COUPLING DISCONTINUOUS GALERKIN AND MIXED FINITE ELEMENT DISCRETIZATIONS USING MORTAR FINITE ELEMENTS

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Abstract. Discontinuous Galerkin (DG) and mixed finite element (MFE) methods are two popular methods that possess local mass conservation. In this paper we investigate DG-DG and DG-MFE domain decomposition couplings using mortar finite elements to impose weak continuity of fluxes and pressures on the interface. The subdomain grids need not match and the mortar grid may be much coarser, giving a two-scale method. Convergence results in terms of the fine subdomain scale h and the coarse mortar scale H are established for both types of couplings. In addition, a non-overlapping parallel domain decomposition algorithm is developed, which reduces the coupled system to an interface mortar problem. The properties of the interface operator are analyzed.

Keywords. discontinuous Galerkin, mixed finite element, flow in porous media, mortar finite element, interface problem

1. Introduction

In modeling flow and reactive transport in porous media, it is important to employ algorithms that preserve mathematical properties of physical systems, such as local mass conservation and continuity of fluxes. In addition, geological media such as aquifers and petroleum reservoirs exhibit a high level of spatial variability at a multiplicity of scales, from the size of individual grains or pores, to facies, stratigraphic and hydrologic units, up to sizes of formations. Two methods that are well-suited to subsurface modeling are the mixed finite element (MFE) and discontinuous Galerkin (DG) methods. Common features of these methods are local conservation of mass and accurate treatment of rough coefficients and grids.

MFE and related methods have been very popular in the porous media modeling community. They provide accurate approximation for both the pressure and the velocity. A number of approaches have been developed to eliminate the velocity and reduce the MFE method to a cell-centered or face-centered algebraic pressure system with a substantially smaller dimension, see, e.g., [40, 7, 6]. For single phase flow, the reduced system in most cases is symmetric and positive definite, allowing for the use of efficient solvers. These reduction techniques, however, apply in most cases to low order MFE methods on relatively structured grids and may lead to deterioration in the accuracy on highly irregular or unstructured grids.

DG methods are finite element methods that use discontinuous approximations. Examples of these schemes include the Bassy-Rebay method [9], the Local Discontinuous Galerkin (LDG) [3, 20] methods, the Oden-Babuška-Baumann (OBB-DG) [32] method and interior penalty Galerkin methods [23, 37, 44]. DG methods are of particular interest for multiscale problems because they 1) support local approximations of high order and are capable of delivering exponential rates of convergence; 2) are robust and non-oscillatory in the presence of high gradients; 3) are implementable on unstructured and even non-matching grids and can thus treat highly heterogeneous porous media. On the negative side, because of the number of unknowns, DG solvers can be expensive.

Non-overlapping domain decomposition is a useful approach for spatial coupling/decoupling. A subsurface flow example is the multiblock mortar MFE methodology described in [4, 34, 35, 45]. The governing equations hold locally on the subdomains and physically driven matching conditions are imposed on block interfaces in a numerically stable and accurate way using mortar finite element spaces. References on the mortar approach for other discretizations include [12, 11, 46, 10] for conforming Galerkin and [24] for finite volume elements. Domain decomposition solvers and preconditioners for mortar discretizations have been developed in [28, 1, 2, 27, 33].

Some computational advantages of the multiblock approach are as follows: 1) *multiphysics*, different physical processes/mathematical models in different parts of the domain may be coupled in a single simulation: 2) *multinumerics*, different numerical techniques may be employed on different subdomains; 3) *multiscale resolution and adaptivity,* highly refined regions or fine scale models may be coupled with more coarsely discretized regions, and dynamic grid adaptivity may be performed locally on each block; 4) *multidomains,* highly irregular domains may be described as unions of more regular and locally discretized subdomains with the possibility of having interfaces with non-matching grids; and 5) *parallelism*, the approach leads to domain decomposition algorithms with near optimal computational load balance and minimal communication overhead.

Couplings of DG and MFE methods have been previously studied in the literature. In [39], a DG-MFE coupling is introduced, which uses two Lagrange multipliers to impose continuity of fluxes and pressures. A method for coupling LDG and MFE is developed in [21] by choosing appropriate numerical fluxes on interface edges.

In [5], a multiscale mortar mixed finite element method was introduced for modeling Darcy flow. There, the continuity of the flux is imposed via mortar finite elements on a coarse grid scale, while the equations in the coarse elements (or subdomains) are discretized on a fine grid scale.

In this paper we develop mortar couplings of DG with DG or MFE methods, using possibly different scales in the mortar and subdomain grids. Such couplings allow for 1) the flexibility of applying DG to subdomains where general grids are required for treating pinchouts, discrete faults and fractures, and highly variable full permeability tensors; 2) developing a mortar domain decomposition parallel DG solver via reduction to an interface problem and employing conjugate gradient or GMRES for its solution; efficient interface preconditioners such as balancing could be developed [30, 22, 33]; 3) applying the MFE method, which has substantially fewer unknowns than DG, in regions with relatively smooth or structured grids; 4) achieving model reduction through multiscale approximations.

We study mortar couplings of type DG-DG and DG-MFE, based on four different DG formulations, the OBB-DG [32], the nonsymmetric interior penalty Galerkin (NIPG) [38], the symmetric interior penalty Galerkin (SIPG) [8, 44, 41, 43], and the incomplete interior penalty Galerkin (IIPG) [41, 23, 43]. The mortar variable has a meaning of pressure and it is used as a Lagrange multiplier to impose weak continuity of normal velocities and subdomain pressures on the interface. This is achieved via a Robin-type matching condition, which involves a flux jump term and a penalized pressure jump term. Our approach differs from the one in [39], where two Lagrange multipliers are used and the method cannot be reduced to an interface problem.

The paper is organized as follows. In the next section we introduce the model problem and set up some notation. In Section 3 we develop and analyze DG-DG mortar couplings. In particular, we establish equivalence between the DG weak formulation and the partial differential equation, existence and uniqueness for the discrete solution, and convergence estimates. The error estimates are derived in terms of h and H, the discretization parameters for the subdomain and mortar spaces, respectively. We also develop a parallel non-overlapping domain decomposition algorithm for the solution of the algebraic system based on a reduction of the algebraic system to an interface mortar problem. Similar results are obtained in Section 4 for DG-MFE mortar couplings. We end with some conclusions in Section 5.

2. Problem statement and notation

2.1. **Model Equations.** Let the domain be $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_{12} \subset \mathbb{R}^d$, $d = 1$, 2 or 3. Although for simplicity we only present the method for two subdomains, our results easily extend to geometrically nonconforming domain decompositions with finite number of subdomains. Similarly, our results can be generalized to more general boundary conditions than (2.2) below, such as Dirichlet or mixed Dirichlet-Neumann boundary conditions. We consider the following equation, which can be used to model a single-phase flow process in porous media:

$$-\nabla \cdot \mathbf{K}\nabla p = f \quad \text{in } \Omega, \tag{2.1}$$

$$-\mathbf{K}\nabla p \cdot \mathbf{n} = g \quad \text{on } \partial\Omega. \tag{2.2}$$

The above system can also be written in a mixed form:

$$\begin{aligned} \mathbf{u} &= -\mathbf{K}\nabla p \quad \text{in}\,\Omega, \\ \nabla \cdot \mathbf{u} &= f \quad \text{in}\,\Omega, \\ \mathbf{u} \cdot \mathbf{n} &= g \quad \text{on}\,\partial\Omega. \end{aligned}$$

Here $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$. We assume that the data f belongs to $L^2(\Omega)$, g belongs to $L^2(\partial\Omega)$ and they both satisfy the compatibility condition

$$\int_\Omega f dx = \int_{\partial\Omega} g d\sigma. \tag{2.3}$$

With (2.3), system (2.1)-(2.2) defines p uniquely up to an additive constant. The conductivity $\mathbf{K}$ is assumed to be uniformly symmetric positive definite and bounded from above. In Ω_1, we use a DG formulation; in Ω_2 we use either a DG or a mixed formulation, with the matching on the interface being achieved by a multiplier.

Throughout the paper we will use the following standard notation. For $D \subset \mathbb{R}^d$, we denote the norm in the Hilbert space $H^s(D)$ by $\|\cdot\|_{s,D}$. Consequently, $\|\cdot\|_{0,D}$ will denote the norm in $L^2(D)$. We may omit the subscript D if $D = \Omega$. We denote by C a generic positive constant, independent of h and H, that may not have the same value at different occurrences. In addition, we denote by ϵ a fixed positive constant that may be chosen arbitrarily small.

Let $\mathcal{E}_h(\Omega_i)$ be a non-degenerate partition of Ω_i, $i = 1, 2$, composed of line segments if $d = 1$, triangles or quadrilaterals if $d = 2$, or tetrahedra, prisms or hexahedra if $d = 3$. If MFE discretization is used in Ω_2, we only consider affine elements there. The partitions do not need to match on the interface Γ_{12}. If SIPG, NIPG, or IIPG is used in Ω_i, we allow $\mathcal{E}_h(\Omega_i)$ to be non-conforming by refining the mesh in some of the elements. We denote $\mathcal{E}_h(\Omega) = \mathcal{E}_h(\Omega_1) \cup \mathcal{E}_h(\Omega_2)$.

Here h is the maximum element diameter for the mesh. The non-degeneracy requirement (also called regularity) is that the element is convex and that there exists $\rho > 0$ such that, if h_j is the diameter of $E_j \in \mathcal{E}_h(\Omega)$, then each of the sub-triangles (for $d = 2$) or sub-tetrahedra (for $d = 3$) of element E_j contains a ball of radius ρh_j in its interior. If E_j is a triangle (or a tetrahedron) then its sub-triangles (or sub-tetrahedra) coincide with E_j. The set of all interior points ($d = 1$), edges ($d = 2$) or faces ($d = 3$) within $\mathcal{E}_h(\Omega_i)$ is denoted by $\Gamma_h(\Omega_i)$. Let $\Gamma_h(\Omega) = \Gamma_h(\Omega_1) \cup \Gamma_h(\Omega_2)$. On each element face $\gamma \in \Gamma_h(\Omega)$, a unit normal vector $\mathbf{n}$ is chosen once and for all. This could be done by numbering the elements of $\mathcal{E}_h(\Omega_i)$ and directing $\mathbf{n}$ from E_k to E_l if E_k and E_l are adjacent and $k < l$. On $\partial\Omega$, the normal vector $\mathbf{n}$ coincides with the outward unit normal vector $\mathbf{n}_{\partial\Omega}$. On Γ_{12}, the unit normal vector $\mathbf{n}$ is chosen as $\mathbf{n} = \mathbf{n}_{\partial\Omega_1} = -\mathbf{n}_{\partial\Omega_2}$. On Γ_{12} we introduce a mortar finite element partition Γ_H, where H is the maximum diameter of mortar elements. We allow for Γ_H to be different from the traces of the subdomain grids on Γ_{12}.

Let E_i and E_j be two adjacent elements in $\mathcal{E}_h(\Omega)$ with $i < j$ and let $\gamma = \partial E_i \cap \partial E_j \in \Gamma_h(\Omega)$; then $\mathbf{n}$ is exterior to E_i. We denote the average and jump on γ for an element-wise smooth function ϕ by

$$\{\phi\} := \frac{1}{2}((\phi|_{E_i})|_\gamma + (\phi|_{E_j})|_\gamma), \qquad [\phi] := (\phi|_{E_i})|_\gamma - (\phi|_{E_j})|_\gamma .$$

The following functional spaces will be used in weak formulations of our problem:

$$\begin{aligned} X(\Omega_i) &:= \left\{q \in L^2(\Omega_i) : \forall E \in \mathcal{E}_h(\Omega_i),\, q|_E \in H^s(E)\right\}, \quad s > \frac{3}{2}, \quad i = 1 \text{ and } 2, \\ \mathbf{V}_0(\Omega_2) &:= \{\mathbf{v} \in H(\mathrm{div};\Omega_2) : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_2 \setminus \Gamma_{12}\}, \\ W(\Omega_2) &:= L^2(\Omega_2), \quad \Lambda := H^{\frac{1}{2}}(\Gamma_{12}). \end{aligned}$$

Here $\mathbf{v}\cdot\mathbf{n}$ is defined in a weak sense, i.e., in the dual of $H_{00}^{\frac{1}{2}}(\partial\Omega_2 \setminus \Gamma_{12})$, where $H_{00}^{\frac{1}{2}}(D)$ is the interpolation space between $L^2(D)$ and $H_0^1(D)$ [29]. The space $X(\Omega_i)$ is equipped with the norm

$$\|\!|\cdot|\!\|_{s,\Omega_i} = \left(\sum_{E\in\mathcal{E}_h(\Omega_i)} \|\cdot\|_{s,E}^2\right)^{\frac{1}{2}}. \tag{2.4}$$

For the DG discretization we will use the finite element spaces

$$X_h(\Omega_i) := \left\{q_h \in L^2(\Omega_i) : \forall E \in \mathcal{E}_h(\Omega_i),\, q_h|_E \in \mathbb{P}_r(E)\right\}, \quad i = 1, 2, \quad r \geq 1.$$

For the MFE discretization in Ω_2 we will use any of the usual mixed spaces, including the RTN spaces [36, 31], BDM spaces [16], BDFM spaces [15], BDDF spaces [14], or CD spaces [18]. We denote these spaces by $\mathbf{V}_h(\Omega_2)\times W_h(\Omega_2)$ where $\mathbf{V}_h(\Omega_2)\times W_h(\Omega_2) \subset H(\mathrm{div};\Omega_2)\times L^2(\Omega_2)$. To enforce the boundary condition on $\partial\Omega_2 \setminus \Gamma_{12}$, we set

$$\mathbf{V}_{h,0}(\Omega_2) = \mathbf{V}_h(\Omega_2) \cap \mathbf{V}_0(\Omega_2).$$

On an element E, the restriction of $\mathbf{V}_h(\Omega_2)$ is denoted by $\mathbf{V}_h(E)$. We assume that the velocity space $\mathbf{V}_h(E)$ contains $(\mathbb{P}_m(E))^d$, $m \geq 0$, with normal components on each edge (face) in $\mathbb{P}_m(\gamma)$, and that the pressure space $W_h(E)$ contains $\mathbb{P}_l(E)$. In all cases $l = m$ or $l = m - 1$, when $m \geq 1$.

On the interface we will use a mortar finite element space to approximate the pressure and impose weakly continuity of flux and pressure:

$$\Lambda_H := \left\{\mu_H \in L^2(\Gamma_{12}) : \forall \tau \in \Gamma_H,\, \mu_H|_\tau \in \mathbb{P}_{\bar{r}}(\tau)\right\}, \quad \bar{r} \geq 1.$$

In the above, r, m, and $\bar{r}$ are possibly different constants. We note that all results in this paper hold if Λ_H is replaced by its continuous version

$$\Lambda_H^c := \left\{\mu_H \in C^0(\Gamma_{12}) : \forall \tau \in \Gamma_H,\, \mu_H|_\tau \in \mathbb{P}_{\bar{r}}(\tau)\right\}, \quad \bar{r} \geq 1.$$

3. Coupling DG with DG using a mortar space

From now on, we assume that the tensor $\mathbf{K}$ is sufficiently smooth in each element, so that the trace $\mathbf{K}\nabla p\cdot\mathbf{n}$ is well defined on element faces. In this section we consider coupled schemes involving DG discretizations in both Ω_1 and Ω_2 and matching conditions on the interface imposed through a mortar finite element space.

3.1. **Weak formulation.** We define bilinear forms and linear functionals for the DG scheme in Ω_i, $i = 1, 2$:

$$
\begin{aligned}
B_i(p, q) := & \sum_{E \in \mathcal{E}_h(\Omega_i)} \int_E \mathbf{K}\nabla p \cdot \nabla q \, dx - \sum_{\gamma \in \Gamma_h(\Omega_i)} \int_\gamma \{\mathbf{K}\nabla p \cdot \mathbf{n}\} [q] \, d\sigma \\
& - s_{\text{form}} \sum_{\gamma \in \Gamma_h(\Omega_i)} \int_\gamma \{\mathbf{K}\nabla q \cdot \mathbf{n}\} [p] \, d\sigma - \int_{\Gamma_{12}} \mathbf{K}\nabla p \cdot \mathbf{n}_{\partial\Omega_i} q|_{\Omega_i} \, d\sigma \\
& - \bar{s}_{\text{form}} \int_{\Gamma_{12}} \mathbf{K}\nabla q \cdot \mathbf{n}_{\partial\Omega_i} p|_{\Omega_i} \, d\sigma + \sum_{\gamma \in \Gamma_h(\Omega_i)} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma [p] \, [q] \, d\sigma + \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau p|_{\Omega_i} q|_{\Omega_i} d\sigma,
\end{aligned}
\tag{3.1}
$$

$$
L_i(q; \lambda) := \int_{\Omega_i} f \, q \, dx - \int_{\partial\Omega_i \setminus \Gamma_{12}} g \, q \, d\sigma - \bar{s}_{\text{form}} \int_{\Gamma_{12}} \mathbf{K}\nabla q \cdot \mathbf{n}_{\partial\Omega_i} \lambda \, d\sigma + \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau q|_{\Omega_i} \lambda \, d\sigma. \tag{3.2}
$$

Here $s_{\text{form}} = -1$ for NIPG or OBB-DG, $s_{\text{form}} = 1$ for SIPG, and $s_{\text{form}} = 0$ for IIPG. The penalty parameter is a discrete positive function that takes the constant value σ_γ on an interior element face γ and σ_τ on a mortar element τ. We let $\sigma_\gamma \equiv 0$ for OBB-DG and assume $0 < \sigma_\gamma^0 \le \sigma_\gamma \le \sigma_\gamma^1$ for SIPG, NIPG, and IIPG. For all methods we assume $0 < \sigma_\tau^0 \le \sigma_\tau \le \sigma_\tau^1$. We will also show solvability for all schemes if $\sigma_\tau \equiv 0$, assuming condition (A.1) holds for either $X_h(\Omega_1)$ or $X_h(\Omega_2)$, but will provide no convergence analysis in this case.

We take $\bar{s}_{\text{form}} = -1$ for all methods. This leads to an easy control of the terms involving integrals on Γ_{12}. The choices 0 or 1 for $\bar{s}_{\text{form}}$ are also possible. However, in these cases, the weights in the last terms of $B_i(\cdot, \cdot)$ and $L_i(\cdot; \cdot)$ must be modified. More details are given in Remark 3.1 and Remark 3.3.

The weak formulation is: find $p \in L^2(\Omega)$ with $p|_{\Omega_i} \in X(\Omega_i)$ for $i = 1, 2$, and $\lambda \in \Lambda$ such that

$$
B_i(p, q) = L_i(q; \lambda) \quad \forall \, q \in X(\Omega_i), \quad i = 1, \, 2, \tag{3.3}
$$

$$
-\int_{\Gamma_{12}} [\mathbf{K}\nabla p \cdot \mathbf{n}] \, \mu d\sigma + \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau \sum_{i=1,2} (p|_{\Omega_i} - \lambda) \mu d\sigma = 0 \quad \forall \mu \in \Lambda. \tag{3.4}
$$

3.2. **Equivalence.** We next show that any solution of the mortar DG weak formulation satisfies the original problem. It is easy to check that the converse is true, provided the solution of the original problem is sufficiently smooth.

Theorem 3.1. *If (p, λ) is a solution of (3.3)-(3.4), then p satisfies (2.1)-(2.2) in the sense of distributions.*

Proof. We first consider the domain Ω_1. For any fixed $E \in \mathcal{E}_h(\Omega_1)$, taking $q \in C_0^\infty(E)$ in (3.3), we easily see that (2.1) holds within E. We next consider two adjacent elements $E_1 \subset \Omega_1$ and $E_2 \subset \Omega_1$ with an interface γ. Letting $q \in H_0^2(E_1 \cup \gamma \cup E_2)$ in (3.3), we have

$$
\begin{aligned}
& \int_{E_1} \mathbf{K}\nabla p \cdot \nabla q dx + \int_{E_2} \mathbf{K}\nabla p \cdot \nabla q dx - s_{\text{form}} \int_\gamma \{\mathbf{K}\nabla q \cdot \mathbf{n}\} [p] \, d\sigma \\
= \; & \int_{E_1 \cup E_2} f q dx = -\int_{E_1} \nabla \cdot (\mathbf{K}\nabla p) q dx - \int_{E_2} \nabla \cdot (\mathbf{K}\nabla p) q dx \\
= \; & \sum_{i=1,2} \left(\int_{E_i} \mathbf{K}\nabla p \cdot \nabla q dx - \int_{\partial E_i} \mathbf{K}\nabla p \cdot \mathbf{n}_{\partial E_i} q d\sigma \right),
\end{aligned}
$$

where we have used the fact that $-\nabla \cdot (\mathbf{K}\nabla p) = f$ in E, $\forall E \in \mathcal{E}_h(\Omega_1)$. Therefore

$$
s_{\text{form}} \int_\gamma \{\mathbf{K}\nabla q \cdot \mathbf{n}\} [p] \, d\sigma = \int_\gamma [\mathbf{K}\nabla p \cdot \mathbf{n}] \, q d\sigma. \tag{3.5}
$$

For OBB-DG, NIPG and SIPG, we have $s_{\text{form}} \neq 0$ and we can choose $q \in H_0^2(E_1 \cup \gamma \cup E_2)$ such that q is zero on γ and $\{\mathbf{K}\nabla q \cdot \mathbf{n}\}$ is arbitrary in $H_{00}^{\frac{1}{2}}(\gamma)$. Then (3.5) implies that $[p] = 0$ on γ. When running over all interior faces of Ω_1, this means that p belongs to $H^1(\Omega_1)$. In turn, we can choose $q \in H_0^2(E_1 \cup \gamma \cup E_2)$ such that $q|_\gamma$ is arbitrary in $H_0^{\frac{3}{2}}(\gamma)$; then (3.5) implies that the jump $[\mathbf{K}\nabla p \cdot \mathbf{n}] = 0$ on γ. Thus $\mathbf{K}\nabla p$ belongs to $H(\text{div}; \Omega_1)$ and therefore, the interior equation $-\nabla \cdot \mathbf{K}\nabla p = f$ holds globally in Ω_1. The same conclusion holds in Ω_2.

If $s_{\text{form}} = 0$ (i.e., if we use IIPG), then (3.5) directly implies that $[\mathbf{K}\nabla p \cdot \mathbf{n}] = 0$ on γ. Next, choosing for $i = 1, 2$, q in (3.3) such that $q|_{E_i} \in H^2(E_i)$ with q and $\nabla q \cdot \mathbf{n}$ both zero on $\partial(E_i) \setminus \gamma$, q arbitrary on $H_0^{\frac{3}{2}}(\gamma)$ and $q = 0$ elsewhere, we see that (3.3) reduces to

$$\frac{\sigma_\gamma}{h_\gamma} \int_\gamma [p]\,[q]\,d\sigma = 0,$$

which implies that $[p] = 0$ on γ. Then we conclude as above that p belongs to $H^1(\Omega_1)$ and (2.1) is satisfied in Ω_1 for the four DG versions. Clearly, the same result is true in Ω_2.

Now, substituting this information into (3.3) with $i = 1, 2$, we obtain that p and λ satisfy, for any $q \in X(\Omega_i)$

$$\begin{aligned}\int_{\Gamma_{12}} \mathbf{K}\nabla q \cdot \mathbf{n}_{\partial\Omega_i} p|_{\Omega_i}\,d\sigma + \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\int_\tau (pq)|_{\Omega_i}\,d\sigma &= \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\int_\tau q|_{\Omega_i}\,\lambda\,d\sigma \\ &- \int_{\partial\Omega_i\setminus\Gamma_{12}} (g + \mathbf{K}\nabla p \cdot \mathbf{n}_{\partial\Omega_i}) q\,d\sigma + \int_{\Gamma_{12}} \mathbf{K}\nabla q \cdot \mathbf{n}_{\partial\Omega_i}\lambda\,d\sigma.\end{aligned} \tag{3.6}$$

To recover the boundary condition (2.2), let E be an element of $\mathcal{E}_h(\Omega_i)$ adjacent to $\partial\Omega_i \setminus \Gamma_{12}$, and $\gamma = \partial E \cap (\partial\Omega_i \setminus \Gamma_{12})$. Taking $q|_{\Omega_i\setminus\overline{E}} = 0$ and $q|_E \in H^2(E)$ with $q = 0$ and $\nabla q \cdot \mathbf{n} = 0$ on $\partial E \setminus \gamma$, (3.6) reduces to

$$-\int_\gamma \mathbf{K}\nabla p \cdot \mathbf{n}\,q\,d\sigma = \int_\gamma g\,q\,d\sigma.$$

Since the trace of q is arbitrary in $H_0^{\frac{3}{2}}(\gamma)$, we conclude that $-\mathbf{K}\nabla p \cdot \mathbf{n} = g$ on γ. Therefore (2.2) is satisfied on $\partial\Omega_i \setminus \Gamma_{12}$, and (3.6) becomes

$$\begin{aligned}\int_{\Gamma_{12}} \mathbf{K}\nabla q \cdot \mathbf{n}_{\partial\Omega_i} p|_{\Omega_i}\,d\sigma + \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\int_\tau (pq)|_{\Omega_i}\,d\sigma &= \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\int_\tau q|_{\Omega_i}\,\lambda\,d\sigma \\ &+ \int_{\Gamma_{12}} \mathbf{K}\nabla q \cdot \mathbf{n}_{\partial\Omega_i}\lambda\,d\sigma.\end{aligned} \tag{3.7}$$

Finally, we turn to the interface Γ_{12}. For $i = 1, 2$, let E be an element of $\mathcal{E}_h(\Omega_i)$ adjacent to Γ_{12}, and $\gamma = \partial E \cap \Gamma_{12}$. Taking $q|_{\Omega_i\setminus\overline{E}} = 0$ and $q|_E \in H^2(E)$ with $q = 0$ and $\nabla q \cdot \mathbf{n} = 0$ on $\partial E \setminus \gamma$, (3.7) becomes

$$\sum_{\tau\in\Gamma_H, \tau\cap\gamma\neq\emptyset} \left(\int_\tau \mathbf{K}\nabla q \cdot \mathbf{n}_{\partial\boldsymbol{\Omega}_\mathbf{i}}(p|_{\Omega_i} - \lambda) d\sigma + \frac{\sigma_\tau}{H_\tau}\int_\tau (p|_{\Omega_i} - \lambda) q|_{\Omega_i}\,d\sigma\right) = 0.$$

By choosing $\mathbf{K}\nabla q \cdot \mathbf{n}$ and q arbitrarily in the interior of γ, we show that $p|_{\Omega_i} = \lambda$ on γ, and consequently for $i = 1, 2$, $p|_{\Omega_i} = \lambda$ on Γ_{12}. This implies in particular that $p \in H^1(\Omega)$. The matching condition (3.4) now becomes $-\int_{\Gamma_{12}} [\mathbf{K}\nabla p \cdot \mathbf{n}]\,\mu d\sigma = 0$, which implies $\mathbf{K}\nabla p \cdot \mathbf{n}$ is continuous across Γ_{12}. Thus we have (2.1) over the entire domain Ω. $\square$

3.3. **Discretization.** The mortar DG–DG finite element scheme is: find $(p_h|_{\Omega_1}, p_h|_{\Omega_2}, \lambda_H) \in X_h(\Omega_1) \times X_h(\Omega_2) \times \Lambda_H$ such that

$$B_i(p_h, q_h) = L_i(q_h; \lambda_H) \quad \forall q_h \in X_h(\Omega_i), \quad i = 1, 2, \tag{3.8}$$

$$\int_{\Gamma_{12}} [\mathbf{K}\nabla p_h \cdot \mathbf{n}]\, \mu_H d\sigma = \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau \sum_{i=1}^{2} (p_h|_{\Omega_i} - \lambda_H)\mu_H d\sigma, \ \forall \mu_H \in \Lambda_H. \tag{3.9}$$

In the analysis we shall use the following inequalities, which hold if the penalty parameter σ_γ^0 is chosen to be sufficiently large.

Lemma 3.1. *For $i = 1, 2$, let $\mathcal{E}_h(\Omega_i)$ be non-degenerate. Then, with each γ in $\Gamma_h(\Omega_i)$ we can associate a positive number σ_γ such that the following inequality holds for all $q_h \in X_h(\Omega_i)$ and all $q \in X(\Omega_i)$:*

$$\left|\sum_{\gamma\in\Gamma_h(\Omega_i)} \int_\gamma \{\mathbf{K}\nabla q_h \cdot \mathbf{n}\}\, [q]\, d\sigma\right| \le \frac{1}{8} |||\mathbf{K}^{\frac{1}{2}}\nabla q_h|||^2_{0,\Omega_i} + \frac{1}{8} \sum_{\gamma\in\Gamma_h(\Omega_i)} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma [q]^2\, d\sigma. \tag{3.10}$$

Proof. For each E in $\mathcal{E}_h(\Omega_i)$, let $\lambda_E^{\max}$, resp. $\lambda_E^{\min}$, denote the maximum, resp. minimum, of the eigenvalues of $\mathbf{K}$ on E. By assumption, $\lambda_E^{\max} \le k_{\max}$ and $\lambda_E^{\min} \ge k_{\min} > 0$, with $k_{\max}$ and $k_{\min}$ independent of h and E. Let $\gamma = \partial E_1 \cap \partial E_2$ belong to $\Gamma_h(\Omega_i)$ and let us expand

$$\|\{\mathbf{K}\nabla q_h \cdot \mathbf{n}\}\|_{0,\gamma} \le \frac{1}{2} \sum_{j=1,2} \|\mathbf{K}\nabla q_h|_{E_j}\|_{0,\gamma} \le \frac{1}{2} \sum_{j=1,2} \lambda_{E_j}^{\max} \|\nabla q_h|_{E_j}\|_{0,\gamma}.$$

By reverting to the reference element $\hat{E}$, and using the equivalence of norms on a finite-dimensional space on $\hat{E}$, there exists a constant $\hat{c}$, independent of h, such that

$$\|\{\mathbf{K}\nabla q_h \cdot \mathbf{n}\}\|_{0,\gamma} \le \frac{\hat{c}}{2} \sum_{j=1,2} \lambda_{E_j}^{\max} \Big(\frac{|\gamma|}{|E_j|}\Big)^{\frac{1}{2}} \|\nabla q_h\|_{0,E_j} \le \frac{\hat{c}}{2} \sum_{j=1,2} \frac{\lambda_{E_j}^{\max}}{\sqrt{\lambda_{E_j}^{\min}}} \Big(\frac{|\gamma|}{|E_j|}\Big)^{\frac{1}{2}} \|\mathbf{K}^{\frac{1}{2}}\nabla q_h\|_{0,E_j}. \tag{3.11}$$

Therefore

$$\left|\sum_{\gamma\in\Gamma_h(\Omega_i)} \int_\gamma \{\mathbf{K}\nabla q_h \cdot \mathbf{n}\}[q] d\sigma\right| \le \frac{\hat{c}}{2} \sum_{\gamma\in\Gamma_h(\Omega_i)} \sum_{j=1,2} \frac{\lambda_{E_j}^{\max}}{\sqrt{\lambda_{E_j}^{\min}}} \Big(\frac{|\gamma|}{|E_j|}\Big)^{\frac{1}{2}} \|\mathbf{K}^{\frac{1}{2}}\nabla q_h\|_{0,E_j} \|[q]\|_{0,\gamma}.$$

Applying Young's inequality with parameter $\epsilon > 0$, this becomes

$$\begin{aligned}\left|\sum_{\gamma\in\Gamma_h(\Omega_i)} \int_\gamma \{\mathbf{K}\nabla q_h \cdot \mathbf{n}\}[q] d\sigma\right| &\le \frac{\epsilon}{4} \sum_{\gamma\in\Gamma_h(\Omega_i)} \sum_{j=1,2} \|\mathbf{K}^{\frac{1}{2}}\nabla q_h\|^2_{0,E_j} \\ &+ \frac{\hat{c}^2}{4\epsilon} \sum_{\gamma\in\Gamma_h(\Omega_i)} \sum_{j=1,2} \frac{(\lambda_{E_j}^{\max})^2}{\lambda_{E_j}^{\min}} \frac{|\gamma|}{|E_j|} \|[q]\|^2_{0,\gamma}.\end{aligned} \tag{3.12}$$

It is easy to see that in the first sum an element E is counted at most L times, where L is a fixed number that depends on the type of elements used; for instance $L = 4$ in the case of tetrahedra. Therefore, choosing $\epsilon = \frac{1}{2L}$, we can take

$$\sigma_\gamma = 4L\hat{c}^2 \sum_{j=1,2} \frac{|\gamma| h_\gamma}{|E_j|} \frac{(\lambda_{E_j}^{\max})^2}{\lambda_{E_j}^{\min}}, \tag{3.13}$$

a quantity that is bounded above and below independently of h, owing to the non-degeneracy of the mesh. With this choice, (3.12) implies (3.10). □

We next analyze the solvability of the system (3.8)-(3.9) . This is a square finite dimensional system and existence is equivalent to uniqueness. Let $f = 0$ and $g = 0$. Take $q_h = p_h$ in (3.8) and sum over the two subdomains Ω_1 and Ω_2 to obtain

$$\begin{aligned}(3.14)\qquad &\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \mathbf{K}\nabla p_h\cdot\nabla p_h dx-(1+s_{\text{form}})\sum_{\gamma\in\Gamma_h(\Omega)}\int_\gamma\{\mathbf{K}\nabla p_h\cdot\mathbf{n}\}[p_h]\,d\sigma\\ &+\sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma[p_h]^2\,d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\sum_{i=1}^2 p_h|_{\Omega_i}^2\,d\sigma\\ =\;&\int_{\Gamma_{12}}[\mathbf{K}\nabla p_h\cdot\mathbf{n}]\,\lambda_H d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\sum_{i=1}^2 p_h|_{\Omega_i}\lambda_H d\sigma.\end{aligned}$$

Summation of (3.14) and (3.9) with $\mu_H = \lambda_H$ leads to

$$\begin{aligned}(3.15)\qquad &|||\mathbf{K}^{\frac12}\nabla p_h|||_{0,\Omega}^2-(1+s_{\text{form}})\sum_{\gamma\in\Gamma_h(\Omega)}\int_\gamma\{\mathbf{K}\nabla p_h\cdot\mathbf{n}\}[p_h]\,d\sigma\\ &+\sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma[p_h]^2\,d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\sum_{i=1}^2\left(p_h|_{\Omega_i}-\lambda_H\right)^2d\sigma=0.\end{aligned}$$

First consider the case $0 < \sigma_\tau^0 \le \sigma_\tau \le \sigma_\tau^1$. For OBB-DG and NIPG, we have

$$(3.16)\qquad |||\mathbf{K}^{\frac12}\nabla p_h|||_{0,\Omega}^2+\sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma[p_h]^2\,d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\sum_{i=1}^2\left(p_h|_{\Omega_i}-\lambda_H\right)^2d\sigma=0.$$

Since $\mathbf{K}$ is positive definite in Ω, the above equation implies that $\nabla p_h = 0$ in each $E \in \mathcal{E}_h(\Omega)$ and $p_h|_{\Omega_i} = \lambda_H$ on Γ_{12}, i.e., p_h is continuous across Γ_{12} For NIPG, we see, in addition, that p_h is continuous in Ω_1 and Ω_2, and therefore p_h is a constant over the entire domain Ω and λ_H is the same constant on Γ_{12}.

For OBB-DG and $i = 1, 2$, (3.8) now implies

$$(3.17)\qquad \sum_{\gamma\in\Gamma_h(\Omega_i)}\int_\gamma\{\mathbf{K}\nabla q_h\cdot\mathbf{n}\}[p_h]\,d\sigma=0$$

Let $\gamma \in \Gamma_h(\Omega_i)$ and let $\gamma \subset \partial E$, $E \in \mathcal{E}_h(\Omega_i)$. If $r \ge 2$, we can construct q_h such that $q_h|_{\Omega\setminus\overline{E}} = 0$, $q_h|_E \in \mathbb{P}_r$, $\int_\gamma \mathbf{K}\nabla q_h \cdot \mathbf{n} d\sigma = 1$, $\int_{\gamma'} \mathbf{K}\nabla q_h \cdot \mathbf{n} d\sigma = 0$ for $\gamma' \subset \partial E \setminus \gamma$ (see [38]). Equation (3.17) implies that $[p_h] = 0$ on γ. Hence p_h is continuous in Ω_i, $i = 1$ and 2, and therefore p_h a constant over the entire domain Ω and λ_H is the same constant on Γ_{12}.

For SIPG and IIPG, assuming that σ_γ^0 is sufficiently large, we employ the inequality (3.10) in Ω_1 and Ω_2 with $q_h = q = p_h$ and conclude from (3.15) that

$$(3.18)\qquad 0\ge\frac12|||\mathbf{K}^{\frac12}\nabla p_h|||_{0,\Omega}^2+\frac12\sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma[p_h]^2\,d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\sum_{i=1}^2\left(p_h|_{\Omega_i}-\lambda_H\right)^2d\sigma.$$

We then conclude that p_h is a constant over the entire Ω and λ_H is the same constant on Γ_{12}.

Let us now consider the case $\sigma_\tau = 0$; we see from (3.8), using (3.17), (3.16) or (3.18), that for all schemes we have

$$(3.19)\qquad \int_{\Gamma_{12}}\mathbf{K}\nabla q_h|_{\Omega_i}\cdot\mathbf{n}_{\partial\Omega_i}(p_h|_{\Omega_i}-\lambda_H)d\sigma=0,\quad i=1\text{ and }2.$$

If the mortar compatibility condition (A.1) is satisfied for $X_h(\Omega_1)$, we have $\lambda_H = p_h|_{\Omega_1}$ from (3.19) with $i = 1$. Since λ_H and $p_h|_{\Omega_2}$ are both constants, we further conclude, from

(3.19) with $i=2$, that $p_h|_{\Omega_1}$, $p_h|_{\Omega_2}$ and λ_H must be the same constant. This concludes the argument for $\sigma_\tau = 0$.

We have shown for all schemes that the null space of the linear system (3.8)-(3.9) is the constant vector. Owing to the compatibility condition (2.3), the right-hand side for $q_h = 1$ is $\int_\Omega f\,dx - \int_{\partial\Omega} g\,d\sigma = 0$. Hence the solution exists and is unique up to an additive constant. We therefore have proved the following solvability theorem.

Theorem 3.2. *For OBB-DG, we assume that $r \geq 2$. For SIPG and IIPG, we assume that σ^0_γ is sufficiently large. We make no assumption for NIPG. Then the scheme (3.8)-(3.9) possesses a solution (p_h, λ_H) unique up to an additive constant that is the same for p_h and λ_H. The same conclusion holds if $\sigma_\tau = 0$, assuming that the compatibility condition (A.1) holds for either $i = 1$ or 2.*

3.4. **Convergence of the DG-DG schemes.** Now, we use an interpolant $\hat{p}$ of p that has particular properties on the elements adjacent to the interface Γ_{12}. On the other elements E, we take $\hat{p}$ to be the interpolant constructed in [38] in the case of OBB-DG, or simply take $\hat{p}$ to be the $L^2(E)$-projection of p for the DG methods with interior penalties. More precisely, $\hat{p}|_{\Omega_i} \in X_h(\Omega_i)$, $i = 1, 2$, for all $E \in \mathcal{E}_h(\Omega)$, $\hat{p}|_E$ is exact on $\mathbb{P}_r$ and for all E adjacent to Γ_{12}

$$\int_\gamma \mathbf{K}\nabla\hat{p}\cdot\mathbf{n} = \int_\gamma \mathbf{K}\nabla p\cdot\mathbf{n}, \quad \forall\gamma \subset \partial E \cap \Gamma_{12}. \tag{3.20}$$

In the case $r \geq 2$ such an interpolant is constructed in [38]. In fact, that interpolant satisfies (3.20) on all element sides $\gamma \subset \partial E$. In the case $r = 1$, we augment (3.20) with the conditions

$$\int_{\gamma_i} \mathbf{K}\nabla\hat{p}\cdot\mathbf{n}d\sigma = \int_{\gamma_i} \mathbf{K}\nabla p\cdot\mathbf{n}d\sigma, \quad \gamma_i \subset \partial E \setminus \Gamma_{12},\ i = 1,\dots,d-k \tag{3.21}$$

$$\int_E \hat{p}dx = \int_E pdx. \tag{3.22}$$

where k is the number of sides that E shares with Γ_{12}. Note that condition (3.21) is empty if $k = d$. It is easy to see that, if $\mathbf{K}$ is a constant on E, (3.20)-(3.22) define $\hat{p}$ uniquely and that $\hat{p}$ is exact for linears. If $\mathbf{K}$ is not a constant on E, an extension similar to the one in [38] can be used. In all cases, $\hat{p}$ has optimal approximation properties, namely on each $E \in \mathcal{E}_h(\Omega)$,

$$|p - \hat{p}|_{k,E} \leq C h_E^{r+1-k}|p|_{r+1,E}, \quad k = 0,\ 1,\ 2. \tag{3.23}$$

Note that, for $\gamma \subset \partial E$, the trace inequality [8] $|\phi|_{k,\gamma} \leq C(h_E^{\frac{1}{2}}|\phi|_{k+1,E} + h_E^{-\frac{1}{2}}|\phi|_{k,E})$ implies

$$|p - \hat{p}|_{k,\gamma} \leq C h_E^{r+\frac{1}{2}-k}|p|_{r+1,E}, \quad k = 0,\ 1. \tag{3.24}$$

More generally, we have the following lemma.

Lemma 3.2. *Let $\mathcal{E}_h(\Omega_i)$ be non-degenerate, $i = 1, 2$. Then there exists a constant C, independent of h, such that for all p in $H^s(\Omega_i)$, $s > \frac{3}{2}$,*

$$\left(\sum_{\gamma\in\Gamma_h(\Omega_i)} h_\gamma \| \{\mathbf{K}\nabla(p-\hat{p})\cdot\mathbf{n}\} \|^2_{0,\gamma}\right)^{\frac{1}{2}} \leq C\, h^{\mu-1}|p|_{\mu,\Omega_i}, \quad \mu = \min\left(r+1, s\right). \tag{3.25}$$

Proof. Consider a side γ adjacent to an element E. The result follows by switching to the reference element, applying a trace theorem and using (3.23):

$$h_\gamma\|\mathbf{K}\nabla(p-\hat{p})|_E\cdot\mathbf{n}\|^2_{0,\gamma} \leq C\frac{h_\gamma|\gamma|}{|E|}(\lambda_E^{\max})^2\left(|p-\hat{p}|^2_{1,E} + h_E^{2(s-1)}|p-\hat{p}|^2_{s,E}\right) \leq C\, h_E^{2(\mu-1)}|p|^2_{\mu,E}.$$

□

Theorem 3.3. *Let p be a solution of (2.1)-(2.2). Let (p_h, λ_H) be a solution of (3.8)-(3.9). We assume that $p \in H^s(\Omega)$ for some real number $s > \frac{3}{2}$ and that σ_γ^0 is sufficiently large for SIPG and IIPG. Then there exists a constant C, independent of h and H, such that*

$$\|\mathbf{K}^{\frac{1}{2}} \nabla (p_h - p)\|_{0,\Omega} + \sqrt{\sum_{\gamma \in \Gamma_h(\Omega)} \frac{\sigma_\gamma}{h_\gamma} \|[p_h]\|_{0,\gamma}^2} + \sqrt{\sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \sum_{i=1,2} \left\| p_h|_{\Omega_i} - \lambda_H \right\|_{0,\tau}^2}$$
$$\leq C \left(h^{\mu-1} \left(\frac{H}{h}\right)^{\frac{1}{2}} + H^{\bar{\mu}-\frac{1}{2}} \right), \quad \mu = \min(r+1, s), \ \bar{\mu} = \min\left(\bar{r}+1, s - \frac{1}{2}\right).$$

Proof. Since by assumption p is smooth enough, we let $\bar{p} \in \Lambda_H$ be the continuous nodal interpolant of p and define

$$\eta := \lambda_H - p, \quad \eta^I := p - \bar{p}, \quad \eta^A := \lambda_H - \bar{p} = \eta + \eta^I.$$

Define

$$\xi := p_h - p, \quad \xi^I := p - \hat{p}, \quad \xi^A := p_h - \hat{p} = \xi + \xi^I.$$

The interpolant $\bar{p}$ satisfies [19]

$$|p - \bar{p}|_{k,\Gamma_{12}} \leq C H^{\bar{\mu}-k} |p|_{\bar{\mu}+1/2,\Omega_i}, \ 0 \leq k \leq 1, \tag{3.26}$$

where the bound for fractional k is obtained by interpolation between $L^2(\Gamma_{12})$ and $H^1(\Gamma_{12})$.

Subtracting the weak formulation (3.3) from the finite element scheme (3.8) and choosing $q_h = \xi^A$, we obtain

$$B_i(\xi^A, \xi^A) \ = \ L_i(\xi^A; \lambda_H) - L_i(\xi^A; p) + B_i(\xi^I, \xi^A), \quad i = 1, 2.$$

Summation over the two subdomains leads to

$$\begin{aligned} \sum_{i=1}^{2} B_i(\xi^A, \xi^A) &= \sum_{i=1}^{2} \big(B_i(\xi^I, \xi^A) + L_i(\xi^A; \lambda_H) - L_i(\xi^A; p) \big) \\ &= \sum_{i=1}^{2} B_i(\xi^I, \xi^A) + \int_{\Gamma_{12}} \left[\mathbf{K} \nabla \xi^A \cdot \mathbf{n} \right] \eta d\sigma + \sum_{i=1}^{2} \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau \xi^A|_{\Omega_i} \eta d\sigma. \end{aligned} \tag{3.27}$$

Similarly, subtracting (3.4) from (3.9), with $\mu = \mu_H = \eta^A$ and noting that p and $\mathbf{K}\nabla p \cdot \mathbf{n}$ are continuous across the interface Γ_{12}, we obtain

$$\begin{aligned} &\int_{\Gamma_{12}} \left[\mathbf{K} \nabla \xi^A \cdot \mathbf{n} \right] \eta^A d\sigma - \sum_{i=1,2} \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau (\xi^A|_{\Omega_i} - \eta^A) \eta^A d\sigma \\ &= \int_{\Gamma_{12}} \left[\mathbf{K} \nabla \xi^I \cdot \mathbf{n} \right] \eta^A d\sigma - \sum_{i=1,2} \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau (\xi^I|_{\Omega_i} - \eta^I) \eta^A d\sigma. \end{aligned} \tag{3.28}$$

Summing (3.27) and (3.28) results in

$$\begin{aligned} &\sum_{i=1}^{2} B_i(\xi^A, \xi^A) - \sum_{i=1,2} \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau \xi^A|_{\Omega_i} \eta^A d\sigma - \sum_{i=1,2} \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau (\xi^A|_{\Omega_i} - \eta^A) \eta^A d\sigma \\ &= \sum_{i=1}^{2} B_i(\xi^I, \xi^A) - \int_{\Gamma_{12}} \left[\mathbf{K} \nabla \xi^A \cdot \mathbf{n} \right] \eta^I d\sigma + \int_{\Gamma_{12}} \left[\mathbf{K} \nabla \xi^I \cdot \mathbf{n} \right] \eta^A d\sigma \\ &\quad - \sum_{i=1,2} \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau \xi^A|_{\Omega_i} \eta^I d\sigma - \sum_{i=1,2} \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau (\xi^I|_{\Omega_i} - \eta^I) \eta^A d\sigma. \end{aligned} \tag{3.29}$$

We expand the first term on the left-hand side of (3.29):

$$\sum_{i=1}^{2} B_i(\xi^A,\xi^A) = \|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|_{0,\Omega}^2 - (1+s_{\text{form}})\sum_{\gamma\in\Gamma_h(\Omega)}\int_\gamma \left\{\mathbf{K}\nabla\xi^A\cdot\mathbf{n}\right\}\left[\xi^A\right]d\sigma + \sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma \left[\xi^A\right]^2 d\sigma + \sum_{i=1,2}\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \xi^A\big|_{\Omega_i}^2 d\sigma.$$

We denote by L_{ErrEqu} and R_{ErrEqu} the left-hand and right-hand sides of (3.29), respectively. An algebraic manipulation yields

$$L_{\text{ErrEqu}} = \|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|_{0,\Omega}^2 - (1+s_{\text{form}})\sum_{\gamma\in\Gamma_h(\Omega)}\int_\gamma \left\{\mathbf{K}\nabla\xi^A\cdot\mathbf{n}\right\}\left[\xi^A\right]d\sigma + \sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma \left[\xi^A\right]^2 d\sigma + \sum_{i=1,2}\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \left(\xi^A\big|_{\Omega_i} - \eta^A\right)^2 d\sigma.$$

For NIPG and OBB-DG, the second term in L_{ErrEqu} vanishes, leaving only the coercive terms. For SIPG and IIPG, we employ the inequality (3.10) with $q_h = q = \xi^A$ to conclude

$$L_{\text{ErrEqu}} \geq \frac{1}{2}\|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|_{0,\Omega}^2 + \frac{1}{2}\sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma \left[\xi^A\right]^2 d\sigma + \frac{1}{2}\sum_{i=1,2}\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \left(\xi^A\big|_{\Omega_i} - \eta^A\right)^2 d\sigma.$$

We now consider the right-hand side of (3.29). Expanding its first term as

$$\sum_{i=1}^{2} B_i(\xi^I,\xi^A) = \sum_{E\in\mathcal{E}_h(\Omega)}\int_E \mathbf{K}\nabla\xi^I\cdot\nabla\xi^A dx - \sum_{\gamma\in\Gamma_h(\Omega)}\int_\gamma \left\{\mathbf{K}\nabla\xi^I\cdot\mathbf{n}\right\}\left[\xi^A\right]d\sigma - s_{\text{form}}\sum_{\gamma\in\Gamma_h(\Omega)}\int_\gamma \left\{\mathbf{K}\nabla\xi^A\cdot\mathbf{n}\right\}\left[\xi^I\right]d\sigma - \int_{\Gamma_{12}}\left[\mathbf{K}\nabla\xi^I\cdot\mathbf{n}\xi^A\right]d\sigma + \int_{\Gamma_{12}}\left[\mathbf{K}\nabla\xi^A\cdot\mathbf{n}\xi^I\right]d\sigma + \sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma \left[\xi^I\right]\left[\xi^A\right]d\sigma + \sum_{i=1,2}\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau (\xi^I\xi^A)|_{\Omega_i}d\sigma,$$

and using the fact that η^I and η^A are uniquely defined on Γ_{12}, we have

$$R_{\text{ErrEqu}} = \sum_{E\in\mathcal{E}_h(\Omega)}\int_E \mathbf{K}\nabla\xi^I\cdot\nabla\xi^A dx - \sum_{\gamma\in\Gamma_h(\Omega)}\int_\gamma \left\{\mathbf{K}\nabla\xi^I\cdot\mathbf{n}\right\}\left[\xi^A\right]d\sigma - s_{\text{form}}\sum_{\gamma\in\Gamma_h(\Omega)}\int_\gamma \left\{\mathbf{K}\nabla\xi^A\cdot\mathbf{n}\right\}\left[\xi^I\right]d\sigma + \sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma \left[\xi^I\right]\left[\xi^A\right]d\sigma - \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \sum_{i=1,2}(\xi^I\big|_{\Omega_i} - \eta^I)(\xi^A\big|_{\Omega_i} - \eta^A)d\sigma - \int_{\Gamma_{12}}\left[\mathbf{K}\nabla\xi^A\cdot\mathbf{n}(\eta^I-\xi^I)\right]d\sigma + \int_{\Gamma_{12}}\left[\mathbf{K}\nabla\xi^I\cdot\mathbf{n}(\eta^A-\xi^A)\right]d\sigma =: \sum_{i=1}^{7} T_i.$$

We now bound each term T_i of R_{ErrEqu}. We first bound T_2 for NIPG, SIPG, and IIPG:

$$|T_2| \leq \epsilon\sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma \left[\xi^A\right]^2 d\sigma + Ch^{2\mu-2}.$$

For OBB-DG we use an argument from [38] and the property (3.20) of $\hat{p}$, which holds for all element sides. On each side $\gamma = \partial E_1\cap\partial E_2$, let $c_\gamma = c_\gamma^1 - c_\gamma^2$, where c_γ^i is the mean value of

ξ^A on E_i. We have

$$|T_2| = \left| \sum_{\gamma\in\Gamma_h(\Omega)} \int_\gamma \left\{\mathbf{K}\nabla\xi^I\cdot\mathbf{n}\right\}([\xi^A]-c_\gamma)d\sigma \right|$$
$$\le \sum_{\gamma\in\Gamma_h(\Omega)} \left(\frac{\tilde\epsilon}{h_\gamma}\|[\xi^A]-c_\gamma\|_{0,\gamma}^2 + Ch_\gamma\|\{\mathbf{K}\nabla\xi^I\cdot\mathbf{n}\}\|_{0,\gamma}^2 \right) \le \epsilon\|\mathbf{K}^{\frac12}\nabla\xi^A\|_{0,\Omega}^2 + Ch^{2\mu-2},$$

where we used that

$$\|[\xi^A]-c_\gamma\|_{0,\gamma} \le \|\xi^A|_{E_1}-c_\gamma^1\|_{0,\gamma} + \|\xi^A|_{E_2}-c_\gamma^2\|_{0,\gamma} \le Ch_\gamma^{\frac12}(|\xi^A|_{1,E_1}+|\xi^A|_{1,E_2}).$$

We continue with bounds on the rest of the terms T_i of R_{ErrEqu} for all methods:

$$\begin{aligned}
|T_1| &\le \epsilon\|\mathbf{K}^{\frac12}\nabla\xi^A\|_{0,\Omega}^2 + C\|\mathbf{K}^{\frac12}\nabla\xi^I\|_{0,\Omega}^2 \le \epsilon\|\mathbf{K}^{\frac12}\nabla\xi^A\|_{0,\Omega}^2 + Ch^{2\mu-2},\\
|T_3| &\le \tilde\epsilon \sum_{\gamma\in\Gamma_h(\Omega)} h_\gamma \int_\gamma \left\{\mathbf{K}\nabla\xi^A\cdot\mathbf{n}\right\}^2 d\sigma + C\sum_{\gamma\in\Gamma_h(\Omega)} \frac{1}{h_\gamma}\int_\gamma \left[\xi^I\right]^2 d\sigma\\
&\le \epsilon\|\mathbf{K}^{\frac12}\nabla\xi^A\|_{0,\Omega}^2 + Ch^{2\mu-2},\\
|T_4| &\le \epsilon \sum_{\gamma\in\Gamma_h(\Omega)} \frac{\sigma_\gamma}{h_\gamma}\int_\gamma \left[\xi^A\right]^2 d\sigma + Ch^{2\mu-2},\\
|T_5| &\le \epsilon \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\int_\tau \sum_{i=1,2} \left(\xi^A\big|_{\Omega_i}-\eta^A\right)^2 d\sigma + C\sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\int_\tau \sum_{i=1,2} \xi^I\big|_{\Omega_i}^2 d\sigma\\
&\quad + C\sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\int_\tau \sum_{i=1,2} \left(\eta^I\right)^2 d\sigma\\
&\le \epsilon \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\int_\tau \sum_{i=1,2} \left(\xi^A\big|_{\Omega_i}-\eta^A\right)^2 d\sigma + Ch^{2\mu-1}H^{-1} + CH^{2\bar\mu}H^{-1}\\
&\le \epsilon \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\int_\tau \sum_{i=1,2} \left(\xi^A\big|_{\Omega_i}-\eta^A\right)^2 d\sigma + Ch^{2\mu-2} + CH^{2\bar\mu-1}.
\end{aligned}$$

To handle term T_6, we use the special properties (3.20)-(3.22) of the interpolant $\hat p$ on the interface Γ_{12}. We define

$$T_{6,i} \;:=\; -\int_{\Gamma_{12}} \mathbf{K}\nabla\xi^A\cdot\mathbf{n}_{\partial\Omega_i}(\eta^I-\xi^I)\big|_{\Omega_i} d\sigma, \quad i=1,\,2.$$

Obviously, we have $T_6 = T_{6,1}+T_{6,2}$. We bound $T_{6,1}$, suppressing the subscript $\partial\Omega_1$ for simplicity; the bound for $T_{6,2}$ is similar. By using the same argument as for bounding T_3, is easy to see that

$$|T_{6,1}| \le T_A + T_B + \tilde\epsilon\|\mathbf{K}^{\frac12}\nabla\xi^A\|_{0,\Omega_1}^2 + Ch^{2\mu-2},$$

where

$$T_A := \int_{\Gamma_{12}} \left|\mathbf{K}\nabla\xi^I\cdot\mathbf{n}\eta^I\right| d\sigma, \quad T_B := \int_{\Gamma_{12}} \left|\mathbf{K}\nabla\xi\cdot\mathbf{n}\eta^I\right| d\sigma.$$

To bound term T_A, let $P_{h,1}$ be the L^2-projection onto the space of piecewise constants on $\mathcal{E}_h(\Omega_1)\cap\Gamma_{12}$, use (3.20) and revert to the reference element to recover the $H^{\frac12}$ norm in the second factor below; this gives

$$T_A = \int_{\Gamma_{12}} \left|\mathbf{K}\nabla(p-\hat p)\cdot\mathbf{n}(\eta^I-P_{h,1}\eta^I)\right| d\sigma \le C\left(\sum_{\gamma\in\Gamma_{12}} h_\gamma\|\mathbf{K}\nabla(p-\hat p)\cdot\mathbf{n}\|_{0,\gamma}^2\right)^{\frac12}\left(\sum_{\gamma\in\Gamma_{12}} \left\|\eta^I\right\|_{\frac12,\gamma}^2\right)^{\frac12}.$$

Then we apply (3.25) to the first factor and (3.26) with $k = \frac{1}{2}$ to the second factor; this gives

$$T_A \leq Ch^{\mu-1}|p|_{\mu,\Omega_1} H^{\bar{\mu}-\frac{1}{2}} |p|_{\bar{\mu}+\frac{1}{2},\Omega_1} \leq Ch^{2\mu-2} + CH^{2\bar{\mu}-1}.$$

To bound term T_B, we first subtract (3.3) from (3.8) and take the test function to be a piecewise constant on $\mathcal{E}_h(\Omega_1)$, to obtain the discrete mass balance equation

$$\begin{aligned} -\sum_{\gamma\in\Gamma_h(\Omega_1)} \int_\gamma \{\mathbf{K}\nabla(p_h - p)\cdot\mathbf{n}\}\,[q]\,d\sigma - \int_{\Gamma_{12}} \mathbf{K}\nabla(p_h-p)\cdot\mathbf{n}q|_{\Omega_1} d\sigma \\ + \sum_{\gamma\in\Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma}\int_\gamma [p_h]\,[q]\,d\sigma + \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \left(p_h|_{\Omega_1} - \lambda_H\right) q d\sigma = 0. \end{aligned} \tag{3.30}$$

Let $\tilde{\eta}^I = P_1(\eta^I)$, where $P_1 \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_{12}); H^1(\Omega_1))$ is an extension operator, and let $\mathcal{Q}_h$ be the L^2-projection onto the space of piecewise constants on $\mathcal{E}_h(\Omega_1)$. Using (3.30) and the continuity of the trace of $\tilde{\eta}^I$ across interior interfaces, the term T_B has the following expression

$$\begin{aligned} &\int_{\Gamma_{12}} \mathbf{K}\nabla\xi\cdot\mathbf{n}\eta^I d\sigma \\ =& \sum_{\gamma\in\Gamma_h(\Omega_1)} \int_\gamma \{\mathbf{K}\nabla(p_h-p)\cdot\mathbf{n}\}\left[\tilde{\eta}^I\right] d\sigma + \int_{\Gamma_{12}} \mathbf{K}\nabla(p_h-p)\cdot\mathbf{n}\tilde{\eta}^I d\sigma \\ =& \sum_{\gamma\in\Gamma_h(\Omega_1)} \int_\gamma \{\mathbf{K}\nabla(p_h-p)\cdot\mathbf{n}\}\left[\tilde{\eta}^I - \mathcal{Q}_h\tilde{\eta}^I\right] d\sigma + \int_{\Gamma_{12}} \mathbf{K}\nabla(p_h-p)\cdot\mathbf{n}(\tilde{\eta}^I - \mathcal{Q}_h\tilde{\eta}^I) d\sigma \\ &+ \sum_{\gamma\in\Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma}\int_\gamma [p_h]\left[\mathcal{Q}_h\tilde{\eta}^I\right] d\sigma + \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \left(p_h|_{\Omega_1} - \lambda_H\right)\mathcal{Q}_h\tilde{\eta}^I d\sigma =: \sum_{i=1}^4 T_{B,i}. \end{aligned}$$

To bound the first term, observe that by mapping to the reference element, using a trace theorem and the regularity of $\mathcal{E}_h(\Omega_1)$, we have on any segment γ adjacent to an element E

$$\|\tilde{\eta}^I - \mathcal{Q}_h\tilde{\eta}^I\|_{0,\gamma} \leq Ch_\gamma^{\frac{1}{2}}|\tilde{\eta}^I|_{1,E}.$$

Applying the continuity of the extension operator P_1 gives

$$|T_{B,1}| \leq C\left(\sum_{\gamma\in\Gamma_h(\Omega_1)} h_\gamma \|\mathbf{K}\nabla(p_h-p)\|_{0,\gamma}^2\right)^{\frac{1}{2}} \|\eta^I\|_{\frac{1}{2},\Gamma_{12}}.$$

For the first factor we write $p_h - p = \xi^A + \xi^I$ and apply (3.11), (3.25), and (3.26), thus deriving

$$\begin{aligned} |T_{B,1}| &\leq CH^{\bar{\mu}-\frac{1}{2}}|p|_{\bar{\mu}+\frac{1}{2},\Omega_1}\left(h^{\mu-1}|p|_{\mu,\Omega_1} + \|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|_{0,\Omega_1}\right) \\ &\leq \epsilon\|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|_{0,\Omega_1}^2 + C(h^{2\mu-2} + H^{2\bar{\mu}-1}), \end{aligned}$$

with a similar bound for $|T_{B,2}|$. The remaining terms are bounded as follows:

$$\begin{aligned} |T_{B,3}| &= \left|\sum_{\gamma\in\Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma}\int_\gamma [p_h]\left[\mathcal{Q}_h\tilde{\eta}^I - \tilde{\eta}^I\right] d\sigma\right| \leq \epsilon \sum_{\gamma\in\Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma}\|[p_h]\|_{0,\gamma}^2 + CH^{2\bar{\mu}-1}, \\ |T_{B,4}| &\leq \left|\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \left(p_h|_{\Omega_1} - \lambda_H\right)\left(\mathcal{Q}_h\tilde{\eta}^I - \tilde{\eta}^I\right) d\sigma\right| + \left|\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \left(p_h|_{\Omega_1} - \lambda_H\right)\tilde{\eta}^I d\sigma\right| \end{aligned}$$

$$\leq \quad \epsilon \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \left\| p_h|_{\Omega_1} - \lambda_H \right\|_{0,\tau}^2 + C(H^{2\bar{\mu}-2}h + H^{2\bar{\mu}-1}).$$

Combining the results for T_A and $T_{B,i}$, we obtain

$$|T_{6,1}| \leq \epsilon \|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|_{0,\Omega_1}^2 + \epsilon \sum_{\gamma\in\Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma} \|[p_h]\|_{0,\gamma}^2 + \epsilon \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \left\| \xi^A\big|_{\Omega_1} - \eta^A \right\|_{0,\tau}^2 + C(h^{2\mu-2} + H^{2\bar{\mu}-1}),$$

where we have also used (3.24) and (3.26). The last term T_7 can be bounded as follows:

$$\begin{aligned} |T_7| \quad &\leq \quad \left(\sum_{i=1,2}\sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \|\xi^A|_{\Omega_i} - \eta^A\|_{0,\tau}^2\right)^{\frac{1}{2}} \left(\sum_{i=1,2}\sum_{\tau\in\Gamma_H} \frac{H_\tau}{\sigma_\tau} \|\mathbf{K}\nabla\xi^I|_{\Omega_i}\cdot\mathbf{n}\|_{0,\tau}^2\right)^{\frac{1}{2}} \\ &\leq \quad \epsilon \sum_{i=1,2}\sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \| \,\xi^A\big|_{\Omega_i} - \eta^A\|_{0,\tau}^2 + Ch^{2\mu-3}H. \end{aligned}$$

Thus

$$\begin{aligned} &\|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|_{0,\Omega_1}^2 + \sum_{\gamma\in\Gamma_h(\Omega)} \frac{\sigma_\gamma}{h_\gamma} \left\|[\xi^A]\right\|_{0,\gamma}^2 + \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \sum_{i=1,2} \left\| \xi^A\big|_{\Omega_i} - \eta^A \right\|_{0,\tau}^2 \\ &\leq Ch^{2\mu-2}\left(\frac{H}{h}\right) + CH^{2\bar{\mu}-1}. \end{aligned}$$

The proof is completed by using the triangle inequality. □

Remark 3.1. It is also possible to choose $\bar{s}_{\text{form}} = 0$ or 1, independently of s_{form}. For example, taking $\bar{s}_{\text{form}} = 1$ in SIPG preserves the symmetry of the method. These choices, however, require fine scale related weights in the interface penalty terms of $B_i(\cdot,\cdot)$ and $L_i(\cdot;\cdot)$, in order to control the terms involving integrals on Γ_{12}. More precisely, the last terms of $B_i(\cdot,\cdot)$ and $L_i(\cdot;\cdot)$ in (3.8)-(3.9) need to be replaced by

$$\text{(3.31)} \qquad \sum_{\gamma\in\Gamma_{h,i}} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma p_h|_{\Omega_i} q_h|_{\Omega_i} d\sigma \quad \text{and} \quad \sum_{\gamma\in\Gamma_{h,i}} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma q_h|_{\Omega_i} \lambda_H \, d\sigma,$$

respectively. Here, $\Gamma_{h,i}$ denotes the trace of $\mathcal{E}_h(\Omega_i)$ on Γ_{12}. Similarly, the interface pressure continuity term in (3.4) should be replaced by

$$\sum_{i=1,2}\sum_{\gamma\in\Gamma_{h,i}} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma (p_h|_{\Omega_i} - \lambda_H)\mu_H d\sigma.$$

Then, the terms involving integrals on Γ_{12} can be controlled by the terms in (3.31) and $\|\cdot\|_{0,\Omega_i}$, via the inequality, for all $q_h \in X_h(\Omega_i)$ and all $\mu \in L^2(\Gamma_{12})$,

$$\text{(3.32)} \qquad \left| \int_{\Gamma_{12}} \mathbf{K}\nabla q_h \cdot \mathbf{n}\, \mu d\sigma \right| \leq \frac{1}{8} \|\mathbf{K}^{\frac{1}{2}}\nabla q_h\|_{0,\Omega_i}^2 + \frac{1}{8} \sum_{\gamma\in\Gamma_{h,i}} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma \mu^2 d\sigma,$$

assuming the weights σ_γ are sufficiently large. Note that (3.32) can be shown in a way similar to (3.10). It can then be seen that all modified mortar DG methods are well posed, i.e., Theorem 3.2 holds for $\bar{s}_{\text{form}} = 0$, 1, or -1. Moreover, under the assumptions of Theorem 3.3, following its argument, we can show that there exists a constant C, independent of h and H, such that

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(p_h-p)\|_{0,\Omega}+\sqrt{\sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\|[p_h]\|_{0,\gamma}^2}+\sqrt{\sum_{i=1,2}\sum_{\gamma\in\Gamma_{h,i}}\frac{\sigma_\gamma}{h_\gamma}\left\|p_h|_{\Omega_i}-\lambda_H\right\|_{0,\gamma}^2}$$
$$\leq C\left(h^{\mu-1}+H^{\bar\mu-\frac{1}{2}}\left(\frac{H}{h}\right)^{\frac{1}{2}}\right),\quad \mu=\min(r+1,s)\,,\ \bar\mu=\min\left(\bar r+1,s-\frac{1}{2}\right).$$

Note that if $H=O(h)$ the above bound and the estimate from Theorem 3.3 provide the same (optimal) asymptotic convergence. However, in the multiscale case $H=O(h^\alpha)$, $0<\alpha<1$, the bound from Theorem 3.3 is better.

3.5. **The interface operator and the reduced problem.** In the following we present a non-overlapping domain decomposition algorithm, which involves the reduction of the coupled system to an interface problem in the mortar space. We are motivated by the algorithms developed in [26, 4].

Let us split $L_i(q,\lambda)$ into a sum of two terms:

$$L_i(q;\lambda)=l_i(q)+b_i(\lambda,q),$$

where

$$l_i(q) := \int_{\Omega_i} fq\,dx-\int_{\partial\Omega_i\setminus\Gamma_{12}} gq\,d\sigma, \tag{3.33}$$

$$b_i(\lambda,q) := -\bar s_{\rm form}\int_{\Gamma_{12}}\mathbf{K}\nabla q\cdot\mathbf{n}_{\partial\Omega_i}\lambda d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau q|_{\Omega_i}\lambda\,d\sigma$$
$$:= \int_{\Gamma_{12}}\mathbf{K}\nabla q\cdot\mathbf{n}_{\partial\Omega_i}\lambda d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau q|_{\Omega_i}\lambda\,d\sigma, \tag{3.34}$$

since we take $\bar s_{\rm form}=-1$. We comment on the choices $\bar s_{\rm form}=0$ or 1 in Remark 3.3.

Define a bilinear form $d_H: L^2(\Gamma_{12})\times L^2(\Gamma_{12})\mapsto\mathbb{R}$ for $\lambda,\mu\in L^2(\Gamma_{12})$ by

$$d_H(\lambda,\mu):=\sum_{i=1}^{2}\left(\int_{\Gamma_{12}}\mathbf{K}\,\nabla p_h^*(\lambda)|_{\Omega_i}\cdot\mathbf{n}_{\partial\Omega_i}\mu d\sigma-\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau(p_h^*(\lambda)|_{\Omega_i}-\lambda)\mu d\sigma\right), \tag{3.35}$$

where $p_h^*(\lambda)|_{\Omega_i}\in X_h(\Omega_i)$, $i=1,2$ is the solution of

$$B_i\left(p_h^*(\lambda),q_h\right) = b_i\left(\lambda,q_h\right),\quad q_h\in X_h(\Omega_i),\ i=1\text{ and }2. \tag{3.36}$$

Define a linear functional $g_H: L^2(\Gamma_{12})\mapsto\mathbb{R}$ by

$$g_H(\mu):=-\sum_{i=1}^{2}\left(\int_{\Gamma_{12}}\mathbf{K}\,\nabla\bar p_h|_{\Omega_i}\cdot\mathbf{n}_{\partial\Omega_i}\mu d\sigma-\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\bar p_h|_{\Omega_i}\,\mu d\sigma\right),$$

where $\bar p_h|_{\Omega_i}\in X_h(\Omega_i)$, $i=1,2$ solves

$$B_i\left(\bar p_h,q_h\right) = l_i\left(q_h\right),\quad q_h\in X_h(\Omega_i),\ i=1\text{ and }2.$$

It is easy to see that the solution (p_h,λ_H) of the DG-DG scheme (3.8)-(3.9) satisfies

$$d_H\left(\lambda_H,\mu_H\right)=g_H\left(\mu_H\right),\quad \mu_H\in\Lambda_H, \tag{3.37}$$

with

$$p_h = p_h^*(\lambda_H)+\bar p_h.$$

We now analyze the properties of the bilinear form $d_H(\cdot,\cdot)$ for the various DG schemes. We first let $q_h = p_h^*(\mu)$ in (3.36) for some $\mu \in L^2(\Gamma_{12})$ to obtain

$$B_i\left(p_h^*(\lambda), p_h^*(\mu)\right) = b_i\left(\lambda, p_h^*(\mu)\right), \quad i = 1 \text{ and } 2. \tag{3.38}$$

Using (3.35), (3.34), and (3.38), we have

$$\begin{aligned} d_H(\lambda,\mu) &= \sum_{i=1}^{2} b_i(\mu, p_h^*(\lambda)) - 2\sum_{i=1}^{2}\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau p_h^*(\lambda)|_{\Omega_i}\mu\, d\sigma + 2\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \lambda\mu d\sigma \\ &= \sum_{i=1}^{2} B_i\left(p_h^*(\mu), p_h^*(\lambda)\right) - 2\sum_{i=1}^{2}\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau p_h^*(\lambda)|_{\Omega_i}\mu\, d\sigma + 2\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \lambda\mu d\sigma. \end{aligned} \tag{3.39}$$

Note that the above representation implies that the interface bilinear form $d_H(\cdot,\cdot)$ is non-symmetric for all DG versions.

To show coercivity, using (3.39) and (3.1), we obtain

$$\begin{aligned} d_H(\lambda,\lambda) &= \|\mathbf{K}^{\frac{1}{2}}\nabla p_h^*(\lambda)\|_{0,\Omega}^2 - (1+s_{\text{form}})\sum_{\gamma\in\Gamma_h(\Omega)}\int_\gamma \{\mathbf{K}\nabla p_h^*(\lambda)\cdot\mathbf{n}\}\,[p_h^*(\lambda)]\, d\sigma \\ &\quad + \sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma [p_h^*(\lambda)]^2\, d\sigma + \sum_{i=1,2}\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \left(p_h^*(\lambda)|_{\Omega_i} - \lambda\right)^2 d\sigma \\ &\ge \frac{1}{2}\|\mathbf{K}^{\frac{1}{2}}\nabla p_h^*(\lambda)\|_{0,\Omega}^2 + \frac{1}{2}\sum_{\gamma\in\Gamma_h(\Omega)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma [p_h^*(\lambda)]^2\, d\sigma \\ &\quad + \sum_{i=1,2}\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \left(p_h^*(\lambda)|_{\Omega_i} - \lambda\right)^2 d\sigma, \end{aligned} \tag{3.40}$$

where we have used (3.10) for the inequality when $s_{\text{form}} = 0$ or 1. Therefore the interface bilinear form $d_H(\cdot,\cdot)$ is positive semi-definite on $L^2(\Gamma_{12})$. The argument in the solvability Theorem 3.2 implies that $d_H(\lambda,\lambda) = 0$ only if $\lambda = $ constant. We summarize our results below.

Theorem 3.4. *Let the assumptions of Theorem 3.2 hold. For all DG versions, the interface bilinear form $d_H(\cdot,\cdot)$ is non-symmetric and positive semi-definite on $L^2(\Gamma_{12})$, with the kernel consisting of the constant functions.*

Remark 3.2. If Dirichlet boundary condition is imposed on a part of $\partial\Omega$, Γ_D, such that $|\Gamma_D| > 0$, then $d_H(\cdot,\cdot)$ is positive definite on $L^2(\Gamma_{12})$.

Remark 3.3. For a general choice of $\bar{s}_{\text{form}}$, coercivity of $d_H(\cdot,\cdot)$ can be shown for all modified schemes introduced in Remark 3.1, using inequality (3.32). Recall that taking $\bar{s}_{\text{form}} = 1$ in SIPG gives symmetric forms $B_i(\cdot,\cdot)$. It is easy to see that in this case $d_H(\cdot,\cdot)$ is also symmetric. In particular, using (3.35) and (3.34) with modified penalty terms, and (3.38), we obtain

$$\begin{aligned} d_H(\lambda,\mu) &= -\sum_{i=1}^{2}\left(b_i(\mu, p_h^*(\lambda)) + \sum_{\gamma\in\Gamma_{h,i}}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma \lambda\mu d\sigma\right) \\ &= -\sum_{i=1}^{2}\left(B_i\left(p_h^*(\mu), p_h^*(\lambda)\right) + \sum_{\gamma\in\Gamma_{h,i}}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma \lambda\mu d\sigma\right) = d_H(\mu,\lambda). \end{aligned}$$

Therefore in this case the conjugate gradient method can be employed for solving (3.37).

4. Coupling DG with MFE using a mortar space

4.1. **Weak formulation.** We now consider the case where DG formulation is used in Ω_1 and a mixed formulation is used in Ω_2, the matching at the interface being achieved by a mortar multiplier. Recall that $W(\Omega_2) = L^2(\Omega_2)$, $\Lambda = H^{\frac{1}{2}}(\Gamma_{12})$, and $B_i(\cdot,\cdot)$ and $L_i(\cdot;\cdot)$ are defined in (3.1) and (3.2) respectively. In this section, we only use the definitions in Ω_1 and drop the subscripts for simplicity. That is, we denote $B_1(\cdot,\cdot)$ and $L_1(\cdot;\cdot)$ simply by $B(\cdot,\cdot)$ and $L(\cdot;\cdot)$.

Let $\mathbf{u}_g$ be the restriction to Ω_2 of an extension of g satisfying $\mathbf{u}_g \in H(\text{div};\Omega)$ and $\mathbf{u}_g \cdot \mathbf{n} = g$ on $\partial\Omega$. The coupled DG-mixed weak formulation is: find $p \in L^2(\Omega)$ such that $p|_{\Omega_1} \in X(\Omega_1)$, $p|_{\Omega_2} \in W(\Omega_2)$, $\mathbf{u} \in \mathbf{V}_0(\Omega_2) + \mathbf{u}_g$, and $\lambda \in \Lambda$, such that

$$B(p,q) = L(q;\lambda), \quad \forall q \in X(\Omega_1), \tag{4.1}$$

$$\int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}\cdot\mathbf{v}dx - \int_{\Omega_2} p\,\nabla\cdot\mathbf{v}dx = -\int_{\Gamma_{12}} \mathbf{v}\cdot\mathbf{n}_{\partial\Omega_2}\lambda d\sigma, \quad \forall\,\mathbf{v}\in\mathbf{V}_0(\Omega_2), \tag{4.2}$$

$$\int_{\Omega_2} \nabla\cdot\mathbf{u}\,q\,dx = \int_{\Omega_2} f\,q\,dx, \quad \forall\,q\in W(\Omega_2), \tag{4.3}$$

$$-\int_{\Gamma_{12}} (\mathbf{K}\nabla p|_{\Omega_1}\cdot\mathbf{n} + \mathbf{u}\cdot\mathbf{n})\mu d\sigma + \sum_{\tau\in\Gamma_H}\int_\tau \frac{\sigma_\tau}{H_\tau}(p|_{\Omega_1} - \lambda)\mu d\sigma = 0, \quad \forall \mu\in\Lambda. \tag{4.4}$$

As in Section 3, we take $\bar{s}_{\text{form}} = -1$. The choices $\bar{s}_{\text{form}} = 0$ or 1 are discussed in Remarks 4.2 and 4.4.

4.2. **Equivalence.**

Theorem 4.1. *If $(\mathbf{u}, p, \lambda)$ is a solution of (4.1)-(4.4), then p satisfies (2.1)-(2.2) in the sense of distributions. Conversely, if p is a sufficiently smooth solution of (2.1)-(2.2), then there exists $\mathbf{u}$ and λ such that $(\mathbf{u}, p, \lambda)$ solves (4.1)-(4.4).*

Proof. Using the same arguments as in the DG-DG case, we conclude that $p|_{\Omega_1} \in H^1(\Omega_1)$ satisfies (2.1) in Ω_1 and (2.2) on $\partial\Omega_1 \setminus \Gamma_{12}$, and that $p|_{\Omega_1} = \lambda$ on Γ_{12}.

We now take $\mathbf{v} \in (C_0^\infty(\Omega_2))^d$ and (4.2) becomes

$$\int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}\cdot\mathbf{v}dx - \int_{\Omega_2} p\nabla\cdot\mathbf{v}dx = 0,$$

which implies $\mathbf{u} = -\mathbf{K}\nabla p$ in Ω_2. With $q \in C_0^\infty(\Omega_2)$, (4.3) implies $\nabla\cdot\mathbf{u} = f$ in Ω_2. Hence (2.1) is satisfied in Ω_2. We have forced that $\mathbf{u}\cdot\mathbf{n} = g$ on $\partial\Omega_2 \setminus \Gamma_{12}$, which results in the satisfaction of (2.2) on $\partial\Omega_2 \setminus \Gamma_{12}$.

Taking $\mathbf{v} \in \left(H^1(\Omega_2)\right)^d$ with $\mathbf{v} = 0$ on $\partial\Omega_2 \setminus \Gamma_{12}$, (4.2) becomes

$$\begin{aligned} -\int_{\Gamma_{12}} \mathbf{v}\cdot\mathbf{n}_{\partial\Omega_2}\lambda d\sigma &= -\int_{\Omega_2} \nabla p\cdot\mathbf{v}dx - \int_{\Omega_2} p\nabla\cdot\mathbf{v}dx \\ &= -\int_{\partial\Omega_2} \mathbf{v}\cdot\mathbf{n}_{\partial\Omega_2} p d\sigma = -\int_{\Gamma_{12}} \mathbf{v}\cdot\mathbf{n}_{\partial\Omega_2} p d\sigma. \end{aligned}$$

Since the trace of $\mathbf{v}$ can be arbitrarily chosen in $H_{00}^{\frac{1}{2}}(\Gamma_{12})^d$, we conclude that $p|_{\Omega_2} = \lambda$ on Γ_{12}. Hence p has the same trace on both sides of Γ_{12} and therefore p belongs to $H^1(\Omega)$.

Finally, on Γ_{12}, (4.4) gives $-\,\mathbf{K}\nabla p|_{\Omega_1}\cdot\mathbf{n} = \mathbf{u}\cdot\mathbf{n} = -\,\mathbf{K}\nabla p|_{\Omega_2}\cdot\mathbf{n}$; that is, $\mathbf{K}\nabla p\cdot\mathbf{n}$ has the same trace on both sides of Γ_{12}. Therefore (2.1) is satisfied in the entire domain Ω. $\square$

4.3. **Discretization.** We recall that the DG and MFE approximation spaces were introduced in Section 2. Let $\Pi_h \in \mathcal{L}(H^1(\Omega_2)^d; \mathbf{V}_h(\Omega_2))$ be the standard MFE interpolation operator satisfying on any $E \in \mathcal{E}_h(\Omega_2)$ [17]

$$\int_E \nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u})\, q_h \, dx = 0, \quad \forall\, q_h \in W_h(E), \tag{4.5}$$

$$\int_\gamma (\mathbf{u} - \Pi_h \mathbf{u}) \cdot \mathbf{n}\ \mathbf{v}_h \cdot \mathbf{n}\, d\sigma, \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h(E), \quad \forall \gamma \in \partial E \tag{4.6}$$

$$\|\Pi_h \mathbf{u}\|_{H(\mathrm{div};E)} \le C \|\mathbf{u}\|_{1,E}, \tag{4.7}$$

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{0,E} \le C h^{m+1} |\mathbf{u}|_{m+1,E}. \tag{4.8}$$

These properties imply that for all MFE spaces under consideration [17]

$$\nabla \cdot \mathbf{V}_{h,0}(\Omega_2) = W_h(\Omega_2). \tag{4.9}$$

More precisely, we have the following inf-sup condition.

Lemma 4.1. *Let $\mathcal{E}_h(\Omega_2)$ be non-degenerate. For any p_h in $W_h(\Omega_2)$, there exists $\mathbf{v}_h$ in $\mathbf{V}_{h,0}(\Omega_2)$ such that*

$$\nabla \cdot \mathbf{v}_h = p_h, \quad \text{in } \Omega_2,$$

and a constant C independent of $\mathbf{v}_h$, p_h and h such that

$$\|\mathbf{v}_h\|_{H(\mathrm{div};\Omega_2)} + \|\mathbf{v}_h \cdot \mathbf{n}\|_{0,\Gamma_{12}} \le C \|p_h\|_{0,\Omega_2}. \tag{4.10}$$

Proof. First, we extend p_h by a constant function in Ω_1 so that its mean-value is zero in Ω. Let $\tilde{p}_h$ denote the extended function. Then

$$\|\tilde{p}_h\|_{0,\Omega} \le C \|p_h\|_{0,\Omega_2}.$$

As $\tilde{p}_h$ has mean-value zero in Ω, there exists $\mathbf{v}$ in $H^1_0(\Omega)^d$ such that (cf.[25])

$$\nabla \cdot \mathbf{v} = \tilde{p}_h, \quad \text{in } \Omega,$$

and

$$|\mathbf{v}|_{1,\Omega} \le C \|\tilde{p}_h\|_{0,\Omega} \le C \|p_h\|_{0,\Omega_2}. \tag{4.11}$$

Take $\mathbf{v}_h = \Pi_h \mathbf{v}$. Then we easily derive from (4.5)-(4.7) and the regularity of $\mathbf{v}$ that the restriction of $\mathbf{v}_h$ to Ω_2 belongs to $\mathbf{V}_{h,0}(\Omega_2)$ and satisfies (4.10). □

Let $\mathbf{u}_{h,g}$ be an adequate approximation of $\mathbf{u}_g$ in $\mathbf{V}_h(\Omega_2)$. The finite element mortar DG-MFE discretization is to find $(p_h|_{\Omega_1}, p_h|_{\Omega_2}, \mathbf{u}_h, \lambda_H)$ in $X_h(\Omega_1) \times W_h(\Omega_2) \times (\mathbf{V}_{h,0}(\Omega_2) + \mathbf{u}_{h,g}) \times \Lambda_H$, such that the following equations hold for all $(q_h|_{\Omega_1}, q_h|_{\Omega_2}, \mathbf{v}_h, \mu_H)$ in $X_h(\Omega_1) \times W_h(\Omega_2) \times \mathbf{V}_{h,0}(\Omega_2) \times \Lambda_H$:

$$B(p_h, q_h) = L(q_h; \lambda_H), \tag{4.12}$$

$$\int_{\Omega_2} \mathbf{K}^{-1} \mathbf{u}_h \cdot \mathbf{v}_h dx = \int_{\Omega_2} p_h \nabla \cdot \mathbf{v}_h dx - \int_{\Gamma_{12}} \mathbf{v}_h \cdot \mathbf{n}_{\partial\Omega_2} \lambda_H d\sigma, \tag{4.13}$$

$$\int_{\Omega_2} \nabla \cdot \mathbf{u}_h q_h dx = \int_{\Omega_2} f q_h dx, \tag{4.14}$$

$$\int_{\Gamma_{12}} \mathbf{u}_h \cdot \mathbf{n} \mu_H d\sigma = -\int_{\Gamma_{12}} \mathbf{K} \nabla p_h|_{\Omega_1} \cdot \mathbf{n} \mu_H d\sigma + \sum_{\tau \in \Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau (p_h|_{\Omega_1} - \lambda_H) \mu_H d\sigma. \tag{4.15}$$

Note that both $\mathbf{u}_g$ and $\mathbf{u}_{h,g}$ are only introduced for theoretical reasons and in practice we only need to approximate g.

We now address existence and uniqueness of the solution of the above system. This is a square finite dimensional system and existence is equivalent to uniqueness. Let $f = 0$ and $\mathbf{u}_{h,g} = \mathbf{0}$. Taking $q_h = p_h$ in (4.12), we have

$$\sum_{E\in\mathcal{E}_h(\Omega_1)} \int_E \mathbf{K}\nabla p_h \cdot \nabla p_h dx - (1+s_{\text{form}}) \sum_{\gamma\in\Gamma_h(\Omega_1)} \int_\gamma \{\mathbf{K}\nabla p_h \cdot \mathbf{n}\} [p_h] \, d\sigma + \sum_{\gamma\in\Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma [p_h]^2 \, d\sigma + \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau p_h|_{\Omega_1}^2 d\sigma = \int_{\Gamma_{12}} \mathbf{K}\nabla p_h|_{\Omega_1} \cdot \mathbf{n}\lambda_H d\sigma + \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau p_h|_{\Omega_1} \lambda_H d\sigma. \tag{4.16}$$

Taking $\mathbf{v}_h = \mathbf{u}_h$ in (4.13) and $q_h = p_h$ in (4.14), we have

$$\int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_h \cdot \mathbf{u}_h dx = -\int_{\Gamma_{12}} \mathbf{u}_h \cdot \mathbf{n}_{\partial\Omega_2} \lambda_H d\sigma = \int_{\Gamma_{12}} \mathbf{u}_h \cdot \mathbf{n}\lambda_H d\sigma. \tag{4.17}$$

Taking $\mu_H = \lambda_H$ in (4.15), we obtain

$$\int_{\Gamma_{12}} \mathbf{u}_h \cdot \mathbf{n}\lambda_H d\sigma = -\int_{\Gamma_{12}} \mathbf{K}\nabla p_h|_{\Omega_1} \cdot \mathbf{n}\lambda_H d\sigma + \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau (p_h|_{\Omega_1} - \lambda_H)\lambda_H d\sigma. \tag{4.18}$$

Summation of (4.16), (4.17) and (4.18) leads to

$$\|\mathbf{K}^{\frac{1}{2}}\nabla p_h\|_{0,\Omega_1}^2 - (1+s_{\text{form}}) \sum_{\gamma\in\Gamma_h(\Omega_1)} \int_\gamma \{\mathbf{K}\nabla p_h \cdot \mathbf{n}\} [p_h] \, d\sigma + \|\mathbf{K}^{-\frac{1}{2}}\mathbf{u}_h\|_{0,\Omega_2}^2 + \sum_{\gamma\in\Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma [p_h]^2 \, d\sigma + \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau} \int_\tau (p_h|_{\Omega_1} - \lambda_H)^2 d\sigma = 0. \tag{4.19}$$

First consider the case $0 < \sigma_\tau^0 \le \sigma_\tau \le \sigma_\tau^1$. As in Section 3.3, we easily derive from (4.19) that for OBB-DG and NIPG, $\mathbf{u}_h = \mathbf{0}$ in Ω_2, p_h is a constant in Ω_1 and λ_H is the same constant on Γ_{12}. The same conclusion holds for SIPG and IIPG, by applying inequality (3.10).

Now, as $\mathbf{u}_h$ is zero, (4.13) becomes

$$\forall \mathbf{v}_h \in \mathbf{V}_{h,0}(\Omega_2)\,, \int_{\Omega_2} p_h \nabla\cdot \mathbf{v}_h dx \;=\; \int_{\Gamma_{12}} \mathbf{v}_h \cdot \mathbf{n}_{\partial\Omega_2} \lambda_H d\sigma. \tag{4.20}$$

Let $\overline{p_h}$ denote the mean-value of p_h in Ω_2: $\overline{p_h} = \frac{1}{|\Omega_2|}\int_{\Omega_2} p_h dx$. Then (4.20) implies that for all $\mathbf{v}_h$ in $\mathbf{V}_{h,0}(\Omega_2)$ with $\mathbf{v}_h \cdot \mathbf{n} = 0$ on $\partial\Omega_2$,

$$\int_{\Omega_2} (p_h - \overline{p_h})\nabla\cdot \mathbf{v}_h dx = 0.$$

As $p_h - \overline{p_h}$ belongs to $W_h(\Omega_2)$ and has mean-value zero, the argument of Lemma 4.1 implies that there exists $\mathbf{v}_h$ in $\mathbf{V}_h(\Omega_2)$ with $\mathbf{v}_h \cdot \mathbf{n} = 0$ on $\partial\Omega_2$ such that

$$\int_{\Omega_2} (p_h - \overline{p_h})\nabla\cdot \mathbf{v}_h dx = \|p_h - \overline{p_h}\|_{0,\Omega_2}^2.$$

Therefore p_h is also constant in Ω_2 and (4.20) implies

$$\forall \mathbf{v}_h \in \mathbf{V}_{h,0}(\Omega_2)\,, \int_{\Gamma_{12}} \mathbf{v}_h \cdot \mathbf{n}(\lambda_H - p_h|_{\Omega_2}) d\sigma = 0. \tag{4.21}$$

Since λ_H is constant and coincides with the trace of p_h coming from Ω_1, we infer from (4.21) that it also coincides with the trace of p_h coming from Ω_2. Thus p_h is constant in Ω.

There remains the case $\sigma_\tau = 0$. With the information we have so far, (4.12) implies

$$\int_{\Gamma_{12}} \mathbf{K}\nabla q_h|_{\Omega_1} \cdot \mathbf{n}(\lambda_H - p_h|_{\Omega_1}) d\sigma = 0. \tag{4.22}$$

If the mortar compatibility condition (A.2) is satisfied, since p_h is constant in Ω_2, (4.21) implies that $\lambda_H = p_h|_{\Omega_2}$. Thus, λ_H is constant and (4.22) implies readily that p_h has the same trace on both sides of Γ_{12}. If the mortar compatibility condition (A.1) holds, as p_h is constant in Ω_1, then (4.22) implies that $\lambda_H = p_h|_{\Omega_1}$. Hence λ_H is constant and (4.21) implies again that $\lambda_H = p_h|_{\Omega_2}$ and therefore p_h is constant in Ω.

We have shown for all schemes that $\mathbf{u}_h$ is unique and that the null space of the linear system (4.12)-(4.15) for p_h and λ_H is the constant vector. The compatibility condition (2.3) implies that the right-hand side is orthogonal to the null space and therefore the solution exists and is unique up to an additive constant for p_h and λ_H. We have proved the following solvability theorem.

Theorem 4.2. *For OBB-DG, we assume that $r \geq 2$. For SIPG and IIPG, we assume that σ_γ^0 is sufficiently large. No assumption is needed for NIPG. Then the scheme (4.12)-(4.15) possesses a solution $(p_h, \mathbf{u}_h, \lambda_H)$ unique up to an additive constant that is the same for p_h and λ_H. The same conclusion holds if $\sigma_\tau = 0$, assuming that either the compatibility condition (A.1) for $i = 1$ or (A.2) holds.*

4.4. **Convergence.** We define the interpolant $\hat{p}$ of p such that $\hat{p}$ in Ω_2 is the L^2-projection and $\hat{p}$ in Ω_1 is defined as in the previous section. Then, on any $E \in \mathcal{E}_h(\Omega_2)$, $\hat{p}$ satisfies

$$\|p - \hat{p}\|_{0,E} \leq C h_E^{l+1} |p|_{l+1,E}. \tag{4.23}$$

For $\bar{p}$, we take again the continuous nodal interpolant of p in Λ_H. Now, we choose $\mathbf{u}_{h,g}$. On any $\gamma \subset \partial E \cap (\partial\Omega_2 \setminus \Gamma_{12})$, considering that $\mathbf{u} \cdot \mathbf{n} = g$, we define $P_h g$ by

$$P_h g = (\Pi_h \mathbf{u} \cdot \mathbf{n})|_\gamma.$$

Since by construction, $(\Pi_h \mathbf{u} \cdot \mathbf{n})|_\gamma$ does not depend on the interior values of $\mathbf{u}$, $P_h g$ only depends on g. Then we can take for $\mathbf{u}_{h,g}$ any function in $\mathbf{V}_h(\Omega_2)$ such that $\mathbf{u}_{h,g} = P_h g$ on $\partial\Omega_2 \setminus \Gamma_{12}$. All results derived below are independent of this choice and depend only on $P_h g$. To prove the convergence theorem below, we need the following trace inequality. The proof is a variant of those derived by Brenner in [13], see also [42]. It is stated in Ω_1, but it is valid in any connected Lipschitz domain.

Theorem 4.3. *Let $\mathcal{E}_h(\Omega_1)$ be non-degenerate and let Γ be any portion of $\partial\Omega_1$ with positive measure. Assume that $0 < \sigma_\gamma^0 \leq \sigma_\gamma \leq \sigma_\gamma^1$. Then there exists a constant C, independent of h such that for all functions q_h in $X_h(\Omega_1)$, the following trace inequality holds:*

$$\int_{\Gamma_{12}} q_h^2 d\sigma \leq C \left(\|\mathbf{K}^{\frac{1}{2}} \nabla q_h\|_{0,\Omega_1}^2 + \sum_{\gamma \in \Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma [q_h]^2 \, d\sigma + \left| \int_\Gamma q_h \, d\sigma \right|^2 \right). \tag{4.24}$$

We proceed with the convergence analysis of the coupled DG-MFE methods. We only consider the cases of NIPG, SIPG, or IIPG, since Theorem 4.3 does not apply in the case of OBB-DG. We comment on that case in Remark 4.1.

Theorem 4.4. *Let $\mathcal{E}_h(\Omega_i)$ be non-degenerate, $i = 1, 2$. Let $p \in H^s(\Omega)$, $s \geq 2$, be a solution of (2.1)-(2.2) and let $\mathbf{u} = -\mathbf{K}\nabla p$. Let $(p_h, \mathbf{u}_h, \lambda_H)$ be a solution of (4.12)-(4.15), where NIPG, SIPG, or IIPG is used in (4.12). We assume that σ_γ^0 is sufficiently large for SIPG and IIPG. Then there exists a constant C, independent of h and H, such that*

$$\|\mathbf{K}^{\frac{1}{2}} \nabla (p_h - p)\|_{0,\Omega_1} + \sqrt{\sum_{\gamma \in \Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma} \|[p_h]\|_{0,\gamma}^2} + \left\| \mathbf{K}^{-\frac{1}{2}} (\mathbf{u}_h - \mathbf{u}) \right\|_{0,\Omega_2}$$

$$+\sqrt{\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\left\|p_h|_{\Omega_1}-\lambda_H\right\|_{0,\tau}^2}\le C\left(h^{\mu-1}\left(\frac{H}{h}\right)^{\frac{1}{2}}+H^{\bar{\mu}-\frac{1}{2}}+h^{\nu}\right),$$

where $\mu=\min(r+1,s)$, $\bar{\mu}=\min\left(\bar{r}+1,s-\frac{1}{2}\right)$, *and* $\nu=\min\left(m+1,s-\frac{3}{2}\right)$.

Proof. In addition to the error variables η, η^I, η^A, ξ, ξ^I, and ξ^A introduced in Section 3.4, we define

$$\chi:=\mathbf{u}_h-\mathbf{u},\quad \chi^I:=\mathbf{u}-\Pi_h\mathbf{u},\quad \chi^A:=\mathbf{u}_h-\Pi_h\mathbf{u}=\chi+\chi^I,$$

and observe that owing to the choice of $\mathbf{u}_{h,g}$, χ^A belongs to $\mathbf{V}_{h,0}(\Omega_2)$. Subtracting the weak formulation (4.1) from the finite element scheme (4.12), and choosing $q_h=\xi^A$, we obtain

$$\begin{aligned}B(\xi^A,\xi^A)&=B(\xi^I,\xi^A)+L(\xi^A;\lambda_H)-L(\xi^A;p)\\&=B(\xi^I,\xi^A)+\int_{\Gamma_{12}}\mathbf{K}\nabla\xi^A|_{\Omega_1}\cdot\mathbf{n}\eta d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\xi^A|_{\Omega_1}\eta d\sigma.\end{aligned}\tag{4.25}$$

Subtracting from the mixed finite element scheme (4.13)-(4.14) the corresponding weak formulation (4.2)-(4.3), we obtain the error equations for all $(\mathbf{v}_h,q_h)$ in $\mathbf{V}_{h,0}(\Omega_2)\times W_h(\Omega_2)$

$$\int_{\Omega_2}\mathbf{K}^{-1}\chi\cdot\mathbf{v}_h\,dx=\int_{\Omega_2}\xi^A\,\nabla\cdot\mathbf{v}_h\,dx-\int_{\Gamma_{12}}\mathbf{v}_h\cdot\mathbf{n}_{\partial\Omega_2}\eta d\sigma,\tag{4.26}$$

$$\int_{\Omega_2}\nabla\cdot\chi^A\,q_h\,dx=0,\tag{4.27}$$

where we have used the properties (4.9) and (4.5). Note that (4.27) and (4.9) imply

$$\nabla\cdot\chi^A=0.\tag{4.28}$$

Taking $\mathbf{v}_h=\chi^A$ in the equations above and noting that $\mathbf{n}=-\mathbf{n}_{\partial\Omega_2}$, we have

$$\int_{\Omega_2}\mathbf{K}^{-1}\chi\cdot\chi^A dx-\int_{\Gamma_{12}}\chi^A\cdot\mathbf{n}\,\eta\,d\sigma=0.\tag{4.29}$$

Similarly, subtracting the matching condition (4.4) from its finite element formulation (4.15), and taking $\mu_H=\eta^A$, we obtain

$$\int_{\Gamma_{12}}\chi\cdot\mathbf{n}\eta^A d\sigma=-\int_{\Gamma_{12}}\mathbf{K}\nabla\xi|_{\Omega_1}\cdot\mathbf{n}\eta^A d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau(\xi|_{\Omega_1}-\eta)\eta^A d\sigma.\tag{4.30}$$

Summation of (4.25), (4.29) and (4.30) yields

$$\begin{aligned}B(\xi^A,\xi^A)&+\int_{\Omega_2}\mathbf{K}^{-1}\chi\cdot\chi^A dx-\int_{\Gamma_{12}}\chi^A\cdot\mathbf{n}\eta d\sigma+\int_{\Gamma_{12}}\chi\cdot\mathbf{n}\eta^A d\sigma\\&=B(\xi^I,\xi^A)+\int_{\Gamma_{12}}\mathbf{K}\nabla\xi^A|_{\Omega_1}\cdot\mathbf{n}\eta d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\xi^A\big|_{\Omega_1}\eta d\sigma\\&\quad-\int_{\Gamma_{12}}\mathbf{K}\nabla\xi|_{\Omega_1}\cdot\mathbf{n}\eta^A d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau(\xi|_{\Omega_1}-\eta)\eta^A d\sigma.\end{aligned}$$

Rearranging terms results in

$$
\begin{aligned}
B(\xi^A,\xi^A) &+ \|\mathbf{K}^{-\frac{1}{2}}\chi^A\|^2_{0,\Omega_2} + \sum_{\tau\subset\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau(\xi^A\big|_{\Omega_1}-\eta^A)^2 d\sigma - \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau(\xi^A\big|_{\Omega_1})^2 d\sigma \\
&= B(\xi^I,\xi^A) - \int_{\Gamma_{12}}\mathbf{K}\nabla\xi^A|_{\Omega_1}\cdot\mathbf{n}\eta^I d\sigma - \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \xi^A\big|_{\Omega_1}\eta^I d\sigma \\
&- \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau(\xi^I\big|_{\Omega_1}-\eta^I)\eta^A d\sigma + \int_{\Gamma_{12}}\mathbf{K}\nabla\xi^I\big|_{\Omega_1}\cdot\mathbf{n}\eta^A d\sigma \\
&+ \int_{\Omega_2}\mathbf{K}^{-1}\chi^I\cdot\chi^A dx - \int_{\Gamma_{12}}\chi^A\cdot\mathbf{n}\eta^I d\sigma + \int_{\Gamma_{12}}\chi^I\cdot\mathbf{n}\eta^A d\sigma.
\end{aligned}
\tag{4.31}
$$

We denote by L_{ErrEqu} the left-hand side of (4.31), and apply an algebraic manipulation to obtain

$$
\begin{aligned}
L_{\mathrm{ErrEqu}} &= \|\!|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|\!|^2_{0,\Omega_1} - (1+s_{\mathrm{form}})\sum_{\gamma\in\Gamma_h(\Omega_1)}\int_\gamma\left\{\mathbf{K}\nabla\xi^A\cdot\mathbf{n}\right\}\left[\xi^A\right]d\sigma \\
&+ \sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma\left[\xi^A\right]^2 d\sigma + \|\mathbf{K}^{-\frac{1}{2}}\chi^A\|^2_{0,\Omega_2} + \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau(\xi^A\big|_{\Omega_1}-\eta^A)^2 d\sigma.
\end{aligned}
$$

For NIPG and OBB-DG, the second term in L_{ErrEqu} vanishes, leaving only the coercive terms. For SIPG and IIPG, we employ the inequality (3.10) with $q_h = q = \xi^A$ to conclude that

$$
\begin{aligned}
L_{\mathrm{ErrEqu}} \;\geq\; & \frac{3}{4}\|\!|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|\!|^2_{0,\Omega_1} + \frac{3}{4}\sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma\left[\xi^A\right]^2 d\sigma \\
&+ \left\|\mathbf{K}^{-\frac{1}{2}}\chi^A\right\|^2_{0,\Omega_2} + \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\left(\xi^A\big|_{\Omega_1}-\eta^A\right)^2 d\sigma.
\end{aligned}
$$

We now consider the right-hand side of (4.31), which is denoted by R_{ErrEqu}. Expanding the first term as

$$
\begin{aligned}
B(\xi^I,\xi^A) &= \sum_{E\in\mathcal{E}_h(\Omega_1)}\int_E\mathbf{K}\nabla\xi^I\cdot\nabla\xi^A dx - \sum_{\gamma\in\Gamma_h(\Omega_1)}\int_\gamma\left\{\mathbf{K}\nabla\xi^I\cdot\mathbf{n}\right\}\left[\xi^A\right]d\sigma \\
&- s_{\mathrm{form}}\sum_{\gamma\in\Gamma_h(\Omega_1)}\int_\gamma\left\{\mathbf{K}\nabla\xi^A\cdot\mathbf{n}\right\}\left[\xi^I\right]d\sigma - \int_{\Gamma_{12}}\mathbf{K}\nabla\xi^I|_{\Omega_1}\cdot\mathbf{n}\xi^A|_{\Omega_1}d\sigma \\
&+ \int_{\Gamma_{12}}\mathbf{K}\nabla\xi^A|_{\Omega_1}\cdot\mathbf{n}\xi^I|_{\Omega_1}d\sigma + \sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma\left[\xi^I\right]\left[\xi^A\right]d\sigma + \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\xi^I|_{\Omega_1}\xi^A|_{\Omega_1}d\sigma,
\end{aligned}
$$

we have

$$
\begin{aligned}
R_{\mathrm{ErrEqu}} &= \sum_{E\in\mathcal{E}_h(\Omega_1)}\int_E\mathbf{K}\nabla\xi^I\cdot\nabla\xi^A dx - \sum_{\gamma\in\Gamma_h(\Omega_1)}\int_\gamma\left\{\mathbf{K}\nabla\xi^I\cdot\mathbf{n}\right\}\left[\xi^A\right]d\sigma \\
&-s_{\mathrm{form}}\sum_{\gamma\in\Gamma_h(\Omega_1)}\int_\gamma\left\{\mathbf{K}\nabla\xi^A\cdot\mathbf{n}\right\}\left[\xi^I\right]d\sigma + \sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma\left[\xi^I\right]\left[\xi^A\right]d\sigma \\
&+ \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau(\xi^I\big|_{\Omega_1}-\eta^I)(\xi^A\big|_{\Omega_1}-\eta^A)d\sigma \\
&+ \int_{\Omega_2}\mathbf{K}^{-1}\chi^I\cdot\chi^A dx - \int_{\Gamma_{12}}\chi^A\cdot\mathbf{n}\eta^I d\sigma + \int_{\Gamma_{12}}\chi^I\cdot\mathbf{n}\eta^A d\sigma
\end{aligned}
$$

$$-\int_{\Gamma_{12}} \mathbf{K}\nabla\xi^A|_{\Omega_1}\cdot\mathbf{n}\left(\eta^I-\xi^I|_{\Omega_1}\right)d\sigma+\int_{\Gamma_{12}}\mathbf{K}\nabla\xi^I\big|_{\Omega_1}\cdot\mathbf{n}\left(\eta^A-\xi^A|_{\Omega_1}\right)d\sigma=:\sum_{i=1}^{10}T_i.$$

We now bound each term in R_{ErrEqu}. We skip terms T_1 through T_5 because they have the same bounds as in the proof of Theorem 3.3. Next, (4.8) implies:

$$|T_6|\le\epsilon\left\|\mathbf{K}^{-\frac{1}{2}}\chi^A\right\|^2_{0,\Omega_2}+C\left\|\mathbf{K}^{-\frac{1}{2}}\chi^I\right\|^2_{0,\Omega_2}\le\epsilon\left\|\mathbf{K}^{-\frac{1}{2}}\chi^A\right\|^2_{0,\Omega_2}+Ch^{2\nu}.$$

To bound term T_7, let $\tilde\eta^I=P_2(\eta^I)$, where $P_2\in\mathcal{L}(H^{\frac{1}{2}}(\Gamma_{12});H^1(\Omega_2))$ is the analogue of P_1. Considering that $\chi^A\cdot\mathbf{n}$ vanishes on $\partial\Omega_2\setminus\Gamma_{12}$, we can write

$$|T_7|\le\left|\int_{\partial\Omega_2}(\chi^A\cdot\mathbf{n})\tilde\eta^I d\sigma\right|\le\|\chi^A\cdot\mathbf{n}\|_{-\frac{1}{2},\partial\Omega_2}\|\tilde\eta^I\|_{\frac{1}{2},\partial\Omega_2}\le C\left\|\chi^A\right\|_{H(\mathrm{div};\Omega_2)}\left\|\eta^I\right\|_{\frac{1}{2},\Gamma_{12}}.$$

Then using (4.28) and (3.26), we obtain

$$|T_7|\le\epsilon\left\|\mathbf{K}^{-\frac{1}{2}}\chi^A\right\|^2_{0,\Omega_2}+CH^{2\bar\mu-1}.$$

To estimate T_8, we split it into

$$\int_{\Gamma_{12}}(\chi^I\cdot\mathbf{n})\eta^A d\sigma=\int_{\Gamma_{12}}(\chi^I\cdot\mathbf{n})(\eta^A-\xi^A)d\sigma+\int_{\Gamma_{12}}(\chi^I\cdot\mathbf{n})\xi^A d\sigma,$$

and we consider first the last term. Let $\overline{\xi^A}=\frac{1}{|\Gamma_{12}|}\int_{\Gamma_{12}}\xi^A d\sigma$. The approximation property (4.6) of Π_h implies that

$$\int_{\Gamma_{12}}(\chi^I\cdot\mathbf{n})\xi^A d\sigma=\int_{\Gamma_{12}}(\chi^I\cdot\mathbf{n})(\xi^A-\overline{\xi^A})d\sigma.$$

As $\xi^A-\overline{\xi^A}$ has mean-value zero on Γ_{12} and $\overline{\xi^A}$ is a constant, the trace Theorem 4.3 yields:

$$\left|\int_{\Gamma_{12}}(\chi^I\cdot\mathbf{n})\xi^A d\sigma\right|\le C\|\chi^I\cdot\mathbf{n}\|_{0,\Gamma_{12}}\left(\|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|^2_{0,\Omega_1}+\sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\|[\xi^A]\|^2_{0,\gamma}\right)^{\frac{1}{2}}.$$

As far as the first term is concerned, we write

$$\left|\int_{\Gamma_{12}}(\chi^I\cdot\mathbf{n})(\eta^A-\xi^A)d\sigma\right|\le\left(\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\|\eta^A-\xi^A\|^2_{0,\,\tau}\right)^{\frac{1}{2}}\left(\sum_{\tau\in\Gamma_H}\frac{H_\tau}{\sigma_\tau}\|\chi^I\cdot\mathbf{n}\|^2_{0,\,\tau}\right)^{\frac{1}{2}}.$$

Collecting these inequalities and using the approximation properties of Π_h, we derive:

$$|T_8|\le\epsilon\sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma\left[\xi^A\right]^2d\sigma+\epsilon\|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|^2_{\Omega_1}+\epsilon\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\left(\xi^A\big|_{\Omega_1}-\eta^A\right)^2d\sigma+Ch^{2\nu}.$$

Term T_9 can be bounded using the argument for $T_{6,1}$ in Theorem 3.3:

$$|T_9|\le\epsilon\|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|^2_{0,\Omega_1}+\epsilon\sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\|[p_h]\|^2_{0,\gamma}+\epsilon\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\left\|\xi^A\big|_{\Omega_1}-\eta^A\right\|^2_{0,\tau}+C(h^{2\mu-2}+H^{2\bar\mu-1}).$$

The estimate of the last term is the same as for T_7 in Theorem 3.3:

$$|T_{10}|\le\epsilon\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\left\|\eta^A-\xi^A|_{\Omega_1}\right\|^2_{0,\tau}+Ch^{2\mu-3}H.$$

Finally, we combine all terms to conclude

$$\|\mathbf{K}^{\frac{1}{2}}\nabla\xi^A\|_{0,\Omega_1}^2 + \sum_{\gamma\in\Gamma_h(\Omega_1)} \frac{\sigma_\gamma}{h_\gamma}\left\|\left[\xi^A\right]\right\|_{0,\gamma}^2 + \left\|\mathbf{K}^{-\frac{1}{2}}\chi^A\right\|_{0,\Omega_2}^2 + \sum_{\tau\in\Gamma_H} \frac{\sigma_\tau}{H_\tau}\left\|\xi^A\big|_{\Omega_1} - \eta^A\right\|_{0,\tau}^2 \le C\left(h^{2\mu-2}\left(\frac{H}{h}\right) + H^{2\bar{\mu}-1} + h^{2\nu}\right).$$

An application of the triangle inequality completes the proof. □

Remark 4.1. The difficulty in coupling OBB-DG with MFE in the above proof lies in the estimate of term T_8. The term is bounded using the special trace inequality from Theorem 4.3, which involves penalized jump terms on interior edges (faces). An alternative approach to handle T_8 is to construct a special MFE interpolant $\tilde{\Pi}\mathbf{u}$ satisfying

$$\forall\mu_H\in\Lambda_H\,,\ \int_{\Gamma_{12}}(\mathbf{u}-\tilde{\Pi}\mathbf{u})\cdot\mathbf{n}\mu_H\,d\sigma = 0.$$

This can be done assuming a specific relation between the mortar grid and the MFE grid. Due to lack of space, we do not pursue this approach further here.

Remark 4.2. The cases $\bar{s}_{\text{form}} = 0$ or 1 can be treated as in Remark 3.1. It can be shown for the modified DG-MFE methods that, under the assumptions of Theorem 4.4, there exists a constant C, independent of h and H, such that

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(p_h-p)\|_{\Omega_1} + \sqrt{\sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\|[p_h]\|_{0,\gamma}^2} + \left\|\mathbf{K}^{-\frac{1}{2}}(\mathbf{u}_h-\mathbf{u})\right\|_{0,\Omega_2} + \sqrt{\sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\left\|p_h|_{\Omega_1}-\lambda_H\right\|_{0,\gamma}^2} \le C\left(h^{\mu-1} + H^{\bar{\mu}-\frac{1}{2}}\left(\frac{H}{h}\right)^{\frac{1}{2}} + h^{\nu}\right),$$

$$\|p_h-p\|_{0,\Omega_2} \le C\left(h^{\mu-1} + H^{\bar{\mu}-\frac{1}{2}}\left(\frac{H}{h}\right)^{\frac{1}{2}} + h^{\nu} + h^{\bar{\nu}}\right),$$

where $\mu = \min(r+1,s)$, $\bar{\mu} = \min\left(\bar{r}+1, s-\frac{1}{2}\right)$, $\nu = \min\left(m+1, s-\frac{3}{2}\right)$, $\bar{\nu} = \min(l+1,s)$.

4.4.1. *Convergence for the MFE pressure.* To estimate the error on the pressure computed by the MFE method in Ω_2, we start again with the error equation (4.26). By virtue of the inf-sup condition in Lemma 4.1, we can choose in (4.26) $\mathbf{v}_h \in \mathbf{V}_{h,0}(\Omega_2)$ such that:

$$\int_{\Omega_2}(p_h-\hat{p})\nabla\cdot\mathbf{v}_h\,dx = \|p_h-\hat{p}\|_{0,\Omega_2}^2,$$

$$\|\mathbf{v}_h\|_{H(\text{div};\Omega_2)} + \|\mathbf{v}_h\cdot\mathbf{n}\|_{0,\Gamma_{12}} \le C\|p_h-\hat{p}\|_{0,\Omega_2}.$$

Therefore,

$$\|p_h-\hat{p}\|_{0,\Omega_2} \le C\left(\|\mathbf{K}^{-\frac{1}{2}}(\mathbf{u}_h-\mathbf{u})\|_{0,\Omega_2} + \|\lambda_H-p\|_{0,\Gamma_{12}}\right).$$

The first term on the right is bounded in Theorem 4.4. For the second term on the right we have

$$\|\lambda_H-p\|_{0,\Gamma_{12}} \le \|\lambda_H-p_h|_{\Omega_1}\|_{0,\Gamma_{12}} + \|p_h|_{\Omega_1}-p\|_{0,\Gamma_{12}}.$$

The first term on right above is bounded in Theorem 4.4. Finally, choosing the undetermined coefficient in p_h such that $\int_{\Gamma_{12}}(p-p_h|_{\Omega_1})\,d\sigma = 0$, and applying Theorem 4.3, the last term above can be controlled by terms bounded in Theorem 4.4.

Thus, using a triangle inequality and (4.23), we derive the following theorem.

Theorem 4.5. *Let the assumptions of Theorem 4.4 hold and let* $\int_{\Gamma_{12}}(p-p_h)\,d\sigma=0$. *Then there exists a constant* C, *independent of* h *and* H, *such that*

$$\|p_h-p\|_{0,\Omega_2}\le C\left(h^{\mu-1}\left(\frac{H}{h}\right)^{\frac{1}{2}}+H^{\bar{\mu}-\frac{1}{2}}+h^{\nu}+h^{\bar{\nu}}\right),$$

where $\mu=\min\left(r+1,s\right)$, $\bar{\mu}=\min\left(\bar{r}+1,s-\frac{1}{2}\right)$, $\nu=\min\left(m+1,s-\frac{3}{2}\right)$, $\bar{\nu}=\min\left(l+1,s\right)$.

Remark 4.3. When Ω is decomposed into several subdomains, then the last step in the proof of Theorem 4.5 involves the estimate of $p-p_h$ over the union of several interfaces, say $\Gamma=\cup_{1\le i\le \ell}\Gamma_i$. The statement of Theorem 4.3 can be extended to this case. Therefore, it suffices to choose the undetermined constant in p_h such that $\int_\Gamma(p_h-p)d\sigma=0$.

4.5. **The interface operator and the reduced problem.** Recall the definition of $l_i(\cdot)$ and $b_i(\cdot,\cdot)$ in (3.33) and (3.34), respectively. We now define a bilinear form $d_H : L^2(\Gamma_{12})\times L^2(\Gamma_{12})\mapsto\mathbb{R}$ for $\lambda,\mu\in L^2(\Gamma_{12})$ by $d_H(\lambda,\mu):=\sum_{i=1}^2 d_{H,i}(\lambda,\mu)$,

$$\begin{aligned}
d_{H,1}(\lambda,\mu) &:= \int_{\Gamma_{12}}\mathbf{K}\,\nabla p_h^*(\lambda)|_{\Omega_1}\cdot\mathbf{n}_{\partial\Omega_1}\mu d\sigma-\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau(p_h^*(\lambda)|_{\Omega_1}-\lambda)\mu d\sigma,\\
d_{H,2}(\lambda,\mu) &:= -\int_{\Gamma_{12}}\mathbf{u}_h^*(\lambda)|_{\Omega_2}\cdot\mathbf{n}_{\partial\Omega_2}\mu d\sigma,
\end{aligned}$$

where $p_h^*(\lambda)\in X_h(\Omega_1)$ is the solution of

$$B_1\left(p_h^*(\lambda),q_h\right) = b_1\left(\lambda,q_h\right),\quad \forall q_h\in X_h(\Omega_1), \tag{4.32}$$

and $\mathbf{u}_h^*(\lambda)$ is the first component of the solution $(\mathbf{u}_h^*(\lambda),p_h^*(\lambda))\in\mathbf{V}_{h,0}(\Omega_2)\times W_h(\Omega_2)$ of

$$\int_{\Omega_2}\mathbf{K}^{-1}\mathbf{u}_h^*(\lambda)\cdot\mathbf{v}_hdx=\int_{\Omega_2}p_h^*(\lambda)\nabla\cdot\mathbf{v}_hdx-\int_{\Gamma_{12}}\mathbf{v}_h\cdot\mathbf{n}_{\partial\Omega_2}\lambda d\sigma,\ \ \forall\mathbf{v}_h\in\mathbf{V}_{h,0}(\Omega_2),$$

$$\int_{\Omega_2}\nabla\cdot\mathbf{u}_h^*(\lambda)q_hdx=0,\quad\forall q_h\in W_h(\Omega_2). \tag{4.33}$$

Define a linear functional $g_H : L^2(\Gamma_{12})\mapsto\mathbb{R}$ by $g_H(\mu):=\sum_{i=1}^2 g_{H,i}(\mu)$,

$$\begin{aligned}
g_{H,1}(\mu) &:= -\int_{\Gamma_{12}}\mathbf{K}\,\nabla\bar{p}_h|_{\Omega_1}\cdot\mathbf{n}_{\partial\Omega_1}\mu d\sigma+\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\bar{p}_h|_{\Omega_1}\,\mu d\sigma,\\
g_{H,2}(\mu) &:= \int_{\Gamma_{12}}\bar{\mathbf{u}}_h|_{\Omega_2}\cdot\mathbf{n}_{\partial\Omega_2}\mu d\sigma,
\end{aligned}$$

where $\bar{p}_h\in X_h(\Omega_1)$ solves

$$B_1\left(\bar{p}_h,q_h\right) = l_1\left(q_h\right),\quad\forall q_h\in W_h(\Omega_1),$$

and $\bar{\mathbf{u}}_h$ is the first component of the solution $(\bar{\mathbf{u}}_h,\bar{p}_h)\in(\mathbf{V}_{h,0}(\Omega_2)+\mathbf{u}_{h,g})\times W_h(\Omega_2)$ of

$$\begin{aligned}
\int_{\Omega_2}\mathbf{K}^{-1}\bar{\mathbf{u}}_h\cdot\mathbf{v}_hdx &= \int_{\Omega_2}\bar{p}_h\nabla\cdot\mathbf{v}_hdx,\quad\forall\mathbf{v}_h\in\mathbf{V}_{h,0}(\Omega_2)\\
\int_{\Omega_2}\nabla\cdot\bar{\mathbf{u}}_h\,q_hdx &= \int_{\Omega_2}f\,q_h\,dx,\quad\forall q_h\in W_h(\Omega_2).
\end{aligned}$$

It is easy to see that the solution $(p_h,\mathbf{u}_h,\lambda_H)$ of the DG-MFE scheme (4.12)-(4.15) satisfies

$$d_H\left(\lambda_H,\mu_H\right)=g_H\left(\mu_H\right),\quad\mu_H\in\lambda_H,$$

with

$$p_h=p_h^*(\lambda_H)+\bar{p}_h,\quad\text{on }\Omega_1\text{ and }\Omega_2;\qquad\mathbf{u}_h=\mathbf{u}_h^*(\lambda_H)+\bar{\mathbf{u}}_h,\quad\text{on}\Omega_2.$$

We now analyze the properties of the bilinear form $d_H(\cdot,\cdot)$ for the various DG-MFE schemes. Taking $q_h = p_h^*(\mu)$ in (4.32) and $\mathbf{v}_h = \mathbf{u}_h^*(\mu)$ in (4.33) for some $\mu \in L^2(\Gamma_{12})$, we obtain

$$B_1\left(p_h^*(\lambda), p_h^*(\mu)\right) = b_1\left(\lambda, p_h^*(\mu)\right) \tag{4.34}$$

and

$$\int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_h^*(\lambda)\cdot\mathbf{u}_h^*(\mu)\,dx = -\int_{\Gamma_{12}} \mathbf{u}_h^*(\mu)\cdot\mathbf{n}_{\partial\Omega_2}\lambda\,d\sigma. \tag{4.35}$$

Using (3.34), (4.34) and (4.35), we have the following representation of $d_H(\cdot,\cdot)$:

$$\begin{aligned} d_H(\lambda,\mu) &= b_1(\mu, p_h^*(\lambda)) - 2\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau p_h^*(\lambda)|_{\Omega_1}\mu\,d\sigma \\ &\quad + \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \lambda\,\mu\,d\sigma - \int_{\Gamma_{12}} \mathbf{u}_h^*(\lambda)|_{\Omega_2}\cdot\mathbf{n}_{\partial\Omega_2}\mu\,d\sigma \\ &= B_1(p_h^*(\mu), p_h^*(\lambda)) - 2\sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau p_h^*(\lambda)|_{\Omega_1}\mu\,d\sigma \\ &\quad + \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \lambda\,\mu\,d\sigma + \int_{\Omega_2}\mathbf{K}^{-1}\mathbf{u}_h^*(\mu)\mathbf{u}_h^*(\lambda)\,dx. \end{aligned}$$

By an argument similar to that used in proving (3.40), using the above representation and (3.1), we obtain

$$\begin{aligned} d_H\left(\lambda,\lambda\right) &\geq \frac{3}{4}|\!|\!|\mathbf{K}^{\frac{1}{2}}\nabla p_h^*(\lambda)|\!|\!|^2_{\Omega_1} + \frac{3}{4}\sum_{\gamma\in\Gamma_h(\Omega_1)}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma [p_h^*(\lambda)]^2\,d\sigma \\ &\quad + \sum_{\tau\in\Gamma_H}\frac{\sigma_\tau}{H_\tau}\int_\tau \left(p_h^*(\lambda)|_{\Omega_1} - \lambda\right)^2 d\sigma + \|\mathbf{K}^{-\frac{1}{2}}\mathbf{u}_h^*(\lambda)\|^2_{0,\Omega_2}, \end{aligned}$$

implying that $d_H(\cdot,\cdot)$ is positive semi-definite on $L^2(\Gamma_{12})$. The argument in the solvability Theorem 4.2 implies that the kernel of $d_H(\cdot,\cdot)$ consists of the constant functions. We have obtained the following result.

Theorem 4.6. *Let the assumptions of Theorem 4.2 hold. For all four versions of coupled DG-MFE methods, the interface bilinear form $d_H(\cdot,\cdot)$ is non-symmetric and positive semi-definite on $L^2(\Gamma_{12})$, with the kernel consisting of the constant functions.*

Remark 4.4. In the case of $\bar{s}_{\text{form}} = 0$ or 1, coercivity of $d_H(\cdot,\cdot)$ can be shown for all modified DG-MFE schemes introduced in Remark 4.2, using inequality (3.32). It is easy to see that $d_H(\cdot,\cdot)$ is symmetric for SIPG with $\bar{s}_{\text{form}} = 1$.

5. Discussion and conclusions

We have developed a multiscale formulation for coupling DG with DG and DG with MFE using mortar spaces. The method is based on imposing weak continuity of flux and pressure via a Robin-type matching condition with penalized pressure jump. Although the formulations described in this paper are for two subdomains, the results can be extended to geometrically nonconforming domain decompositions with finite number of subdomains.

Our mortar formulation can be viewed as a two level domain decomposition solver via reduction to an interface problem. By choosing the special case of continuous approximating functions in the subdomains, this approach allows the coupling of continuous Galerkin (CG) with DG, CG with MFE, and CG with CG. The latter represents a new mortar domain decomposition algorithm for CG.

Appendix A. Mortar compatibility conditions

Here we define the mortar compatibility conditions needed for solvability of the methods with $\sigma_\tau = 0$.

Definition. (Mortar compatibility conditions) We say that a DG space $X_h(\Omega_i)$ is compatible with a mortar space Λ_H if, for any $\mu_H \in \Lambda_H$,

$$\int_{\Gamma_{12}} \mathbf{K}\nabla q_h \cdot \mathbf{n}\, \mu_H d\sigma = 0, \quad \forall q_h \in X_h(\Omega_i) \ \Rightarrow \mu_H = 0. \tag{A.1}$$

We say that the MFE space $\mathbf{V}_{h,0}(\Omega_2)$ is compatible with the mortar space Λ_H if, for any $\mu_H \in \Lambda_H$,

$$\int_{\Gamma_{12}} \mathbf{v}_h \cdot \mathbf{n}\, \mu_H d\sigma = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_{h,0}(\Omega_2) \ \Rightarrow \mu_H = 0. \tag{A.2}$$

Note that (A.1) is imposed only if $\sigma_\tau = 0$. For DG-MFE methods, either (A.1) or (A.2) is needed. For matching meshes, one can choose Γ_H to be the trace of the subdomain grids. In this case a sufficient condition for (A.1) is $r \geq \bar{r} + 1$, and a sufficient condition for (A.2) is $m \geq \bar{r}$. For nonmatching meshes, if a compatibility condition is needed, it is imposed only on one of the subdomains, allowing flexibility for the mesh and the finite element space in the other subdomain.

The compatibility condition (A.2) limits the richness of the mortar space. It has been studied in [47, 4, 33] and it has been shown to hold for very general nonmatching configurations of mortar and subdomain grids and spaces.

Below we give some examples of spaces on nonmatching meshes satisfying the compatibility condition (A.1) for $d = 2$. For simplicity we assume that $\mathbf{K}$ is a constant tensor in each element.

Proposition A.1. *Let $\mathbf{K}$ be constant in each element. Assume that $d = 2$, $r = 2$, and that each element of Γ_H contains at least two element faces from $\mathcal{E}_h(\Omega_1) \cap \Gamma_{12}$. Then the DG space $X_h(\Omega_1)$ and the mortar space Λ_H with $\bar{r} = 2$ or $\bar{r} = 3$ satisfy* (A.1).

Proof. Consider a mortar element $\tau \in \Gamma_H$ and assume that $\tau \supset \gamma_1 \cup \gamma_2$, where γ_1 and γ_2 are the edges of two distinct elements $E_1 \subset \mathcal{E}_h(\Omega_1)$ and $E_2 \subset \mathcal{E}_h(\Omega_1)$. Since $\mathbf{K}$ is non-singular (because $\mathbf{K}$ is positive definite), $\mathbf{Kn}$ is not zero. Noting that $\mathbf{K}\nabla q_h \cdot \mathbf{n} = \nabla q_h \cdot \mathbf{Kn}$ (because $\mathbf{K}$ is symmetric) and q_h ranges over $\mathbb{P}_2(\gamma_i)$, we conclude that $\mathbf{K}\nabla q_h \cdot \mathbf{n}$ ranges over $\mathbb{P}_1(\gamma_i)$.

First let us consider $\bar{r} = 2$ for Λ_H and take $\mu_H \in \mathbb{P}_2(\tau)$. The orthogonality condition in (A.1) yields

$$\int_{\gamma_i} p_1 \mu_H d\sigma = 0\,, \forall p_1 \in \mathbb{P}_1(\gamma_i)\,,\ i = 1, 2.$$

In particular, $\int_{\gamma_1} \mu_H d\sigma = \int_{\gamma_2} \mu_H d\sigma = 0$. Therefore, μ_H has a root inside γ_1, say α_1, and a root inside γ_2, say α_2. Therefore $\mu_H = C(x - \alpha_1)(x - \alpha_2)$. Choosing $p_1 = (x - \alpha_1)$ in γ_1, we have $\int_{\gamma_1} C(x - \alpha_1)^2 (x - \alpha_2) d\sigma = 0$. Since $(x - \alpha_2)$ cannot change sign in γ_1, we must have $C = 0$; consequently $\mu_H = 0$ and (A.1) holds.

We now let $\bar{r} = 3$ and take $\mu_H \in \mathbb{P}_3(\tau)$. Then, as before, we have two distinct roots $\alpha_1 \in \gamma_1$ and $\alpha_2 \in \gamma_2$. Since $\mu_H \in \mathbb{P}_3(\tau)$, we know that μ_H has another real root, say α_3, and then $\mu_H = C(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$. Taking $p_1 = (x - \alpha_1)$ in γ_1, we have $\int_{\gamma_1} C(x - \alpha_1)^2 (x - \alpha_2)(x - \alpha_3) d\sigma = 0$. Since $(x - \alpha_2)$ cannot change sign in γ_1, we must have either $C = 0$ or $\alpha_3 \in \gamma_1$. Taking $p_1 = (x - \alpha_2)$ in γ_2, we conclude similarly that either $C = 0$ or $\alpha_3 \in \gamma_2$. As γ_1 and γ_2 are disjoint, we must have $C = 0$; consequently $\mu_H = 0$ and (A.1) holds. □

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