^{-1})\|\nabla u_1^{h,i}\|^2 \\ &\leq 2e^{(\frac{2T}{1-k})}(\|u_0^s\|^2 + k\Sigma_{i=1}^{n+1}\|f(t_i)\|^2).\end{aligned} \tag{3.13}$$

□

We use the result of Lemma 3.2 in the following

THEOREM 3.3 (Stability). *Let u_1^h, u_2^h satisfy (1.5). Let $f \in L^2(0,T;H^{-1}(\Omega))$. Then for $n = 0,...,N-1$: $u_1^{h,n+1}, u_2^{h,n+1}$ are bounded and*

$$\|u_2^{h,n+1}\|^2 + \frac{2h^2}{(h+Re^{-1})^2}\|u_1^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_2^{h,i}\|^2 \tag{3.14}$$
$$\leq (1+\frac{2h^2}{(h+Re^{-1})^2})\|u_0^s\|^2$$
$$+(1+\frac{h^2}{(h+Re^{-1})^2})\frac{2}{h+Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2.$$

Also, if $f \in L^2(0,T;L^2(\Omega))$ and the time constraint T is finite, then there exists a constant $C = C(T)$ such that

$$\|u_2^{h,n+1}\|^2 + \frac{2h^2}{(h+Re^{-1})^2}\|u_1^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_2^{h,i}\|^2 \tag{3.15}$$
$$\leq C(\|u_0^s\|^2 + k\Sigma_{i=1}^{n+1}\|f(t_i)\|^2).$$

It follows from (3.15) that both approximations u_1^h and u_2^h are bounded at any time level and for any viscosity, provided that the initial approximation and the forcing term are L^2-integrable.

The rest of the section is devoted to the proof of Theorem 3.3.

Proof.

Take $v^h = u_2^{h,n+1} \in V^h$ in the second equation of (1.5). This gives

$$\frac{1}{2k}(\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2) + (h+Re^{-1})\|\nabla u_2^{h,n+1}\|^2 \leq (f(t_{n+1}), u_2^{h,n+1}) \tag{3.16}$$
$$+h(\nabla u_1^{h,n+1}, \nabla u_2^{h,n+1}).$$

The Cauchy-Schwarz and Young's inequalities give

$$\frac{1}{2k}(\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2) + (h+Re^{-1})\|\nabla u_2^{h,n+1}\|^2 \tag{3.17}$$
$$\leq \frac{1}{h+Re^{-1}}\|f(t_{n+1})\|_{-1}^2 + \frac{h+Re^{-1}}{4}\|\nabla u_2^{h,n+1}\|^2$$
$$+\frac{h^2}{h+Re^{-1}}\|\nabla u_1^{h,n+1}\|^2 + \frac{h+Re^{-1}}{4}\|\nabla u_2^{h,n+1}\|^2.$$

Multiply (3.17) by $2k$ and simplify to obtain

$$\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2 + (h+Re^{-1})k\|\nabla u_2^{h,n+1}\|^2 \tag{3.18}$$
$$\leq \frac{2}{h+Re^{-1}}k\|f(t_{n+1})\|_{-1}^2 + \frac{2h^2}{h+Re^{-1}}k\|\nabla u_1^{h,n+1}\|^2.$$

Summing over all time levels leads to

$$\|u_2^{h,n+1}\|^2 + k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_2^{h,i}\|^2 \tag{3.19}$$
$$\leq \|u_0^s\|^2 + \frac{2}{h+Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2$$
$$+\frac{2h^2}{(h+Re^{-1})^2}k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_1^{h,i}\|^2.$$

Inserting the bound on $k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_1^{h,i}\|^2$ from the stability result (3.8) in (3.19) gives

$$\|u_2^{h,n+1}\|^2 + (h+Re^{-1})k\Sigma_{i=1}^{n+1}\|\nabla u_2^{h,i}\|^2 \tag{3.20}$$
$$\leq \|u_0^s\|^2 + \frac{2}{h+Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2$$
$$+\frac{2h^2}{(h+Re^{-1})^2}(\|u_0^s\|^2 - \|u_1^{h,n+1}\|^2 + \frac{1}{h+Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2).$$

Thus

$$\|u_2^{h,n+1}\|^2 + \frac{2h^2}{(h+Re^{-1})^2}\|u_1^{h,n+1}\|^2 + (h+Re^{-1})k\Sigma_{i=1}^{n+1}\|\nabla u_2^{h,i}\|^2 \tag{3.21}$$
$$\leq (1+\frac{2h^2}{(h+Re^{-1})^2})\|u_0^s\|^2$$
$$+(1+\frac{h^2}{(h+Re^{-1})^2})\frac{2}{h+Re^{-1}}k\Sigma_{i=1}^{n+1}\|f(t_i)\|_{-1}^2.$$

This proves the first statement of Theorem 3.3. To conclude, consider (3.16); as in (3.9)-(3.10), use the Young's inequalities differently at different time levels to obtain

$$\frac{\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2}{2k} + (h+Re^{-1})\|\nabla u_2^{h,n+1}\|^2 \tag{3.22}$$
$$\leq k\|f(t_{n+1})\|^2 + \frac{1}{4k}\|u_2^{h,n+1}\|^2$$
$$+\frac{h^2}{2(h+Re^{-1})}\|\nabla u_1^{h,n+1}\|^2 + \frac{h+Re^{-1}}{2}\|\nabla u_2^{h,n+1}\|^2,$$

and

$$\frac{\|u_2^{h,i+1}\|^2 - \|u_2^{h,i}\|^2}{2k} + (h+Re^{-1})\|\nabla u_2^{h,i+1}\|^2 \tag{3.23}$$
$$\leq \frac{1}{2}\|f(t_{i+1})\|^2 + \frac{1}{2}\|u_2^{h,i+1}\|^2$$
$$+\frac{h^2}{2(h+Re^{-1})}\|\nabla u_1^{h,i+1}\|^2 + \frac{h+Re^{-1}}{2}\|\nabla u_2^{h,i+1}\|^2,$$
$$\text{for } \forall i = 0, 1, .., n-1.$$

Sum (3.23) over all time levels and add to (3.22); multiply by $4k$ to obtain

$$\|u_2^{h,n+1}\|^2 - 2\|u_0^s\|^2 + 2k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_2^{h,i}\|^2 \tag{3.24}$$
$$\leq 2k\Sigma_{i=1}^{n}\|u_2^{h,i}\|^2 + 4k^2\|f(t_{n+1})\|^2$$
$$+2k\Sigma_{i=1}^{n}\|f(t_i)\|^2 + \frac{2h^2}{(h+Re^{-1})^2}k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_1^{h,i}\|^2.$$

Insert the bound on $k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_1^{h,i}\|^2$ from (3.13) into (3.24) and simplify. For the finite time constraint T, the discrete Gronwall's lemma yields

$$\|u_2^{h,n+1}\|^2 + \frac{2h^2}{(h+Re^{-1})^2}\|u_1^{h,n+1}\|^2 + 2k\Sigma_{i=1}^{n+1}(h+Re^{-1})\|\nabla u_2^{h,i}\|^2 \tag{3.25}$$
$$\leq (2e^{(\frac{2T}{1-k})} + 4e^{(\frac{4T}{1-k})}\frac{h^2}{(h+Re^{-1})^2})[\|u_0^s\|^2 + k\Sigma_{i=1}^{n}\|f(t_i)\|^2].$$

□

The result of Theorem 3.3, combined with the result of Proposition 3.1, proves the unconditional stability of both $u_1^{h,i}$ and $u_2^{h,i}$ for any $i \geq 0$.

4. Error estimates. In this section we explore the error estimates in approximating the NSE velocity u by the Artificial Viscosity approximation u_1 and the Correction Step approximation u_2. The results agree with the general theory of the defect correction methods: $\|u-u_1^h\|_{energy-norm} \leq C(h^m+h)$, $\|u-u_2^h\|_{energy-norm} \leq C(h^m+h^2)$, where the velocity approximations u_1^h and u_2^h are sought in the finite-element space of piecewise polynomials of degree m.

In the error analysis we shall use the error estimate of the Stokes projection (2.3).

Proposition 4.1 (Error estimate for Stokes Projection). *Suppose the discrete inf-sup condition (2.1) holds. Then the error in the Stokes Projection satisfies*

$$(h+Re^{-1})\|\nabla(u-\tilde{u})\|^2 \leq C[(h+Re^{-1})\inf_{v^h\in V^h}\|\nabla(u-v^h)\|^2 \\ +(h+Re^{-1})^{-1}\inf_{q^h\in Q^h}\|p-q^h\|^2], \tag{4.1}$$

where C is a constant independent of h and Re.

Proof. Decompose the projection error $e = u-\tilde{u}$ into $e = u - I(u) - (\tilde{u} - I(u)) = \eta - \phi$, where $\eta = u - I(u)$, $\phi = \tilde{u} - I(u)$, and $I(u)$ approximates u in V^h. Take $v^h = \phi \in V^h$ in (2.3). This gives

$$(h+Re^{-1})\|\nabla\phi\|^2 = (h+Re^{-1})(\nabla\eta,\nabla\phi) \\ -(p-q^h,\nabla\cdot\phi). \tag{4.2}$$

Since $\Omega \subset \mathbb{R}^d$, we have $\|\nabla\cdot\phi\|^2 \leq d\|\nabla\phi\|^2$.

Applying the Cauchy-Schwarz and Young's inequalities to (4.2) gives

$$(h+Re^{-1})\|\nabla\phi\|^2 \leq 2(h+Re^{-1})\|\nabla\eta\|^2 \\ +2d(h+Re^{-1})^{-1}\inf_{q^h\in Q^h}\|p-q^h\|^2. \tag{4.3}$$

Since $I(u)$ is an approximation of u in V^h, we can take infimum over V^h. The proof is concluded by applying the triangle inequality. □

The following constants (depending upon Ω and u) are introduced in order to simplify the notation.

Definition 4.2. *Let*

$$C_u := \|u(x,t)\|_{L^\infty(0,T;L^\infty(\Omega))}, \\ C_{\nabla u} := \|\nabla u(x,t)\|_{L^\infty(0,T;L^\infty(\Omega))},$$

and introduce $\tilde{C}$, satisfying

$$\inf_{v\in V^h}\|\nabla(u-v)\| \leq C\inf_{v\in X^h}\|\nabla(u-v)\| \leq C_1 h^m\|u\|_{H^{m+1}} \leq \tilde{C}h^m.$$

Also, using the constant $C(\Omega)$ from Lemma 2.3, we define

$$\bar{C} := 1728C^4(\Omega).$$

The main results of this section are presented in the following theorem:

Theorem 4.3 (Error estimates). *Let $f \in L^2(0,T;H^{-1}(\Omega))$, let u_1^h, u_2^h satisfy (1.5),*

$$k \le \frac{h + Re^{-1}}{4C_u^2 + 2(h+Re^{-1})C_{\nabla u} + 2\bar{C}\tilde{C}^4(h+Re^{-1})^{-2}h^{4m}},$$
$$u \in L^2(0,T;H^{m+1}(\Omega)) \bigcap L^\infty(0,T;L^\infty(\Omega)), \nabla u \in L^\infty(0,T;L^\infty(\Omega)),$$
$$u_t \in L^2(0,T;H^{m+1}(\Omega)), u_{tt} \in L^2(0,T;L^2(\Omega)), p \in L^2(0,T;H^m(\Omega)).$$

Then there exists a constant $C = C(\Omega, T, u, p, f, h+Re^{-1})$, such that

$$\max_{1\le i\le N} \|u(t_i) - u_1^{h,i}\| + \left(k \sum_{i=1}^{n+1} (h+Re^{-1}) \|\nabla(u(t_i) - u_1^{h,i})\|^2 \right)^{1/2} \le C(h^m + h + k),$$

and

$$\max_{1\le i\le N} \|u(t_i) - u_2^{h,i}\| + \left(k \sum_{i=1}^{n+1} (h+Re^{-1}) \|\nabla(u(t_i) - u_2^{h,i})\|^2 \right)^{1/2} \le C(h^m + h^2 + hk + k).$$

Hence, the second (Correction) step of the method gives an approximation of the true solution, that is improved by (roughly) an order of h compared to the first step (Artificial Viscosity) approximation.

The goal of this section is to prove Theorem 4.3 - that is, that the method is of the first order in time and that the order of the approximation in space depends upon the step of defect correction procedure.

Proof. By Taylor expansion, $\frac{u(t_{n+1})-u(t_n)}{k} = u_t(t_{n+1}) - k\rho^{n+1}$, where $\rho^{n+1} = u_{tt}(t_{n+\theta})$, for some $\theta \in [0,1]$. The variational formulation of the NSE, followed by the equations (1.5), gives for $u \in X, p \in Q, u_1, u_2 \in X^h, p_1, p_2 \in Q^h, \forall v \in V^h$

$$\begin{aligned}(\frac{u(t_{n+1})-u(t_n)}{k}, v) + (h+Re^{-1})(\nabla u(t_{n+1}), \nabla v) + b^*(u(t_{n+1}), u(t_{n+1}), v) \\ -(p(t_{n+1}), \nabla\cdot v) = (f(t_{n+1}), v) + h(\nabla u(t_{n+1}), \nabla v) - k(\rho^{n+1}, v),\end{aligned} \tag{4.4}$$

$$\begin{aligned}(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, v) + (h+Re^{-1})(\nabla u_1^{h,n+1}, \nabla v) + b^*(u_1^{h,n+1}, u_1^{h,n+1}, v) \\ -(p_1^{h,n+1}, \nabla\cdot v) = (f(t_{n+1}), v),\end{aligned} \tag{4.5}$$

$$\begin{aligned}(\frac{u_2^{h,n+1} - u_2^{h,n}}{k}, v) + (h+Re^{-1})(\nabla u_2^{h,n+1}, \nabla v) + b^*(u_2^{h,n+1}, u_2^{h,n+1}, v) \\ -(p_2^{h,n+1}, \nabla\cdot v) = (f(t_{n+1}), v) + h(\nabla u_1^{h,n+1}, \nabla v).\end{aligned} \tag{4.6}$$

Subtract (4.5) from (4.4). Introduce the error in the AV approximation $e_1^i := u(t_i) - u_1^{h,i}, \forall i$. This gives

$$\begin{aligned}(\frac{e_1^{n+1} - e_1^n}{k}, v) + (h+Re^{-1})(\nabla e_1^{n+1}, \nabla v) \\ +[b^*(u(t_{n+1}), u(t_{n+1}), v) - b^*(u_1^{h,n+1}, u_1^{h,n+1}, v)] \\ -((p(t_{n+1}) - p_1^{h,n+1}), \nabla\cdot v) = h(\nabla u(t_{n+1}), \nabla v) - k(\rho^{n+1}, v).\end{aligned} \tag{4.7}$$

Adding and subtracting $b^*(u_1^{h,n+1}, u(t_{n+1}), v)$ to the nonlinear terms in (4.7) gives

$$\begin{aligned}b^*(u(t_{n+1}), u(t_{n+1}), v) - b^*(u_1^{h,n+1}, u_1^{h,n+1}, v) \\ = b^*(e_1^{n+1}, u(t_{n+1}), v) + b^*(u_1^{h,n+1}, e_1^{n+1}, v).\end{aligned} \tag{4.8}$$

Decompose the error

$$e_1^i = u(t_i) - u_1^{h,i} = u(t_i) - \tilde{u}^i + \tilde{u}^i - u_1^{h,i} = \eta_1^i - \phi_1^{h,i}, \quad (4.9)$$
$$\text{where } \tilde{u}^i \in V^h \text{ is some projection of } u(t_i) \text{ into } V^h,$$
$$\text{and } \eta_1^i = u(t_i) - \tilde{u}^i,\ \phi_1^{h,i} = u_1^{h,i} - \tilde{u}^i, \phi_1^{h,i} \in V^h, \forall i.$$

Take $v = \phi_1^{h,n+1} \in V^h$ in (4.7) and use (4.8). Using also $b^*(\cdot, \phi_1^{h,n+1}, \phi_1^{h,n+1}) = 0$ and $V^h \perp Q^h$, we obtain

$$\begin{aligned}
&(\frac{\eta_1^{n+1} - \eta_1^n}{k}, \phi_1^{h,n+1}) - (\frac{\phi_1^{h,n+1} - \phi_1^{h,n}}{k}, \phi_1^{h,n+1}) \quad (4.10)\\
&+(h + Re^{-1})(\nabla\eta_1^{n+1}, \nabla\phi_1^{h,n+1}) - (h + Re^{-1})\|\nabla\phi_1^{h,n+1}\|^2\\
&+b^*(\eta_1^{n+1}, u(t_{n+1}), \phi_1^{h,n+1}) - b^*(\phi_1^{h,n+1}, u(t_{n+1}), \phi_1^{h,n+1})\\
&+b^*(u_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1}) - (p(t_{n+1}) - q^{h,n+1}, \nabla \cdot \phi_1^{h,n+1})\\
&= h(\nabla u(t_{n+1}), \nabla\phi_1^{h,n+1}) - k(\rho^{n+1}, \phi_1^{h,n+1}).
\end{aligned}$$

Apply the Cauchy-Schwarz and Young's inequalities to (4.10). Since $\|\nabla \cdot \phi_1^{h,n+1}\|^2 \leq d\|\nabla\phi_1^{h,n+1}\|^2$ for $\forall\epsilon > 0$

$$\begin{aligned}
&\frac{\|\phi_1^{h,n+1}\|^2 - \|\phi_1^{h,n}\|^2}{2k} + (h + Re^{-1})\|\nabla\phi_1^{h,n+1}\|^2 \quad (4.11)\\
&\leq \epsilon(h + Re^{-1})\|\nabla\phi_1^{h,n+1}\|^2 + \frac{1}{4\epsilon(h + Re^{-1})}\|\frac{\eta_1^{n+1} - \eta_1^n}{k}\|_{-1}^2\\
&+\epsilon(h + Re^{-1})\|\nabla\phi_1^{h,n+1}\|^2 + \frac{(h + Re^{-1})}{4\epsilon}\|\nabla\eta_1^{n+1}\|^2\\
&+|b^*(\eta_1^{n+1}, u(t_{n+1}), \phi_1^{h,n+1})| + |b^*(\phi_1^{h,n+1}, u(t_{n+1}), \phi_1^{h,n+1})|\\
&+|b^*(u_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})|\\
&+\epsilon(h + Re^{-1})\|\nabla\phi_1^{h,n+1}\|^2 + \frac{d}{4\epsilon(h + Re^{-1})}\inf_{q^h \in Q^h}\|p(t_{n+1}) - q^{h,n+1}\|^2\\
&+\epsilon(h + Re^{-1})\|\nabla\phi_1^{h,n+1}\|^2 + \frac{h^2}{4\epsilon(h + Re^{-1})}\|\nabla u(t_{n+1})\|^2\\
&+\epsilon(h + Re^{-1})\|\nabla\phi_1^{h,n+1}\|^2 + \frac{1}{4\epsilon(h + Re^{-1})}k^2\|\rho^{n+1}\|_{-1}^2.
\end{aligned}$$

We bound the nonlinear terms on the right-hand side of (4.11), starting now with the first one. Use the bound (2.4), the regularity of u and Young's inequality to obtain

$$\begin{aligned}
|b^*(\eta_1^{n+1}, u(t_{n+1}), \phi_1^{h,n+1})| &\leq \epsilon(h + Re^{-1})\|\nabla\phi_1^{h,n+1}\|^2 \quad (4.12)\\
&+C\frac{1}{h + Re^{-1}}\|\nabla\eta_1^{n+1}\|^2.
\end{aligned}$$

The second nonlinear term can be bounded, using the definition of $b^*(\cdot, \cdot, \cdot)$ and the regularity of u. This gives

$$\begin{aligned}
&|b^*(\phi_1^{h,n+1}, u(t_{n+1}), \phi_1^{h,n+1})| \leq \frac{C_{\nabla u}}{2}\|\phi_1^{h,n+1}\|^2 + \frac{C_u}{2}(|\phi_1^{h,n+1}|, |\nabla\phi_1^{h,n+1}|) \quad (4.13)\\
&\leq \frac{C_{\nabla u}}{2}\|\phi_1^{h,n+1}\|^2 + \epsilon(h + Re^{-1})\|\nabla\phi_1^{h,n+1}\|^2 + \frac{C_u^2}{16\epsilon(h + Re^{-1})}\|\phi_1^{h,n+1}\|^2.
\end{aligned}$$

For the third nonlinear term of (4.11), use the error decomposition to obtain

$$
\begin{aligned}
|b^*(u_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})| &\le |b^*(u(t_{n+1}), \eta_1^{n+1}, \phi_1^{h,n+1})| \\
+|b^*(\eta_1^{n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})| &+ |b^*(\phi_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})|.
\end{aligned} \tag{4.14}
$$

Use the regularity of u and the inequality (2.4) to bound the first two terms on the right-hand side of (4.14). Applying Lemma 2.3 to the third term gives

$$
|b^*(\phi_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})| \le C(\Omega)\|\nabla \phi_1^{h,n+1}\|^{3/2}\|\phi_1^{h,n+1}\|^{1/2}\|\eta_1^{n+1}\|. \tag{4.15}
$$

We apply the Young's inequality to (4.15) with $p = \frac{4}{3}$ and $q = 4$. Finally it follows from (4.14) that

$$
\begin{aligned}
|b^*(u_1^{h,n+1}, \eta_1^{n+1}, \phi_1^{h,n+1})| &\le \epsilon(h + Re^{-1})\|\nabla \phi_1^{h,n+1}\|^2 \\
&+\frac{C}{h+Re^{-1}}(\|\nabla \eta_1^{n+1}\|^2 + \|\nabla \eta_1^{n+1}\|^4) \\
&+\frac{27C^4(\Omega)}{64\epsilon^3(h+Re^{-1})^3}\|\nabla \eta_1^{n+1}\|^4\|\phi_1^{h,n+1}\|^2,
\end{aligned} \tag{4.16}
$$

where $C(\Omega)$ is the constant from Lemma 2.3 .

Take $\epsilon = \frac{1}{16}$ in (4.11). Using the bounds (4.12)-(4.16), we obtain

$$
\begin{aligned}
&\frac{\|\phi_1^{h,n+1}\|^2 - \|\phi_1^{h,n}\|^2}{2k} + \frac{h+Re^{-1}}{2}\|\nabla \phi_1^{h,n+1}\|^2 \\
&\le \frac{C}{h+Re^{-1}}\|\frac{\eta_1^{n+1} - \eta_1^n}{k}\|_{-1}^2 \\
&+C(h+Re^{-1})\|\nabla \eta_1^{n+1}\|^2 \\
&+\frac{C}{h+Re^{-1}}\inf_{q^h\in Q^h}\|p(t_{n+1}) - q^{h,n+1}\|^2 \\
&+\frac{C}{h+Re^{-1}}h^2\|\nabla u(t_{n+1})\|^2 \\
&+\frac{C}{h+Re^{-1}}k^2\|\rho^{n+1}\|_{-1}^2 + \frac{C}{h+Re^{-1}}(\|\nabla \eta_1^{n+1}\|^2 + \|\nabla \eta_1^{n+1}\|^4) \\
&+(\frac{1}{2}C_{\nabla u} + \frac{C_u^2}{h+Re^{-1}} + \frac{\bar{C}}{(h+Re^{-1})^3}\|\nabla \eta_1^{n+1}\|^4)\|\phi_1^{h,n+1}\|^2.
\end{aligned} \tag{4.17}
$$

Sum (4.17) over all time levels and multiply by $2k$. It follows from the regularity assumptions of the theorem that

$$
k\sum_{i=0}^{n}\|\rho^{i+1}\|_{-1}^2 \le Ck\sum_{i=0}^{n}\|\rho^{i+1}\|^2 \le C.
$$

Therefore we obtain

$$\|\phi_1^{h,n+1}\|^2 + (h+Re^{-1})k\sum_{i=0}^{n}\|\nabla\phi_1^{h,i+1}\|^2 \le \|\phi_1^{h,0}\|^2 \tag{4.18}$$
$$+\frac{2C}{h+Re^{-1}}k\sum_{i=0}^{n}[\|\frac{\eta_1^{i+1}-\eta_1^i}{k}\|_{-1}^2 + (h+Re^{-1})^2\|\nabla\eta_1^i\|^2$$
$$+\|\nabla\eta_1^i\|^2 + \|\nabla\eta_1^i\|^4 + \inf_{q^h\in Q^h}\|p(t_i)-q^{h,i}\|^2 + h^2 + k^2]$$
$$+k\sum_{i=0}^{n}(C_{\nabla u} + \frac{2C_u^2}{h+Re^{-1}} + \frac{2\bar{C}}{(h+Re^{-1})^3}\|\nabla\eta_1^{i+1}\|^4)\|\phi_1^{h,i+1}\|^2.$$

Take $\tilde{u}^i$ in the error decomposition (4.9) to be the L^2-projection of $u(t_i)$ into V^h, for $i \ge 1$. Take $\tilde{u}^0$ to be u_0^s. This gives $\phi_1^{h,0} = 0$ and $e_1^0 = \eta_1^0$. Also it follows from Proposition 4.1 that $\|\nabla\eta_1^0\| \le Ch^m$; under the assumptions of the theorem the discrete Gronwall's lemma gives

$$\|\phi_1^{h,n+1}\|^2 + (h+Re^{-1})k\sum_{i=0}^{n}\|\nabla\phi_1^{h,i+1}\|^2 \tag{4.19}$$
$$\le \frac{C}{h+Re^{-1}}k\sum_{i=0}^{n}[\|\frac{\eta_1^{i+1}-\eta_1^i}{k}\|_{-1}^2 + \|\nabla\eta_1^i\|^2$$
$$+\|\nabla\eta_1^i\|^4 + \inf_{q^h\in Q^h}\|p(t_i)-q^{h,i}\|^2 + h^2 + k^2].$$

Using the error decomposition and the triangle inequality, we obtain

$$\|e_1^{n+1}\| \le \|\eta_1^{n+1}\| + \|\phi_1^{h,n+1}\|, \tag{4.20}$$
$$\|e_1^{n+1}\|^2 \le 2\|\eta_1^{n+1}\|^2 + 2\|\phi_1^{h,n+1}\|^2,$$
$$\|\nabla e_1^{i+1}\|^2 \le 2\|\nabla\eta_1^{i+1}\|^2 + 2\|\nabla\phi_1^{h,i+1}\|^2,$$
$$k\sum_{i=0}^{n}(h+Re^{-1})\|\nabla e_1^{i+1}\|^2$$
$$\le 2k\sum_{i=0}^{n}(h+Re^{-1})\|\nabla\phi_1^{h,i+1}\|^2 + 2k\sum_{i=0}^{n}(h+Re^{-1})\|\nabla\eta_1^{i+1}\|^2.$$

Then it follows from (4.19),(4.20) that

$$\|e_1^{n+1}\|^2 + k\sum_{i=0}^{n}(h+Re^{-1})\|\nabla e_1^{i+1}\|^2 \tag{4.21}$$
$$\le \frac{C}{h+Re^{-1}}k\sum_{i=0}^{n}[\|\frac{\eta_1^{i+1}-\eta_1^i}{k}\|_{-1}^2 + \|\nabla\eta_1^i\|^2$$
$$+\|\nabla\eta_1^i\|^4 + \inf_{q^h\in Q^h}\|p(t_i)-q^{h,i}\|^2 + h^2 + k^2].$$

Use the approximation properties of X^h, Q^h. Since the mesh nodes do not depend upon the time level, it follows from (2.5),(2.6) that

$$k\sum_{i=0}^{n}\|\frac{\eta_1^{i+1}-\eta_1^i}{k}\|_{-1}^2 \le Ck\sum_{i=0}^{n}\|\frac{\eta_1^{i+1}-\eta_1^i}{k}\|^2 \le Ch^{2m}, \tag{4.22}$$

$$k\sum_{i=0}^{n}\|\nabla\eta_1^i\|^2 \le Ch^{2m},$$

$$k\sum_{i=0}^{n}\inf_{q^h\in Q^h}\|p(t_i)-q^{h,i}\|^2 \le Ch^{2m}.$$

Hence, we obtain from (4.21),(4.22) that

$$\|u(t_{n+1})-u_1^{h,n+1}\|^2 + k\sum_{i=0}^{n}(h+Re^{-1})\|\nabla(u(t_{n+1})-u_1^{h,n+1})\|^2 \tag{4.23}$$
$$\le \frac{C}{h+Re^{-1}}[h^{2m}+h^2+k^2],$$
$$\text{where } C=C(\Omega,T,u,p,f).$$

This proves the first statement of the theorem.

Now subtract (4.6) from (4.4). Introduce the error in the Correction Step approximation $e_2^i := u(t_i)-u_2^{h,i}, \forall i$. This gives

$$(\frac{e_2^{n+1}-e_2^n}{k},v)+(h+Re^{-1})(\nabla e_2^{n+1},\nabla v) \tag{4.24}$$
$$+[b^*(u(t_{n+1}),u(t_{n+1}),v)-b^*(u_2^{h,n+1},u_2^{h,n+1},v)]$$
$$-((p(t_{n+1})-p_2^{h,n+1}),\nabla\cdot v)=h(\nabla e_1^{n+1},\nabla v)-k(\rho^{n+1},v).$$

Note that (4.24) differs from (4.7) only in the first term on the right-hand side. Using the Cauchy-Schwarz and Young's inequality, we obtain that for any $\epsilon>0$

$$|h(\nabla e_1^{n+1},\nabla v)| \le \epsilon(h+Re^{-1})\|\nabla v\|^2+\frac{1}{4\epsilon(h+Re^{-1})}h^2\|\nabla e_1^{n+1}\|^2. \tag{4.25}$$

Therefore,

$$k\sum_{i=0}^{n}|h(\nabla e_1^{n+1},\nabla v)| \le k\sum_{i=0}^{n}\epsilon(h+Re^{-1})\|\nabla v\|^2 \tag{4.26}$$
$$+\frac{1}{4\epsilon(h+Re^{-1})^2}h^2k\sum_{i=0}^{n}(h+Re^{-1})\|\nabla e_1^{n+1}\|^2.$$

Using the bound on $k\sum_{i=0}^{n}(h+Re^{-1})\|\nabla e_1^{n+1}\|^2$ from (4.23), we obtain

$$k\sum_{i=0}^{n}|h(\nabla e_1^{n+1},\nabla v)| \le k\sum_{i=0}^{n}\epsilon(h+Re^{-1})\|\nabla v\|^2 \tag{4.27}$$
$$+\frac{C}{(h+Re^{-1})^3}[h^{2m+2}+h^4+h^2k^2].$$

Decompose the error

$$e_2^i=u(t_i)-u_2^{h,i}=u(t_i)-\tilde{u}^i+\tilde{u}^i-u_2^{h,i}=\eta_2^i-\phi_2^{h,i}, \tag{4.28}$$
$$\text{where } \eta_2^i=u(t_i)-\tilde{u}^i,\ \phi_2^{h,i}=u_2^{h,i}-\tilde{u}^i, \phi_2^{h,i}\in V^h, \forall i.$$

To conclude, repeat the proof of the first statement of the theorem, replacing $u_1^h, e_1, \phi_1^h, \eta_1$ by $u_2^h, e_2, \phi_2^h, \eta_2$, respectively, and using (4.27). Note that the term $\frac{C}{h+Re^{-1}}h^2$ on the right-hand side of (4.23), which was obtained from the bound on $h(\nabla u(t_{n+1}), \nabla v)$, is now replaced by $\frac{C}{(h+Re^{-1})^3}[h^{2m+2}+h^4+h^2k^2]$. Hence, we obtain

$$\|u(t_{n+1}) - u_2^{h,n+1}\|^2 + k\sum_{i=0}^{n}(h+Re^{-1})\|\nabla(u(t_{n+1}) - u_2^{h,n+1})\|^2 \tag{4.29}$$
$$\leq \frac{C}{(h+Re^{-1})^3}[h^{2m}+h^4+h^2k^2+k^2],$$
$$\text{where } C = C(\Omega, T, u, p, f).$$

□

This completes the proof of Theorem 4.3. Thus, we have derived the error estimates, that agree with the general theory of the defect correction methods. Namely, the Correction Step approximation u_2^h is improved by an order of h, compared to the Artificial Viscosity approximation u_1^h.
Next we shall prove stability and derive the error estimates for the pressure.

5. Pressure. This section gives the proof of stability and the convergence rates for pressure approximations p_1^h and p_2^h.

For the pressure analysis we shall need the bounds on discrete time derivatives $\|\frac{e_1^{n+1}-e_1^n}{k}\|$ and $\|\frac{e_2^{n+1}-e_2^n}{k}\|$. For pressure stability it is enough to bound these quantities by a constant, but a more subtle estimate is needed for proving the convergence rates. We start by proving this estimate as a theorem.

Throughout this section we use the error decomposition $e_j^i = u(t_i) - u_j^{h,i} = \eta_j^i - \phi_j^{h,i}$, $j = 1,2$, $i = 1,...,n$, introduced in (4.9),(4.28).

Also, taking $\tilde{u}^i = u_0^s$ on the initial time level gives $\phi_1^{h,0} = \phi_2^{h,0} = 0$ and $e_1^0 = \eta_1^0$, $e_2^0 = \eta_2^0$. It follows from Proposition 4.1 that $\|\nabla\eta_1^0\| \leq Ch^m$ and $\|\nabla\eta_2^0\| \leq Ch^m$.

Theorem 5.1. *Let the regularity assumptions of Theorem 4.3 be satisfied. Let*

$$p_t \in L^2(0,T;H^m(\Omega)), u_{ttt} \in L^2(0,T;L^2(\Omega)).$$

Also let $k \leq \min(h, (h+Re^{-1})^3)$. Then for any time level $n \geq 0$

$$\|\frac{e_1^{n+1}-e_1^n}{k}\| + (k\sum_{i=1}^{n}(h+Re^{-1})\|\nabla(\frac{e_1^{i+1}-e_1^i}{k})\|^2)^{1/2} \leq C(h^m+h+k),$$

and

$$\|\frac{e_2^{n+1}-e_2^n}{k}\| + (k\sum_{i=1}^{n}(h+Re^{-1})\|\nabla(\frac{e_2^{i+1}-e_2^i}{k})\|^2)^{1/2} \leq C(h^m+h^2+hk+k).$$

Proof. Start with the proof of the bound for $\|\frac{\phi_1^{h,n+1}-\phi_1^{h,n}}{k}\|$. Consider (4.7) with (4.8) for $n \geq 1$

$$(\frac{e_1^{n+1}-e_1^n}{k}, v) + (h+Re^{-1})(\nabla e_1^{n+1}, \nabla v) \tag{5.1}$$
$$+b^*(e_1^{n+1}, u(t_{n+1}), v) + b^*(u_1^{h,n+1}, e_1^{n+1}, v)$$
$$-((p(t_{n+1}) - p_1^{h,n+1}), \nabla\cdot v) = h(\nabla u(t_{n+1}), \nabla v) - k(\rho^{n+1}, v),$$
$$\text{where } k\rho^{n+1} = u_t(t_{n+1}) - \frac{u(t_{n+1})-u(t_n)}{k}.$$

Take $v = \frac{\phi_1^{h,n+1} - \phi_1^{h,n}}{k} =: s^{h,n+1} \in V^h$ in (5.1). Then consider (5.1) at the previous time level and make exactly the same choice $v = s^{h,n+1} \in V^h$. Subtract the equations, using the Taylor expansion to simplify the last term on the right-hand side. We obtain

$$\begin{aligned} k(\frac{\eta_1^{n+1} - 2\eta_1^n + \eta_1^{n-1}}{k^2}, s^{h,n+1}) - (s^{h,n+1} - s^{h,n}, s^{h,n+1}) \\ +(h + Re^{-1})k(\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k}), \nabla s^{h,n+1}) - (h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 \\ +b^*(e_1^{n+1}, u(t_{n+1}), s^{h,n+1}) + b^*(u_1^{h,n+1}, e_1^{n+1}, s^{h,n+1}) \\ -b^*(e_1^n, u(t_n), s^{h,n+1}) - b^*(u_1^{h,n}, e_1^n, s^{h,n+1}) \\ -k(\frac{(p(t_{n+1}) - p_1^{h,n+1}) - (p(t_n) - p_1^{h,n})}{k}, \nabla \cdot s^{h,n+1}) \\ = hk(\nabla(\frac{u(t_{n+1}) - u(t_{n+1})}{k}), \nabla s^{h,n+1}) - Ck^2(\rho_t^{n+1}, s^{h,n+1}), \\ \text{where } \rho_t^{n+1} = u_{ttt}(t_{n+\theta}) \text{ for some } \theta \in [0,1]. \end{aligned} \tag{5.2}$$

Consider the nonlinear terms of (5.2). Adding and subtracting $b^*(e_1^n, u(t_{n+1}), s^{h,n+1})$ and $b^*(u_1^{h,n+1}, e_1^n, s^{h,n+1})$ gives

$$\begin{aligned} b^*(e_1^{n+1}, u(t_{n+1}), s^{h,n+1}) - b^*(e_1^n, u(t_n), s^{h,n+1}) \\ +b^*(u_1^{h,n+1}, e_1^{n+1}, s^{h,n+1}) - b^*(u_1^{h,n}, e_1^n, s^{h,n+1}) \\ = [b^*(e_1^{n+1}, u(t_{n+1}), s^{h,n+1}) - b^*(e_1^n, u(t_{n+1}), s^{h,n+1}) \\ +b^*(e_1^n, u(t_{n+1}), s^{h,n+1}) - b^*(e_1^n, u(t_n), s^{h,n+1})] \\ +[b^*(u_1^{h,n+1}, e_1^{n+1}, s^{h,n+1}) - b^*(u_1^{h,n+1}, e_1^n, s^{h,n+1}) \\ +b^*(u_1^{h,n+1}, e_1^n, s^{h,n+1}) - b^*(u_1^{h,n}, e_1^n, s^{h,n+1})]. \end{aligned} \tag{5.3}$$

Use the error decomposition (4.9). Since $b^*(\cdot, s^{h,n+1}, s^{h,n+1}) = 0$, it follows from (5.3) that

$$\begin{aligned} b^*(e_1^{n+1}, u(t_{n+1}), s^{h,n+1}) - b^*(e_1^n, u(t_n), s^{h,n+1}) \\ +b^*(u_1^{h,n+1}, e_1^{n+1}, s^{h,n+1}) - b^*(u_1^{h,n}, e_1^n, s^{h,n+1}) \\ = kb^*(\frac{\eta_1^{n+1} - \eta_1^n}{k}, u(t_{n+1}), s^{h,n+1}) - kb^*(s^{h,n+1}, u(t_{n+1}), s^{h,n+1}) \\ +kb^*(e_1^{n+1}, \frac{u(t_{n+1}) - u(t_n)}{k}, s^{h,n+1}) + kb^*(u_1^{h,n+1}, \frac{\eta_1^{n+1} - \eta_1^n}{k}, s^{h,n+1}) \\ +kb^*(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, e_1^n, s^{h,n+1}). \end{aligned} \tag{5.4}$$

Use the regularity of u and the Cauchy-Schwarz and Young's inequalities to obtain the bounds on the terms in (5.4). It follows from (2.4) that for any $\epsilon > 0$

$$\begin{aligned} k|b^*(\frac{\eta_1^{n+1} - \eta_1^n}{k}, u(t_{n+1}), s^{h,n+1})| \\ \leq \epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 + \frac{C}{h + Re^{-1}}k\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2. \end{aligned} \tag{5.5}$$

For the second term on the right-hand side of (5.4) use the regularity of u and the

Cauchy-Schwarz and Young's inequalities to obtain

$$k|b^*(s^{h,n+1}, u(t_{n+1}), s^{h,n+1})| \leq \epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 \\ + \frac{C}{h + Re^{-1}} C_u^2 k\|s^{h,n+1}\|^2 + \frac{1}{2} C_{\nabla u} k\|s^{h,n+1}\|^2. \tag{5.6}$$

The third nonlinear term on the right-hand side of (5.4) is bounded by

$$k|b^*(e_1^{n+1}, \frac{u(t_{n+1}) - u(t_n)}{k}, s^{h,n+1})| \\ \leq \epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 + \frac{C}{h + Re^{-1}} k\|\nabla e_1^{n+1}\|^2. \tag{5.7}$$

For the fourth nonlinear term, add and subtract $u(t_{n+1})$ to the first term of the trilinear form. Using (2.4) and Lemma 2.3 leads to

$$k|b^*(u_1^{h,n+1}, \frac{\eta_1^{n+1} - \eta_1^n}{k}, s^{h,n+1})| \leq 2\epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 \\ + \frac{C}{h + Re^{-1}} k\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2 + Ck\|e_1^{n+1}\|\|\nabla e_1^{n+1}\|\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2. \tag{5.8}$$

For the fifth term add and subtract $u(t_{n+1})$ to the first term of the trilinear form to obtain

$$k|b^*(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, e_1^n, s^{h,n+1})| \leq k|b^*(\frac{u(t_{n+1}) - u(t_n)}{k}, e_1^n, s^{h,n+1})| \\ + k|b^*(\frac{\eta_1^{n+1} - \eta_1^n}{k}, e_1^n, s^{h,n+1})| + k|b^*(s^{h,n+1}, e_1^n, s^{h,n+1})|. \tag{5.9}$$

Apply the result of Lemma 2.3 to the last trilinear form in (5.9) and use the Young's inequality with $p = \frac{4}{3}$ and $q = 4$. This gives

$$k|b^*(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, e_1^n, s^{h,n+1})| \\ \leq 3\epsilon(h + Re^{-1})k\|\nabla s^{h,n+1}\|^2 + \frac{C}{h + Re^{-1}} k\|\nabla e_1^n\|^2 \\ + \frac{C}{h + Re^{-1}} k\|\nabla e_1^n\|^2\|\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k})\|^2 + \frac{C}{(h + Re^{-1})^3} k\|\nabla e_1^n\|^4\|s^{h,n+1}\|^2. \tag{5.10}$$

Apply the Cauchy-Schwarz and Young's inequalities to (5.2), using the bounds (5.4)-

(5.10) for the nonlinear terms. This gives

$$
\begin{aligned}
\frac{\|s^{h,n+1}\|^2 - \|s^{h,n}\|^2}{2} &+ (h+Re^{-1})k\|\nabla s^{h,n+1}\|^2 \qquad (5.11)\\
&\leq 13\epsilon(h+Re^{-1})k\|\nabla s^{h,n+1}\|^2\\
+\frac{C}{h+Re^{-1}}k\|\frac{\eta_1^{n+1}-2\eta_1^n+\eta_1^{n-1}}{k^2}\|_{-1}^2 &+ C(h+Re^{-1})k\|\nabla(\frac{\eta_1^{n+1}-\eta_1^n}{k})\|^2\\
&+\frac{C}{h+Re^{-1}}k\inf_{q^h\in Q^h}\|\frac{p(t_{n+1})-p(t_n)}{k}-\frac{q^{h,n+1}-q^{h,n}}{k}\|^2\\
+\frac{C}{h+Re^{-1}}k[\|\nabla(\frac{\eta_1^{n+1}-\eta_1^n}{k})\|^2 &+ \|\nabla e_1^n\|^2 + \|\nabla(\frac{\eta_1^{n+1}-\eta_1^n}{k})\|^2\|\nabla e_1^n\|^2]\\
&+Ck\|e_1^n\|^2\|\nabla e_1^n\|^2 + Ck\|\nabla(\frac{\eta_1^{n+1}-\eta_1^n}{k})\|^4\\
+\frac{C}{h+Re^{-1}}k\cdot h^2\|\nabla(\frac{u(t_{n+1})-u(t_n)}{k})\|^2 &+ \frac{C}{h+Re^{-1}}k\cdot k^2\|\rho_t^{n+1}\|_{-1}^2\\
&+C(C_{\nabla u}+\frac{C_u^2}{h+Re^{-1}}+\frac{1}{(h+Re^{-1})^3}\|\nabla e_1^n\|^4)k\|s^{h,n+1}\|^2.
\end{aligned}
$$

Since $u_{ttt} \in L^2(0,T;L^2(\Omega))$, we have

$$
k\sum_{i=0}^{n}\|\rho_t^{i+1}\|_{-1}^2 \leq Ck\sum_{i=0}^{n}\|\rho_t^{i+1}\|^2 \leq C.
$$

It follows from the assumption $k \leq h$ and the result of Theorem 4.3 that

$$
\begin{aligned}
\max_i \|\nabla e_1^i\| &\leq C,\\
\max_i \|\nabla e_2^i\| &\leq C.
\end{aligned}
$$

Take $\epsilon = \frac{1}{26}$ in (5.11), simplify, multiply both sides of the inequality by 2 and sum over all time levels $n \geq 1$ to obtain

$$
\begin{aligned}
\|s^{h,n+1}\|^2 + (h+Re^{-1})k\sum_{i=1}^{n}\|\nabla s^{h,i+1}\|^2 &\leq \|s^{h,1}\|^2 \qquad (5.12)\\
+\frac{C}{h+Re^{-1}}k\sum_{i=1}^{n}[\|&\frac{\eta_1^{i+1}-2\eta_1^i+\eta_1^{i-1}}{k^2}\|_{-1}^2\\
+(h+Re^{-1})^2\|\nabla(\frac{\eta_1^{i+1}-\eta_1^i}{k})\|^2 &+ \|\nabla(\frac{\eta_1^{i+1}-\eta_1^i}{k})\|^2\\
+(h+Re^{-1})&\|\nabla(\frac{\eta_1^{i+1}-\eta_1^i}{k})\|^4\\
+\inf_{q^h\in Q^h}\|\frac{p(t_{i+1})-p(t_i)}{k}-\frac{q^{h,i+1}-q^{h,i}}{k}\|^2 &+ h^2 + k^2]\\
+\frac{C}{(h+Re^{-1})^2}k\sum_{i=1}^{n}(h+Re^{-1})\|\nabla e_1^i\|^2 &+ Ck\sum_{i=1}^{n}\|e_1^i\|^2\\
+Ck\sum_{i=1}^{n}(C_{\nabla u}+\frac{C_u^2}{h+Re^{-1}}+\frac{1}{(h+Re^{-1})^3}&\|\nabla e_1^i\|^4)\|s^{h,i+1}\|^2.
\end{aligned}
$$

Consider the error decomposition (4.9). Take $\tilde{u}^i$ to be the L^2 projection of $u(t_i)$ into V^h, for all $i \geq 1$. Since the mesh nodes do not depend upon the time level, it follows from the approximation properties of X^h, Q^h and the regularity of u, p that

$$k\sum_{i=1}^{n}\|\frac{\eta_1^{i+1}-2\eta_1^i+\eta_1^{i-1}}{k^2}\|_{-1}^2 \leq Ck\sum_{i=1}^{n}\|\frac{\eta_1^{i+1}-2\eta_1^i+\eta_1^{i-1}}{k^2}\|^2 \leq Ch^{2m}, \quad (5.13)$$

$$k\sum_{i=1}^{n}\|\nabla(\frac{\eta_1^{i+1}-\eta_1^i}{k})\|^2 \leq Ch^{2m},$$

$$k\sum_{i=1}^{n}\|\nabla(\frac{\eta_1^{i+1}-\eta_1^i}{k})\|^4 \leq Ch^{4m},$$

$$k\sum_{i=1}^{n}\inf_{q^h\in Q^h}\|\frac{p(t_{i+1})-p(t_i)}{k}-\frac{q^{h,i+1}-q^{h,i}}{k}\|^2 \leq Ch^{2m}.$$

Using (5.13) and (4.23), we derive from (5.12) that

$$\begin{aligned}\|s^{h,n+1}\|^2 + (h+Re^{-1})k\sum_{i=1}^{n}\|\nabla s^{h,i+1}\|^2 \leq \|s^{h,1}\|^2 \quad (5.14)\\ +C[h^{2m}+h^2+k^2]\\ +Ck\sum_{i=1}^{n}(C_{\nabla u}+\frac{C_u^2}{h+Re^{-1}}+\frac{1}{(h+Re^{-1})^3}\|\nabla e_1^i\|^4)\|s^{h,i+1}\|^2.\end{aligned}$$

Take $\tilde{u}^0 = u_0^s$ on the initial time level. This gives $\phi_1^{h,0} = 0$ and $e_1^0 = \eta_1^0 = u_0 - u_0^s$.

For the bound on $\|s^{h,1}\|^2 = \|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2$, consider (5.1) at $n = 0$ and take $v = \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}$. This gives

$$\begin{aligned}(\frac{e_1^1-e_1^0}{k},\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + (h+Re^{-1})(\nabla e_1^1, \nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})) \quad (5.15)\\ +b^*(e_1^1, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(u_1^{h,1}, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\\ -((p(t_1)-p_1^{h,1}), \nabla\cdot(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}))\\ = h(\nabla u(t_1), \nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})) - k(\rho^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}).\end{aligned}$$

Rewrite the left-hand side of (5.15) so that we could use the properties of the modified Stokes projection (2.3)

$$\begin{aligned}(\frac{e_1^1-e_1^0}{k},\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + (h+Re^{-1})k(\nabla(\frac{e_1^1-e_1^0}{k}), \nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})) \quad (5.16)\\ +b^*(e_1^1, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(u_1^{h,1}, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\\ +(h+Re^{-1})(\nabla e_1^0, \nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})) - ((p(t_1)-p_1^{h,1}), \nabla\cdot(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}))\\ = h(\nabla u(t_1), \nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})) - k(\rho^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}).\end{aligned}$$

Since $\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k} \in V^h$ and $p_1^{h,1} \in Q^h$, it follows from the choice of initial approximation $\tilde{u}^0$ and from (2.3) that

$$(h+Re^{-1})(\nabla e_1^0, \nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})) - ((p(t_1)-p_1^{h,1}), \nabla\cdot(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})) = 0. \quad (5.17)$$

Hence, using the Cauchy-Schwarz and Young's inequalities, we derive from (5.16) and (5.17) that for any ϵ, $\epsilon_1 > 0$

$$\begin{aligned}
&\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 \qquad (5.18)\\
&\le \epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + C\|\frac{\eta_1^1-\eta_1^0}{k}\|^2\\
&+\epsilon(h+Re^{-1})k\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + C(h+Re^{-1})k\|\nabla(\frac{\eta_1^1-\eta_1^0}{k})\|^2\\
&+\epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + Ck^2\|\rho^1\|^2 + \epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2\\
&+\frac{C}{h+Re^{-1}}h^2k\|\nabla(\frac{u(t_1)-u_0}{k})\|^2 + \epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + Ch^2\|\Delta u_0\|^2\\
&+kb^*(\frac{e_1^1-e_1^0}{k}, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(e_1^0, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\\
&+b^*(u(t_1), e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(\phi_1^{h,1}, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\\
&-b^*(\eta_1^1, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}).
\end{aligned}$$

Using the fact that $\phi_1^{h,0} = 0$, we obtain $b^*(\cdot, \phi_1^{h,1}, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) = 0$. The nonlinear terms in (5.18) are bounded by applying Cauchy-Schwarz and Young's inequalities. We obtain

$$\begin{aligned}
&kb^*(\frac{e_1^1-e_1^0}{k}, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(e_1^0, u(t_1), \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) \qquad (5.19)\\
&+b^*(u(t_1), e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}) + b^*(\phi_1^{h,1}, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\\
&-b^*(\eta_1^1, e_1^1, \frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\\
&\le \frac{C}{h+Re^{-1}}k\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + \epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2\\
&+\epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 + \frac{C}{h+Re^{-1}}k\|\nabla(\frac{\eta_1^1-\eta_1^0}{k})\|^2\\
&+\epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 + \frac{C}{(h+Re^{-1})^3}k\|\nabla\eta_1^1\|^4\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2\\
&+2\epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + C\|\eta_1^0\|^2 + C\|\nabla\eta_1^0\|^2\\
&+2\epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + C\|\eta_1^1\|^2 + C\|\nabla\eta_1^1\|^2\\
&+\epsilon_1\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2 + Ch^{-2}\|\nabla\eta_1^1\|^4.
\end{aligned}$$

The inequalities (5.18)-(5.19) give

$$\begin{aligned}
&\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2+(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 \\
&\le (8\epsilon_1+\frac{C}{h+Re^{-1}}k+\frac{C}{(h+Re^{-1})^3}k\|\nabla\eta_1^1\|^4)\|\frac{\phi_1^{h,1}-\phi_1^0}{k}\|^2 \\
&+5\epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_1^1-\phi_1^0}{k})\|^2 \\
&+C[\|\frac{\eta_1^1-\eta_1^0}{k}\|^2+(h+Re^{-1})k\|\nabla(\frac{\eta_1^1-\eta_1^0}{k})\|^2 \\
&+k^2\|\rho^1\|^2+\frac{1}{h+Re^{-1}}h^2k\|\nabla(\frac{u(t_1)-u(t_0)}{k})\|^2+h^2\|\Delta u_0\|^2 \\
&+\frac{1}{h+Re^{-1}}k\|\nabla(\frac{\eta_1^1-\eta_1^0}{k})\|^2+\|\eta_1^1\|^2+\|\nabla\eta_1^1\|^2+h^{-2}\|\nabla\eta_1^1\|^4].
\end{aligned} \tag{5.20}$$

It follows from the approximation properties of X^h, Q^h that

$$\begin{aligned}
&\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2+(h+Re^{-1})k\|\nabla(\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k})\|^2 \\
&\le C[h^{2m}+h^2+k^2],
\end{aligned} \tag{5.21}$$

and the triangle inequality gives

$$\begin{aligned}
&\|\frac{e_1^1-e_1^0}{k}\|^2+(h+Re^{-1})k\|\nabla(\frac{e_1^1-e_1^0}{k})\|^2 \\
&\le C[h^{2m}+h^2+k^2].
\end{aligned} \tag{5.22}$$

Insert the bound on $\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|^2$ into (5.14). The restriction on the time step k allows to apply discrete Gronwall's lemma. This leads to

$$\begin{aligned}
&\|\frac{\phi_1^{h,n+1}-\phi_1^{h,n}}{k}\|^2+(h+Re^{-1})k\sum_{i=1}^{n}\|\nabla(\frac{\phi_1^{h,i+1}-\phi_1^{h,i}}{k})\|^2 \\
&\le C[h^{2m}+h^2+k^2].
\end{aligned} \tag{5.23}$$

Using the triangle inequality we obtain

$$\begin{aligned}
&\|\frac{e_1^{n+1}-e_1^n}{k}\|^2+(h+Re^{-1})k\sum_{i=1}^{n}\|\nabla(\frac{e_1^{i+1}-e_1^i}{k})\|^2 \\
&\le C[h^{2m}+h^2+k^2].
\end{aligned} \tag{5.24}$$

This result proves the first statement of Theorem 5.1.

For the bound on $\|\frac{\phi_2^{h,n+1}-\phi_2^{h,n}}{k}\|$ consider (4.24), $n\ge 1$. Following the proof above, take $v=\frac{\phi_2^{h,n+1}-\phi_2^{h,n}}{k}=: s_2^{h,n+1}$, then consider (4.24) at the previous time level, make the same choice $v = s_2^{h,n+1}$ and subtract the two equations. This leads to

$$
\begin{aligned}
& k(\frac{\eta_2^{n+1} - 2\eta_2^n + \eta_2^{n-1}}{k^2}, s_2^{h,n+1}) - (s_2^{h,n+1} - s_2^{h,n}, s_2^{h,n+1}) \\
&+(h + Re^{-1})k(\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k}), \nabla s_2^{h,n+1}) - (h + Re^{-1})k\|\nabla s_2^{h,n+1}\|^2 \\
&+b^*(e_2^{n+1}, u(t_{n+1}), s_2^{h,n+1}) + b^*(u_2^{h,n+1}, e_2^{n+1}, s_2^{h,n+1}) \\
&-b^*(e_2^n, u(t_n), s_2^{h,n+1}) - b^*(u_2^{h,n}, e_2^n, s_2^{h,n+1}) \\
&-k(\frac{(p(t_{n+1}) - p_2^{h,n+1}) - (p(t_n) - p_2^{h,n})}{k}, \nabla \cdot s_2^{h,n+1}) \\
&= hk(\nabla(\frac{e_1^{n+1} - e_1^n}{k}), \nabla s_2^{h,n+1}) - Ck^2(\rho_t^{n+1}, s_2^{h,n+1}).
\end{aligned} \tag{5.25}
$$

The nonlinear terms in (5.25) are bounded in the same manner as those in (4.24), with s^h, ϕ_1^h and η_1 replaced by s_2^h, ϕ_2^h and η_2. Using these bounds and the Cauchy-Schwarz and Young's inequalities, we obtain from (5.25) that

$$
\begin{aligned}
& \frac{\|s_2^{h,n+1}\|^2 - \|s_2^{h,n}\|^2}{2} + (h + Re^{-1})k\|\nabla s_2^{h,n+1}\|^2 \\
&\le 13\epsilon(h + Re^{-1})k\|\nabla s_2^{h,n+1}\|^2 \\
&+\frac{C}{h + Re^{-1}}k\|\frac{\eta_2^{n+1} - 2\eta_2^n + \eta_2^{n-1}}{k^2}\|_{-1}^2 + C(h + Re^{-1})k\|\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k})\|^2 \\
&+\frac{C}{h + Re^{-1}}k \inf_{q^h \in Q^h} \|\frac{p(t_{n+1}) - p(t_n)}{k} - \frac{q^{h,n+1} - q^{h,n}}{k}\|^2 \\
&+\frac{C}{h + Re^{-1}}k[\|\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k})\|^2 + \|\nabla e_2^n\|^2 + \|\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k})\|^2\|\nabla e_2^n\|^2] \\
&+Ck\|e_2^n\|^2\|\nabla e_2^n\|^2 + Ck\|\nabla(\frac{\eta_2^{n+1} - \eta_2^n}{k})\|^4 \\
&+\frac{C}{h + Re^{-1}}k \cdot h^2\|\nabla(\frac{e_1^{n+1} - e_1^n}{k})\|^2 + \frac{C}{h + Re^{-1}}k \cdot k^2\|\rho_t^{n+1}\|_{-1}^2 \\
&+C(C_{\nabla u} + \frac{C_u^2}{h + Re^{-1}} + \frac{1}{(h + Re^{-1})^3}\|\nabla e_2^n\|^4)k\|s_2^{h,n+1}\|^2.
\end{aligned} \tag{5.26}
$$

It follows from the assumption $k \le h$ and the result of Theorem 4.3 that $\max_i \|\nabla e_2^i\| \le C$.

Take $\epsilon = \frac{1}{26}$ in (5.26), simplify, multiply both sides of (5.26) by 2 and sum over all time levels $n \ge 1$. The bound on $(h + Re^{-1})k\sum_{i=1}^n \|\nabla(\frac{e_1^{i+1} - e_1^i}{k})\|^2$ is obtained from (5.24). Using the approximation properties of X^h, Q^h and the triangle inequality, we obtain

$$
\begin{aligned}
& \|s_2^{h,n+1}\|^2 + (h + Re^{-1})k\sum_{i=1}^n \|\nabla s_2^{h,i+1}\|^2 \le \|s_2^{h,1}\|^2 \\
&+C[h^{2m} + h^4 + h^2k^2 + k^2] \\
&+Ck\sum_{i=1}^n (C_{\nabla u} + \frac{C_u^2}{h + Re^{-1}} + \frac{1}{(h + Re^{-1})^3}\|\nabla e_2^i\|^4)\|s_2^{h,i+1}\|^2.
\end{aligned} \tag{5.27}
$$

The bound on $\|s_2^{h,1}\| = \|\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k}\|$ is obtained in the same manner as the bound on $\|\frac{\phi_1^{h,1}-\phi_1^{h,0}}{k}\|$. Consider (4.24) at $n=0$ and take $v = s_2^{h,1} = \frac{\phi_2^{h,1}-\phi_2^{h,0}}{k}$. This leads to

$$\begin{aligned}
&\|\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k})\|^2 \qquad (5.28)\\
&\le (8\epsilon_1 + \frac{C}{h+Re^{-1}}k + \frac{C}{(h+Re^{-1})^3}k\|\nabla\eta_2^1\|^4)\|\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k}\|^2\\
&\quad +5\epsilon(h+Re^{-1})k\|\nabla(\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k})\|^2\\
&\quad +C[\|\frac{\eta_2^1-\eta_2^0}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{\eta_2^1-\eta_2^0}{k})\|^2\\
&\quad +k^2\|\rho^1\|^2 + \frac{1}{h+Re^{-1}}h^2k\|\nabla(\frac{e_1^1-e_1^0}{k})\|^2 + h^2\|\Delta\eta_1^0\|^2\\
&\quad +\frac{1}{h+Re^{-1}}k\|\nabla(\frac{\eta_2^1-\eta_2^0}{k})\|^2 + \|\eta_2^1\|^2 + \|\nabla\eta_2^1\|^2 + h^{-2}\|\nabla\eta_2^1\|^4].
\end{aligned}$$

Use the bound on $(h+Re^{-1})k\|\nabla(\frac{e_1^1-e_1^0}{k})\|^2$ from (5.22). It follows from the approximation properties of X^h, Q^h and the triangle inequality, that

$$\begin{aligned}
&\|\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k})\|^2 \qquad (5.29)\\
&\le C[h^{2m}+h^4+h^2k^2+k^2]
\end{aligned}$$

and

$$\begin{aligned}
&\|\frac{e_2^1-e_2^0}{k}\|^2 + (h+Re^{-1})k\|\nabla(\frac{e_2^1-e_2^0}{k})\|^2 \qquad (5.30)\\
&\le C[h^{2m}+h^4+h^2k^2+k^2].
\end{aligned}$$

Insert the bound on $\|\frac{\phi_2^{h,1}-\phi_2^{h,0}}{k}\|^2$ into (5.27). The restriction on the time step k allows to apply discrete Gronwall's lemma. This leads to

$$\begin{aligned}
&\|\frac{\phi_2^{h,n+1}-\phi_2^{h,n}}{k}\|^2 + (h+Re^{-1})k\sum_{i=1}^{n}\|\nabla(\frac{\phi_2^{h,i+1}-\phi_2^{h,i}}{k})\|^2 \qquad (5.31)\\
&\le C[h^{2m}+h^4+h^2k^2+k^2].
\end{aligned}$$

Using the triangle inequality we obtain

$$\begin{aligned}
&\|\frac{e_2^{n+1}-e_2^n}{k}\|^2 + (h+Re^{-1})k\sum_{i=1}^{n}\|\nabla(\frac{e_2^{i+1}-e_2^i}{k})\|^2 \qquad (5.32)\\
&\le C[h^{2m}+h^4+h^2k^2+k^2].
\end{aligned}$$

This completes the proof of Theorem 5.1. □

5.1. Stability of the Pressure. The stability of the pressure approximations p_1^h and p_2^h follows from the discrete inf-sup condition (2.1). The required bound on the time derivative of velocity is obtained under the assumptions of Theorem 5.1.

THEOREM 5.2. *Let $f \in L^2(0,T;H^{-1}(\Omega))$. Let p_1^h and p_2^h satisfy the equations (1.5) and let the assumptions of Theorem 5.1 be satisfied. Then there exists a constant $C = C(T, f, h + Re^{-1}, u_0^s)$ s.t.*

$$k \sum_{i=0}^{n} \|p_1^{h,i+1}\| \leq C$$

and

$$k \sum_{i=0}^{n} \|p_2^{h,i+1}\| \leq C$$

Proof. Consider the first equation of (1.5). It holds true for $\forall v^h \in X^h$. Apply the Cauchy-Schwarz inequality, divide both sides of the inequality by $\|\nabla v^h\|$ and regroup the terms, leaving only the pressure term on the left-hand side. Using Lemma 2.1 gives

$$\frac{(p_1^{h,n+1}, \nabla \cdot v^h)}{\|\nabla v^h\|} \leq \|\frac{u_1^{h,n+1} - u_1^{h,n}}{k}\|_{-1} + (h + Re^{-1})\|\nabla u_1^{h,n+1}\| \tag{5.33}$$
$$+ M\|\nabla u_1^{h,n+1}\|^2 + \|f(t_{n+1})\|_{-1}.$$

It follows from (5.33) and the discrete LBB condition (2.1) that

$$\beta^h \|p_1^{h,n+1}\| \leq \|\frac{u_1^{h,n+1} - u_1^{h,n}}{k}\|_{-1} + (h + Re^{-1})\|\nabla u_1^{h,n+1}\| \tag{5.34}$$
$$+ M\|\nabla u_1^{h,n+1}\|^2 + \|f(t_{n+1})\|_{-1}.$$

Decompose the first term on the right-hand side of (5.34), using the error decomposition and the triangle inequality. This gives

$$\|\frac{u_1^{h,n+1} - u_1^{h,n}}{k}\|_{-1} \leq \|\frac{u(t_{n+1}) - u(t_n)}{k}\|_{-1} + \|\frac{e_1^{n+1} - e_1^n}{k}\|_{-1}. \tag{5.35}$$

Multiply both sides of (5.34) by k and sum over the time levels. Using (5.35), we obtain

$$\beta^h k \sum_{i=0}^{n} \|p_1^{h,i+1}\| \leq k \sum_{i=0}^{n} \|\frac{u(t_{i+1}) - u(t_i)}{k}\|_{-1} + k \sum_{i=0}^{n} \|\frac{e_1^{i+1} - e_1^i}{k}\|_{-1} \tag{5.36}$$
$$+ (h + Re^{-1}) k \sum_{i=0}^{n} \|\nabla u_1^{h,i+1}\| + Mk \sum_{i=0}^{n} \|\nabla u_1^{h,i+1}\|^2 + k \sum_{i=0}^{n} \|f(t_{i+1})\|_{-1}.$$

The discrete Hölder's inequality gives

$$k \sum_{i=0}^{n} \|\nabla u_1^{h,i+1}\| = k \sum_{i=0}^{n} \|\nabla u_1^{h,i+1}\| \cdot 1 \tag{5.37}$$
$$\leq (k \sum_{i=0}^{n} \|\nabla u_1^{h,i+1}\|^2)^{\frac{1}{2}} \cdot (k \sum_{i=0}^{n} 1^2)^{\frac{1}{2}} = C(k \sum_{i=0}^{n} \|\nabla u_1^{h,i+1}\|^2)^{\frac{1}{2}}.$$

Similarly,

$$k\sum_{i=0}^{n}\|\frac{e_1^{i+1}-e_1^i}{k}\|_{-1}\leq C(k\sum_{i=0}^{n}\|\frac{e_1^{i+1}-e_1^i}{k}\|_{-1}^2)^{\frac{1}{2}}\leq C(k\sum_{i=0}^{n}\|\frac{e_1^{i+1}-e_1^i}{k}\|^2)^{\frac{1}{2}} \quad (5.38)$$

The stability bound on $k\sum_{i=0}^{n}\|\nabla u_1^{h,i+1}\|^2$ is obtained from Lemma 3.2. Using (5.38) and Theorem 5.1, it follows from (5.36) that

$$\beta^h k\sum_{i=0}^{n}\|p_1^{h,i+1}\|\leq C[\frac{1}{h+Re^{-1}}\|u_0^s\|^2+\frac{1}{(h+Re^{-1})^2}k\sum_{i=0}^{n}\|f(t_{i+1})\|_{-1}^2]. \quad (5.39)$$

Hence, if the forcing term f is sufficiently smooth, the pressure approximation p_1^h is stable.

Next, consider the second equation of (1.5). Apply the Cauchy-Schwarz inequality, divide both sides of the inequality by $\|\nabla v^h\|$ and regroup the terms. Following the outline of the proof above, we obtain

$$\begin{aligned}\beta^h k\sum_{i=0}^{n}\|p_2^{h,i+1}\|\leq{}& k\sum_{i=0}^{n}\|\frac{u(t_{i+1})-u(t_i)}{k}\|_{-1}+k\sum_{i=0}^{n}\|\frac{e_2^{i+1}-e_2^i}{k}\|_{-1} \\ &+(h+Re^{-1})k\sum_{i=0}^{n}\|\nabla u_2^{h,i+1}\|+Mk\sum_{i=0}^{n}\|\nabla u_2^{h,i+1}\|^2 \\ &+hk\sum_{i=0}^{n}\|\nabla u_1^{h,i+1}\|+k\sum_{i=0}^{n}\|f(t_{i+1})\|_{-1}.\end{aligned} \quad (5.40)$$

Use the discrete Hölder's inequality as in (5.37). It follows from Theorem 5.1 and Theorem 3.3 that

$$\beta^h k\sum_{i=0}^{n}\|p_2^{h,i+1}\|\leq C[\frac{1}{h+Re^{-1}}\|u_0^s\|^2+\frac{1}{(h+Re^{-1})^2}k\sum_{i=0}^{n}\|f(t_{i+1})\|_{-1}^2]. \quad (5.41)$$

□

5.2. Error estimates for the pressure. In this section we estimate the error in pressure approximations $\|p(t_i)-p_1^{h,i}\|$ and $\|p(t_i)-p_2^{h,i}\|$. The results are obtained by using the inf-sup condition (2.1) and the result of Theorem 5.1. The main result of the section is

THEOREM 5.3 (Pressure Convergence Rates). *Let $u, p, u_1^h, p_1^h, u_2^h, p_2^h$ satisfy the equations (4.4)-(4.6). Let the assumptions of Theorem 5.1 be satisfied. Then, for* $\forall n\geq 0$

$$k\sum_{i=0}^{n}\|p(t_{i+1})-p_1^{h,i+1}\|\leq C[h^m+h+k] \quad (5.42)$$

and

$$k\sum_{i=0}^{n}\|p(t_{i+1})-p_2^{h,i+1}\|\leq C[h^m+h^2+hk+k]. \quad (5.43)$$

Proof. Decompose the error in the pressure approximation

$$p(t_{n+1}) - p_1^{h,n+1} = (p(t_{n+1}) - q^{h,n+1}) - (p_1^{h,n+1} - q^{h,n+1}) \qquad (5.44)$$
$$=: \gamma_1^{n+1} - \psi_1^{h,n+1},$$

where $q^{h,n+1}$ is some projection of $p(t_{n+1})$ into Q^h. Thus, $\psi_1^{h,n+1} \in Q^h$.

Divide both sides of (5.1) by $\|\nabla v\|$ and regroup the terms. Use the result of Lemma 2.1 and the Cauchy-Schwarz inequality to obtain

$$\frac{(\psi_1^{h,n+1}, \nabla\cdot v)}{\|\nabla v\|} \le \|\frac{e_1^{n+1} - e_1^n}{k}\|_{-1} \qquad (5.45)$$
$$+(h + Re^{-1})\|\nabla e_1^{n+1}\| + M\|\nabla u(t_{n+1})\|\|\nabla e_1^{n+1}\| + M\|\nabla u_1^{h,n+1}\|\|\nabla e_1^{n+1}\|$$
$$+ \inf_{q^h\in Q^h} \|p(t_{n+1}) - q^{h,n+1}\| + h\|\nabla u(t_{n+1})\| + k\|\rho^{n+1}\|_{-1}.$$

Apply the discrete inf-sup condition. Multiply both sides of (5.45) by k and sum over all time levels. Decomposing $u_1^{h,n+1} = u(t_{n+1}) - e_1^{n+1}$ gives

$$\beta^h k\sum_{i=0}^{n}\|\psi_1^{h,i+1}\| \le k\sum_{i=0}^{n}\|\frac{e_1^{i+1} - e_1^i}{k}\|_{-1} \qquad (5.46)$$
$$+(h + Re^{-1})k\sum_{i=0}^{n}\|\nabla e_1^{i+1}\| + 2M\max_{0\le i\le n+1}\|\nabla u(t_i)\|k\sum_{i=0}^{n}\|\nabla e_1^{i+1}\|$$
$$+Mk\sum_{i=0}^{n}\|\nabla e_1^{i+1}\|^2 + hk\sum_{i=0}^{n}\|\nabla u(t_{i+1})\|$$
$$+k\sum_{i=0}^{n}\inf_{q^h\in Q^h}\|p(t_{i+1}) - q^{h,i+1}\| + k\cdot k\sum_{i=0}^{n}\|\rho^{i+1}\|_{-1}.$$

Applying the discrete Hölder's inequality and the triangle inequality and using Theorem 5.1 and Theorem 4.3 proves (5.42).

Next, subtract (4.6) from (4.4). This gives for any $v \in X^h$

$$(\frac{e_2^{n+1} - e_2^n}{k}, v) + (h + Re^{-1})(\nabla e_2^{n+1}, \nabla v) \qquad (5.47)$$
$$+b^*(e_2^{n+1}, u(t_{n+1}), v) + b^*(u_2^{h,n+1}, e_2^{n+1}, v)$$
$$-((p(t_{n+1}) - p_2^{h,n+1}), \nabla\cdot v) = h(\nabla e_1^{n+1}, \nabla v) - k(\rho^{n+1}, v).$$

Following the proof above, we obtain

$$\beta^h k\sum_{i=0}^{n}\|\psi_2^{h,n+1}\| \le k\sum_{i=0}^{n}\|\frac{e_2^{i+1} - e_2^i}{k}\|_{-1} \qquad (5.48)$$
$$+(h + Re^{-1})k\sum_{i=0}^{n}\|\nabla e_2^{i+1}\| + 2M\max_{0\le i\le n+1}\|\nabla u(t_i)\|k\sum_{i=0}^{n}\|\nabla e_2^{i+1}\|$$
$$+Mk\sum_{i=0}^{n}\|\nabla e_2^{i+1}\|^2 + hk\sum_{i=0}^{n}\|\nabla e_1^{i+1}\|$$
$$+k\sum_{i=0}^{n}\inf_{q^h\in Q^h}\|p(t_{i+1}) - q^{h,i+1}\| + k\cdot k\sum_{i=0}^{n}\|\rho^{i+1}\|_{-1}.$$

Applying the discrete Hölder's inequality and the triangle inequality and using Theorem 5.1 and Theorem 4.3 leads to (5.43). ☐

6. Computational Tests. We test the convergence rates for a two-dimensional problem with a known exact solution. Consider the Chorin's vortex decay problem in the unit square $\Omega = (0,1)^2$. Take

$$u = \begin{pmatrix} -\cos(\pi x)\sin(\pi y) exp(-2\pi^2 t/Re) \\ \sin(\pi x)\cos(\pi y) exp(-2\pi^2 t/Re) \end{pmatrix}, \tag{6.1}$$
$$p = -\frac{1}{4}(\cos(2\pi x) + \cos(2\pi y)) exp(-4\pi^2 t/Re),$$

and then the right-hand side f and initial condition u_0 are computed such that (6.1) satisfies (1.4).

In order to reduce the influence of the time discretization error, the time step is taken to be very small: $\Delta t = O(h^3)$.

For $Re = 1$, $Re = 100000$ and final time $T = 1/320$, the calculated convergence rates in Tables 6.1-6.4 confirm what is predicted by Theorem 4.3 for (P_2, P_1) discretization in space.

TABLE 6.1
AV approximation. $Re = 1$.

h	$\|\| u - u_1^h \|\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\| u - u_1^h \|\|_{L^2(0,T;H^1(\Omega))}$	rate
1/4	0.000295318	-	0.0111291	-
1/8	5.77794E-05	2.35	0.00280563	1.99
1/16	2.28146E-05	1.34	0.000756655	1.89
1/32	0.000011235	1.02	0.000244007	1.63

TABLE 6.2
Correction Step approximation. $Re = 1$.

h	$\|\| u - u_2^h \|\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\| u - u_2^h \|\|_{L^2(0,T;H^1(\Omega))}$	rate
1/4	0.00027283	-	0.0110347	-
1/8	3.56252E-05	2.94	0.00271592	2.02
1/16	4.55025E-06	2.97	0.000665649	2.03
1/32	5.77583E-07	2.98	0.000164297	2.02

The convergence rate of $\|u - u_2^h\|_{L^2(0,T;L^2(\Omega))}$, predicted by Theorem 4.3, appears to be improvable in the case of moderate Reynolds' number. However, for the flow with sufficiently large Reynolds' number, the computed rates agree with those predicted by the theorem.

TABLE 6.3
AV approximation. $Re = 100000$.

h	$\|\| u - u_1^h \|\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\| u - u_1^h \|\|_{L^2(0,T;H^1(\Omega))}$	rate
1/4	0.000339015	-	0.00534596	-
1/8	7.39569E-05	2.2	0.00104601	2.35
1/16	3.19763E-05	1.21	0.00025783	2.02
1/32	1.62156E-05	0.98	9.19028E-05	1.49

Table 6.4
Correction Step approximation. Re = 100000.

h	$\|\| u - u_2^h \|\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\| u - u_2^h \|\|_{L^2(0,T;H^1(\Omega))}$	rate
1/4	0.000300427	-	0.00525358	-
1/8	0.000040032	2.91	0.000975526	2.43
1/16	5.94795E-06	2.75	0.000190267	2.36
1/32	1.37357E-06	2.11	4.26364E-05	2.16

7. Comparison of the approaches. For many years, it has been widely believed that (1.2) can be directly imported into implicit time discretizations of flow problems in the obvious way: discretize in time, given $u^h(t_{OLD})$, the quasistatic flow problem for $u^h(t_{NEW})$ is solved by DCM of the form (1.2). In this section we compare this approach and our method (1.5), applied to the same problem.

Apply both methods to the one-dimensional singularly perturbed equation

$$u_t - \epsilon u_{xx} + u_x = f$$

Our method leads to the coupled pair of equations

$$\begin{aligned} \frac{u_{1,i}^{n+1} - u_{1,i}^{n}}{\Delta t} - (\epsilon + h)\frac{u_{1,i-1}^{n+1} - 2u_{1,i}^{n+1} + u_{1,i+1}^{n+1}}{h^2} + \frac{u_{1,i+1}^{n+1} - u_{1,i-1}^{n+1}}{2h} \\ = f_i^{n+1}, \\ \frac{u_{2,i}^{n+1} - u_{2,i}^{n}}{\Delta t} - (\epsilon + h)\frac{u_{2,i-1}^{n+1} - 2u_{2,i}^{n+1} + u_{2,i+1}^{n+1}}{h^2} + \frac{u_{2,i+1}^{n+1} - u_{2,i-1}^{n+1}}{2h} \\ = f_i^{n+1} - h\frac{u_{1,i-1}^{n+1} - 2u_{1,i}^{n+1} + u_{1,i+1}^{n+1}}{h^2}, \end{aligned}$$

whereas the other method gives

$$\begin{aligned} \frac{u_{1,i}^{n+1} - u_{2,i}^{n}}{\Delta t} - (\epsilon + h)\frac{u_{1,i-1}^{n+1} - 2u_{1,i}^{n+1} + u_{1,i+1}^{n+1}}{h^2} + \frac{u_{1,i+1}^{n+1} - u_{1,i-1}^{n+1}}{2h} = f_i^{n+1}, \\ \frac{u_{2,i}^{n+1} - u_{2,i}^{n}}{\Delta t} - (\epsilon + h)\frac{u_{2,i-1}^{n+1} - 2u_{2,i}^{n+1} + u_{2,i+1}^{n+1}}{h^2} + \frac{u_{2,i+1}^{n+1} - u_{2,i-1}^{n+1}}{2h} \\ = f_i^{n+1} - h\frac{u_{1,i-1}^{n+1} - 2u_{1,i}^{n+1} + u_{1,i+1}^{n+1}}{h^2}. \end{aligned}$$

Consider $0 \le x \le 1$, $0 \le t \le 1$, $u(0,t) = 0$, $u(1,t) = 25$, $u(x,0) = 0$, $f(x,t) = tx^2 + 20x$, $\epsilon = 0.000001$.

As one could have predicted, if we let the time interval be fixed and reasonably big ($\Delta t = 0.1$) and decrease the space-interval, both methods give almost the same results, since they mainly differ in treating the time-derivative. But if we fix Δh and monotonically decrease Δt, we immediately see the oscillations of the solution, obtained by the alternative method.

Figures Fig.7.1-Fig.7.4 show the solution, obtained by our method (denoted by the solid line) and the solution, obtained by the alternative approach (dashed line on the graphs). The spacial mesh is fixed (with $\Delta h = 0.01$) and the time step Δt decreases to zero (see the captions).

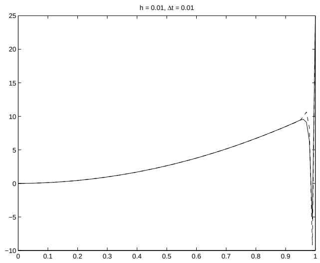

FIG. 7.1. $\Delta t = 0.01$

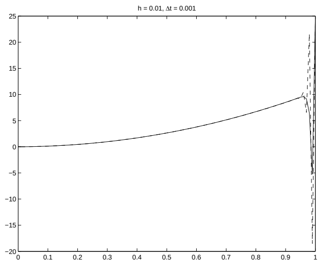

FIG. 7.2. $\Delta t = 0.001$

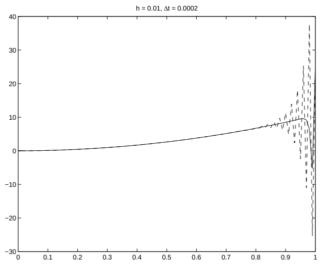

FIG. 7.3. $\Delta t = 0.0002$

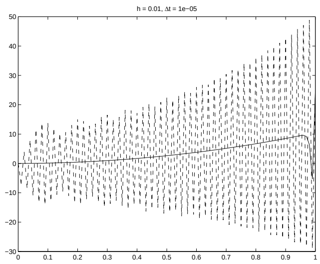

FIG. 7.4. $\Delta t = 0.00001$

As we see, although this alternative approach uses the Correction Step approximation of the true solution on each time level (instead of the AV approximation), the results are worse even for a simple one-dimensional problem with the bounded domain and bounded right-hand side.

We conclude the comparison of the methods by applying them to the Navier-

Stokes equations in $\mathbb{R}^2$. Consider the Chorin's vortex decay problem in the square $\Omega = (-\frac{1}{2}, \frac{1}{2})^2$ with

$$f(x,y,t) = \begin{pmatrix} \frac{1}{2}\pi \sin(2\pi x) exp(-4\pi^2 t/Re) \\ \frac{1}{2}\pi \sin(2\pi y) exp(-4\pi^2 t/Re) \end{pmatrix} \tag{7.1}$$

and $Re = 10^5$. The final time is taken to be $T = 1/10$ and the mesh diameter is fixed at $h = 1/4$. As the time step Δt is decreased, the error estimates, obtained by the DCM (1.5), do not change - see the following table.

Table 7.1
DCM. $Re = 100000$, $T = 1/10$, $h = 1/4$

Δt	$\lVert u - u_2^h \rVert_{L^2(0,T;L^2(\Omega))}$	$\lVert u - u_2^h \rVert_{L^2(0,T;H^1(\Omega))}$
10^{-3}	0.00682	0.0585
10^{-4}	0.00682	0.0585
10^{-5}	0.00682	0.0585

At the same time, applying the alternative approach we obtain

Table 7.2
ALTERNATIVE APPROACH. $Re = 100000$, $T = 1/10$, $h = 1/4$

Δt	$\lVert u - u_2^h \rVert_{L^2(0,T;L^2(\Omega))}$	$\lVert u - u_2^h \rVert_{L^2(0,T;H^1(\Omega))}$
10^{-3}	0.01019	0.1104
10^{-4}	0.01449	0.1759
10^{-5}	0.01582	0.2076

Hence, in the alternative approach the error increases as Δt tends to zero.

We have seen from Figures Fig.7.1-Fig.7.4 that the alternative approach gives worse results than the DCM, when solving the convection diffusion equation. Comparing the Tables 7.1-7.2, we conclude that the Defect Correction Method (1.5) also behaves better, when applied to a more difficult Navier-Stokes problem.

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