Convergence Analysis and Computational Testing of the Finite Element Discretization of the Navier-Stokes Alpha Model

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Abstract

This report performs a complete analysis of convergence and rates of convergence of finite element approximations of the Navier-Stokes-α (NS-α) regularization of the NSE, under a zero-divergence constraint on the velocity, to the true solution of the NSE. Convergence of the *discrete* NS-α approximate velocity to the true Navier-Stokes velocity is proved and rates of convergence derived, under no-slip boundary conditions. Generalization of the results herein to periodic boundary conditions is evident. 2D experiments are performed, verifying convergence and predicted rates of convergence. It is shown that the α-FE solutions converge at the theoretical limit of $O(h^2)$ when choosing $\alpha = h$, in the H^1 norm. Convergence in L^2 is shown to approach $O(h^3)$, but may plateau below the optimal rate. Furthermore, in the case of flow over a step the NS-α model is shown to resolve vortex separation in the recirculation zone.

1 Introduction

Regularizations of the Navier-Stokes equations (NSE) were first introduced as a theoretical tool in 1934 by J. Leray [20, 21] and have since been shown to have some attractive computational properties, [11, 12]. The Leray and related regularizations replace one of its velocities in the non-linear term of the NSE with a filtered velocity $\overline{u}$. In particular, choosing $\overline{u} = (I - \alpha^2 \Delta)^{-1} u$ gives the Leray-α regularization:

$$u_t + \overline{u} \cdot \nabla u - \nu \triangle u + \nabla p = f \text{ and } \nabla \cdot u = 0\,, \text{ in } \Omega \times (0, T]. \tag{1.1}$$

The solution to (1.1) is smooth, and converges as $\alpha \to 0$ modulo a subsequence to a weak solution of the NSE [20, 21]. Higher order regularizations using deconvolution have shown higher order convergence properties [1]. If a similar regularization is used when the NSE is in rotational form the model which we study herein results:

$$\left.\begin{aligned} u_t + (\nabla \times u) \times \overline{u} - \nu \Delta u + \nabla p = f, &\text{ in } \Omega \times (0, T] \\ \overline{u} - \alpha^2 \Delta \overline{u} + \nabla \lambda = u, &\text{ in } \Omega \times (0, T] \\ \nabla \cdot \overline{u} = \nabla \cdot u = 0, &\text{ in } \Omega \times (0, T] \\ u(x, 0) = u_0(x), &\text{ in } \Omega \\ \overline{u} = u = 0 &\text{ on } \partial\Omega \times (0, T] \end{aligned}\right\}. \tag{1.2}$$

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Solving for $p, \lambda \in L_0^2(\Omega)$, this corresponds to the usual α-model with an added constraint, and coincides in the case of periodic boundary conditions where in fact $\lambda \equiv 0$. The NS-α model is an outgrowth of the work by Camassa and Holm (e.g. [6, 7, 8]) to model turbulent flow. In summary, the Camassa-Holm equations were shown to have an energy spectrum closely resembling the Kolmogorov model associated with the NSE. The equations are physically derivable, and sequences $\{\alpha_n\} \to 0$ yield solutions converging (weakly) to a weak solution of the NSE. The energy cascade in the inertial range follows the expected wavenumber scaling $k^{-5/3}$ for $k\alpha < 1$, but then scales as k^{-3} for $k\alpha > 1$. This results in a dissipation wavenumber cutoff of order $Re^{1/2}$ for NS-α, smaller than the corresponding $Re^{3/4}$ for NS-α, which translates to a DNS computational cost for NS-α of roughly $(NSE\ cost)^{2/3}$. Thus NS-α is expected to provide a useful model for many physical flows.

In practice, finite element approximations using the rotation form of the NSE suffers a degradation in accuracy when the gradient of the pressure is large compared to second derivatives of the velocity. Addition of a *grad-div* stabilization term helps to reduce errors in this case, (see [19]). Let $X = \mathring{H}^1(\Omega)$ and $Q = L_0^2(\Omega)$. If $X_h \subset X$ and $Q_h \subset Q$ are finite element spaces, then the finite element discretization of the NS-α model studied herein is find $u_h, \overline{u_h}^h \in X_h$ and $p_h, \lambda_h \in Q_h$ satisfying:

$$\left.\begin{aligned} &(u_{h,t}, v_h) + \nu(\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) \\ &\qquad + ((\nabla \times u_h) \times \overline{u_h}^h, v_h) + \gamma(\nabla \cdot u_h, \nabla \cdot v_h) = (f, v_h), \forall v_h \in X_h \\ &(\overline{u_h}^h, w_h) + \alpha^2(\nabla \overline{u_h}^h, \nabla w_h) - (\lambda_h, \nabla \cdot w_h) = (u_h, w_h)\ \forall w_h \in X_h \\ &\qquad (\nabla \cdot \overline{u_h}^h, \mu_h) = 0, \forall \mu_h \in Q_h \\ &\qquad (\nabla \cdot u_h, q_h) = 0, \forall q_h \in Q_h \end{aligned}\right\} . \tag{1.3}$$

An equivalent problem is formulated over discretely divergence free functions $V_h = \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0, \forall q_h \in Q_h\}$. The equivalent problem is then posed as:
Find $u_h, \overline{u_h}^h \in V_h$ satisfying:

$$\begin{aligned} (u_{h,t}, v_h) + \nu(\nabla u_h, \nabla v_h) &+ ((\nabla \times u_h) \times \overline{u_h}^h, v_h) \\ &+ \gamma(\nabla \cdot u_h, \nabla \cdot v_h) = (f, v_h), \forall v_h \in V_h \end{aligned} \tag{1.4}$$

and

$$(\overline{u_h}^h, w_h) + \alpha^2(\nabla \overline{u_h}^h, \nabla w_h) = (u_h, w_h)\ \forall w_h \in V_h \tag{1.5}$$

See Section 2 and [22] for details. The computational expense of the discrete filtering operation (1.5) is small compared to solving (1.4), especially considering the the condition number of the associated linear system in (1.5) is $O\left(\frac{\alpha^2}{h^2} + 1\right)$ [22]. Herein we study convergence of u_h to a solution of the NSE. Since NS-α is a regularization of the NSE, it seems appropriate to study convergence as both $\alpha, h \to 0$ to the NSE solution. An alternate convergence problem would be to study convergence to the continuous NS-α solution as $h \to 0$, (Çağlar), [5]. This leaves the problem of characterizing the dependence of the continuum NS-α model's solution upon α, for small α.

The issue of boundary conditions for the NS-α model is an interesting question. Work by Chen, Foias, Holm, Olson, Titi and Wynne, [6], shows that some statistics of pipe flows can be well fit using a Navier-slip condition, while Fried and Gurtin [9] remark that the friction values required violate thermodynamic constraints on the model. Based on this work of Fried and Gurtin, an interesting, alternative numerical approach was formulated by Kim, Dolbow and Fried, [15], treating the NS-α equations as a higher order problem with accompanying boundary conditions. It was shown a non-conforming finite element method based on C^0 basis functions can be implemented, using stabilization techniques on finite element interfaces to enforce continuity of higher derivatives. Another computationally efficient treatment of boundary conditions for LES models is known as "Near Wall Modelling" (NWM), where boundary conditions for the filtered variables are derived using a closure model near the wall, ensuring certain physical conditions hold. One interesting example was proposed by Borggaard and Iliescu, [4], estimating the filtered boundary conditions at each time step using an explicit update derived via an NWM model. Treatment of boundary conditions is still an open question and no-slip boundary conditions are chosen for tests herein. The numerical analysis (Section 3) is easily extendable to any linear, well posed boundary conditions.

The relationship between the filter radius α and the mesh size h will be examined in the error analysis. The solution to (1.2) is known to be globally regular with periodic boundary conditions [7], or for zero boundary conditions lies in $L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ with bounds proportional to α^{-1}, [5]. Since it is a regularized approximation of the NSE as $\alpha \to 0$, the value of α chosen in computations should effect resolution of some non-smooth or transitional flow behavior. In the computational section, we perform direct convergence rate verification for three flows: one in the unit circle with zero boundary conditions, the unit square with no-slip and periodic boundary conditions. A comparison of NS-α computations against direct finite element discretization of the NSE in the case of flow over a step [10] is performed. Previous work indicates problems resolving separation of vortices in the recirculation region behind the step for some regularizations [1] or when using the rotational NSE [19]. Computations performed by Layton, Manica, Neda and Rebholz [18] using the NS-α model failed to resolve the characteristic vortex street produced by flow around a cylinder while not imposing $\nabla \cdot u = 0$. In this report the extra divergence-free constraint is imposed to avoid such problems, and the computational implications discussed.

2 Notation and Preliminaries

This section provides the necessary preliminary definitions and lemmas for the convergence analysis of the discrete NS-α model. The Sobolev space $H^k = W^{k,2}$ is equipped with the usual norm $\|\cdot\|_k$, and semi-norm $|\cdot|_k$, for $1 \leq k < \infty$, e.g. Adams [2]. The L^2 norm is denoted by $\|\cdot\|$, and the L^∞ norm by $\|\cdot\|_\infty$. For functions $v(x,t)$ defined for $t \in (0,T]$ we define the norms $(1 \leq m < \infty, 1 \leq k \leq \infty)$:

$$\|\mathrm{v}\|_{\infty,k} = \underset{0<t<T}{ess\ sup}\ \|\mathrm{v}(\cdot,\mathrm{t})\|_k$$

$$\|\mathrm{v}\|_{m,k} = \left(\int_0^T \|\mathrm{v}\|_k^m dt \right)^{1/m}$$

The convergence analysis will be proved in the case of zero boundary conditions, although generalization to the case of periodic boundary conditions (with the usual zero spacial mean condition) is clear. For internal flows, we take the domain to be $\Omega \subset \mathbb{R}^d$, for

$d = 2$ or 3. The corresponding velocity spaces are:

$$\begin{aligned} X \equiv H_0^1 &= \left\{ v \in H^1(\Omega)^d : \ v|_{\partial\Omega} = 0 \right\} \\ Q \equiv L_0^2 &= \left\{ q \in L^2(\Omega) : \ \int_\Omega q \ dx = 0 \right\} \\ V &= \left\{ v \in X : \ \int_\Omega q \, \nabla \cdot v \, dx = 0, \ \forall q \in Q \right\} \end{aligned}$$

We assume conforming finite element spaces $X_h \subset X$ and $Q_h \subset Q$, satisfying the uniform LBB^h condition:

$$\underset{q_h \in Q_h}{inf} \ \underset{v_h \in X^h}{sup} \frac{(q_h, \nabla \cdot v_h)}{\|\nabla v_h\| \|q_h\|} \geq \beta > 0, \forall h > 0.$$

The continuous and discrete differential filters are central to the analysis of the convergence of the FEM applied to the NS-α model.

Definition 2.1 (Continuous differential filter)**.** *Let $\phi \in L^2(\Omega)$ and $\alpha > 0$. The filtering operation on ϕ is denoted by $\overline{\phi}$. Here $(\overline{\phi}, \lambda) \in (V \times Q)$ is the unique solution to:*

$$-\alpha^2 \Delta \overline{\phi} \ + \ \overline{\phi} \ + \ \nabla \lambda \ = \ \phi \tag{2.1}$$

$$\nabla \cdot \overline{\phi} \ = \ 0. \tag{2.2}$$

Definition 2.2 (Discrete differential filter [22])**.** *Given $\phi \in L^2(\Omega)$ and $\alpha > 0$. The discrete differential filter of ϕ, $\overline{\phi}^h \in V_h$, is the unique solution to:*

$$(\phi, v_h) = (\overline{\phi}^h, v_h) \ + \ \alpha^2 (\nabla \overline{\phi}^h, \nabla v_h), \forall v_h \in V_h \tag{2.3}$$

Furthermore, define $A_h^{-1} : L^2(\Omega) \to V_h$ by A_h^{-1} by $A_h^{-1}\phi = \overline{\phi}^h, \forall \phi \in L^2(\Omega)$.

We can understand the operator A_h in terms of the L^2-projection and the discrete Laplacian.

Definition 2.3 (L^2-projection Π^h and Discrete Laplacian Δ^h)**.** *Let $\phi \in L^2(\Omega)$. Then we denote the L^2-projection of ϕ onto V_h by $\Pi^h \phi$, the unique solution to:*

$$(\phi - \Pi^h \phi, v_h) = 0, \forall v_h \in V_h. \tag{2.4}$$

Given $\psi \in X$. Let $\psi_h \in V_h$ be the unique solution to:

$$(\psi_h, v_h) = -(\nabla \psi, \nabla v_h), \forall v_h \in V_h. \tag{2.5}$$

Then the discrete Laplacian $\Delta^h : X \to V_h$ is defined by $\Delta^h \psi = \psi_h$.

We then have $A_h = \Pi^h - \alpha^2 \Delta^h$, and $\overline{\phi}^h = A_h^{-1} (\Pi^h \phi)$. In order to bound error terms on X_h in a useful way for the convergence proof, we make use of the next identities.

Lemma 2.1. *For $\phi_h \in V_h$ the following identities hold:*

$$(\phi_h, \overline{\phi_h}^h) = \|\overline{\phi_h}^h\|^2 + \alpha^2 \|\nabla \overline{\phi_h}^h\|^2 \tag{2.6}$$

$$(\nabla \phi_h, \nabla \overline{\phi_h}^h) = \|\nabla \overline{\phi_h}^h\|^2 + \alpha^2 \|\Delta^h \overline{\phi_h}^h\|^2 \tag{2.7}$$

Proof. The proof of (2.6) follows immediately from choosing $v_h = \overline{\phi_h}^h$ in (2.3) and (2.5). Notice that from (2.3) and (2.5) we can also write:

$$(\phi_h - (\overline{\phi_h}^h - \alpha^2 \Delta^h \overline{\phi_h}^h), v_h) = 0, \forall v_h \in V^h$$

or, using $A_h = \Pi^h - \alpha^2 \Delta^h$,

$$(\phi_h - A_h \overline{\phi_h}^h, v_h) = 0, \forall v_h \in V^h$$

Now, by choosing $v_h = \phi_h - A_h \overline{\phi_h}^h$, we have:

$$\|\phi_h - A_h \overline{\phi_h}^h\|^2 = 0$$

and hence $\phi_h = \overline{\phi_h}^h - \alpha^2 \Delta^h \overline{\phi_h}^h$. Note since $\overline{\phi_h}^h, \phi_h \in V_h$ we can further write $\nabla \phi_h = \nabla \overline{\phi_h}^h - \alpha^2 \nabla \Delta^h \overline{\phi_h}^h$ a.e., and we have:

$$(\nabla \phi_h, \nabla v_h) = (\nabla \overline{\phi_h}^h, \nabla v_h) - \alpha^2 (\nabla \Delta^h \overline{\phi_h}^h, \nabla v_h)$$

Therefore, choosing $v_h = \overline{\phi_h}^h$ and applying (2.4) we get:

$$(\nabla \phi_h, \nabla \overline{\phi_h}^h) = (\nabla \overline{\phi_h}^h, \nabla \overline{\phi_h}^h) + \alpha^2 (\Delta^h \overline{\phi_h}^h, \Delta^h \overline{\phi_h}^h), \forall v_h \in V^h$$

thus proving (2.6).

□

We need to be able to bound the L^2 norms of functions and their derivatives on V_h in terms of (2.5)-(2.6). This motivates the next lemma by Rebholz and Miles [24].

Lemma 2.2. *Let $X_h \subset X$ be a finite element space with no-slip or periodic boundary conditions, satisfying the inverse inequality:*

$$\|\nabla \phi_h\| \leq \frac{C}{h} \|\phi_h\|, \qquad \forall \phi_h \in X^h \tag{2.8}$$

Assume $0 < \alpha \leq C\,h < \infty$. Then there exists C_0 independent of h such that for all $\phi_h \in V_h \subset X^h$

$$(\phi_h, \overline{\phi_h}^h) \leq \|\phi_h\|^2 \leq C\,(\phi_h, \overline{\phi_h}^h) \tag{2.9}$$

$$(\nabla \phi_h, \nabla \overline{\phi_h}^h) \leq \|\nabla \phi_h\|^2 \leq C\,(\nabla \phi_h, \nabla \overline{\phi_h}^h) \tag{2.10}$$

Remark 2.1. *Typical finite element spaces in use satisfy the inverse estimate (2.7), when used as a locally quasi-uniform mesh, e.g. [25].*

The convergence analysis requires a variety of bounds for trilinear terms.

Lemma 2.3. *For $a, b, c \in X$ and $d =$ 1,2,3:*

$$\begin{aligned}
(a \times (\nabla \times b), c) &\leq C\,\|a\|^{1/2} \|\nabla a\|^{1/2} \|\nabla b\| \|\nabla c\| && (2.11)\\
(a \times (\nabla \times b), c) &\leq C\,\|\nabla a\| \|\nabla b\| \|c\|^{1/2} \|\nabla c\|^{1/2} && (2.12)\\
(a \times (\nabla \times b), c) &\leq \|\nabla \times b\|_\infty \|a\| \|c\| && (2.13)\\
(a \cdot \nabla b, c) &\leq C\,\|a\|^{1/2} \|\nabla a\|^{1/2} \|\nabla b\| \|\nabla c\| && (2.14)\\
(a \cdot \nabla b, c) &\leq C\,\|\nabla a\| \|\nabla b\| \|c\|^{1/2} \|\nabla c\|^{1/2} && (2.15)\\
(a \cdot b, \nabla \cdot c) &\leq C\,\|a\|^{1/2} \|\nabla a\|^{1/2} \|\nabla b\| \|\nabla c\| && (2.16)\\
(a \cdot b, \nabla \cdot c) &\leq C\,\|\nabla a\| \|b\|^{1/2} \|\nabla b\|^{1/2} \|\nabla c\| && (2.17)
\end{aligned}$$

Proof. See e.g. [2, 7].

□

The following vector identities will also be employed. Let $a, b, c \in X$: $a(x), b(x), c(x) \in \mathbb{R}^3$ for all $x \in \Omega^d$, then:

$$(Triple\ Product) \qquad a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) \tag{2.18}$$

$$\nabla \times (a \times b) = b \cdot \nabla a - a \cdot \nabla b + a(\nabla \cdot b) - b(\nabla \cdot a) \tag{2.19}$$

$$(a \times b) \cdot a = (a \times b) \cdot b = 0 \tag{2.20}$$

$$a \times (\nabla \times b) = (\nabla \times a) \times b - a \cdot \nabla b - b \cdot \nabla a + \nabla(a \cdot b) \tag{2.21}$$

3 Convergence of the FEM

The convergence analysis will use bounds on the terms in (2.6) and (2.7) in order to subsequently bound $\|\phi_h\|$ and $\|\nabla \phi_h\|$ on V^h. The stability of the FEM with respect to the terms in (2.6), (2.7) must first be established.

Lemma 3.1 (Stability)**.** *Let $u_h \in V^h$ satisfy (1.4), and assume the finite element space X^h has no-slip or periodic boundary conditions, and satisfies the inverse inequality (2.8). Let $0 < \alpha \leq C\,h < \infty$. Then $\exists M > 0$ such that if $0 < \gamma \leq M\nu$, u_h satisfies:*

$$\|\overline{u_h}^h\|^2 + \alpha^2 \|\nabla \overline{u_h}^h\|^2 + \nu \int_0^T \left\{ \|\nabla \overline{u_h}^h\|^2 + \alpha^2 \|\Delta^h \overline{u_h}^h\|^2 \right\} dt + \gamma \int_0^T \|\nabla \cdot u_h\|^2 \, dt \leq C \tag{3.1}$$

$$\|u_h\|^2 + \nu \int_0^T \|\nabla u_h\|^2 \ dt + \gamma \int_0^T \|\nabla \cdot u_h\|^2 \, dt \leq C \tag{3.2}$$

where $C = C(u_0, \nu, f, T)$ is independent of α, h, γ.

Proof. Choose $v_h = \overline{u_h}^h$ in (1.4). Then it follows:

$$(\frac{\partial}{\partial t} u_h, \overline{u_h}^h) + \nu(\nabla u_h, \nabla \overline{u_h}^h) + \gamma(\nabla \cdot u_h, \nabla \cdot \overline{u_h}^h) = (f, \overline{u_h}^h)$$

Here (2.20) is used to eliminate the trilinear term. After rewriting $(\nabla \cdot u_h, \nabla \cdot \overline{u_h}^h) = (\nabla \cdot u_h, \nabla \cdot u_h) + \alpha^2(\nabla \cdot u_h, \nabla \cdot \Delta^h \overline{u_h}^h)$ and rearranging terms,

$$(\frac{\partial}{\partial t} u_h, \overline{u_h}^h) + \nu(\nabla u_h, \nabla \overline{u_h}^h) + \gamma(\nabla \cdot u_h, \nabla \cdot u_h) = (f, \overline{u_h}^h) - \gamma\alpha^2(\nabla \cdot u_h, \nabla \cdot \Delta^h \overline{u_h}^h).$$

Note that $(\frac{\partial}{\partial t} u_h, \overline{u_h}^h) = \frac{1}{2}\frac{d}{dt}(u_h, \overline{u_h}^h)$. Apply Lemma 2.1, followed by Hölder's and Young's inequalities to get:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big\{\|\overline{u_h}^h\|^2 + \alpha^2\|\nabla \overline{u_h}^h\|^2\Big\} + \nu\Big\{\|\nabla \overline{u_h}^h\|^2 + \alpha^2 \|\Delta^h \overline{u_h}^h\|^2\Big\} + \gamma\|\nabla \cdot u_h\|^2 \\
\leq \|f\|_{-1}\|\nabla \overline{u_h}^h\| + \gamma\alpha^2 \|\nabla \cdot u_h\| \|\nabla \cdot \Delta^h \overline{u_h}^h\| \\
\leq \frac{1}{2\nu}\|f\|_{-1}^2 + \frac{\nu}{2}\|\nabla \overline{u_h}^h\|^2 + C_1\,\gamma\alpha^2 h^{-1}\|\nabla \cdot u_h\| \|\Delta^h \overline{u_h}^h\| \\
\leq \frac{1}{2\nu}\|f\|_{-1}^2 + \frac{\nu}{2}\|\nabla \overline{u_h}^h\|^2 + C_2\,\gamma\alpha\|\nabla \cdot u_h\| \|\Delta^h \overline{u_h}^h\| \\
\leq \frac{1}{2\nu}\|f\|_{-1}^2 + \frac{\nu}{2}\|\nabla \overline{u_h}^h\|^2 + C_3\,\gamma\frac{\alpha^2}{2}\|\Delta^h \overline{u_h}^h\|^2 + \frac{\gamma}{2}\|\nabla \cdot u_h\|^2.
\end{aligned}$$

Here the inverse inequality (2.8) and assumption $\alpha \le Ch$ were used. Taking $M = \frac{1}{C_3}$ and $\gamma \le M\nu$,

$$\frac{1}{2}\frac{d}{dt}\Big\{\|\overline{u_h}^h\|^2 + \alpha^2\|\nabla\overline{u_h}^h\|^2\Big\} + \nu\Big\{\|\nabla\overline{u_h}^h\|^2 + \alpha^2\|\Delta^h\overline{u_h}^h\|^2\Big\} + \gamma\|\nabla\cdot u_h\|^2$$
$$\le \frac{1}{2\nu}\|f\|_{-1}^2 + \frac{\nu}{2}\|\nabla\overline{u_h}^h\|^2 + \frac{\nu\,\alpha^2}{2}\|\Delta^h\overline{u_h}^h\|^2 + \frac{\gamma}{2}\|\nabla\cdot u_h\|^2.$$

Subsume all terms on the right hand side except $\frac{1}{2\nu}\|f\|_{-1}^2$ and multiply through by 2:

$$\frac{d}{dt}\Big\{\|\overline{u_h}^h\|^2 + \alpha^2\|\nabla\overline{u_h}^h\|^2\Big\} + \nu\Big\{\|\nabla\overline{u_h}^h\|^2 + \alpha^2\|\Delta^h\overline{u_h}^h\|^2\Big\} + \gamma\|\nabla\cdot u_h\|^2 \le \frac{1}{\nu}\|f\|_{-1}^2$$

Integrate in time to yield (3.1). Lemma 2.2 may be applied to (3.1) by assumption, yielding (3.2).

□

The choice of scaling $\gamma = O(\nu)$ is expected to improve accuracy of solutions to (1.4)-(1.5) compared to $\gamma = 0$, when $\nu << 1$. For details, see Section 2.3 in [19].

Corollary 3.1. *$u_h : [0,T] \to V_h$ exists.*

Proof. Select a basis of (V_h, Q_h) and expand (u_h, p_h) in terms of basis functions in (1.4),(1.5). This reduces (1.4),(1.5) to a finite system of ODE's. Lemma 3.1 shows the corresponding solutions cannot blow up in finite time.

□

We now proceed to prove convergence of the FEM for the NS-α model. In the analysis of the following theorem, the regularity assumptions arise due to the problem of characterizing convergence of the solution of the NS-α model to the solution of the NSE; known results are not sufficient for the convergence proof.

Theorem 3.1. *Let $X^h \subset X$ be a conforming finite element space satisfying (2.8). Assume $u(x,\cdot) \in V$ is a strong solution of the NSE, satisfying:*

$$\nabla \times u \in L^2(0,T;\ L^\infty(\Omega)). \tag{3.3}$$

Furthermore, assume $\exists\ C_1 > 0,\ C_2 > 0$ independent of α, h such that:

$$0 < C_2 h \le \alpha \le C_1 h \tag{3.4}$$
$$0 < \alpha \le 1 \tag{3.5}$$

and the differential filter of u satisfies $\overline{u} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ (α-independent). Then $\exists M > 0$ such that if $0 < \gamma \le M\nu$, the solution $u_h \in V^h$ to (1.4)-(1.5) satisfies:

$$\sup_{0\le t\le T}\|u-u_h\|^2 + \nu\int_0^T \|\nabla(u-u_h)\|^2\,dt + \gamma\int_0^T \|\nabla\cdot(u-u_h)\|^2\,dt \le 2\sup_{0\le t\le T}\ \inf_{v_h\in V^h}\Big\{\|u-v_h\|^2\Big\}$$
$$+C\|u(0)-u_0\|^2 + C\inf_{\substack{v_h\in V^h\\ q_h\in Q_h}}\Big\{\int_0^T \{\|p-q_h\|^2 + \|\nabla(u-v_h)\|^2 + \|\frac{\partial}{\partial t}(u-v_h)\|^2\}dt\Big\}$$
$$+C\Big\{\inf_{v_h\in V^h}\Big(\int_0^T \|\nabla(u-v_h)\|^4 dt\Big)^{\frac{1}{2}} + \alpha^4\int_0^T \|\Delta\overline{u}\|^2 dt + h^{2k+2}\int_0^T |\overline{u}|_{k+1}^2 dt\Big\}$$

Proof. Subtract (1.4) from the weak formulation of the rotation NSE, to get the error equation:

$$
\begin{aligned}
&(\frac{\partial}{\partial t}(u-u_h), v_h) + \nu(\nabla(u-u_h), \nabla v_h) - \gamma(\nabla\cdot u_h, \nabla\cdot v_h) \\
&\qquad = (p, \nabla\cdot v_h) + ((\nabla\times u_h)\times \overline{u_h}^h, v_h) - ((\nabla\times u)\times u, v_h), \forall v_h \in V^h
\end{aligned}
$$

Add in $-(q_h, \nabla\cdot v_h) = 0$ and $\gamma(\nabla\cdot u, \nabla\cdot v_h) = 0$:

$$
\begin{aligned}
&(\frac{\partial}{\partial t}(u-u_h), v_h) + \nu(\nabla(u-u_h), \nabla v_h) + \gamma(\nabla\cdot(u-u_h), \nabla\cdot v_h) \\
&\quad = (p-q_h, \nabla\cdot v_h) + ((\nabla\times u_h)\times \overline{u_h}^h, v_h) - ((\nabla\times u)\times u, v_h), \forall (v_h, q_h) \in (V^h, Q^h).
\end{aligned}
$$

Let $\tilde{u} \in V^h$ be arbitrary, and split $u - u_h = (u-\tilde{u}) + (\tilde{u} - u_h) = \eta + \phi_h$. Insert this in above and rearrange terms to get:

$$
\left.
\begin{aligned}
&(\frac{\partial}{\partial t}\phi_h, v_h) + \nu(\nabla\phi_h, \nabla v_h) + \gamma(\nabla\cdot\phi_h, \nabla\cdot v_h) \\
&\qquad = (p-q_h, \nabla\cdot v_h) - (\frac{\partial\eta}{\partial t}, v_h) - \nu(\nabla\eta, \nabla v_h) \\
&\qquad + ((\nabla\times u_h)\times\overline{u_h}^h, v_h) - ((\nabla\times u)\times u, v_h) - \gamma(\nabla\cdot\eta, \nabla\cdot v_h)
\end{aligned}
\right\} \tag{3.6}
$$

To deal with the trilinear terms, we begin by writing $\nabla\times u = \nabla\times u_h - \nabla\times(u_h - u)$ and re-write the two trilinear terms as:

$$((\nabla\times u_h)\times\overline{u_h}^h, v_h) - ((\nabla\times u)\times u, v_h) = ((\nabla\times u_h)\times(\overline{u_h}^h - u), v_h) - ((\nabla\times(u_h-u))\times u, v_h).$$

Inserting this back into (3.6) and choosing $v_h = \overline{\phi_h}^h$ we obtain:

$$
\begin{aligned}
&(\frac{\partial}{\partial t}\phi_h, \overline{\phi_h}^h) + \nu(\nabla\phi_h, \nabla\overline{\phi_h}^h) + \gamma(\nabla\cdot\phi_h, \nabla\cdot\overline{\phi_h}^h) \\
&\qquad = (p-q_h, \nabla\cdot\overline{\phi_h}^h) - (\frac{\partial\eta}{\partial t}, \overline{\phi_h}^h) - \nu(\nabla\eta, \nabla\overline{\phi_h}^h) - \gamma(\nabla\cdot\eta, \nabla\cdot\overline{\phi_h}^h) \\
&\qquad + ((\nabla\times u_h)\times(\overline{u_h}^h - u), \overline{\phi_h}^h) - ((\nabla\times(u_h-u))\times u, \overline{\phi_h}^h)
\end{aligned}
$$

Insert $\nabla\cdot\overline{\phi_h}^h = \nabla\cdot\phi_h + \alpha^2\nabla\cdot\Delta^h\overline{\phi_h}^h$ on the left and subtract $\alpha^2\gamma(\nabla\cdot\phi_h, \nabla\cdot\Delta^h\overline{\phi_h}^h)$ to the right:

$$
\begin{aligned}
&(\frac{\partial}{\partial t}\phi_h, \overline{\phi_h}^h) + \nu(\nabla\phi_h, \nabla\overline{\phi_h}^h) + \gamma(\nabla\cdot\phi_h, \nabla\cdot\phi_h) \\
&\qquad = (p-q_h, \nabla\cdot\overline{\phi_h}^h) - (\frac{\partial\eta}{\partial t}, \overline{\phi_h}^h) - \nu(\nabla\eta, \nabla\overline{\phi_h}^h) - \alpha^2\gamma(\nabla\cdot\phi_h, \nabla\cdot\Delta^h\overline{\phi_h}^h) \\
&\qquad - \gamma(\nabla\cdot\eta, \nabla\cdot\overline{\phi_h}^h) + ((\nabla\times u_h)\times(\overline{u_h}^h - u), \overline{\phi_h}^h) - ((\nabla\times(u_h-u))\times u, \overline{\phi_h}^h).
\end{aligned}
$$

Apply Lemma 2.1 to the LHS and Hölder's inequality to the first five terms on the RHS.

This yields:

$$\left.\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{\|\overline{\phi_h}^h\|^2+\alpha^2\|\nabla\overline{\phi_h}^h\|^2\Big\}+\nu\Big\{\|\nabla\overline{\phi_h}^h\|^2+\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2\Big\}+\gamma\|\nabla\cdot\phi_h\|^2\\&\qquad\le\|p-q_h\|\|\nabla\cdot\overline{\phi_h}^h\|+\|\frac{\partial\eta}{\partial t}\|\|\overline{\phi_h}^h\|+\nu\|\nabla\eta\|\|\nabla\overline{\phi_h}^h\|\\&\qquad+\alpha^2\gamma\|\nabla\cdot\phi_h\|\|\nabla\cdot\Delta^h\overline{\phi_h}^h\|+\gamma\|\nabla\cdot\eta\|\|\nabla\cdot\overline{\phi_h}^h\|\\&\qquad+((\nabla\times u_h)\times(\overline{u_h}^h-u),\overline{\phi_h}^h)-((\nabla\times(u_h-u))\times u,\overline{\phi_h}^h)\end{aligned}\right\}\tag{3.7}$$

The following bounds follow from applying Young's inequality, the inverse inequality (2.8) and $\|\nabla\cdot w\|\le C\|\nabla w\|$:

$$\begin{aligned}\gamma\|\nabla\cdot\overline{\phi_h}^h\|\|\nabla\cdot\eta\|&\le C\|\nabla\eta\|^2+\frac{\nu}{12}\|\nabla\overline{\phi_h}^h\|^2\\\gamma\alpha^2\|\nabla\cdot\Delta^h\overline{\phi_h}^h\|\|\nabla\cdot\phi_h\|&\le\frac{C^*\gamma\alpha^2}{h}\|\Delta^h\overline{\phi_h}^h\|\|\nabla\cdot\phi_h\|\\&\le\frac{(C^*)^2\gamma\alpha^4}{2h^2}\|\Delta^h\overline{\phi_h}^h\|^2+\frac{\gamma}{2}\|\nabla\cdot\phi_h\|^2\end{aligned}\tag{3.8}$$

Choose $M=\frac{\nu}{3(C_1\cdot C^*)^2}$, and $0<\gamma\le M$. Combined with the assumption $\alpha\le C_1h$, (3.8) is used to rewrite (3.7):

$$\left.\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{\|\overline{\phi_h}^h\|^2+\alpha^2\|\nabla\overline{\phi_h}^h\|^2\Big\}+\nu\Big\{\|\nabla\overline{\phi_h}^h\|^2+\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2\Big\}+\gamma\|\nabla\cdot\phi_h\|^2\\&\qquad\le\|p-q_h\|\|\nabla\cdot\overline{\phi_h}^h\|+\|\frac{\partial\eta}{\partial t}\|\|\overline{\phi_h}^h\|+\nu\|\nabla\eta\|\|\nabla\overline{\phi_h}^h\|\\&\qquad+\frac{\nu\alpha^2}{6}\|\Delta^h\overline{\phi_h}^h\|^2+\frac{\gamma}{2}\|\nabla\cdot\phi_h\|^2+C\|\nabla\eta\|^2+\frac{\nu}{12}\|\nabla\overline{\phi_h}^h\|^2\\&\qquad+((\nabla\times u_h)\times(\overline{u_h}^h-u),\overline{\phi_h}^h)-((\nabla\times(u_h-u))\times u,\overline{\phi_h}^h)\end{aligned}\right\}\tag{3.9}$$

Consider the terms in (3.9). Split the last term using $u_h-u=-\eta-\phi_h$ to get:

$$((\nabla\times(u_h-u))\times u,\overline{\phi_h}^h)=-((\nabla\times\eta)\times u,\overline{\phi_h}^h)-((\nabla\times\phi_h)\times u,\overline{\phi_h}^h)\tag{3.10}$$

The two RHS terms will be treated separately. Bound the first term by applying (2.12) and then Young's inequality as follows:

$$\left.\begin{aligned}-((\nabla\times\eta)\times u,\overline{\phi_h}^h)&\le C\|\nabla\eta\|\|\nabla u\|\|\overline{\phi_h}^h\|^{1/2}\|\nabla\overline{\phi_h}^h\|^{1/2}\\&\le\frac{\nu}{24}\|\nabla\overline{\phi_h}^h\|^2+C\|\nabla\eta\|^{4/3}\|\nabla u\|^{4/3}\|\overline{\phi_h}^h\|^{2/3}\\&\le\frac{\nu}{24}\|\nabla\overline{\phi_h}^h\|^2+C\Big\{\|\nabla\eta\|^2+\|\nabla u\|^4\|\overline{\phi_h}^h\|^2,\Big\}\end{aligned}\right\}\tag{3.11}$$

using $ab\le\frac{2}{3}a^{3/2}+\frac{1}{3}b^3$. This two-step application of Young's inequality will be commonly employed. To bound the second term on the RHS of (3.10), write $\phi_h=\overline{\phi_h}^h-\alpha^2\Delta^h\overline{\phi_h}^h$ (see Lemma 3.1 proof) and apply (2.12). This gives:

$$\begin{aligned}-((\nabla\times\phi_h)\times u,\overline{\phi_h}^h)&=-((\nabla\times\overline{\phi_h}^h)\times u,\overline{\phi_h}^h)+\alpha^2((\nabla\times\Delta^h\overline{\phi_h}^h)\times u,\overline{\phi_h}^h)\\&\le C\Big\{\|\nabla\overline{\phi_h}^h\|^{3/2}\|\overline{\phi_h}^h\|^{1/2}\|\nabla u\|\\&+\alpha^2\|\nabla\Delta^h\overline{\phi_h}^h\|\|\overline{\phi_h}^h\|^{1/2}\|\nabla\overline{\phi_h}^h\|^{1/2}\|\nabla u\|\Big\}\end{aligned}$$

Here we apply Young's inequality twice:

$$\left.\begin{aligned} -((\nabla\times\phi_h)\times u, \overline{\phi_h}^h) &\le \frac{\nu}{24}\|\nabla\overline{\phi_h}^h\|^2 + C\Big\{\|\nabla u\|^4\|\overline{\phi_h}^h\|^2 \\ &\qquad + \alpha^{8/3}\|\nabla\Delta^h\overline{\phi_h}^h\|^{4/3}\|\overline{\phi_h}^h\|^{2/3}\|\nabla u\|^{4/3}\Big\} \\ &\le \frac{\nu}{24}\|\nabla\overline{\phi_h}^h\|^2 + C\,\epsilon\|\overline{\phi_h}^h\|^2\|\nabla u\|^4 + \frac{\nu}{6}\alpha^4\,\epsilon^{-1/2}\|\nabla\Delta^h\overline{\phi_h}^h\|^2 \end{aligned}\right\} \tag{3.12}$$

We cannot subsume $\|\nabla\Delta^h\overline{\phi_h}^h\|^2$ into the LHS of (3.9), or apply a Gronwall technique to treat it. Instead, using the inverse inequality (2.8) and bound (3.4) on α, there follows:

$$\alpha^4\|\nabla\Delta^h\overline{\phi_h}^h\|^2 \le C(\frac{\alpha^2}{h^2})\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2 \le C^*\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2$$

Choose $\epsilon = (C^*)^2$ in (3.12) and applying the above bounds we get:

$$-((\nabla\times\phi_h)\times u, \overline{\phi_h}^h) \le \frac{\nu}{24}\|\nabla\overline{\phi_h}^h\|^2 + \frac{\nu}{6}\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2 + C\|\nabla u\|^4\|\overline{\phi_h}^h\|^2 \tag{3.13}$$

The first two terms of (3.13) can be subsumed on the LHS of (3.9), and the last term will later be absorbed into a constant using Gronwall's inequality. Combining (3.10), (3.11), and (3.13) implies the bound:

$$\left.\begin{aligned} ((\nabla\times(u_h-u))\times u, \overline{\phi_h}^h) &\le \frac{\nu}{12}\|\nabla\overline{\phi_h}^h\|^2 + C\|\nabla\eta\|^2 \\ &\qquad + C\|\nabla u\|^4\|\overline{\phi_h}^h\|^2 + \frac{\nu}{6}\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2 \end{aligned}\right\} \tag{3.14}$$

This provides the necessary bound for the second trilinear term on the RHS of (3.9). Note thus far we have used $\nabla u \in L^4(0,T;\ L^2(\Omega))$, but no more regularity on the velocity. To bound the other trilinear term in (3.9) we first split $u - \overline{u_h}^h = (u-\overline{u}^h) + (\overline{u}^h - \overline{u_h}^h)$. By linearity of the discrete filtering operation, we have

$$\overline{u}^h - \overline{u_h}^h = \overline{u-\tilde{u}}^h + \overline{\tilde{u}-u_h}^h = \overline{\eta}^h + \overline{\phi_h}^h$$

and can therefore write

$$\left.\begin{aligned} ((\nabla\times u_h)\times(\overline{u_h}^h - u), \overline{\phi_h}^h) &= ((u-\overline{u}^h)\times(\nabla\times u_h), \overline{\phi_h}^h) \\ &\qquad + (\overline{\eta}^h\times(\nabla\times u_h), \overline{\phi_h}^h) + (\overline{\phi_h}^h\times(\nabla\times u_h), \overline{\phi_h}^h) \end{aligned}\right\} \tag{3.15}$$

The last term on the RHS of (3.15) is zero by (2.20). To bound the first term, write $u_h = u - (u-u_h)$ to get

$$\left.\begin{aligned} &((u-\overline{u}^h)\times(\nabla\times u_h), \overline{\phi_h}^h) \\ &\quad = ((u-\overline{u}^h)\times(\nabla\times u), \overline{\phi_h}^h) - ((u-\overline{u}^h)\times(\nabla\times(u-u_h)), \overline{\phi_h}^h) \\ &\quad = ((u-\overline{u}^h)\times(\nabla\times u), \overline{\phi_h}^h) + ((u-\overline{u}^h)\times(\nabla\times\eta), \overline{\phi_h}^h) \\ &\qquad\quad - ((u-\overline{u}^h)\times(\nabla\times\phi_h), \overline{\phi_h}^h) \end{aligned}\right\} \tag{3.16}$$

We require special care in bounding the RHS of (3.16). Bounds for $\|u-\overline{u}^h\|$ and $\|\nabla(u-\overline{u}^h)\|$ will be needed. Applying the triangle inequality and then (2.1) we have

$$\|u-\overline{u}^h\| \leq \|u-\overline{u}\| + \|\overline{u}-\overline{u}^h\|$$

$$= \alpha^2\|\Delta\overline{u}\| + \|\overline{u}-\overline{u}^h\|$$

But it has been shown [1] that

$$\|u-\overline{u}^h\| \leq C\left\{\alpha h^k|\overline{u}|_{k+1} + h^{k+1}|\overline{u}|_{k+1}.\right\}$$

Using $\alpha \leq C_1\, h$ gives

$$\|u-\overline{u}^h\| \leq \alpha^2\|\Delta\overline{u}\| + C\, h^{k+1}|\overline{u}|_{k+1}. \tag{3.17}$$

The first term on the RHS of (3.17) is bounded by applying (2.13) and Young's inequality as follows

$$\left.\begin{aligned}((u-\overline{u}^h)\times(\nabla\times u),\overline{\phi_h}^h) &\leq C\,\|\nabla\times u\|_\infty\|u-\overline{u}^h\|\|\overline{\phi_h}^h\|\\ &\leq C\,\|\nabla\times u\|_\infty^2\|\overline{\phi_h}^h\|^2 + \frac{1}{2}\|u-\overline{u}^h\|^2\\ &\leq C\,\|\nabla\times u\|_\infty^2\|\overline{\phi_h}^h\|^2 + C\left\{\alpha^4\|\Delta\overline{u}\|^2 + h^{2k+2}|\overline{u}|_{k+1}^2.\right\}\end{aligned}\right\} \tag{3.18}$$

Here we are using the assumption (3.3), so that later Gronwall's inequality will be used to treat $\|\nabla\times u\|_\infty^2\|\overline{\phi_h}^h\|^2$.

To finish bounding (3.16), we bound $\|\nabla(u-\overline{u}^h)\|$ uniformly, by first using the triangle inequality to write

$$\|\nabla(u-\overline{u}^h)\| \leq \|\nabla(u-\overline{u})\| + \|\nabla(\overline{u}-\overline{u}^h)\|$$

It is known [17] that $\|\nabla(u-\overline{u})\| \leq C\,\|\nabla u\|$, where C is α, h independent. Also, using the same technique described in [1] it can be easily shown that with assumption (3.4) , it follows $\|\nabla(\overline{u}-\overline{u}^h)\| \leq C\, h^k|\overline{u}|_{k+1}$. It is assumed $\overline{u} \in L^2(0,T;\, H^2(\Omega)) \cap L^\infty(0,T;\, H^1(\Omega))$, so we can take k = 1 (at least). Therefore, we have the bound

$$\|\nabla(u-\overline{u}^h)\| \leq C\left\{\|\nabla u\| + h^k|\overline{u}|_{k+1}\right\} \tag{3.19}$$

and we can proceed to finish bounding (3.16). Applying (2.12) and Young's inequality, we get

$$((u-\overline{u}^h)\times(\nabla\times\eta),\overline{\phi_h}^h) \leq C\,\|\nabla(u-\overline{u}^h)\|\|\nabla\eta\|\|\overline{\phi_h}^h\|^{1/2}\|\nabla\overline{\phi_h}^h\|^{1/2}$$

$$\leq C\,\|\nabla u\|\|\nabla\eta\|\|\overline{\phi_h}^h\|^{1/2}\|\nabla\overline{\phi_h}^h\|^{1/2} + C\, h^k|\overline{u}|_{k+1}\|\nabla\eta\|\|\overline{\phi_h}^h\|^{1/2}\|\nabla\overline{\phi_h}^h\|^{1/2}$$

$$\leq \frac{\nu}{24}\|\nabla\overline{\phi_h}^h\|^2 + C\left\{\|\nabla\eta\|^2 + \|\nabla u\|^4\|\overline{\phi_h}^h\|^2 + h^{2k}|\overline{u}|_{k+1}^4\|\overline{\phi_h}^h\|^2\right\}$$

To guarantee integrability in time, take k=0 to get:

$$\left.\begin{aligned}((u-\overline{u}^h)\times(\nabla\times\eta),\overline{\phi_h}^h) &\leq \frac{\nu}{24}\|\nabla\overline{\phi_h}^h\|^2\\ &\quad + C\left\{\|\nabla\eta\|^2 + \left(\|\nabla u\|^4 + \|\nabla\overline{u}\|^4\right)\|\overline{\phi_h}^h\|^2\right\}\end{aligned}\right\} \tag{3.20}$$

Now the last term can be dealt with using Gronwall's inequality near the end of the proof. One more term in (3.16) needs to be bounded. We write $\phi_h = \overline{\phi_h}^h - \alpha^2 \Delta^h \overline{\phi_h}^h$ to get

$$-((u-\overline{u}^h)\times\nabla\times\phi_h, \overline{\phi_h}^h) = -((u-\overline{u}^h)\times\nabla\times\overline{\phi_h}^h, \overline{\phi_h}^h) + \alpha^2((u-\overline{u}^h)\times\nabla\times\Delta^h\overline{\phi_h}^h, \overline{\phi_h}^h)$$

Now apply (2.12) followed by Young's inequality and (3.19), and it follows:

$$\begin{aligned}
-((u-\overline{u}^h)\times\nabla\times\phi_h, \overline{\phi_h}^h) &\leq C\,\|\nabla(u-\overline{u}^h)\|\|\nabla\overline{\phi_h}^h\|^{3/2}\|\overline{\phi_h}^h\|^{1/2} \\
&\quad + C\,\alpha^2\|\nabla(u-\overline{u}^h)\|\|\nabla\Delta^h\overline{\phi_h}^h\|\|\nabla\overline{\phi_h}^h\|^{1/2}\|\overline{\phi_h}^h\|^{1/2} \\
&\leq \frac{\nu}{24}\|\nabla\overline{\phi_h}^h\|^2 + C\,\epsilon\left\{\|\nabla u\|^4 + \|\nabla\overline{u}\|^4\right\}\|\overline{\phi_h}^h\|^2 + \epsilon^{-1/2}\frac{\nu}{6}\alpha^4\|\nabla\Delta^h\overline{\phi_h}^h\|^2
\end{aligned}$$

Applying the inverse inequality (2.8) and choosing ϵ as in (3.12), it follows

$$\left.\begin{aligned}
-((u-\overline{u}^h)\times(\nabla\times\phi_h), \overline{\phi_h}^h) \leq{}& \frac{\nu}{24}\|\nabla\overline{\phi_h}^h\|^2 \\
&+ \frac{\nu}{6}\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2 + \left\{\|\nabla u\|^4 + \|\nabla\overline{u}\|^4\right\}\|\overline{\phi_h}^h\|^2
\end{aligned}\right\} \tag{3.21}$$

Combining (3.18), (3.20) and (3.21) yields a final bound for (3.16):

$$\left.\begin{aligned}
((u-\overline{u}^h)\times(\nabla\times u_h), \overline{\phi_h}^h) \leq{}& C\left\{\|\nabla\times u\|_\infty^2 + \|\nabla u\|^4 + \|\nabla\overline{u}\|^4\right\}\|\overline{\phi_h}^h\|^2 \\
&+ \frac{\nu}{12}\|\nabla\overline{\phi_h}^h\|^2 + \frac{\nu}{6}\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2 \\
&+ C\left\{\|\nabla\eta\|^2 + \alpha^4\|\Delta\overline{u}\|^2 + h^{2k+2}|\overline{u}|_{k+1}^2\right\}
\end{aligned}\right\} \tag{3.22}$$

This subsequently bounds one of the two non-zero terms on the RHS of (3.15), leaving only one term to bound. To start bounding the last term for (3.15), apply vector identity (2.21).

$$\left.\begin{aligned}
(\overline{\eta}^h\times(\nabla\times u_h), \overline{\phi_h}^h) ={}& ((\nabla\times\overline{\eta}^h)\times u_h, \overline{\phi_h}^h) - (\overline{\eta}^h\cdot\nabla u_h, \overline{\phi_h}^h) \\
&- (u_h\cdot\nabla\overline{\eta}^h, \overline{\phi_h}^h) + (\nabla(\overline{\eta}^h\cdot u_h), \overline{\phi_h}^h)
\end{aligned}\right\} \tag{3.23}$$

Next, proceed term by term.

$$\left.\begin{aligned}
((\nabla\times\overline{\eta}^h)\times u_h, \overline{\phi_h}^h) &= ((\nabla\times\overline{\eta}^h)\times(u_h - \overline{u_h}^h), \overline{\phi_h}^h) + ((\nabla\times\overline{\eta}^h)\times\overline{u_h}^h, \overline{\phi_h}^h) \\
&= -\alpha^2((\nabla\times\overline{\eta}^h)\times\Delta^h\overline{u_h}^h, \overline{\phi_h}^h) + ((\nabla\times\overline{\eta}^h)\times\overline{u_h}^h, \overline{\phi_h}^h)
\end{aligned}\right\} \tag{3.24}$$

Applying (2.12) and the inverse inequality (2.8), then Young's inequality, it follows

$$\begin{aligned}
-\alpha^2((\nabla\times\overline{\eta}^h)\times\Delta^h\overline{u_h}^h, \overline{\phi_h}^h) &\leq C\,\alpha^2\|\nabla\eta\|\|\overline{\phi_h}^h\|^{1/2}\|\nabla\overline{\phi_h}^h\|^{1/2}\|\nabla\Delta^h\overline{u_h}^h\| \\
&\leq C\left(\alpha\|\Delta^h\overline{u_h}^h\|\right)\|\nabla\eta\|\|\overline{\phi_h}^h\|^{1/2}\|\nabla\overline{\phi_h}^h\|^{1/2} \\
&\leq \frac{\nu}{48}\|\nabla\overline{\phi_h}^h\|^2 + C\left(\alpha^{4/3}\|\Delta^h\overline{u_h}^h\|^{4/3}\right)\|\nabla\eta\|^{4/3}\|\overline{\phi_h}^h\|^{2/3} \\
&= \frac{\nu}{48}\|\nabla\overline{\phi_h}^h\|^2 + C\,\big(\underbrace{\alpha^{2/3}\|\Delta^h\overline{u_h}^h\|^{2/3}\|\nabla\eta\|^{4/3}}_{a}\big)\big(\underbrace{\alpha^{2/3}\|\Delta^h\overline{u_h}^h\|^{2/3}\|\overline{\phi_h}^h\|^{2/3}}_{b}\big)
\end{aligned}$$

and apply $ab \leq \frac{2}{3}a^{3/2} + \frac{1}{3}b^3$:

$$\left.\begin{aligned} -\alpha^2((\nabla \times \overline{\eta}^h) \times \Delta^h \overline{u_h}^h, \overline{\phi_h}^h) &\leq \frac{\nu}{48}\|\nabla \overline{\phi_h}^h\|^2 \\ &\quad + C\,\alpha^2 \|\Delta^h \overline{u_h}^h\|^2 \|\overline{\phi_h}^h\|^2 + \frac{1}{3}\alpha\|\Delta^h \overline{u_h}^h\|\|\nabla\eta\|^2 \end{aligned}\right\} \tag{3.25}$$

Reference to (3.1) shows $\int_0^t \alpha^2 \|\Delta^h \overline{u_h}^h\|^2 \, dt' \leq C$, independent of α, h. This also creates a way of bounding the last term on the RHS of (3.25), using Hölder's inequality after integration in time:

$$\left.\begin{aligned} \int_0^t \alpha\|\Delta^h \overline{u_h}^h\|\|\nabla\eta\|^2 \, dt' &\leq \Big(\int_0^t \alpha^2 \|\Delta^h \overline{u_h}^h\|^2 dt'\Big)^{1/2} \Big(\int_0^t \|\nabla\eta\|^4 dt'\Big)^{1/2} \\ &\leq C\Big(\int_0^t \|\nabla\eta\|^4 dt'\Big)^{1/2} \end{aligned}\right\} \tag{3.26}$$

To bound the next term in (3.24), apply (2.11) and Young's inequality, along with the uniform bound on $\|\overline{u_h}^h\|$ from Lemma 3.1 to get:

$$\left.\begin{aligned} ((\nabla \times \overline{\eta}^h) \times \overline{u_h}^h, \overline{\phi_h}^h) &\leq C\,\|\overline{u_h}^h\|^{1/2} \|\nabla \overline{u_h}^h\|^{1/2} \|\nabla \overline{\eta}^h\| \|\nabla \overline{\phi_h}^h\| \\ &\leq C\,\|\nabla \overline{u_h}^h\| \|\nabla \overline{\eta}^h\|^2 + \frac{\nu}{48}\|\nabla \overline{\phi_h}^h\|^2 \end{aligned}\right\} \tag{3.27}$$

We've also applied $\|\nabla \overline{\eta}^h\| \leq \|\nabla \eta\|$ so that standard interpolation results will apply to the final result in Theorem 3.1. Next, apply the skew-symmetric property (integrate by parts) $-(\overline{\eta}^h \cdot \nabla u_h, \overline{\phi_h}^h) = (\overline{\eta}^h \cdot \nabla \overline{\phi_h}^h, u_h)$. Then by applying $u_h = (u_h - \overline{u_h}^h) + \overline{u_h}^h$, it follows:

$$-(\overline{\eta}^h \cdot \nabla u_h, \overline{\phi_h}^h) = (\overline{\eta}^h \cdot \nabla \overline{\phi_h}^h, u_h - \overline{u_h}^h) + (\overline{\eta}^h \cdot \nabla \overline{\phi_h}^h, \overline{u_h}^h) \tag{3.28}$$

Now use $u_h - \overline{u_h}^h = -\alpha^2 \Delta^h \overline{u_h}^h$ and apply (2.15) to get:

$$\begin{aligned} (\overline{\eta}^h \cdot \nabla \overline{\phi_h}^h, u_h - \overline{u_h}^h) &= -\alpha^2 (\overline{\eta}^h \cdot \nabla \overline{\phi_h}^h, \Delta^h \overline{u_h}^h) \\ &\leq C\,\alpha^2 \|\nabla \overline{\eta}^h\| \|\nabla \overline{\phi_h}^h\| \|\Delta^h \overline{u_h}^h\|^{1/2} \|\nabla \Delta^h \overline{u_h}^h\|^{1/2} \\ &\leq C\,\alpha^{3/2} \|\Delta^h \overline{u_h}^h\| \|\nabla \overline{\eta}^h\| \|\nabla \overline{\phi_h}^h\| \\ &= C\,\big(\alpha \|\Delta^h \overline{u_h}^h\|\big) \|\nabla \overline{\eta}^h\| \big(\alpha^{1/2} \|\nabla \overline{\phi_h}^h\|^{1/2}\big) \|\nabla \overline{\phi_h}^h\|^{1/2} \\ &\leq C\,\big(\alpha \|\Delta^h \overline{u_h}^h\|\big) \|\nabla \overline{\eta}^h\| \|\overline{\phi_h}^h\|^{1/2} \|\nabla \overline{\phi_h}^h\|^{1/2} \end{aligned}$$

From here, apply bounds exactly as shown to derive (3.25), yielding

$$\left.\begin{aligned} (\overline{\eta}^h \cdot \nabla \overline{\phi_h}^h, u_h - \overline{u_h}^h) &\leq \frac{\nu}{96}\|\nabla \overline{\phi_h}^h\|^2 \\ &\quad + C\,\alpha^2 \|\Delta^h \overline{u_h}^h\|^2 \|\overline{\phi_h}^h\|^2 + \frac{1}{3}\alpha\|\Delta^h \overline{u_h}^h\|\|\nabla\eta\|^2 \end{aligned}\right\} \tag{3.29}$$

Now apply (2.15) and Young's inequality to get

$$\begin{aligned} (\overline{\eta}^h \cdot \nabla \overline{\phi_h}^h, \overline{u_h}^h) &\leq C\,\|\nabla \overline{u_h}^h\|^{1/2} \|\nabla \eta\| \|\nabla \overline{\phi_h}^h\| \\ &\leq C\,\|\nabla \overline{u_h}^h\| \|\nabla \eta\|^2 + \frac{\nu}{96}\|\nabla \overline{\phi_h}^h\|^2 \end{aligned}$$

and combine with (3.28) and (3.29) to get the bound

$$\left.\begin{aligned}-(\overline{\eta}^h \cdot \nabla u_h, \overline{\phi_h}^h) \leq \frac{\nu}{48}\|\nabla\overline{\phi_h}^h\|^2 + C\,\|\nabla\overline{u_h}^h\|\|\nabla\eta\|^2 \\ + C\,\alpha\|\Delta^h\overline{u_h}^h\|\|\nabla\eta\|^2 + C\,\alpha^2\|\Delta^h\overline{u_h}^h\|^2\|\overline{\phi_h}^h\|^2\end{aligned}\right\} \quad (3.30)$$

We bound $-(u_h \cdot \nabla\overline{\eta}^h, \overline{\phi_h}^h)$ the same as in (3.24) - (3.27), by first applying $u_h = (u_h - \overline{u_h}^h) + \overline{u_h}^h$ and the corresponding bounds (2.15). Then from (3.25) and (3.27) we get

$$\left.\begin{aligned}-(u_h \cdot \nabla\overline{\eta}^h, \overline{\phi_h}^h) \leq \frac{\nu}{48}\|\nabla\overline{\phi_h}^h\|^2 + C\left\{\alpha\|\Delta^h\overline{u_h}^h\| + \|\nabla\overline{u_h}^h\|\right\}\|\nabla\eta\|^2 \\ + C\,\alpha^2\|\Delta^h\overline{u_h}^h\|^2\|\overline{\phi_h}^h\|^2\end{aligned}\right\} \quad (3.31)$$

Now, to finish bounding (3.23), we treat the final term by integrating by parts and inserting $u_h = \overline{u_h}^h - \alpha^2\Delta^h\overline{u_h}^h$ to get

$$\begin{aligned}-(\nabla(\overline{\eta}^h \cdot u_h), \overline{\phi_h}^h) &= (\overline{\eta}^h \cdot u_h, \nabla \cdot \overline{\phi_h}^h) \\ &= -\alpha^2(\overline{\eta}^h \cdot \Delta^h\overline{u_h}^h, \nabla \cdot \overline{\phi_h}^h) + (\overline{\eta}^h \cdot \overline{u_h}^h, \nabla \cdot \overline{\phi_h}^h)\end{aligned}$$

Now apply (2.17) to these terms to get bounds exactly as in (3.30).

$$\left.\begin{aligned}-(\nabla(\overline{\eta}^h \cdot u_h), \overline{\phi_h}^h) \leq \frac{\nu}{48}\|\nabla\overline{\phi_h}^h\|^2 + C\left\{\alpha\|\Delta^h\overline{u_h}^h\| + \|\nabla\overline{u_h}^h\|\right\}\|\nabla\eta\|^2 \\ + C\,\alpha^2\|\Delta^h\overline{u_h}^h\|^2\|\overline{\phi_h}^h\|^2\end{aligned}\right\} \quad (3.32)$$

Now combine (3.27) and (3.30) - (3.32) to get a final bound for (3.23):

$$\left.\begin{aligned}-(\overline{\eta}^h \times (\nabla \times u_h), \overline{\phi_h}^h) \leq \frac{\nu}{12}\|\nabla\overline{\phi_h}^h\|^2 + C\left\{\alpha\|\Delta^h\overline{u_h}^h\| + \|\nabla\overline{u_h}^h\|\right\}\|\nabla\eta\|^2 \\ + C\,\alpha^2\|\Delta^h\overline{u_h}^h\|^2\|\overline{\phi_h}^h\|^2\end{aligned}\right\} \quad (3.33)$$

Combining (3.22) and (3.33) provides a bound for (3.15), and in turn with (3.14) provides bounds for (3.9). Therefore, combining all these results, we can re-write (3.9) as follows:

$$\left.\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{\|\overline{\phi_h}^h\|^2 + \alpha^2\|\nabla\overline{\phi_h}^h\|^2\Big\} + \nu\Big\{\|\nabla\overline{\phi_h}^h\|^2 + \alpha^2\|\Delta^h\overline{\phi_h}^h\|^2\Big\} + \gamma\|\nabla\cdot\phi_h\|^2 \\ &\leq \|p - q_h\|\|\nabla\overline{\phi_h}^h\| + \|\frac{\partial\eta}{\partial t}\|\|\overline{\phi_h}^h\| + \nu\|\nabla\eta\|\|\nabla\overline{\phi_h}^h\| \\ &+ \frac{4\nu}{12}\|\nabla\overline{\phi_h}^h\|^2 + \frac{\nu}{2}\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2 + \frac{\gamma}{2}\|\nabla\cdot\phi_h\|^2 \\ &+ C\left\{1 + \alpha\|\Delta^h\overline{u_h}^h\| + \|\nabla\overline{u_h}^h\|\right\}\|\nabla\eta\|^2 \\ &+ C\left\{\|\nabla u\|^4 + \|\nabla\overline{u}\|^4 + \|\nabla\times u\|_\infty^2 + \alpha^2\|\Delta^h\overline{u_h}^h\|^2\right\}\|\overline{\phi_h}^h\|^2 \\ &+ C\left\{\alpha^4\|\Delta\overline{u}\|^2 + h^{2k+2}|\overline{u}|_{k+1}^2\right\}\end{aligned}\right\} \quad (3.34)$$

After applying Young's inequality to (3.34), subtract $\frac{\nu}{2}\Big\{\|\nabla\overline{\phi_h}^h\|^2 + \alpha^2\|\Delta^h\overline{\phi_h}^h\|^2\Big\}$ and $\frac{\gamma}{2}\|\nabla\cdot\phi_h\|^2$

from both sides of (3.34) and integrate in time to get:

$$\frac{1}{2}\Big\{\|\overline{\phi_h}^h(t)\|^2+\alpha^2\|\nabla\overline{\phi_h}^h(t)\|^2\Big\}+\frac{\nu}{2}\int_0^t\Big\{\|\nabla\overline{\phi_h}^h\|^2+\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2\Big\}\,dt'+\frac{\gamma}{2}\int_0^t\|\nabla\cdot\phi_h\|^2\,dt'$$
$$\le\frac{1}{2}\Big\{\|\overline{\phi_h}^h(0)\|^2+\alpha^2\|\nabla\overline{\phi_h}^h(0)\|^2\Big\}+C\int_0^t\{\|p-q_h\|^2+\|\frac{\partial\eta}{\partial t}\|^2\}\,dt'$$
$$+C\int_0^t\{1+\alpha\|\Delta^h\overline{u_h}^h\|+\|\nabla\overline{u_h}^h\|\}\|\nabla\eta\|^2\,dt'$$
$$+C\int_0^t\{\|\nabla u\|^4+\|\nabla\overline{u}\|^4+\|\nabla\times u\|_\infty^2+\alpha^2\|\Delta^h\overline{u_h}^h\|^2\}\|\overline{\phi_h}^h\|^2\,dt'$$
$$+C\int_0^t\{\alpha^4\|\Delta\overline{u}\|^2+h^{2k+2}|\overline{u}|_{k+1}^2\}\,dt'$$

Now multiply through by 2 and apply Gronwall's inequality. The third line is dealt with using the technique shown in (3.26). Recall $\eta=u-\tilde{u},\tilde{u}\in X^h$ arbitrarily chosen. Now call $\tilde{u}=v_h\in X^h$ and we have:

$$\left.\begin{aligned}&\Big\{\|\overline{\phi_h}^h(t)\|^2+\alpha^2\|\nabla\overline{\phi_h}^h(t)\|^2\Big\}+\nu\int_0^t\Big\{\|\nabla\overline{\phi_h}^h\|^2+\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2\Big\}\,dt'\\&\quad+\gamma\int_0^t\|\nabla\cdot\phi_h\|^2\,dt'\le\frac{1}{2}\Big\{\|\overline{\phi_h}^h(0)\|^2+\alpha^2\|\nabla\overline{\phi_h}^h(0)\|^2\Big\}\\&\quad+C\int_0^t\Big\{\|p-q_h\|^2+\|\frac{\partial(u-v_h)}{\partial t}\|^2+\|\nabla(u-v_h)\|^2\Big\}\,dt'\\&\quad+C\Big(\int_0^t\|\nabla\eta\|^4\,dt'\Big)^{1/2}+C\Big\{\alpha^4+h^{2k+2}\int_0^t|\overline{u}|_{k+1}^2\,dt'\Big\}\end{aligned}\right\}\tag{3.35}$$

To derive the final result, begin by applying the triangle inequality:

$$\|u-u_h\|^2+\nu\int_0^t\|\nabla(u-u_h)\|^2\,dt'+\gamma\int_0^t\|\nabla\cdot(u-u_h)\|^2\,dt'$$
$$\le2\|u-v_h\|^2+2\nu\int_0^t\|\nabla(u-v_h)\|^2\,dt'+2\|\phi_h\|^2+2\nu\int_0^t\|\nabla\phi_h\|^2\,dt'$$
$$+2\gamma\int_0^t\|\nabla\cdot\eta\|^2\,dt'+2\gamma\int_0^t\|\nabla\cdot\phi_h\|^2\,dt'$$

Now apply Lemma 2.1 and Lemma 2.2 to get:

$$\left.\begin{aligned}&\|u-u_h\|^2+\nu\int_0^t\|\nabla(u-u_h)\|^2\,dt'+\gamma\int_0^t\|\nabla\cdot(u-u_h)\|^2\,dt'\\&\quad\le2\|u-v_h\|^2+2\nu\int_0^t\|\nabla(u-v_h)\|^2\,dt'+2\gamma\int_0^t\|\nabla\cdot(u-v_h)\|^2\,dt'\\&\quad+C\Big\{\|\overline{\phi_h}^h\|^2+\alpha^2\|\nabla\overline{\phi_h}^h\|^2+\nu\int_0^t\{\|\nabla\overline{\phi_h}^h\|^2+\alpha^2\|\Delta^h\overline{\phi_h}^h\|^2\}\,dt'\Big\}\\&\quad+2\gamma\int_0^t\|\nabla\cdot\phi_h\|^2\,dt'\end{aligned}\right\}\tag{3.36}$$

Using this and (3.35) we can get the final bound needed on (3.36). As in the NSE case, (e.g. [16]), apply the triangle inequality $\|\phi_h(0)\|\le\|(u-u_h)(0)\|+\|(u-v_h)(0)\|$, then take the infimum over $v_h\in V^h,q_h\in Q^h$ and the supremum over $t\in[0,T]$ to get the final result.

□

Corollary 3.2. *Let (X_h, Q_h) be finite element spaces corresponding to Taylor-Hood elements, with a locally quasi-uniform family of meshes. Choose $\alpha = h$ for each mesh. If $u(\cdot, t)$ is a solution of the NSE satisfying (3.3), then the corresponding NS-α approximations converge at the rate $O(h^2)$ in the H^1 norm.*

Proof. Using Theorem 3.1, this is a standard exercise in finite element analysis, (e.g. [16, 25]).

4 2-D Computational Testing

4.1 Convergence rate verification

In this section we provide computational verification of the convergence rate of the alpha model as predicted by Theorem 3.1. Approximations of known solutions to the Navier-Stokes equations are calculated in three cases: flow around a circle under no-slip boundary conditions, flow on the unit square with no-slip boundary conditions and flow on the unit square with periodic boundary conditions. The convergence using the full Crank-Nicholson method for small time steps should yield convergence rates comparable to those predicted by Theorem 3.1. The fully discrete method used herein follows as Algorithm 4.1.

Algorithm 4.1 (Crank-Nicholson Scheme for NS-α)**.** *Let $\Delta t > 0$, $(w_0, q_0) \in (X_h, Q_h)$, $f \in X^*$ and $M := \frac{T}{\Delta t}$ and $(w_{-1}, q_{-1}) = (w_0, q_0)$. For $n = 0, 1, 2, \cdots, M-1$, find $(w_h^{n+1}, q_h^{n+1}) \in (X_h, Q_h)$ satisfying*

$$\frac{1}{\Delta t}(w_h^{n+1} - w_h^n, v_h) + \frac{1}{2}(\overline{w_h^{n+1}}^h \times \nabla \times w_h^{n+1}, v_h) + \frac{1}{2}(\overline{w_h^n}^h \times \nabla \times w_h^n, v_h) - (q_h^{n+1/2}, \nabla \cdot v_h)$$
$$+ \nu(\nabla w_h^{n+1/2}, \nabla v_h) + \gamma(\nabla \cdot w_h^{n+1/2}, \nabla \cdot v_h) = (f^{n+1/2}, v_h) \ \forall \ v_h \in X_h \quad (4.1)$$
$$(\nabla \cdot w_h^{n+1}, \chi_h) = 0 \ \forall \chi_h \in Q_h$$

All calculations were performed using FreeFEM++ [14], with a triangular Delaunay-Voronoi mesh and Hood-Taylor finite element space. At each time step, the full non-linear problem is solved iteratively as a fixed point problem, using Oseen linearization. The resulting matrix systems were solved using the FreeFEM UMFPACK solver. The error analysis performed in Theorem 3.1 shows that by choosing $\alpha = h$ we should expect to see an effective convergence rate of order h^2. In all cases the scaling $\alpha = h$ was chosen on each mesh, with h calculated as the maximum diameter of a triangle in the mesh. The grad-div stabilization parameter is $\gamma = 1$, which in some cases is larger than the scaling $O(\nu)$ from Section 3. However, in practice the divergence of both $u_h, \overline{u_h}^h$ were found to be very small ($\|\cdot\|_\infty \approx 10^{-18}$) with this choice of γ. Figure 1 shows an example of the mesh and calculated solution. Errors are calculated in $L^\infty(0, T; L^2(\Omega))$ and $L^\infty(0, T; H^1(\Omega))$ with convergence rates calculated using the rule $err \approx C \cdot h^p$. These norms are stronger than necessary for verification of Theorem 3.1.

4.1.1 Circular Domain, No-slip Boundary Conditions

A smooth, divergence free 2-D velocity field in the unit circle with zero boundary conditions was derived. The corresponding pressure and driving force f(x,y) was subsequently obtained from this exact solution. Indeed, choose $u(x, y, t) = 2^{-t}(1 - x^2 - y^2) < y, -x >$ and $p(x, y, t) = -\frac{1}{6}2^{-2t}\left((1 - x^2 - y^2)^3 - \frac{1}{4}\right)$. Then differentiation gives $\nabla \cdot u = 0$ and

$$\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla p =$$

$$2^{-t}\left(ln2(1 - x^2 - y^2) - 8\nu\right) < y, -x >= f(x, y, t)$$

This choice of pressure satisfies $\int_\Omega p \; d\Omega = 0$, and curl(u) is uniformly bounded on (0,T). The viscosity is $\nu = 1$.

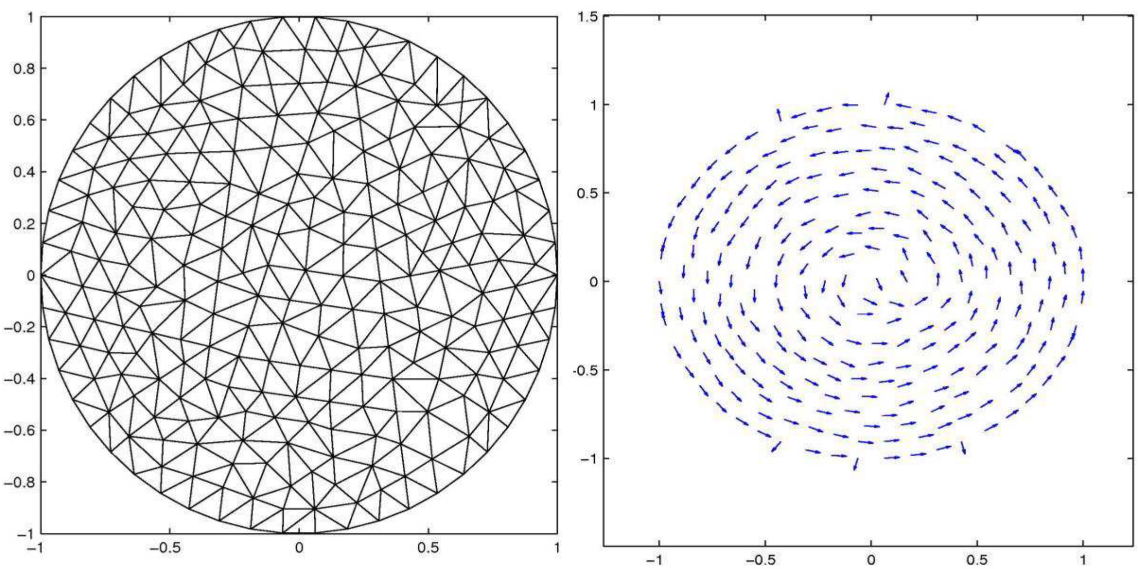

Figure 1: Left: example mesh, Right: normalized velocity solution.

h	$\left\|u - u^h\right\|_{L^2}$	Rate	$\left\|u - u^h\right\|_{H^1}$	Rate
0.393	1.22e-2	—-	1.26e-1	—-
0.191	2.98e-3	2.07	4.05e-2	1.67
0.101	7.32e-4	1.96	1.33e-2	1.55
0.051	1.81e-4	2.32	4.53e-3	1.79
0.030	4.96e-5	2.37	1.71e-3	1.79

Table 1: L^2 **and** H^1 **errors and rates for circular flow.**

Table 1 summarizes the results. There is an error contribution due to the polygonal approximation of the circular boundary, and due to the time discretization. Error contribution due to time discretization with $dt = 0.01$ is assumed to be small, which was determined by decreasing the time-step size until the errors began decreasing slowly. These factors may partially explain why the convergence rate with respect to the H^1 norm are generally below 2. In particular, since the true solution vanishes at the boundary, the error contributions should decrease substantially as the boundary becomes more accurately approximated. Computations using smaller meshes and possibly smaller time steps must be used to verify optimality here.

4.1.2 Square Domain, No-slip Boundary Conditions

An ideal convergence result should be possible using a square domain where boundary approximation is exact. Choosing a solution

$$
\begin{aligned}
u_1(x,y,t) &= x^2(x-1)^2(2y^3-3y^2+y)\\
u_2(x,y,t) &= -y^2(y-1)^2(2x^3-3x^2+x)\\
p(x,y,t) &= 0
\end{aligned}
$$

differentiation gives $\nabla \cdot u = 0$. This choice of pressure satisfies $\int_\Omega p \, d\Omega = 0$, and curl(u) is uniformly bounded on (0,T). The viscosity is $\nu = 10^{-3}$ with final time $T = 1$. A time step size $dt = 5e-3$ was chosen. Table 2 summarizes the results.

h	$\|u-u^h\|_{L^2}$	Rate	$\|u-u^h\|_{H^1}$	Rate
1.088e-1	9.620e-7	—-	4.141e-4	—-
5.657e-2	7.225e-8	3.96	1.131e-4	1.98
2.886e-2	5.096e-9	3.94	2.952e-5	2.00
1.458e-2	4.845e-10	3.45	7.539e-6	2.00

Table 2: $L^\infty(0,T)$ **errors and convergence rates, square domain.**

Interestingly we obtain in $L^\infty(0,T;L^2(\Omega))$ a possible superconvergence for this problem. The decreasing rate may be a result of asymptotically approaching an optimal rate h^3, or an artifact resulting from the fixed time step size being too large to verify the superconvergent rate. However, the optimal convergence rate in $L^\infty(0,T;H^1(\Omega))$ matches exactly the prediction of Theorem 3.1.

4.1.3 Square Domain, Periodic Boundary Conditions

The analysis in this report can be extended to periodic boundary conditions. Convergence rate verification in this case is performed using Taylor-Green vortices on the unit square as a true solution, [13, 3]. Taylor-Green vortices are a solution of the NSE with driving force $f \equiv 0$, given by:

$$
\begin{aligned}
u_1(x,y,t) &= -cos(N\pi x)sin(N\pi y)e^{-2N^2\pi^2\nu t}\\
u_2(x,y,t) &= cos(N\pi y)sin(N\pi x)e^{-2N^2\pi^2\nu t}\\
p(x,y,t) &= -\frac{1}{4}cos(2N\pi x)cos(2N\pi y)e^{-2N^2\pi^2\nu t}.
\end{aligned}
$$

We choose $N = 2$. The viscosity parameter is $\nu = 10^{-2}$ with final time $T = 1$. Table 3 summarizes the results using Algorithm 4.1.

h	$\|u-u^h\|_{L^2}$	Rate	$\|u-u^h\|_{H^1}$	Rate
1.28565e-1	5.94047e-2	—-	9.51165e-1	—-
6.73435e-2	1.87157e-2	1.786	3.16293e-1	1.703
3.44930e-2	4.20793e-3	2.231	7.64096e-2	2.123
1.74594e-2	9.45320e-4	2.193	1.77568e-2	2.143

Table 3: $L^\infty(0,T)$ **errors and convergence rates, periodic boundary conditions.**

The optimal convergence rate h^2 in $L^\infty(0,T;H^1(\Omega))$ is achieved, again verifying Theorem 3.1. In $L^\infty(0,T;L^2(\Omega))$ a suboptimal rate is observed. Given the similar convergence rates for the circular flow it must be concluded that some special properties exist for the problem of flow on the unit square with no-slip boundary conditions which resulted in the superoptimal convergence rate observed there. Computational evidence suggests in general only a suboptimal rate is possible in $L^\infty(0,T;L^2(\Omega))$. This does not contradict the analysis herein.

4.2 Prediction of Coherent Vortices

In this section Algorithm 4.1 is used to approximate solutions of the NSE for the forward-backward facing step problem (see [10] for details). For Reynolds numbers $500 \leq Re \leq 700$ the solution of the NSE for this problem features a recirculation zone behind the step, and vortices separate and detach, propagating downstream over time. It is known that using Leray regularization for this problem with filtering radius scaling $\alpha = h$ retards vortex separation, [1]. There is a question of scaling the filter parameter near walls but is beyond this report. The filter is chosen to be a constant on the domain, $\alpha = 0.05(h + h_{min})$, with h_{min} being the diameter of the smallest triangle, thus corresponding to an order of magnitude less than the average of the maximum and minimum triangle diameters. With this choice of filter scale, using shape regular elements Theorem 3.1 still holds. Less regularity will be imposed by the NS-α model in the recirculation zone compared to choosing $\alpha = h$, improving prediction of vortex separation.

A non-uniform mesh is chosen, refined near the step as shown in Figure 2. Calculations were performed for $\nu = 1/600$ with grad-div stabilization parameter $\gamma = 1$ and time step $dt = 0.005$. Figure 3 shows the streamline plot generated using the NS-α model at time $t = 32$. For comparison, the calculations were repeated using the standard finite element discretization of the NSE and convective form of the non-linearity on the same mesh, shown in Figure 4. In both cases two separate vortices are resolved, as expected at this time step. The NSE discretization predicts a small vortex adherent to the back wall of the step and larger one immediately adjacent, downstream. The vortices predicted by the NS-α model are elongated and generally larger compared to those predicted by the NSE discretization. Evidently the NS-α model predicts a faster vortex shedding kinetic, as well as a snowball type growth of vortex diameter while propagating downstream. Calculations using smaller meshes for this problem are computationally expensive and beyond this report, hence no comment can be made regarding the overall quality of calculations for the step problem using NS-α. However, the formation and separation of distinct vortices is clear and demonstrates to some extent the ability to predict time dependent behavior using NS-α.

Remark 4.1. *An important observation is that repeating the step problem calculations using the standard NS-α model without imposing the extra constraint $\nabla \cdot u = 0$ yields a numerical method which fails to converge during the nonlinear solve. Using both fixed-point and Newton's methods, no meaningful output is obtained. It is critical using NS-α to impose $\nabla \cdot \overline{u} = \nabla \cdot u = 0$ for practical computations.*

5 Conclusions

Finite element discretizations of the continuous-time NS-α model have a convergence rate $O(h^2)$ in the H^1 norm if we choose $\alpha = h$, consistent with Theorem 3.1. This has been

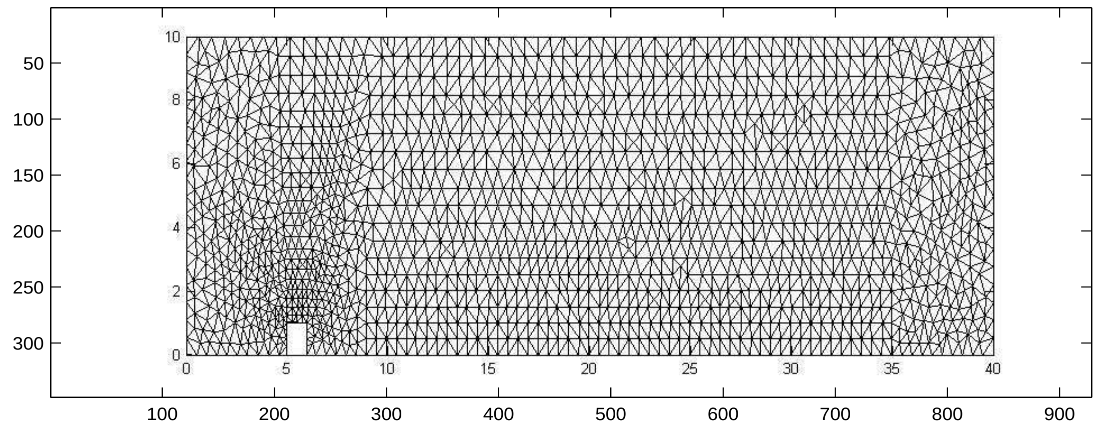

Figure 2: Example mesh, moderately refined near the step.

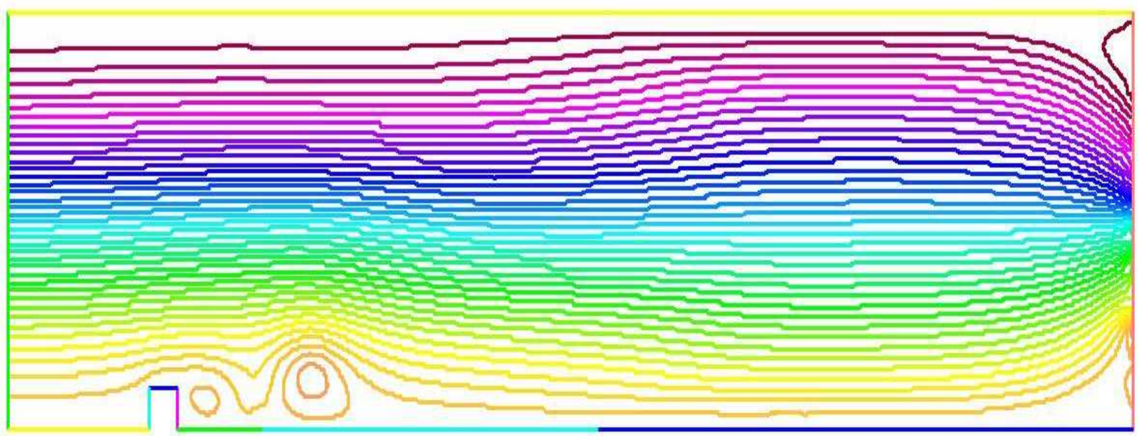

Figure 3: NS-α approximation streamline plot, $t = 32$.

observed in 2D experiments herein. Further study is needed to draw conclusions as to optimality of the NS-α model in the L^2 norm. It has been demonstrated that the model qualitatively predicts time dependent behavior, using the constraint $\nabla \cdot u = 0$.

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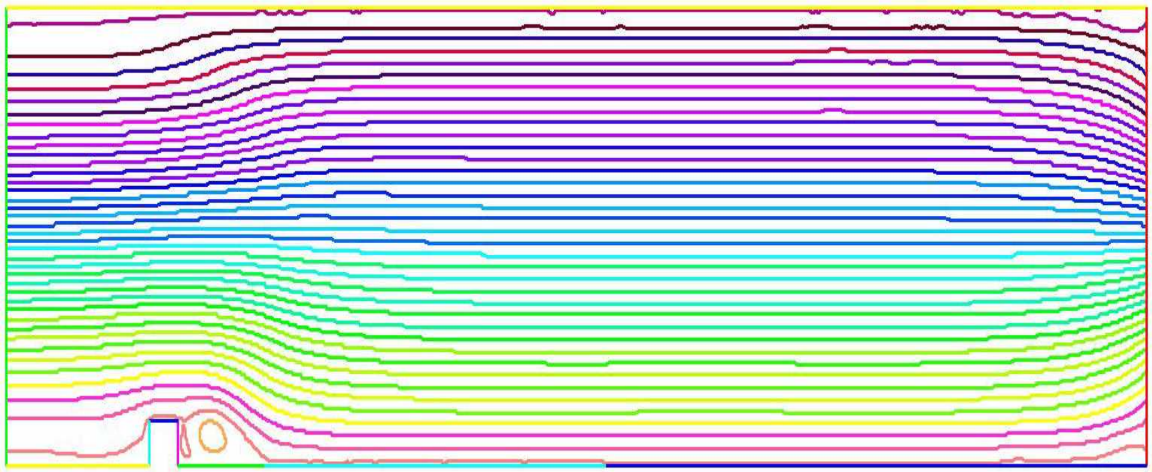

Figure 4: NSE finite element approximation streamline plot, $t = 32$.

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