A congruence problem for polyhedra

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April 18, 2007

Abstract

It is well known that to determine a triangle up to congruence requires 3 measurements: three sides, two sides and the included angle, or one side and two angles. We consider various generalizations of this fact to two and three dimensions. In particular we consider the following question: given a convex polyhedron P, how many measurements are required to determine P up to congruence?

We show that in general the answer is that the number of measurements required is equal to the number of edges of the polyhedron. However, for many polyhedra fewer measurements suffice; in the case of the cube we show that nine carefully chosen measurements are enough.

We also prove a number of analogous results for planar polygons. In particular we describe a variety of quadrilaterals, including all rhombi and all rectangles, that can be determined up to congruence with only four measurements, and we prove the existence of n-gons requiring only n measurements. Finally, we show that one cannot do better: for any ordered set of n distinct points in the plane one needs at least n measurements to determine this set up to congruence.

An appendix by David Allwright shows that the set of twelve face-diagonals of the cube fails to determine the cube up to conjugacy. Allwright gives a classification of all hexahedra with all face-diagonals of equal length.

1 Introduction

We discuss a class of problems about the congruence or similarity of three dimensional polyhedra. The basic format is the following:

Problem 1.1. Given two polyhedra in $\mathbf{R}^3$ which have the same combinatorial structure (e.g. both are hexahedra with four-sided faces), determine whether a given set of measurements is sufficient to ensure that the polyhedra are congruent or similar.

We will make this more specific by specifying what sorts of measurements will be allowed. For example, in much of the paper, allowed measurements will include distances between pairs of vertices, angles between edges, and angles between two intersecting face diagonals (possibly on different faces with a common vertex) or between a face diagonal and an edge. One motivation for these choices is given below.

In two dimensions this was a fundamental question answered by Euclidean geometers, as (we hope) every student who takes geometry in high school learns. If the lengths of the corresponding sides of two triangles are equal, then the triangles are congruent. The same is true if the lengths of two corresponding sides and the included angles of two triangles are equal. The extension to other shapes is not often discussed, but we will have some remarks about the planar problem as well. It is surprising to us that beyond the famous theorem of Cauchy discussed below, we have been unable to find much discussion of these problems in the literature, though we think it is almost certain that they have been considered in the past. We would be appreciative if any reader can point us to relevant results.

Our approach will usually be to look at the problem locally. If the two polyhedra are almost congruent, and agree in a certain set of measurements, are they congruent? At first glance this looks like a basic question in what is known as rigidity theory, but a little thought shows that it is different. In rigidity theory, attention is paid to relative positions of vertices, viewing these as connected by inextensible rods which are hinged at their ends and so can rotate relative to each other, subject to constraints imposed by the overall structure of rods. In our problem there is the additional constraint that in any movement of the vertices, the combinatorial structure of the polyhedron cannot change. In particular, any vertices that were coplanar before the movement must be coplanar after the movement. This feature seems to us to produce an interesting area of study.

Our original motivation for considering this problem came from a very practical question encountered by one of us (SPH). If one attempts to make solid wooden models of interesting polyhedra, using standard woodworking equipment, it is natural to want to check how accurate these models are.[1] As a mathematician one may be attracted first to the Platonic solids, and of these, the simplest to make appears to be the cube. (The regular tetrahedron looks harder, because non-right angles seem harder to cut accurately.)

It is possible to purchase lengths of wood with a square cross section, called "turning squares" because they are mostly used in lathes. To make a cube, all one has to do is use a saw to cut off the correct length piece from a turning square. Of course, one has to do so in a plane perpendicular to the planes of sides of the turning square. It is obvious that there are several degrees of freedom, meaning several ways to go wrong. The piece cut off could be the wrong length, or you could cut at the wrong angle, or perhaps the cross section wasn't square to begin with. So, you start measuring to see how well you have done.

In this measurement, though, it seems reasonable to make some assumptions. The basic one of interest here is that the saw cuts off a planar slice. You also assume that this was true at the sawmill where the turning square was made. So you assume that you have a hexahedron – a polyhedron with six faces, all of which are quadrilaterals. Do you have a cube? At this point you are not asking a question addressed by standard rigidity theory.

One's first impulse may be to measure all of the edges of the hexahedron, with the thought that if these are equal, then it is indeed a cube. This is quickly seen to be false, because the faces could be rhombi. Another intriguing possibility that we considered early on is that measuring the twelve face diagonals might suffice. However, David Allwright showed that again there are non-cubical hexahedra with constant face diagaonals. We include his work as an appendix to the current paper. Clearly some other configuration of measurements, perhaps including angles, is necessary. From the practical point of view, measuring twelve edges is already time consuming, and one hopes to get away with fewer measurements, rather than needing more.

In our experience most people, even most mathematicians, who are presented with this problem do not see an answer immediately. Apparently the cube is harder than it looks, and so one would like a simpler problem to get some clues. The (regular) tetrahedron comes to mind, and so one asks how many measurements are required to establish that a polyhedron with four triangular faces is a tetrahedron.

Now we turn to Cauchy's theorem.

Theorem 1.2 (Cauchy, 1839). *Two convex polyhedra with corresponding congruent and similarly situated faces have equal corresponding dihedral angles.*

If we measure the six edges of our triangular faced object, and find them equal, then we have established congruence of the faces of our object to those of a tetrahedron. Cauchy's theorem tells us that the dihedral angles are the same and this implies the desired congruence.

For a tetrahedron this result is pretty obvious, but for the two other Platonic solids with triangular faces, namely the octahedron and the icosahedron, it is less so. Hence Cauchy's theorem is of practical value to the woodworker, and shows that for these objects, the number of edges is at least an upper bound for the number of measurements necessary to prove congruence. From now on we will let E denote the number of edges of our polyhedron, and while we're at it, let V denote the number of vertices and F the number of faces. We will only consider simply connected polyhedra, so that Euler's formula, $V + F = E + 2$, holds.

It is not hard to give an example showing the necessity of convexity in Cauchy's result, but it is one where the two polyhedra being compared are, in some sense, far apart. What was not easy was to determine if convexity was necessary for rigidity. Can a nonconvex polyhedron with all faces triangular be perturbed smoothly through a family of noncongruent polyhedra while keeping the lengths of all edges constant? The

[1] The usual method of constructing a polyhedron is by folding a paper shell.

answer is yes, as was proved in a famous paper by R. Connelly in 1979.[3] This seems to be the major advance in the field since Cauchy.

But this doesn't help us with a cube. Since the answer is not obvious, we started by considering a square. We can approach this algebraically by assuming that one vertex of the square is at $(0,0)$ in the plane. Without loss of generality we can also take one edge along the x-axis, going from $(0,0)$ to $(x_1,0)$ for some $x_1 > 0$. The remaining vertices are then (x_2,x_3) and $(x_{4,}x_5)$ and this leads to the conclusion that to determine five unknowns, we need five equations, and so five measurements. For example, we could measure the four sides of the square and one vertex angle, or we could measure a diagonal instead of the angle.

We then use this to study the cube. Five measurements show that one face is a square. Only four more are needed to specify an adjacent face, because of the common edge, and the three more for one of the faces adjoining the first two. The requirement that this is a hexahedron then implies that we have determined the cube completely, with twelve measurements. This is a satisfying result because it shows that E measurements suffice for a cube as well as for the triangular faced Platonic solids.

However, the reader may notice that we have not proved the necessity of twelve measurements, only the sufficiency. One of the most surprising developments for us in this work was that in fact, twelve are not necessary. It is possible to determine a cube with nine measurements of distances and face angles. The reason, even more surprisingly, is that only four measurements can determine a square, rather than five as seemed so obvious in the argument above.

We will give the algorithm that determines a square in four measurements in the final section of the paper, which contains a number of remarks about congruence of polygons. For now, we proceed with developing a general method for polyhedra. This method will also handle similarity problems, where the shape of the polyhedron is specified up to a scale change. In determining similarity, only angle measurements are involved. As the reader might expect, in general this can be done with $E-1$ measurements, with one additional length required to get congruence.

2 E measurements suffice

In this section we prove that for a *convex* polyhedron P with E edges, there is a set of E measurements that, at least locally, suffices to determine P up to congruence. To avoid any ambiguity we begin with a precise definition of convex polyhedron.

Definition 2.1. A *convex polyhedron* is a subset P of $\mathbf{R}^3$ which is bounded, does not lie in any plane, and can be expressed as a finite intersection of closed half-spaces—that is, sets of the form $\{\,(x,y,z) \in \mathbf{R}^3 \mid ax+by+cz+d \geq 0\,\}$ with $(a,b,c) \neq (0,0,0)$.

The vertices, edges and faces of a convex polyhedron P can be conveniently specified as intersections of P with suitable closed half-spaces. For example, a *face* of P is an intersection $P \cap H$ of P with some closed half-space H such that $P \cap H$ lies entirely within some plane but is not contained in any line, and edges and vertices can be defined similarly.

The original problem refers to two polyhedra with the same 'combinatorial structure', so we give a notion of 'abstract polyhedron', which isolates the combinatorial data embodied in a convex polyhedron.

Definition 2.2. The underlying *abstract polyhedron* of a convex polyhedron P is the triple $(\mathcal{V}_P, \mathcal{F}_P, \mathcal{I}_P)$, where $\mathcal{V}_P$ is the set of vertices of P, $\mathcal{F}_P$ is the set of faces of P, and $\mathcal{I}_P \subset \mathcal{V}_P \times \mathcal{F}_P$ is the incidence relation between vertices and faces; that is, (v,f) is in $\mathcal{I}_P$ if and only if the vertex v lies on the face f.

Thus to say that two polyhedra P and Q have the same combinatorial structure is to say that their underlying abstract polyhedra are *isomorphic*; that is, there are bijections $\beta_V \colon \mathcal{V}_P \to \mathcal{V}_Q$ and $\beta_F \colon \mathcal{F}_P \to \mathcal{F}_Q$ that respect the incidence relation: (v,f) is in $\mathcal{I}_P$ if and only if $(\beta_V(v), \beta_F(f))$ is in $\mathcal{I}_Q$. Note that we don't record information about the edges; we leave it to the reader to verify that the edge data and incidence relations involving the edges can be recovered from the incidence structure $(\mathcal{V}_P, \mathcal{F}_P, \mathcal{I}_P)$. The cardinality of the set $\mathcal{I}_P$ is twice the number of edges of P, since

$$|\mathcal{I}_P| = \sum_{f\in\mathcal{F}_P} (\text{number of vertices on } f) = \sum_{f\in\mathcal{F}_P} (\text{number of edges on } f)$$

and the latter sum counts each edge of P exactly twice.

For the remainder of this section, we fix a convex polyhedron P and write V, E and F for the number of vertices, edges and faces of P, respectively. Let $\Pi = (\mathcal{V}, \mathcal{F}, \mathcal{I})$ be the underlying abstract polyhedron. We're interested in determining which sets of measurements are sufficient to determine P up to congruence. A natural place to start is with a naive dimension count: how many degrees of freedom does one have in specifying a polyhedron with the same combinatorial structure as P?

Definition 2.3. A *realization* of the abstract polyhedron $\Pi = (\mathcal{V}, \mathcal{F}, \mathcal{I})$ is a pair of functions $(\alpha_\mathcal{V}, \alpha_\mathcal{F})$ where $\alpha_\mathcal{V} \colon \mathcal{V} \to \mathbf{R}^3$ gives a point for each v in $\mathcal{V}$, $\alpha_\mathcal{F} \colon \mathcal{F} \to \{\text{planes in } \mathbf{R}^3\}$ gives a plane for each f in $\mathcal{F}$, and the point $\alpha_\mathcal{V}(v)$ lies on the plane $\alpha_\mathcal{F}(f)$ whenever (v, f) is in $\mathcal{I}$.

Given any convex polyhedron Q together with an isomorphism $\beta \colon (\mathcal{V}_Q, \mathcal{F}_Q, \mathcal{I}_Q) \cong (\mathcal{V}, \mathcal{F}, \mathcal{I})$ of incidence structures we obtain a realization of Π, by mapping each vertex of P to the position of the corresponding (under β) vertex of Q and mapping each face of P to the plane containing the corresponding face of Q. In particular, P itself gives a realization of Π, and when convenient we'll also use the letter P for this realization. Conversely, while not every realization of Π comes from a convex polyhedron in this way, any realization of Π that's *sufficiently close* to P in the natural topology for the space of realizations gives—for example by taking the convex hull of the image of α_V—a convex polyhedron whose underlying abstract polyhedron can be identified with Π. So the number of degrees of freedom is the dimension of the space of realizations of Π in a neighborhood of P.

Now we can count degrees of freedom. There are $3V$ degrees of freedom in specifying $\alpha_\mathcal{V}$ and $3F$ in specifying $\alpha_\mathcal{F}$. So if the $|\mathcal{I}| = 2E$ 'vertex-on-face' conditions are independent in a suitable sense then the space of all realizations of Π should have dimension $3V + 3F - 2E$ or—using Euler's formula—dimension $E + 6$. We must also take the congruence group into account: we have three degrees of freedom available for translations, and a further three for rotations. Thus if we form the quotient of the space of realizations by the action of the congruence group, we expect this quotient to have dimension E. This suggests that E measurements should suffice to pin down P up to congruence.

In the remainder of this section we show how to make the above naive dimension count rigorous, and how to identify specific sets of E measurements that suffice to determine congruence. The main ideas are: first, to use a combinatorial lemma to show that the linearizations of the vertex-on-face conditions are linearly independent at P, allowing us to use the inverse function theorem to show that the space of realizations really does have dimension $E + 6$ near P and to give an infinitesimal criterion for a set of measurements to be sufficient (Theorem 2.7), and second, to use an infinitesimal version of Cauchy's rigidity theorem to identify sufficient sets of measurements.

The various measurements that we're interested in can be thought of as real-valued functions on the space of realizations of Π (defined at least on a neighborhood of P) that are invariant under congruence. We single out one particular type of measurement: given two vertices v and w of P that lie on the same face, the *face measurement* associated to v and w is the function that maps a realization $Q = (\alpha_\mathcal{V}, \alpha_\mathcal{F})$ of Π to the distance from $\alpha_\mathcal{V}(v)$ to $\alpha_\mathcal{V}(w)$. In other words, it corresponds to the measurement of the distance between the vertices of Q corresponding to v and w. We now have the following theorem.

Theorem 2.4. *Let P be a convex polyhedron with underlying abstract polyhedron $(\mathcal{V}, \mathcal{F}, \mathcal{I})$. Then there is a set S of face measurements of P such that (i) S has cardinality E, and (ii) locally near P, the set S completely determines P up to congruence in the following sense: there is a positive real number ε such that for any convex polyhedron Q and isomorphism $\beta \colon (\mathcal{V}, \mathcal{F}, \mathcal{I}) \cong (\mathcal{V}_Q, \mathcal{F}_Q, \mathcal{I}_Q)$ of underlying abstract polyhedra, if*

1. *each vertex v of P is within distance ε of the corresponding vertex $\beta_\mathcal{V}(v)$ of Q, and*

2. $m(Q) = m(P)$ for each measurement m in S,

then Q is congruent to P.

We'll prove this theorem as a corollary of Theorem 2.7 below, which gives conditions for a set of measurements to be sufficient. We first fix some notation. Choose numberings $v_1, \dots, v_V$ and $f_1, \dots, f_F$ of the vertices and faces of Π, and write $(x_i(P), y_i(P), z_i(P))$ for the coordinates of vertex v_i of P. We translate P if necessary to ensure that no plane that contains a face of P passes through the origin. This allows us to give an equation for the plane containing f_j in the form $a_j(P)x + b_j(P)y + c_j(P)z = 1$ for some nonzero triple of real numbers $(a_j(P), b_j(P), c_j(P))$; similarly, for any realization Q of Π that's close enough to P the ith vertex of Q is a triple $(x_i(Q), y_i(Q), z_i(Q)$ and the jth plane of Q can be described by an equation $a_j(Q)x + b_j(Q)y + c_j(Q)z = 1$. Hence the coordinate functions $(x_1, y_1, z_1, x_2, y_2, z_2, \dots, a_1, b_1, c_1, \dots)$ give an embedding into $\mathbf{R}^{3V+3F}$ of some neighborhood of P in the space of realizations of Π.

For every pair (v_i, f_j) in $\mathcal{I}$ a realization Q should satisfy the 'vertex-on-face' condition

$$a_j(Q)x_i(Q) + b_j(Q)y_i(Q) + c_j(Q)z_i(Q) = 1.$$

Let $\phi_{i,j}$ be the function from $\mathbf{R}^{3V+3F}$ to $\mathbf{R}$ defined by

$$\phi_{i,j}(x_1, y_1, z_1, \dots, a_1, b_1, c_1, \dots) = a_j x_i + b_j y_i + c_j z_i - 1,$$

and let $\phi\colon \mathbf{R}^{3V+3F} \to \mathbf{R}^{2E}$ be the vector-valued function whose components are the $\phi_{i,j}$ as (v_i, f_j) runs over all elements of $\mathcal{I}$ (in some fixed order). Then a vector in $\mathbf{R}^{3V+3F}$ gives a realization of Π if and only if it maps to the zero vector under ϕ.

We now show that the functions $\phi_{i,j}$ are independent in a neighborhood of P. Write $D\phi(P)$ for the derivative of ϕ at P; as usual, we regard $D\phi(P)$ as a $2E$-by-$(3V + 3F)$ matrix.

Lemma 2.5. *The derivative $D\phi(P)$ has rank $2E$.*

In more abstract terms, this lemma implies that the space of all realizations of Π is, in a neighborhood of P, a smooth manifold of dimension $3V + 3F - 2E = E + 6$.

Proof. We prove that there are no nontrivial linear relations on the $2E$ rows of $D\phi(P)$. To illustrate the argument, suppose that the vertex v_1 lies on the first three faces and no others. Placing the rows corresponding to $\phi_{1,1}$, $\phi_{1,2}$ and $\phi_{1,3}$ first, and writing simply x_1 for $x_1(P)$ and similarly for the other coordinates, the matrix $D\phi(P)$ has the following structure.

$$\left(\begin{array}{ccc|c||ccccccccc|c}
a_1 & b_1 & c_1 & 0\dots0 & x_1 & y_1 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\dots0 \\
a_2 & b_2 & c_2 & 0\dots0 & 0 & 0 & 0 & x_2 & y_2 & z_2 & 0 & 0 & 0 & 0\dots0 \\
a_3 & b_3 & c_3 & 0\dots0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & z_3 & 0\dots0 \\
\hline
0 & 0 & 0 & & & & & & & & & & & \\
\vdots & \vdots & \vdots & * & & & & & * & & & & & * \\
0 & 0 & 0 & & & & & & & & & & &
\end{array}\right)$$

Here the vertical double bar separates the derivatives for the vertex coordinates from those for the face coordinates. Since the faces f_1, f_2 and f_3 cannot contain a common line, the 3-by-3 submatrix in the top left corner is nonsingular. Since v_1 lies on no other faces, all other entries in the first three columns are zero. Thus any nontrivial linear relation of the rows cannot involve the first three rows. So $D\phi(P)$ has full rank if and only if the matrix obtained from $D\phi(P)$ by deleting the first three rows has full rank—that is, rank $2E - 3$.

Extrapolating from the above, given any vertex that lies on exactly three faces, the three rows corresponding to that vertex may be removed from the matrix $D\phi(P)$, and the new matrix has full rank if and only if $D\phi(P)$ does. The dual statement is also true: by exchanging the roles of vertex and face we see that for any triangular face f we may remove the three rows corresponding to f from $D\phi(P)$ and again the resulting matrix has full rank if and only if the $D\phi(P)$ does. Applying this idea inductively, if *every* vertex of P lies on exactly three faces (as in for example the regular tetrahedron, cube or dodecahedron), or dually if *every* face of P is triangular (as in for example the tetrahedron, octahedron or icosahedron) then the lemma is proved.

For the general case, we define a bipartite graph Γ as follows: Γ has one node for each vertex of Π, and one node for each face of Π. Whenever a vertex v of Π lies on a face f of Π (that is, whenever (v, f) is in $\mathcal{I}$) we introduce an edge of Γ connecting the nodes corresponding to v and f. Thus the edges of Γ are in one-to-one correspondence with the rows of $D\phi(P)$. Since P is convex, the graph Γ is planar; indeed, by choosing a point on each face of P, one can draw the graph Γ directly on the surface of P and then project onto the plane. (The graph Γ is known as the *Levi graph* of the incidence structure $\Pi = (\mathcal{V}, \mathcal{F}, \mathcal{I})$.)

Suppose that Γ has a node n of degree 3. Such a node n corresponds either to a triangular face or to a vertex lying on exactly three faces; in either case, the argument above shows that the rows corresponding to the edges incident with n can be removed from the matrix $D\phi(P)$, and $D\phi(P)$ has full rank if and only if the new matrix has full rank. Remove these rows, together with the corresponding edges of Γ. If the new graph again has a node m of degree 3 or smaller, then by the same argument we can remove the edges incident with m and the corresponding rows of $D\phi(P)$. Inductively, suppose that we can reduce Γ to the empty graph by removing nodes and their attached edges, subject only to the rule that at each step we may only remove a node if it has degree 3 or less. Then $D\phi(P)$ must have rank $2E$.

It remains to show that in fact we *can* always reduce Γ to the empty graph in this way. Note that Γ is a planar, bipartite graph and any subgraph of Γ must automatically also be planar and bipartite. Thus it is enough to show that any nonempty, planar, bipartite graph Γ has a vertex of degree at most 3. A standard consequence of Euler's formula (see, for example, Theorem 16 of [2]) states that the number of edges in a nonempty bipartite planar graph of degree $n \geq 3$ is at most $2(n-2)$. If every vertex of Γ had degree at least 4 then the total number of edges would be at least $2n$, contradicting this result. Hence every nonempty planar bipartite graph has a vertex of degree at most 3, as required. This completes the proof of the lemma. $\square$

We now prove a general criterion for a set of measurements to be sufficient. Given the previous lemma, this criterion is essentially a direct consequence of the inverse function theorem.

Definition 2.6. A *measurement* for P is a smooth function m defined on an open neighborhood of P in the space of realizations of Π, such that m is invariant under rotations and translations.

Given any such measurement m, it follows from Lemma 2.5 that we can extend m to a smooth function on a neighborhood of P in $\mathbf{R}^{3V+3F}$. Then the derivative $Dm(P)$ is a row vector of length $3V + 3F$, well-defined up to a linear combination of the rows in $D\phi(P)$.

Theorem 2.7. *Let S be a finite set of measurements for Π near P. Let $\psi\colon \mathbf{R}^{3V+3F} \to \mathbf{R}^{|S|}$ be the vector-valued function obtained by combining the measurements in S, and write $D\psi(P)$ for its derivative at P, an $|S|$-by-$(3V + 3F)$ matrix whose rows are the derivatives $Dm(P)$ for m in S. Then the matrix*

$$D(\phi, \psi)(P) = \begin{pmatrix} D\phi(P) \\ D\psi(P) \end{pmatrix}$$

has rank at most $3E$, and if it has rank exactly $3E$ then the measurements in S are sufficient to determine congruence: that is, for any realization Q of Π, sufficiently close to P, if $m(Q) = m(P)$ for all m in S then Q is congruent to P.

Proof. Let $Q(t)$ be any smooth one-dimensional family of realizations of Π such that $Q(0) = P$ and $Q(t)$ is congruent to P for all t. Since each $Q(t)$ is a valid realization, $\phi(Q(t)) = 0$ for all t. Differentiating and applying the chain rule at $t = 0$ gives the matrix equation $D\phi(P)Q'(0) = 0$ where $Q'(0)$ is thought of as a column vector of length $3V + 3F$. The same argument applies to the map ψ: since $Q(t)$ is congruent to P for all t, $\psi(Q(t)) = \psi(P)$ is constant and $D\psi(P)Q'(0) = 0$.

We apply this argument first to the three families where $Q(t)$ is P translated t units along the x-axis, y-axis, or z-axis respectively, and second when $Q(t)$ is P rotated by t radians around the x-axis, the y-axis and the z-axis. This gives 6 column vectors that are annihilated by both $D\phi(P)$ and $D\psi(P)$. Writing G for the $(3V+3F)$-by-6 matrix obtained from these column vectors, we have the matrix equation

$$\begin{pmatrix} D\phi(P) \\ D\psi(P) \end{pmatrix} G = 0.$$

It's straightforward to compute G directly; we leave it to the reader to check that the transpose of G is

$$\left(\begin{array}{ccccccc||ccccccc} 1 & 0 & 0 & 1 & 0 & 0 & \dots & -a_1^2 & -a_1b_1 & -a_1c_1 & -a_2^2 & -a_2b_2 & -a_2c_2 & \dots \\ 0 & 1 & 0 & 0 & 1 & 0 & \dots & -a_1b_1 & -b_1^2 & -b_1c_1 & -a_2b_2 & -b_2^2 & -b_2c_2 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & \dots & -a_1c_1 & -b_1c_1 & -c_1^2 & -a_2c_2 & -b_2c_2 & -c_2^2 & \dots \\ 0 & -z_1 & y_1 & 0 & -z_2 & y_2 & \dots & 0 & -c_1 & b_1 & 0 & -c_2 & b_2 & \dots \\ z_1 & 0 & -x_1 & z_2 & 0 & -x_2 & \dots & c_1 & 0 & -a_1 & c_2 & 0 & -a_2 & \dots \\ -y_1 & x_1 & 0 & -y_2 & x_2 & 0 & \dots & -b_1 & a_1 & 0 & -b_2 & a_2 & 0 & \dots \end{array}\right)$$

For the final part of this argument, we introduce a notion of normalization on the space of realizations of Π. We'll say that a realization Q of Π is *normalized* if $v_1(Q) = v_1(P)$, the vector from $v_1(Q)$ to $v_2(Q)$ is in the direction of the positive x-axis, and $v_1(Q)$, $v_2(Q)$ and $v_3(Q)$ all lie in a plane parallel to the xy-plane, with $v_3(Q)$ lying in the positive y-direction from $v_1(P)$ and $v_2(P)$. In terms of coordinates we require that $x_1(P) = x_1(Q) < x_2(Q)$, $y_1(P) = y_1(Q) = y_2(Q) < y_3(Q)$ and $z_1(P) = z_1(Q) = z_2(Q) = z_3(Q)$. Clearly every realization Q of Π is congruent to a unique normalized realization, which we'll refer to as the *normalization* of Q. Note that the normalization operation is a continuous map on a neighborhood of P.

Without loss of generality, rotating P around the origin if necessary, we may assume that P itself is normalized.

The condition that a realization Q be normalized gives six more conditions on the coordinates of Q, corresponding to six extra functions $\chi_1, \dots, \chi_6$, which we use to augment the function $(\phi, \psi)\colon \mathbf{R}^{3V+3F} \to \mathbf{R}^{2E+|S|}$ to a function $(\phi, \psi, \chi)\colon \mathbf{R}^{3V+3F} \to \mathbf{R}^{2E+|S|+6}$. These six functions are simply $\chi_1(Q) = x_1(Q) - x_1(P)$, $\chi_2(Q) = y_1(Q) - y_1(P)$, $\chi_3(Q) = z_1(Q) - z_1(P)$, $\chi_4(Q) = y_2(Q) - y_1(P)$, $\chi_5(Q) = z_2(Q) - z_1(P)$ and $\chi_6(Q) = z_3(Q) - z_1(P)$.

Claim 2.8. The 6-by-6 matrix $D\chi(P)G$ is invertible.

Proof. The matrix for G was given earlier; the product $D\chi(P)G$ is easily verified to be

$$\begin{pmatrix} 1 & 0 & 0 & 0 & z_1 & -y_1 \\ 0 & 1 & 0 & -z_1 & 0 & x_1 \\ 0 & 0 & 1 & y_1 & -x_1 & 0 \\ 0 & 1 & 0 & -z_2 & 0 & x_2 \\ 0 & 0 & 1 & y_2 & -x_2 & 0 \\ 0 & 0 & 1 & y_3 & -x_3 & 0 \end{pmatrix}$$

which has nonzero determinant $(y_3 - y_1)(x_2 - x_1)^2$. $\square$

As a corollary, the columns of G are linearly independent, which proves that the matrix in the statement of the theorem has rank at most $3E$. Similarly, the rows of $D\chi(P)$ must be linearly independent, and moreover no nontrivial linear combination of those rows is a linear combination of the rows of $D\psi(P)$. Hence if $D\psi(P)$ has rank exactly $3E$ then the augmented matrix

$$\begin{pmatrix} D\phi(P) \\ D\psi(P) \\ D\chi(P) \end{pmatrix}$$

has rank $3E+6$. Hence the map (ϕ, ψ, χ) has injective derivative at P, and so by the inverse function theorem the map (ϕ, ψ, χ) itself is injective on a neighborhood of P in $\mathbf{R}^{3V+3F}$.

Now suppose that Q is a polyhedron as in the statement of the theorem. Let R be the normalization of Q. Then $\phi(R) = \phi(Q) = \phi(P) = 0$, $\psi(R) = \psi(Q) = \psi(P)$, and $\chi(R) = \chi(P)$. So if Q is sufficiently close to P, then by continuity of the normalization map R is close to P and hence $R = P$ by the inverse function theorem. So Q is congruent to $R = P$ as required. □

Definition 2.9. Call a set S of measurements *sufficient* for P if the conditions of the above theorem apply: that is, the matrix $D(\phi, \psi)(P)$ has rank $3E$.

Corollary 2.10. *Given a sufficient set S of measurements, there's a subset of S of size E that's also sufficient.*

Proof. Since the matrix $D(\phi, \psi)(P)$ has rank $3E$ by assumption, and the $2E$ rows coming from $D\phi(P)$ are all linearly independent by Lemma 2.5, we can find E rows corresponding to measurements in S such that $D\phi(P)$ together with those E rows has rank $3E$. □

The final ingredient that we need for the proof of Theorem 2.4 is the infinitesimal version of Cauchy's rigidity theorem. We phrase it in terms of the notation and definitions above.

Theorem 2.11. *Let P be a convex polyhedron, and suppose that $Q(t)$ is a continuous family of polyhedra specializing to P at $t = 0$. Suppose that the real-valued function $m \circ Q$ is stationary (that is, its derivative vanishes) at $t = 0$ for all face measurements m. Then $Q'(0)$ is in the span of the columns of G above.*

Proof. See Chapter 10, section 1 of [1]. □

Corollary 2.12. *Let S be the collection of all face measurements. Then the matrix $D(\phi, \psi)(P)$ has rank $3E$.*

Proof. The infinitesimal rigidity theorem implies that for any column vector v such that $D(\phi, \psi)(P)v = 0$, v is in the span of the columns of G. Hence the kernel of the map $D(\phi, \psi)(P)$ has dimension exactly 6 and so by the rank-nullity theorem together with Euler's formula the rank of $D(\phi, \psi)(P)$ is $3V + 3F - 6 = 3E$. □

Thus the set of all face measurements is sufficient. Now Theorem 2.4 follows from Corollary 2.12 together with Corollary 2.10.

A computer program has been written in the language Python to implement the approach we have taken to find a sufficient set within a defined class of measurements, such as all the face measurements. This program is attached as Appendix 2 of the online version of this paper, available from the Technical Reports area of the Math Department website at the University of Pittsburgh: http://www.mathematics.pitt.edu/research/technical-reports.php. We hope that the comments within the program are sufficient explanation for those who wish to try it, and we thank Eric Korman for a number of these comments.

As an example we consider a similarity calculation for a dodecahedron. Our allowed measurements will be the set of angles formed by pairs of lines from a vertex to two other points on a face containing that vertex. In Figure 2 we show a set of 29 such angles which our program determines to characterize a dodecahedron up to similarity.

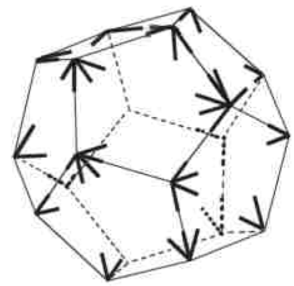

Figure 2. A set of 29 angles which locally determine a dodecahedron up to similarity.

There is no restriction of the method to Platonic solids. Such objects as the truncated icosahedron ('buckyball') present mainly a problem of data entry. Data for a number of other examples can be found in the program listing. Among these examples are several which are non-convex. The program appears to give a result for many of these, but we have not extended the theory beyond the convex case.

3 Congruence of polygons

The theory up to now has been all 'generic', and applies to the 'average' convex polyhedron in $\mathbf{R}^3$. But we will see now that there are many exceptional cases, including even the object which started this study, namely the cube. The results in this section are mostly about polygons in the plane, but it is obvious that they have signficant consequences for polyhedra.

We prove several theorems about the number of measurements necessary to determine an n-gon locally. Most of the results apply to general polygons, or even collections of distinct points in the plane.

Suppose that $A_1 A_2 \cdots A_n$ is a polygon. There are three kinds of simple measurements that are natural to consider:

1. distances $|A_i A_j|$, $i \neq j$
2. angles $\angle A_i A_j A_k$ for i, j, k distinct
3. 'diagonal angles' between $A_i A_j$ and $A_k A_l$.

Other quantities might also be considered, like the distance from A_i to $A_j A_k$, but in practice these would require several measurements.

The next theorem is surely known, but as mentioned before, we have not been able to find references to results of this type. In this theorem, we will assume that $A_1 = (0,0)$, and that $A_2 = (x_2, 0)$ for some $x_2 > 0$. It then becomes obvious that each polygon can be represented by a point in $\mathbf{R}^{2n-3}$, corresponding to the undetermined coordinates $x_2, x_3, y_3, \ldots, x_n, y_n$ with $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$.

Theorem 3.1. *Suppose that $A_1 \cdots A_n$ is a polygon. Then it can be determined locally up to translation and rotation by $2n-3$ measurements of types $1, 2,$ or 3. Moreover, if we view the collection of vertices of a polygon as a point in $\mathbf{R}^{2n-3}$, then the set of polygons requiring fewer than $2n-3$ measurements to determine is of measure zero.*

Proof. The following $2n-3$ measurements suffice: the $n-1$ distances $|A_1A_2|$, $|A_2A_3|, \ldots, |A_{n-1}A_n|$, together with the $n-2$ angles $\angle A_1A_2A_3, \ldots, \angle A_{n-2}A_{n-1}A_n$. For the second statement, observe that any set of $2n-3$ specific measurements is a smooth map from $\mathbf{R}^{2n-3}$ to itself. There is only a finite number of such maps, with measurements chosen from the the types 1,2 3. At a non-critical point, the map has an inverse. The result is then a consequence of Sard's theorem (Chapter 2, Theorem 8 of [4]). □

Observe that instead of using distances and angles, one can use only distances, by subsituting $|A_{i-1}A_{i+1}|$ for $\angle A_{i-1}A_iA_{i+1}$.

It was a surprise to us when we discovered that for some polygons, even convex polygons, fewer than $2n-3$ measurements is enough. This may seem counter-intuitive, especially to those with a background in algebraic geometry. It may appear that a degenerate case of some kind could only result in a given set of conditions becoming insufficient to describe a polygon. For evidence that this is not the case, we observe that while the equation $x^2+y^2=r$ determines a circle if $r>0$, the single equation $x^2+y^2=0$ determines both of the coordinates x and y. The algebraic geometer may wish to keep in mind that we are dealing here with real, not complex, varieties[2].

The biggest surprise was the following:

Example 3.2. Suppose that $ABCD$ is a square, with $|AB|=1$, and $A'B'C'D'$ is any quadrilateral. Then the following **four** conditions guarantee that $A'B'C'D'$ is congruent to $ABCD$:

$$\begin{aligned}|A'B'| &= 1\\ |A'D'| &= 1\\ |A'C'| &= \sqrt{2}\\ \angle B'C'D' &= \frac{\pi}{2}.\end{aligned}$$

Proof. These measurements imply that B' and D' lie on the circle of radius 1 around A', and C' lies on the circle of radius $\sqrt{2}$. With these constraints it is easy to show that the maximum possible value of $\angle B'C'D'$ is $\frac{\pi}{2}$, and this occurs only if $A'B'C'D'$ is a unit square. □

The idea of showing that a particular polygon occurs when some angle or length is maximized within given constraints, and that there is only one such maximum, is at the heart of all of the examples in this section.

Definition 3.3. A polygon is called *fancy* if it can be described locally by fewer than $2n-3$ measurements.

The example above shows that squares are fancy. In fact, they are part of a 3-parameter family of fancy quadrilaterals.

Example 3.4. Suppose that $ABCD$ is a convex quadrilateral with $\angle ABC = \angle CDA = \frac{\pi}{2}$. Then $ABCD$ is fancy. It is determined up to congruence by the following four measurements: $|AB|$, $|AD|$, $|AC|$, and $\angle BCD$.

Proof. Consider an arbitrary quadrilateral $ABCD$ with the same four measurements as the quadrilateral that we are aiming for (note that we are not assuming that the $\angle ABC = \angle CDA = \frac{\pi}{2}$). We can consider the points A and C fixed on the plane. Then the points B and D lie on two fixed circles around A, with radius $|AB|$ and radius $|AD|$. Among all such pairs of points on this circle, on opposite sides of AC, the maximum possible value of $\angle BCD$ is obtained for only one choice of B and D. This is the choice which makes CB and CD tangent to the circle at B and D, and so $\angle ABC$ and $\angle ADC$ are right angles, and the quadrilateral $ABCD$ is congruent to the one we are aiming for.

[2]One of our goals in writing up this paper has been to avoid technical terms from algebraic geometry, but one has just slipped in! We can think of a 'variety' in $\mathbf{R}^m$ as simply the set of points $(t_1, \ldots, t_m)$ satisfying one or more algebraic equations $p(t_1, \ldots, t_m) = 0$, where p is a polynomial. It is a convenient term for the special class of subsets of $\mathbf{R}^m$ which are of interest here.

This family of quadrilaterals includes all rectangles. In a sense it is the biggest possible: one cannot hope for a 4-parameter family requiring just 4 measurements. Another such family is given below. Note that it includes all rhombi except the square.

Example 3.5. Suppose $ABCD$ is a convex quadrilateral, and for given B and D, and given acute angles θ_1, θ_2, choose A and C so that $\angle DAB = \theta_1$, $\angle DCB = \theta_2$, and $|AC|$ is as large as possible. Then this determines a unique quadrilateral, up to congruence, and this quadrilateral has the property that $|AB| = |AD|$ and $|CB| = |CD|$.

□

We leave the proof to the reader. It implies that for the set of quadrilaterals $ABCD$ such that AC is a perpendicular bisector of BD, and the angles $\angle DAB$ and $\angle DCB$ are acute, the measurements $|BD|, |AC|$, $\angle DAB$ and $\angle DCB$ are sufficient to determine $ABCD$.

As David Allwright pointed out to the authors, one can further extend this example to the situation when $\angle DAB + \angle DCB < \frac{\pi}{2}$.

We next address the question of whether there are n-gons which are determined by n measurements for $n > 4$. The answer is yes, for any such n.

Example 3.6. Suppose that $A_1 A_2 \cdots A_n$ is a convex polygon, with

$$\angle A_1 A_2 A_3 = \angle A_1 A_3 A_4 = \cdots = A_1 A_k A_{k+1} = \cdots = A_1 A_{n-1} A_n = \frac{\pi}{2}$$

(Note that many such polygons exist for every n). Then $A_1 A_2 \cdots A_n$ is fancy, moreover the distances $|A_1 A_2|$, $|A_1 A_n|$ and the angles $\angle A_k A_{k+1} A_1$, $2 \leq k \leq n-1$ determine the polygon locally.

Proof. Suppose $\angle A_k A_{k+1} A_1 = \alpha_k$, $2 \leq k \leq (n-1)$. Then because of the Law of Sines for the triangles $\triangle A_1 A_k A_{k+1}$, we obtain the following sequence of inequalities:

$$|A_1 A_n| \leq \frac{|A_1 A_{n-1}|}{\sin \alpha_{n-1}} \leq \frac{|A_1 A_{n-2}|}{\sin \alpha_{n-1} \cdot \sin \alpha_{n-2}} \leq \cdots \leq \frac{|A_1 A_2|}{\sin \alpha_{n-1} \cdot \cdots \cdot \sin \alpha_2}$$

Equality is achieved if and only if all angles $A_1 A_k A_{k+1}$ are right angles, which implies the result.

One can ask whether an even smaller number of measurements might work for some very special polygons. The following theorem shows that in a very strong sense the answer is negative.

Theorem 3.7. *For any ordered set of distinct points $A_1, A_2, \ldots, A_n$ on the plane, one needs at least n distance / angle / diagonal angle measurements to determine this set up to plane isometry.*

Proof. At least one distance measurement is needed, and we can assume that it is $|A_1 A_2|$. We assume that $A_1 A_2$ is fixed, so the positions of the other $n-2$ points determine the set up to isometry. We identify the ordered set of coordinates of these points with a point in $\mathbf{R}^{2(n-2)} = \mathbf{R}^{2n-4}$. The set of ordered sets of all distinct points then corresponds to an open set $U \subset \mathbf{R}^{2n-4}$. Each measurement of type 1, 2, or 3 determines a smooth submanifold $V \subset U$. The following observation is the main idea of the proof.

Lemma 3.8. *Suppose that $x \in V$. Then there exists an affine subspace W of $\mathbf{R}^{2n-4}$, of dimension at least $2n-6$, which contains x and is such that for some open ball B containing x we have $W \cap B \subset V$.*

□

Proof. The proof of this lemma involves several different cases, depending on the kind of measurements used and whether A_1 and/or A_2 are involved. We give some examples, the other cases being similar.

First suppose that the polygon is the unit square, with vertices at $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (1,1)$, and $A_4 = (0,1)$ Suppose that the measurement is the angle $\angle A_2 A_1 A_3$. With A_1 and A_2 fixed, we are free to move A_3 along the line $y = x$, and A_4 arbitrarily. In this case, then, W could be three dimensional, one more than promised by the Lemma.

Second, again with the unit square, suppose that the measurement is the distance from A_3 to A_4. In this case, we can move A_3 and A_4 the same distance along parallel lines. Thus,

$$W = \{(1 + c, 1 + d, c, 1 + d)\}$$

which is the desired two-dimensional affine space.

Finally, suppose that $n \geq 5$ and that the measurement is the angle between A_1A_k and A_lA_m where $1, 2, k, l, m$ are distinct. In this case we can move A_l arbitrarily, giving two free parameters, and we can move A_m so that A_lA_m remains parallel to the original line containing these points. Then the length $|A_lA_m|$ can be changed. This gives a third parameter. Then the length A_1A_k can be changed, giving a fourth, and the remaining $n-5$ points can be moved, giving $2n-10$ more dimensions. In this case, W is $2n-6$ dimensional. We leave other cases to the reader. □

Continuing the proof of Theorem 3.7, we first show that $k \leq n - 3$ measurements is insufficient to determine the points $A_1, \ldots, A_n$. Recall that one measurement, $|A_1A_2|$, was already used. Suppose that a configuration $x \in \mathbf{R}^{2n-4}$ belongs to all the varieties $V_1, \ldots, V_k$. We will prove that x is not an isolated point in $V = \cap_{i=1}^{k} V_i$.

Proof. Denote the affine subspaces obtained from Lemma 3.8 corresponding to V_i and x by W_i. Let $W = W_1 \cap \cdots \cap W_k$. Since each W_i has dimension at least $2n - 6$, its codimension in $\mathbf{R}^{2n-4}$ is less than or equal to 2.[3] Hence

$$\text{codim} W \leq \text{codim } W_1 + \cdots + \text{codim } W_k \leq 2k \leq 2\,(n-3) < 2n-4,$$

so W must have dimension at least 2. Since W is contained in all of the varieties $V_1, .., V_k$, there is a two-dimensional affine subspace of points containing x and satisfying all of the measurements. A neighborhood of x in this subspace is contained in U, showing that x is not isolated.

The case $k = n-2$ is trickier, because the dimensional count does not work in such a simple way. In this case, it is possible that $W = W_1 \cap \cdots \cap W_k = \{x\}$. However, $W' = W_2 \cap \cdots \cap W_k$ must have dimension at least 2.

We now consider the space W_1 in more detail. Denote the measurement defining V_1 by $f_1 : \mathbf{R}^{2n-4} \to \mathbf{R}$. The linearization of f_1 at x is its derivative, $Df_1\,(x)$. For measurements of types 1, 2, or 3 above it is not hard to see that $Df_1\,(x) \neq \mathbf{0}$. If X is the null space of the $1 \times (2n-4)$ matrix $Df_1\,(x)$, then the tangent space of V_1 at x can be defined as the affine subspace $T_x\,(V_1) = x + X$. This has dimension $2n - 5$.

It is possible that $W_1 = T_x\,(V_1)$. This is the case in the first example in the proof of the Lemma above. Then, $\dim W_1 = 2n - 5$. Even if $\dim\,(W') = 2$, we must have $\dim(W) \geq 1$, and since $W \cap U \subset V$, x is not isolated in V.

However in the second example in the Lemma above, (a distance measurement for the square), $\dim W_1 = 2 < \dim T_x\,(V_1) = 3$. Suppose we add as the last two measurement the distance from A_4 to A_2. Then one can easily work out the spaces exactly, and show that $V_1 \cap W'$, when projected onto the first three coordinates (x_3, y_3, x_4), is the intersection of a vertical cylinder with the x_3, x_4 plane. Again, x is not isolated in V.

We have left to consider the general case, where $\dim W_1 = 2n - 6$. (We are assuming that in the lemma we always choose the maximal W.) Since $\dim T_x\,(V_1) = 2n - 5$ and $\dim W' = 2$, $W' \cap T_x\,(V_1)$ must be one dimensional.

We wish to show that x is not isolated in $V_1 \cap W'$. We consider the map $g_1 = f_1|_{W'}$. If $Dg_1 \neq 0$, then the implicit function theorem implies that 0 is not an isolated zero of g_1 in W', proving the theorem.

To show that $Dg_1 \neq 0$ we can use the chain rule. Let $i : W' \to \mathbf{R}^{2n-4}$ be the identity on W'. Then $g_1 = f_1 \circ i$ and so by the chain rule, $Dg_1 = Df_1 \circ i$. If $Dg_1 = 0$, then $(Df_1)\,|_{W'} = 0$, which implies that $W' \subset T_x\,(V_1)$. Since $\dim W' = 2$, this contradicts the earlier assertioin that $W' \cap T_x\,(V_1)$ is one dimensional, completing the proof of the theorem. □

Similar ideas can be used to construct other interesting examples of fancy polygons and polyhedra. The following are worth mentioning.

[3] If V is an affine subspace of $\mathbf{R}^m$, then it is of the form $x + X$, where X is a subspace of $\mathbf{R}^m$. If the orthogonal complement of X in $\mathbf{R}^m$ is Y, then the codimension of V is the dimension of Y.

1. There exist tetrahedra determined by just 5 measurements, instead of the generic 6. It seems unlikely that 4 would work.

2. There exist $5-$vertex convex polyhedra that are characterized by just 5 measurements. Moreover, four of the vertices are on the same plane, but, unlike in the beginning of the paper, we do not have to assume this a priori! To construct such an example, we start with a quadrilateral $ABCD$, as in Example 3.4, with $|AD| < |AB|$. We then add a vertex E outside of the plane $ABCD$, so that $\angle ADE = \angle AEB = \frac{\pi}{2}$. There are obviously many such polyhedra. Now notice that the distances $|AD|$, $|AC|$, and angles $\angle AED$, $\angle ABE$, $\angle BCD$ completely determine the configuration.

3. As pointed out earlier, using four measurements for a square one can determine the cube with just 9 distance / angle measurements. Interestingly, one can also use 10 distance measurements for a cube. If $A_1B_1C_1D_1$ is its base and $A_2B_2C_2D_2$ is a parallel face, then the six distances between A_1, B_1, D_1 and A_2 completely fix the corner. Then $|A_1C_2|$, $|B_2C_2|$, $|D_2C_2|$, and $|C_1C_2|$ determine the cube, because

$$|A_1C_2|^2 \leq |B_2C_2|^2 + |D_2C_2|^2 + |C_1C_2|^2,$$

with the equality only when the segments B_2C_2, D_2C_2, and C_1C_2 are perpendicular to the corresponding faces of the tetrahedron $A_1B_1D_1A_2$.

The last example shows that the problem is not totally trivial even when we only use the distance measurements. It is natural to ask for the smallest number of measurements needed to fix a non-degenerate $n-$gon in the plane. We have the following theorem.

Theorem 3.9. *Suppose $A_1, A_2, \ldots, A_n$ are points in the plane, with no three of them on the same line. Then one needs at least* $\min(2n-3, \frac{3n}{2})$ *distance measurements to determine it up to a plane isometry.*

Proof. Denote by $k(n)$ the smallest number of distance measurements that allows us to fix some non-degenerate configuration as above. We need to show that $k(n) \geq \min(2n-3, \frac{3n}{2})$. We use induction on n. So suppose that the statement is true for all sets of fewer than n points, and that there is a non-degenerate configuration of n points, $A_1, A_2, \ldots, A_n$ that is fixed by k measurements where $k < 2n-3$ and $k < \frac{3n}{2}$. Because $k < \frac{3n}{2}$, there is some point A_i, which is being used in less than 3 measurements. If it is only used in 1 measurement, the configuration is obviously not fixed. So we can assume that it is used in two distance measurements, $|A_iA_1|$ and $|A_iA_2|$. Because A_1, A_2, and A_i are not on the same line, the circle with radius $|A_iA_1|$ around A_1 and the circle with radius $|A_iA_2|$ around A_2 intersect transversally at A_i. Now remove A_i and these two measurements. By the induction assumption, the remaining system of $(n-1)$ points admits a small perturbation. This small perturbation leads to a small perturbation of the original system. Q. E. D.

For $n \leq 7$ the above bound coincides with the upper bound $(2n-3)$ for the minimal number of measurements (see note after Theorem 1). It is likely that for $n \geq 8$, $\frac{3n}{2}$ is the optimal bound, with the regular octagon being the simplest 'distance-fancy' polygon:

Example 3.10. Suppose $A_1A_2A_3 \ldots A_8$ is a regular octagon. Its 8 sides and the diagonals $|A_1A_5|$, $|A_8A_6|$, $|A_2A_4|$, and, finally, $|A_3A_7|$ determine it locally. ($|A_3A_7|$ is, at least locally, the biggest possible with all other distances fixed. In fact, one can show that the distance between the midpoints of A_1A_5 and A_6A_8 has a local maximum when $A_1A_2A_3 \ldots A_8$ is a regular octagon).

There are many open questions in this area, some of which could be relatively easy to answer. For example, we do not know whether either of the results for the cube (9 measurements, or 10 distance measurements) is best possible, and we encourage the reader to either find smaller sets, or provide a proof that these results are optimal.

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A Quadrilateral-faced hexahedrons with all face-diagonals equal

by David Allwright

In this appendix we classify hexahedrons with quadrilateral faces with all face-diagonals of length d. We shall show that such a hexahedron lies in either (a) a 1-parameter family with dihedral symmetry of order 6 and all the faces congruent, or (b) a 2-parameter family with a plane of symmetry and 2 congruent opposite faces joined by 4 symmetric trapezoids. The cube is a special case of both families and in fact has the maximum volume and the maximum surface area. We first construct the two families and then show that the classification is complete.

To construct the first family, let $ABCD$ be a regular tetrahedron with base BCD in the xy-plane, vertex A on the z-axis, and edges of length d. Let l be a line parallel to the base and passing through the z-axis, and choose l such that the rotation of the line segment AB through π about l intersects the line segment CD. This is a single constraint on a 2-parameter family of lines so there is a 1-parameter family of such lines l. Then let A', B', C', D' be the rotations of A, B, C, D through π about l. By choice of l, the points $A'CB'D$ are coplanar and form a convex quadrilateral with both diagonals of length d. But rotation through π about l and rotation through $2\pi/3$ about the z-axis together generate a dihedral group of order 6 permuting these 8 points. Then the images of the quadrilateral $A'CB'D$ under that group are the 6 congruent faces of a hexahedron with all face-diagonals of length d.

To construct the second family, let $ABCD$ be a regular tetrahedron with edge length d, and let P be a plane with AB on one side and CD on the other, and such that the reflection of AB in P intersects CD. This is a single constraint on a 3-parameter family of planes, so there is a 2-parameter family of such planes P. Then let A', B', C', D' be the reflections of A, B, C, D in P. By choice of P, the points $A'CB'D$ form a quadrilateral with both diagonals of length d. Its reflection in P has the same property. The points $AA'C'C$ then have AA' parallel to CC' (both perpendicular to P) so they are coplanar and form a symmetric trapezoid with both diagonals d. The same holds for the other faces that cut P, so again we have a hexahedron with the required property.

To complete the classification we now show than *any* hexahedron with all face-diagonals equal must lie in one of these families. So, suppose we have such a hexahedron, let $\mathbf{a}_0$ be one vertex and let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ be the other ends of the diagonals of the 3 faces meeting at $\mathbf{a}_0$. Then $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are also face-diagonally opposite one another in pairs, so $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are the vertices of a regular tetrahedron, T_a, of edge length d say. In fact, let us choose origin at $\mathbf{a}_0$, scaling $d = \sqrt{2}$ and orient the coordinate axes so that $\mathbf{a}_1 = (0, 1, 1)$, $\mathbf{a}_2 = (1, 0, 1)$ and $\mathbf{a}_3 = (1, 1, 0)$. If we let $\mathbf{b}_i$ be the other vertices of the hexahedron, with $\mathbf{b}_i$ diagonally opposite $\mathbf{a}_i$, then the $\mathbf{b}_i$ also form a tetrahedron, T_b, with edge length d, but of the opposite orientation. So in fact we may write $\mathbf{b}_i = \mathbf{b}_0 - L\mathbf{a}_i$ and L will then be a proper orthogonal matrix. The conditions on the $\mathbf{b}_0$ and L now are that all the faces must be planar, and must not intersect each other except at the edges where they meet. Each face has vertices $\mathbf{a}_i, \mathbf{b}_j, \mathbf{a}_k, \mathbf{b}_l$ where $\{i, j, k, l\}$ are some permutation of $\{0, 1, 2, 3\}$. The condition for face $\mathbf{a}_0, \mathbf{b}_3, \mathbf{a}_1, \mathbf{b}_2$ to be planar is

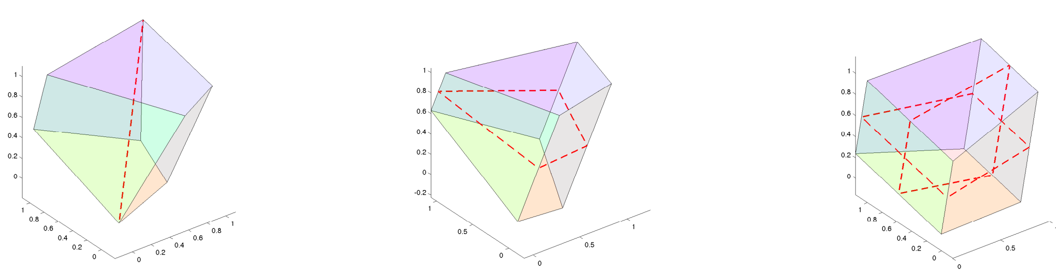
$$A_1: \qquad 0 = [\mathbf{a}_1, \mathbf{b}_0 - L\mathbf{a}_2, \mathbf{b}_0 - L\mathbf{a}_3] = [\mathbf{a}_1, L\mathbf{a}_2, L\mathbf{a}_3] - [\mathbf{b}_0, \mathbf{a}_1, L(\mathbf{a}_2 - \mathbf{a}_3)]. \tag{1}$$

Equally, the condition for the face $\mathbf{b}_0, \mathbf{a}_2, \mathbf{b}_1, \mathbf{a}_3$ to be planar is

$$B_1: \qquad 0 = [\mathbf{b}_0 - \mathbf{b}_1, \mathbf{b}_0 - \mathbf{a}_2, \mathbf{b}_0 - \mathbf{a}_3] = [L\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] - [\mathbf{b}_0, L\mathbf{a}_1, \mathbf{a}_2 - \mathbf{a}_3]. \tag{2}$$

So the 6 planarity conditions are A_1, B_1 and the equations A_2, A_3, B_2, B_3 obtained from them by cyclic permutation of the indices $\{1, 2, 3\}$. Since we now have 6 inhomogeneous linear equations for the 3 components of $\mathbf{b}_0$, consistency of these equations constrains L. It seems computationally easiest to find what these constraints are by representing L in terms of a unit quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, so that $L\mathbf{x} = q\mathbf{x}\overline{q}$. If L is rotation by θ about a unit vector $\mathbf{n}$ then $q = \pm\big(\cos(\tfrac{1}{2}\theta) + \mathbf{n}\sin(\tfrac{1}{2}\theta)\big)$, and we shall choose the representation with $q_0 > 0$. Then a determinant calculation by Mathematica shows that consistency of equations A_1, A_2, B_1, B_2 gives either $q_1 = \pm q_2$ or some $q_i = 0$ for $i \neq 0$. Similarly, by cyclic permutation we deduce that either (a) $q_1 = \pm q_2 = \pm q_3$, or (b) $q_1 = 0$ or $q_2 = 0$ or $q_3 = 0$. Stating this geometrically, in case (a) the axis of the rotation L is parallel to one of the axes of 3-fold rotational symmetry of T_a; while in case (b) it is coplanar with one of the pairs of opposite edges of T_a.

In case (a), if we choose $q_1 = +q_2 = +q_3$ then the solution is parametrised by q_1, $\mathbf{b}_0 = (b, b, b)$ with $b = (1 - 4q_1^2)/(1 - 6q_1^2)$, $\mathbf{b}_1 = \mathbf{b}_0 - L\mathbf{a}_1 = \mathbf{b}_0 - \big(4q_1^2, (q_0 - q_1)^2, (q_0 + q_1)^2\big)$ and $\mathbf{b}_2$, $\mathbf{b}_3$ are obtained from $\mathbf{b}_1$ by cyclic permutation of coordinates. This naturally is the 1-parameter family of solutions constructed earlier with 3-fold rotational symmetry about the $(1, 1, 1)$ direction. The other choices of $\pm$ signs give congruent hexahedrons to this family. The solution remains valid for $-1/\sqrt{12} < q_1 < +1/\sqrt{12}$: at the ends of this range when $q_1 = \pm 1/\sqrt{12}$ the vertex $\mathbf{a}_i$ coincides with $\mathbf{b}_{i\mp 1}$ where those suffices are taken from $\{1, 2, 3\}$ (mod 3), and for $|q_1| > 1/\sqrt{12}$ the polyhedron becomes improper because some pairs of faces intersect. An example (with $q_1 = 0.2$) is shown in Figure 1(a).



(a) (b) (c)

Figure 1: Hexahedrons with all face-diagonals equal.
In case (a) the dashed line is the axis of 3-fold rotational symmetry. In case (b) the dashed lines mark the plane of symmetry. In case (c) the dashed lines mark the 2 planes of symmetry.

In case (b), suppose to be definite we take $q_3 = 0$, so $q_0^2 + q_1^2 + q_2^2 = 1$. Then the coordinates of $\mathbf{b}_0$ are

$$\mathbf{b}_0 = \left(1 + q_0 q_2 + q_1 q_2 - q_2^2 + 2q_1 q_2^2/q_0, q_0^2 - q_0 q_1 + q_1 q_2 + q_2^2 - 2q_1^2 q_2/q_0, q_0^2 + q_0 q_1 - q_0 q_2 + 2q_1 q_2\right). \tag{3}$$

This then forms the 2-parameter family constructed earlier, and it can be checked that the face $\mathbf{a}_0, \mathbf{b}_2, \mathbf{a}_3, \mathbf{b}_1$ is congruent to $\mathbf{b}_0, \mathbf{a}_1, \mathbf{b}_3, \mathbf{a}_2$, and that there is a plane of symmetry P with $\mathbf{b}_{3-i}$ being the reflection of $\mathbf{a}_i$ in P. An example (with $q_1 = 0.2$, $q_2 = 0.25$) is given in Figure 1(b), where the congruent faces are roughly kite-shaped. It remains a valid solution over the region defined by the inequalities

$$|q_1| + |q_2| < \frac{1}{\sqrt{2}}, \qquad \left(1 - (|q_1| + |q_2|)^2\right)\left(1 - 2(|q_1| + |q_2|)^2\right) > 2|q_1 q_2|(|q_1| + |q_2|)^2. \tag{4}$$

On the edge of this region P passes through a vertex and so two of the faces fail to be proper quadrilaterals, and outside this region the polyhedron becomes improper because some pairs of faces intersect.

In case (b) when *two* of the q_i vanish, say $q_2 = q_3 = 0$, then there are *two* planes of symmetry, and the faces are 2 congruent rectangles and 4 congruent symmetric trapezoids, illustrated (for $q_1 = 0.2$) in Figure 1(c).

B Python code

```
from numpy import array, reshape, concatenate, transpose, compress, \
    float64, zeros, ones, eye, cross, inner, outer, sometrue
from numpy.linalg import det, svd
from math import sqrt, acos, pi, cos, sin
from itertools import chain, groupby
from operator import itemgetter

def first(iterable):
    return iterable.next()

def all(conditions):
    for c in conditions:
        if not c:
            return c
    return True

def any(conditions):
    for c in conditions:
        if c:
            return c
    return False

#----------------------------------------
# Numerical linear algebra. We use singular value decomposition to
# determine the nullspace of a matrix.

def norm(v):
    return sqrt(inner(v, v))

def rank(A, eps = 1e-12):
    return sum(svd(A)[1] > 1e-12)

def independent(A, eps = 1e-12):
    """Determine whether the rows of the matrix A are linearly independent"""
    return len(A) == 0 or sum(svd(A)[1] > 1e-12) == len(A)
```

```
def nullspace(A, eps = 1e-12):
    """null space of the given matrix A

    Find a return a basis for the space of vectors v such that Av = 0."""

    _, D, VT = svd(A)

    # the second return value (D) of svd is the list of the entries of
    # the diagonal matrix, in decreasing order; if the original matrix
    # has shape (m, n) then D has length min(m, n). We need to pad it
    # out to length n.

    scale = max(D[0], 1e-3)
    condition_number = min(a for a in D/scale if a > eps)
    if condition_number <= 1e-6:
        # warn that results may be unreliable
        print \
            "Warning: matrix appears ill-conditioned; " + \
            "results may be unreliable." + \
            "Condition number: %s" % condition_number
        print "singular value array: ",D

    return compress(concatenate(
            (D/scale <= eps, [True]*(A.shape[1] - len(D)))), VT, axis=0)

def independent_subset(labelled_rows, A, eps = 1e-12):
    """find an independent subset from a labelled list of vectors

    Given a labelled list of pairs (label, row) where each row is a
    vector of length n, and an initial m-by-n matrix A of rank m,
    return a list of labels corresponding to a maximal subset of the
    given vectors that, together with the rows of A, forms a linearly
    independent set."""

    subset = []
    B = A.copy()
    Bnull = nullspace(B)
    for label, row in labelled_rows:
        if sometrue(abs(inner(Bnull, row)) > eps):
            subset.append(label)
            B = concatenate((B, [row]))
            Bnull = nullspace(B)
    return subset

#----------------------------------------
# Planes, lines and points

# A point in Euclidean 3-space is represented as usual simply as a
# vector of 3 real numbers; it's also occasionally convenient to think
# of it as an element of projective 3-space, identifying [x, y, z]
# with [x : y : z : 1].

# A line in P3 will be represented simply as a pair of points.

# The plane with equation ax + by + cz + d = 0 is represented as a
```

```
# quadruple [a, b, c, d] of real numbers; this representation is
# clearly unique only up to scaling by a nonzero constant. A point
# [x, y, z] lies on a plane [a, b, c, d] if and only if the inner
# product of [x, y, z, 1] with [a, b, c, d] vanishes.

class NoUniqueSolution(Exception):
    pass

def plane_containing_points(points):
    """Find the unique plane containing a given set of points

    Given a list of points in projective 3-space, determine whether they lie
    on a unique plane, and if so return that plane. If not, raise a
    NoUniqueSolution exception."""

    v = nullspace(array(points))
    if len(v) == 1:
        return v[0]
    elif len(v) > 1:
        raise NoUniqueSolution(
            "more than one plane contains the given points")
    elif len(v) < 1:
        raise NoUniqueSolution("points are not coplanar")

def line_containing_points(points):
    v = nullspace(array(points))
    if len(v) == 2:
        return tuple(v)
    elif len(v) > 2:
        raise NoUniqueSolution("more than one line contains the given points")
    elif len(v) < 2:
        raise NoUniqueSolution("points are not collinear")

def point_avoiding_planes(planes):
    """find a point in P3 not lying on any plane in the given list"""

    n = len(planes)+1
    base = []
    for plane_set in concatenate((transpose([eye(3, dtype=float64)]*n, [1, 0, 2]),
                                  -transpose([[range(n)]*3], [1, 2, 0])), 2):
        base.append(first(p for p in plane_set
            if all(independent(base + [p, plane]) for plane in planes)))
    return nullspace(array(base))[0]

#----------------------------------------
# class for representing permutations of finite sets; this is used for
# the two permutations of darts required to specify the combinatorial
# structure of a polyhedron.

class Permutation(object):
    """Permutation of a finite set"""

    def __init__(self, pairlist):
        """initialize from a list of key, value pairs"""
        self.perm_map = dict(pairlist)
        self.domain = set(self.perm_map.keys())
```

```
        if self.domain != set(self.perm_map.values()):
            raise ValueError("map is not a permutation")

    def __eq__(self, other):
        return self.perm_map == other.perm_map

    def __ne__(self, other):
        return not (self == other)

    def __iter__(self):
        return self.perm_map.iteritems()

    def __getitem__(self, key):
        return self.perm_map[key]

    @staticmethod
    def identity(S):
        """identity permutation on the given set"""
        return Permutation((s, s) for s in S)

    def __mul__(self, other):
        if self.domain == other.domain:
            return Permutation((k, self[other[k]])
                               for k in other.perm_map.keys())
        else:
            raise ValueError("different domains in multiplication")

    def __str__(self):
        return "".join(map(str, self.cycles()[0]))

    __repr__ = __str__

    @staticmethod
    def fromcycle(cycles):
        """initialize from a list of cycles"""
        return Permutation((c[i-1],c[i]) for c in cycles for i in range(len(c)))

    def cycles(self):
        """return a list of the cycles of the given permutation, together with
        a function that maps every element of the domain to the cycle
        containing it."""

        p = dict(self.perm_map)
        def cycle_starting_at(p, v):
            cycle = []
            while v in p:
                cycle.append(v)
                v = p.pop(v)
            return tuple(cycle)
        cycle_list = []
        while p:
            cycle_list.append(cycle_starting_at(p, min(p)))

        cycle_containing = dict((e, c) for c in cycle_list for e in c)
        return cycle_list, (lambda x: cycle_containing[x])
```

```
def equivalence_classes(S, pairs):
    """given a set or list S, and a sequence of pairs (s, t) of elements of S,
    compute the equivalence classes of S generated by the relations s ~ t for
    (s, t) in pairs.

    Returns a list of equivalence classes, with each equivalence class
    given as a list of elements of S.

    Example:

    >>> equivalence_classes(range(10), [(1, 4), (2, 6), (2, 3),
                                        (0, 7), (1, 8), (3, 0)])
    [[0, 2, 3, 6, 7], [1, 4, 8], [5], [9]]

    """

    # Algorithm based on that described in 'Numerical Recipes in C,
    # 2nd edition', section 8.6; in turn based on Knuth.

    def root(s):
        while s != parent[s]:
            s = parent[s]
        return s

    parent = dict((s, s) for s in S)
    for s, t in pairs:
        s, t = root(s), root(t)
        if s != t:
            parent[max(s, t)] = min(s, t)

    # by this stage, two elements s and t are in the same equivalence
    # class if and only if they have the same root. So we just
    # have to collect together items with the same root.

    return [map(itemgetter(1), c[1]) for c in
            groupby(sorted((root(s), s) for s in S), itemgetter(0))]

#--------------------------------------------------------

class CombinatorialError(Exception):
    def __init__(self, message, abstract_poly):
        self.message = message
        self.abstract_poly = abstract_poly

    def __str__(self, message, abstract_poly):
        return message

class AbstractPolyhedron(object):
    """Combinatorial data associated to an (oriented) polyhedron.

    The fundamental element of the combinatorial information is the
    'dart'. One dart is associated to each pair (v, f) where v is a
    vertex, f is a face, and v is incident with f. Thus each dart has
    an associated vertex and face, each vertex can be described by the
    set of darts around it, and each face can be described by the set
    of darts on it. Each dart is also naturally associated to an
```

```
edge: in general for a dart d = (v, f) there will be two edges
incident with both v and f; let w be the next vertex from v on f,
moving *counterclockwise* around f, then we associate the edge v-w
to the dart (v, f).

Now the combinatorial data needed are completely described by two
permutations: one that associates to each dart d the next dart
counterclockwise that's associated to the same vertex, and one
that associates to each dart d the next dart counterclockwise
that's associated to the same face.

Attributes of an AbstractPolyhedron instance:

darts : a list of the underlying darts of the polyhedron; these can be
    instances of any class that's equipped with an ordering, for example
    integers or strings.
vertexcycle : a Permutation instance such that for each dart d,
    associated to a vertex v, vertexcycle[d] is the next dart
    around v from d, moving counterclockwise around v.
facecycle: similar to vertexcycle: gives the permutation of darts
    around each face, again counterclockwise.
edgecycle: composition of vertexcycle with facecycle. Sends
    a dart d to the other dart associated to the same edge.

vertices: list of vertices of the polyhedron. Each vertex is
    (currently) represented as a list of darts.
faces: list of faces of the polyhedron
edges: list of edges of the polyhedron
components : list of components of the polyhedron; each component
    is represented as a list of darts. For a connected (nonempty)
    polyhedron, there will be only one component.
euler : the Euler characteristic of the abstract polyhedron

V : function that associates to a given dart d the corresponding
    vertex
F : function associating to each dart d the corresponding face
E : function associating to each dart d the corresponding edge

"""

def __init__(self, vertexcycle, facecycle):
    """given permutations `vertexcycle' and `facecycle' describing
    the vertex and face permutations on a set of darts, validate
    and construct the corresponding abstract (oriented) polyhedron.

    """

    if vertexcycle.domain == facecycle.domain:
        self.darts = vertexcycle.domain
    else:
        raise CombinatorialError("the vertex and face permutations "\
            "should have the same underlying set of darts", self)

    self.edgecycle = vertexcycle * facecycle
    self.facecycle = facecycle
    self.vertexcycle = vertexcycle
```

```
        if self.edgecycle*self.edgecycle != Permutation.identity(self.darts):
            raise CombinatorialError("every cycle of the edge permutation "\
                "should have length <= 2", self)

        self.vertices, self.V = self.vertexcycle.cycles()
        self.faces, self.F = self.facecycle.cycles()
        self.edges, self.E = self.edgecycle.cycles()
        self.components = equivalence_classes(self.darts,
                chain(self.vertexcycle, self.facecycle))

        self.euler = 2 - len(self.darts) - 2*len(self.components) \
            + len(self.vertices) + len(self.edges) + len(self.faces)

        # additional checks that apply to a connected polyhedron,
        # embeddable in R3 with planar faces.
        self.validate()

    # some basic combinatorial operations; others can be defined similarly
    def vertices_on_face(self, f): return map(self.V, f)
    def vertices_on_edge(self, e): return map(self.V, e)
    def faces_around_vertex(self, v): return map(self.F, v)
    def faces_around_edge(self, e): return map(self.F, e)
    def edges_on_face(self, f): return map(self.E, f)
    def edges_around_vertex(self, v): return map(self.E, v)

    def __str__(self):
        return """\
Abstract polyhedron with:
  dart set : %s
  vertex permutation : %s
  face permutation : %s
  edge permutation : %s
""" % (self.darts, self.vertexcycle, self.facecycle, self.edgecycle)

    __repr__ = __str__

    def validate(self):
        # each edge should contain at least two darts; each vertex and face
        # should have at least three.
        bad_edge = any(len(c) < 2 for c in self.edges)
        if bad_edge:
            raise CombinatorialError("edge %s has the wrong length: it "\
                "should involve exactly two darts" % bad_edge, self)

        bad_vertex = any(len(c) < 3 for c in self.vertices)
        if bad_vertex:
            raise CombinatorialError("vertex %s has the wrong length: it "\
                "should involve at least three darts" % bad_vertex, self)

        bad_face = any(len(c) < 3 for c in self.faces)
        if bad_face:
            raise CombinatorialError("face %s has the wrong length: it "\
                "should involve at least three darts" % bad_face, self)

        # verify that the abstract polyhedron is connected
```

```
        if len(self.components) != 1:
            raise CombinatorialError("Disconnected polyhedron. " \
                  " Components: %s" % self.components, self)

class Polyhedron(AbstractPolyhedron):

    # a polyhedron should consist of:
    # (1) an underlying abstract polyhedron
    # (2) an embedding of that abstract polyhedron

    def __init__(self, vertexcycle, facecycle, embedding, allowScaling = False):

        # initialize the combinatorial data
        AbstractPolyhedron.__init__(self, vertexcycle, facecycle)

        # store position information for vertices (in P3)
        vertexeqn = dict(
            (self.V(d), array(pos + [1.]))
            for d, pos in embedding.items())

        # verify that every vertex has had its position given
        # exactly once
        if not (len(vertexeqn) == len(embedding) == len(self.vertices)):
            raise ValueError("missing vertices or repeated vertex "\
                             "in given embedding")

        # find the coordinates of the plane for each face (in P3)
        # this raises an error if the set of given points is not coplanar
        faceeqn = {}
        for f in self.faces:
            vertex_positions = [vertexeqn[v] for v in self.vertices_on_face(f)]
            try:
                faceeqn[f] = plane_containing_points(vertex_positions)
            except NoUniqueSolution:
                raise ValueError("the points of face %s do not lie "
                                 "in a unique plane" % (str(f)))

        # choose a reference point in P3 that doesn't lie on any of the planes
        # and doesn't lie on the plane at infinity
        basept = point_avoiding_planes(faceeqn.values() + [array([0., 0., 0., 1.])])

        # project into R3, shifting so that the reference point goes
        # to (0, 0, 0) Then none of the faces goes through 0, 0, 0, so
        # each face can be represented by a triple a, b, c,
        # corresponding to the equation ax + by + cz + 1 = 0.
        vertexpos = dict((v, p[:3]/p[3] - basept[:3]/basept[3])
                         for v, p in vertexeqn.items())
        facepos = dict((f, e[:3]*basept[3]/inner(e, basept))
                       for f, e in faceeqn.items())

        # now fill in the relationmatrix (= D(phi))
        ambient_dim = 3*(len(self.vertices)+len(self.faces))
        relationmatrix = zeros((len(self.darts), ambient_dim), dtype = float64)
        for i, d in enumerate(self.darts):
            # construct the row corresponding to a vertex--face pair
            v = self.V(d)
```

```python
        f = self.F(d)
        vi = 3*self.vertices.index(v)
        fi = 3*(len(self.vertices)+self.faces.index(f))
        relationmatrix[i, vi:vi+3] = facepos[f]
        relationmatrix[i, fi:fi+3] = vertexpos[v]

    # check to see whether the relation matrix has full rank
    if rank(relationmatrix) != len(relationmatrix):
        print ("Warning: relation matrix has rank %d; "
               "expected rank %d" %
               (rank(relationmatrix), len(relationmatrix)))

    # compute the vectors of the Lie group: shifts, rotations and scalings
    shifts = [eye(3, dtype=float64) for v in self.vertices] + \
             [outer(facepos[f], facepos[f]) for f in self.faces]
    shifts = reshape(transpose(shifts, [1, 0, 2]), (3, ambient_dim))
    rotations = \
        [cross(eye(3, dtype=float64), vertexpos[v]) for v in self.vertices] + \
        [cross(eye(3, dtype=float64), facepos[f]) for f in self.faces]
    rotations = reshape(transpose(rotations, [1, 0, 2]), (3, ambient_dim))
    if allowScaling:
        scalings = [vertexpos[v] for v in self.vertices] + \
                   [−facepos[f] for f in self.faces]
        scalings = reshape(scalings, (1, ambient_dim))
        liealgebra = concatenate((shifts, rotations, scalings))
    else:
        liealgebra = concatenate((shifts, rotations))

    self.euler_characteristic = \
        len(self.faces) + len(self.vertices) − len(self.edges)

    # Polyhedron attributes:

    # darts : list of darts (= vertex−face pairs)
    # vertices : list of darts representing the vertices, one per vertex
    # faces : list of darts representing the faces, one per face
    # liealgebra contains the vectors of the lie algebra
    # vertexcycle[d] : the dart obtained by moving counterclockwise
    # about the vertex associated to d
    # facecycle[d] : the dart obtained by moving counterclockwise
    # about the face associated to d
    # vertexpos[v] : the position of vertex v in R3 (after shifting to
    # make sure that no faces contain 0, 0, 0)
    # facepos[f] : the plane containing face f (after shifting), given
    # by a triple (a, b, c) representing the plane with equation
    # ax + by + cz + 1 = 0.
    # embedding[v] : the original position of vertex v
    # faceeqn[f] : the equation of face f (before shifting), given
    # by a 4−tuple (a, b, c, d) representing the plane with
    # equation ax + by + cz + d = 0.

    self.liealgebra = liealgebra
    self.vertexpos = vertexpos
    self.facepos = facepos
    self.allowScaling = allowScaling
    self.ambient_dim = ambient_dim
```

```
        self.relationmatrix = relationmatrix
        self.vertexeqn = vertexeqn
        self.faceeqn = faceeqn

    def filter_measurements(self, measurementlist):
        measurements_used = independent_subset([(m, m.derivative())
            for m in measurementlist], self.relationmatrix)

        defect = self.ambient_dim - len(self.liealgebra) - \
            len(self.relationmatrix) - len(measurements_used)
        return measurements_used, defect

    def length(self, v1, v2):
        return LengthMeasurement(self, v1, v2)

    def angle(self, v1, v2, v3):
        return AngleMeasurement(self, v1, v2, v3)

    def dihedral(self, f1, f2):
        return DihedralMeasurement(self, f1, f2)

class Measurement(object):
    """Class representing a particular measurement of a
    polyhedron."""

    # we'll subclass this for length, angle and dihedral angle
    # measurements

    # each subclass should implement value(), derivative(), str()

    pass

# Note that the length measurement actually only needs to depend on
# the abstract polyhedron; we need a genuine embedded polyhedron only
# to apply the measurement.

class LengthMeasurement(Measurement):
    """Class representing the measurement of the distance
    between two vertices of a polyhedron."""

    def __init__(self, P, v1, v2):
        if not (v1 in P.vertices and v2 in P.vertices):
            raise ValueError("invalid vertices")

        if P.allowScaling:
            raise ValueError("working up to similarity only; "
                             "length measurements not permitted")

        # P is the polyhedron; v1 and v2 are the relevant vertices
        self.P = P
        self.v1, self.v2 = v1, v2

    def __str__(self):
        return "Length measurement V(%s) -- V(%s)" % (self.v1[0], self.v2[0])

    def value(self):
```

```
        return norm(self.P.vertexpos[self.v1] - self.P.vertexpos[self.v2])

    def derivative(self):
        P = self.P
        p1, p2 = P.vertexpos[self.v1], P.vertexpos[self.v2]
        diff = p1 - p2
        length = norm(diff)

        row = zeros((len(P.vertices) + len(P.faces), 3), dtype=float64)
        row[P.vertices.index(self.v1)] += diff/length
        row[P.vertices.index(self.v2)] -= diff/length
        return reshape(row, (P.ambient_dim,))

class AngleMeasurement(Measurement):
    # cosine of the angle v2-v1-v3
    def __init__(self, P, v1, v2, v3):
        if not all(v in P.vertices for v in [v1, v2, v3]):
            raise ValueError("invalid vertices")
        self.P = P
        self.v1, self.v2, self.v3 = v1, v2, v3

    def __str__(self):
        return "Angle measurement V(%s) -- V(%s) -- V(%s)" % \
            (self.v2[0], self.v1[0], self.v3[0])

    def value(self):
        P = self.P
        (p1, p2, p3) = (P.vertexpos[self.v1],
                        P.vertexpos[self.v2], P.vertexpos[self.v3])
        diff2, diff3 = p2 - p1, p3 - p1
        return inner(diff2, diff3)/norm(diff2)/norm(diff3)

    def derivative(self):
        """vector representing the derivative of the cosine of the angle
        measurement between vectors v2-v1 and v3-v1"""

        P = self.P
        (p1, p2, p3) = (P.vertexpos[self.v1],
                        P.vertexpos[self.v2], P.vertexpos[self.v3])
        diff2, diff3 = p2 - p1, p3 - p1
        a23 = inner(diff2, diff3)
        a22 = inner(diff2, diff2)
        a33 = inner(diff3, diff3)
        M = sqrt(a22 * a33)

        cosangle = a23/M

        dv1 = (diff2*a23/a22 + diff3*a23/a33 - diff2 - diff3)/M
        dv2 = (diff3 - diff2*a23/a22)/M
        dv3 = (diff2 - diff3*a23/a33)/M

        row = zeros((len(P.vertices) + len(P.faces), 3), dtype = float64)
        row[P.vertices.index(self.v1)] += dv1
        row[P.vertices.index(self.v2)] += dv2
        row[P.vertices.index(self.v3)] += dv3
        return reshape(row, (P.ambient_dim,))
```

```
class DihedralMeasurement(Measurement):
    # measurement of the angle between faces f1 and f2

    def __init__(self, P, f1, f2):
        if not all(f in P.faces for f in [f1, f2]):
            raise ValueError("invalid faces")
        self.P = P
        self.f1, self.f2 = f1, f2

    def __str__(self):
        return "Dihedral measurement F(%s) -- F(%s)" % (self.f1[0], self.f2[0])

    def value(self):
        # angle between two faces: the faces have corresponding embeddings
        n1, n2 = self.P.facepos[self.f1], self.P.facepos[self.f2]

        # n1 and n2 are (not necessarily unit) normal vectors to the
        # two faces note that n1 and n2 don't necessarily have
        # corresponding orientations: one of n1 and n2 might point
        # outwards while the other points inwards. We need to fix
        # this. For now we just compute the (cosine of the) angle
        # between n1 and n2.
        return inner(n1, n2)/norm(n1)/norm(n2)

    def derivative(self):
        P = self.P
        n1, n2 = P.facepos[self.f1], P.facepos[self.f2]

        a11 = inner(n1, n1)
        a12 = inner(n1, n2)
        a22 = inner(n2, n2)

        dn1 = (n2 - n1*a12/a11)/sqrt(a11*a22)
        dn2 = (n1 - n2*a12/a22)/sqrt(a11*a22)

        row = zeros((len(P.vertices) + len(P.faces), 3), dtype=float64)
        row[len(P.vertices) + P.faces.index(self.f1)] += dn1
        row[len(P.vertices) + P.faces.index(self.f2)] += dn2
        return reshape(row, (P.ambient_dim,))

#------------------------------------------------
# Output routines

def print_polyhedron(P):
    print "Vertices (number, position, cycle)\n" \
          "------------------------------------------"

    print "".join("%2d " % v[0] +
        "(% 7.4f, % 7.4f, % 7.4f) " %
                tuple(P.vertexeqn[v][:3]/P.vertexeqn[v][3]) +
                str(v) + "\n" for v in P.vertices)

    print "Faces (number, equation, cycle)\n" \
          "--------------------------------------"
    print "".join("%2d " % f[0] +
```

```
        "% 7.4fx + % 7.4fy + % 7.4fz = % 7.4f " % tuple(P.faceeqn[f]) +
        str(f) + "\n" for f in P.faces)

# print results in a meaningful format
def print_results(P, results):
    for r in results:
        print "%s: expected value: % 7.4f" \
            % (r, r.value())

#----------------------------------------
# Platonic solids

tetrahedron = Polyhedron(
    vertexcycle = Permutation.fromcycle([
        [0, 6 ,3], [1, 5, 9], [2, 11, 7], [4, 8, 10]]),
    facecycle = Permutation.fromcycle([range(i, i+3) for i in range(0, 12, 3)]),
    embedding = dict([
        [0, [1., 1., 1.]],
        [1, [1., -1., -1.]],
        [2, [-1., 1., -1.]],
        [4, [-1., -1., 1.]]]))

cube = Polyhedron(
    vertexcycle = Permutation.fromcycle([
        [0, 4, 8], [1, 11, 22], [2, 21, 19], [3, 18, 5],
        [6, 17, 15], [7, 14, 9], [10, 13, 23], [12, 16, 20]]),
    facecycle = Permutation.fromcycle(
        [range(i, i+4) for i in range(0, 24, 4)]),
    embedding = dict([
        [0, [1., 1., 1.]],
        [1, [1., 1., -1.]],
        [2, [1., -1., -1.]],
        [3, [1., -1., 1.]],
        [6, [-1., -1., 1.]],
        [7, [-1., 1., 1.]],
        [10, [-1., 1., -1.]],
        [12, [-1., -1., -1.]]]))

twisted_cube = Polyhedron(
    vertexcycle = Permutation.fromcycle([
        [0, 4, 8], [1, 11, 22], [2, 21, 19], [3, 18, 5],
        [6, 17, 15], [7, 14, 9], [10, 13, 23], [12, 16, 20]]),
    facecycle = Permutation.fromcycle(
        [range(i, i+4) for i in range(0, 24, 4)]),
    embedding = dict([
        [0, [59./51., 451./510., 451./510.]],
        [1, [1., 1., 0.]],
        [2, [13./17., -13./170., -13./170.]],
        [3, [1., 0., 1.]],
        [6, [0., .1, 1.1]],
        [7, [0., 1., 1.]],
        [10, [0., 1.1, 0.1]],
        [12, [0., 0., 0.]]]))

twisted_cube2 = Polyhedron(
    vertexcycle = Permutation.fromcycle([
```

```
            [0, 4, 8], [1, 11, 22], [2, 21, 19], [3, 18, 5],
            [6, 17, 15], [7, 14, 9], [10, 13, 23], [12, 16, 20]]),
        facecycle = Permutation.fromcycle(
            [range(i, i+4) for i in range(0, 24, 4)]),
        embedding = dict([
            [0, [−1.1, −1.1, 1.]],
            [1, [1., 1., −1.]],
            [2, [−1.1, .9, −1.]],
            [3, [1., −1., 1.]],
            [6, [.9, .9, 1.]],
            [7, [−1., 1., 1.]],
            [10, [.9, −1.1, −1.]],
            [12, [−1., −1., −1.]]]))

# rotate 2nd tetrahedron by 45 degrees about z−axis
# transformation: [x, y, z] −> [x cost − y sint, x sint + y cost, z]
theta = 0.3
cost = cos(theta)
sint = sin(theta)
twisted_cube3 = Polyhedron(
        vertexcycle = Permutation.fromcycle([
            [0, 4, 8], [1, 11, 22], [2, 21, 19], [3, 18, 5],
            [6, 17, 15], [7, 14, 9], [10, 13, 23], [12, 16, 20]]),
        facecycle = Permutation.fromcycle(
            [range(i, i+4) for i in range(0, 24, 4)]),
        embedding = dict([
            [0, [1., 1., 1.]],
            [2, [1., −1., −1.]],
            [6, [−1., −1., 1.]],
            [10, [−1., 1., −1.]],
            [1, [cost−sint, sint+cost, −1.]],
            [3, [cost+sint, sint−cost, 1.]],
            [7, [−cost−sint, −sint+cost, 1.]],
            [12,[−cost+sint, −sint−cost,−1.]]]))

octahedron = Polyhedron(
        vertexcycle = Permutation.fromcycle(
            [range(i, i+4) for i in range(0, 24, 4)]),
        facecycle = Permutation.fromcycle([
            [0, 4, 8], [1, 11, 22], [2, 21, 19], [3, 18, 5],
            [6, 17, 15], [7, 14, 9], [10, 13, 23], [12, 16, 20]]),
        embedding = dict([
            [0, [ 1., 0., 0.]],
            [4, [ 0., 0., 1.]],
            [8, [ 0., 1., 0.]],
            [12, [−1., 0., 0.]],
            [16, [ 0., −1., 0.]],
            [20, [ 0., 0., −1.]]]))

t, tr = (sqrt(5)+1.)/2., (sqrt(5)−1.)/2.
dodecahedron = Polyhedron(
        vertexcycle = Permutation.fromcycle([
            [0, 5, 11], [1, 10, 16], [2, 15, 21], [3, 20, 26], [4, 25, 6],
            [7, 29, 58], [8, 57, 54], [9, 53, 12], [13, 52, 49], [14, 48, 17],
            [18, 47, 44], [19, 43, 22], [23, 42, 39], [24, 38, 27], [28, 37, 59],
            [30, 35, 41], [31, 40, 46], [32, 45, 51], [33, 50, 56], [34, 55, 36]]),
```

```
    facecycle = Permutation.fromcycle(
        [range(i, i+5) for i in range(0, 60, 5)]),
    embedding = dict([
        [0, [1., 1., 1.]],
        [1, [t, tr, 0.]],
        [2, [1., 1., −1.]],
        [3, [0., t, −tr]],
        [4, [0., t, tr]],
        [7, [−1., 1., 1.]],
        [8, [−tr, 0., t]],
        [9, [tr, 0., t]],
        [13, [1., −1., 1.]],
        [14, [t, −tr, 0.]],
        [18, [1., −1., −1.]],
        [19, [tr, 0., −t]],
        [23, [−tr, 0., −t]],
        [24, [−1., 1., −1.]],
        [28, [−t, tr, 0.]],
        [30, [−1., −1., −1.]],
        [31, [0., −t, −tr]],
        [32, [0., −t, tr]],
        [33, [−1., −1., 1.]],
        [34, [−t, −tr, 0.]]]),
    allowScaling = True)

icosahedron = Polyhedron(
    vertexcycle = Permutation.fromcycle(
        [range(i, i+5) for i in range(0, 60, 5)]),
    facecycle = Permutation.fromcycle([
        [0, 5, 11], [1, 10, 16], [2, 15, 21], [3, 20, 26], [4, 25, 6],
        [7, 29, 58], [8, 57, 54], [9, 53, 12], [13, 52, 49], [14, 48, 17],
        [18, 47, 44], [19, 43, 22], [23, 42, 39], [24, 38, 27], [28, 37, 59],
        [30, 35, 41], [31, 40, 46], [32, 45, 51], [33, 50, 56], [34, 55, 36]]),
    embedding = dict([
        [0, [1., t, 0.]],
        [5, [0., 1., t]],
        [10, [t, 0., 1.]],
        [15, [t, 0., −1.]],
        [20, [0., 1., −t]],
        [25, [−1., t, 0.]],
        [30, [−1., −t, 0.]],
        [35, [−t, 0., −1.]],
        [40, [0., −1., −t]],
        [45, [1., −t, 0.]],
        [50, [0., −1., t]],
        [55, [−t, 0., 1.]]]))

# a tetrahedron with a notch taken out of the center of one edge. The
# result is a polyhedron with 12 edges, 6 faces and 8 vertices; the
# abstract polyhedron can be realized as a simple polyhedron, but not
# as a convex polyhedron (the corresponding planar graph is not
# 3−connected, so Steinitz' theorem doesn't apply).

notched_tetrahedron = Polyhedron(
    vertexcycle = Permutation.fromcycle([
        [0, 12, 6], [1, 11, 18], [2, 20, 22], [3, 21, 9],
```

```
            [4, 8, 17], [5, 16, 13], [7, 14, 15], [10, 23, 19]]),
        facecycle = Permutation.fromcycle([
            [0, 1, 2, 3, 4, 5], [6, 7, 8, 9, 10, 11],
            [12, 13, 14], [15, 16, 17], [18, 19, 20], [21, 22, 23]]),
        embedding = dict([
            [0, [1., 1., 1.]],
            [1, [0.5, 1., 0.5]],
            [2, [-0.5, 0., 0.5]],
            [3, [-0.5, 1., -0.5]],
            [4, [-1., 1., -1.]],
            [5, [-1., -1., 1.]],
            [7, [1., -1., -1.]],
            [10, [0.5, 0., -0.5]]]))

square_pyramid = Polyhedron(
        vertexcycle = Permutation.fromcycle([
            [0, 3, 6, 9], [1, 11, 13], [2, 12, 4], [5, 15, 7], [8, 14, 10]]),
        facecycle = Permutation.fromcycle([
            [0, 1, 2], [3, 4, 5], [6, 7, 8], [9, 10, 11], [12, 13, 14, 15]]),
        embedding = dict([
            [0, [0., 0., 1.]],
            [1, [1., 1., 0.]],
            [2, [-1., 1., 0.]],
            [5, [-1., -1., 0.]],
            [8, [1., -1., 0.]]]))

# square pyramid with two of the triangular faces equal

degenerate_square_pyramid = Polyhedron(
        vertexcycle = Permutation.fromcycle([
            [0, 3, 6, 9], [1, 11, 13], [2, 12, 4], [5, 15, 7], [8, 14, 10]]),
        facecycle = Permutation.fromcycle([
            [0, 1, 2], [3, 4, 5], [6, 7, 8], [9, 10, 11], [12, 13, 14, 15]]),
        embedding = dict([
            [0, [0., 0., 1.]],
            [1, [1., 1., 0.]],
            [2, [0., 0., 0.]],
            [5, [-1., -1., 0.]],
            [8, [1., -1., 0.]]]))

# torus (embedding as polyhedron with 9 quadrilateral faces)

s3 = sqrt(3.)
hs3 = s3/2.
torus = Polyhedron(
        facecycle = Permutation.fromcycle(
            [range(i, i+4) for i in range(0, 36, 4)]),
        vertexcycle = Permutation.fromcycle([
            [0, 9, 34, 27],
            [1, 26, 31, 4],
            [2, 7, 16, 13],
            [3, 12, 21, 10],
            [5, 30, 35, 8],
            [6, 11, 20, 17],
            [14, 19, 28, 25],
            [15, 24, 33, 22],
```

```
        [18, 23, 32, 29]]),
    embedding = dict([
        [0, [1., 0., 0.]],
        [1, [-.5, hs3, 0.]],
        [2, [-1., s3, 1.]],
        [3, [2., 0., 1.]],
        [5, [-.5, -hs3, 0.]],
        [6, [-1., -s3, 1.]],
        [14, [-1., s3, -1.]],
        [15, [2., 0., -1.]],
        [18, [-1., -s3, -1.]]]))

# szilassi polyhedron; combinatorial information based on the heawood graph

szilassi = AbstractPolyhedron(
    vertexcycle = Permutation.fromcycle(
        [range(i, i+3) for i in range(0, 42, 3)]),
    facecycle = Permutation.fromcycle(
        [[(i + j) % 42 for i in [0, 5, 31, 27, 26, 40]] for j in range(0, 42, 6)]))

#-----------------------------------------
# attempt to find a set of angles that's sufficient for a dodecahedron;
# we can do this using face measurements only

D = dodecahedron
faceangles = []
for f in D.faces:
    vs = D.vertices_on_face(f)
    for i in range(5):
        for j in range(i+1, 5):
            for k in range(j+1, 5):
                faceangles.append(AngleMeasurement(D, vs[i], vs[j], vs[k]))
                faceangles.append(AngleMeasurement(D, vs[j], vs[k], vs[i]))

results, defect = dodecahedron.filter_measurements(faceangles)
print "Regular dodecahedron"
print_polyhedron(dodecahedron)
if defect == 0:
    print "A sufficient set of angle measurements for the dodecahedron"
    print_results(dodecahedron, results)
else:
    print ("The following measurements are independent, "
           "but not infinitesimally sufficient.")
    print "%d more measurements are required" % defect
    print_results(dodecahedron, results)
```