Decoupled scheme with different time step sizes for the evolutionary Stokes-Darcy model

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Abstract

In this report, a partitioned time stepping algorithm which allows different time steps in the fluid region and the porous region is analyzed for the *fully evolutionary* Stokes-Darcy problem. This method requires only one, uncoupled Stokes and Darcy sub-physics and sub-domain solve per time step. Under a modest time step restriction of the form $\triangle t \leq C$ where $C = C(\textit{physical parameters})$ we prove zero-stability of the method. We also derive error estimates. Numerical tests given confirming the convergence theory and demonstrating the computational efficiency of the partitioned method.

Keywords: Stokes-Darcy coupling, decoupled method, different time step.

1 Introduction

The transport of substances coupling between surface water and groundwater is an important problem of great current interest. The essential features of estimating penetration of a plume of pollution from surface water to ground water and remediation thereafter are that (i) the coupled problems in the fluid and porous media sub-regions are both inherently time dependent, (ii) the flows in the two regions act with different characteristic speeds, (iii) the physical processes are sufficiently different that codes optimized for each individual sub-process ultimately will need to be used to solve the coupled problem, and (iv) the large domains involved and the need to compute for several turn-over times to obtain reliable statistics requires calculations over long time intervals for large systems (often arising from relatively coarse meshes). With these issues in mind, we analyze herein

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an asynchronous, uncoupled, partitioned method for the *fully evolutionary* Stokes-Darcy problem. The method allows different time steps in the two subregions (such methods are often called "asynchronous coupling" in geophysics) and requires only one, uncoupled Stokes solve and one Darcy solve per time step (with no iteration or construction of a fully coupled problem). The partitioning is based on simply lagging the interfacial coupling terms following a method analyzed by Mu and Zhu [16], see also [1] for its use in other applications. Connecting the different time steps at the interface is adapts an idea developed by Connors and Howell [6] for atmosphere-ocean coupling in climate models. The essential difficulty of both lagging terms and interpolation between meshes and time steps is doing so without creation of non-physical system energy.

The mathematical model consists of the evolutionary Stokes equations in the fluid region coupled with the evolutionary Darcy equations in the porous medium, [8, 12, 14, 17, 18]. The key part is the interface coupling conditions of conservation of mass across the interface, balance of forces and the (tangential) Beavers-Joseph-Saffman conditions [2]. Consider thus a Stokes flow in Ω_f coupled with a porous media flow in Ω_p, where $\Omega_f, \Omega_p \subset R^d (d = 2 \text{ } or \text{ } 3)$ are bounded domains, $\Omega_f \cap \Omega_p = \emptyset$, and $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$. Denote by $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, $\mathbf{n}_f$ and $\mathbf{n}_p$ the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, respectively, and $\tau_i, i = 1, \cdots, d-1$, the unit tangential vectors on the interface Γ. Note that $\mathbf{n}_p = -\mathbf{n}_f$ *on* Γ, see Figure 1 below.

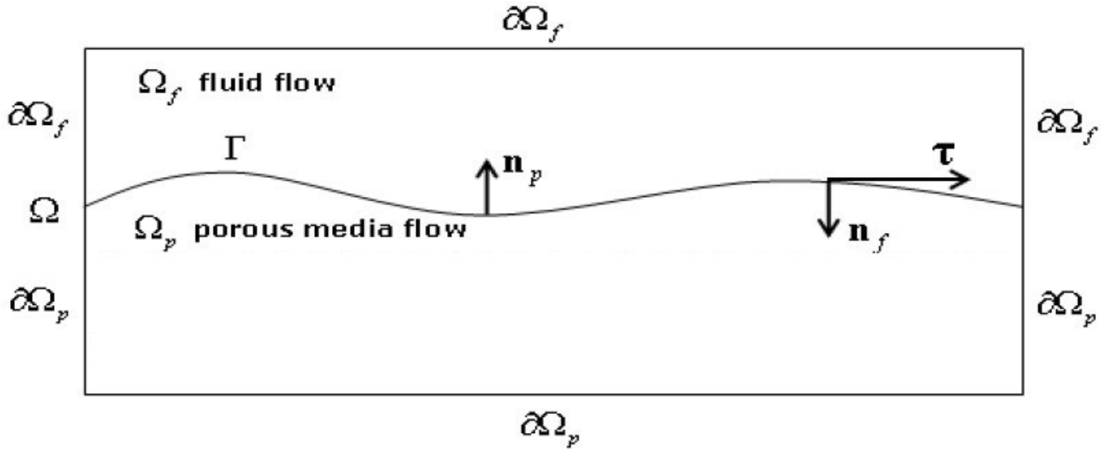

Figure 1: The global domain Ω consisting of the fluid region Ω_f and the porous media region Ω_p, separated by the interface Γ.

Let $T \geq 0$ be a finite time, the fluid flow is governed by the Stokes equations on Ω_f:

$$u_t - \nu\Delta u + \nabla p = f \quad \texttt{in } \Omega_f \times (0, T], \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \texttt{in } \Omega_f \times (0, T], \tag{1.2}$$

$$u(x, 0) = u_0 \quad \texttt{in } \Omega_f, \tag{1.3}$$

$$u = 0 \quad \texttt{on } \partial\Omega_f \setminus \Gamma, \tag{1.4}$$

where $u(x,t)$ represents the velocity of the fluid flow in Ω_f, $p(x,t)$ the pressure, $f(x,t)$ the external force, and ν the kinematic viscosity.

The porous media flow is governed by the following equations on Ω_p:

$$S_0\phi_t + \nabla\cdot q = g \quad \texttt{in } \Omega_p\times(0,T], \tag{1.5}$$
$$q = -K\nabla\phi \quad \texttt{in } \Omega_p\times(0,T], \tag{1.6}$$
$$u_p = \frac{q}{n} \quad \texttt{in } \Omega_p\times(0,T], \tag{1.7}$$
$$\phi(x,0) = \phi_0 \quad \texttt{in } \Omega_p, \tag{1.8}$$
$$\phi = 0 \quad \texttt{on } \partial\Omega_p\setminus\Gamma, \tag{1.9}$$

where ϕ is the piezometric head, q is the specific discharge defined as the volume of the fluid flowing per unit time through a unit cross-sectional area normal to the direction of the flow, ξ is the fluid velocity in Ω_p, S_0 is the specific mass storativity coefficient, K represents the hydraulic conductivity tensor, n is the volumetric porosity, and g is the source term. Note that $\phi = z + \frac{P_p}{\rho g}$, the sum of elevation head plus pressure head, where P_p is the pressure of the fluid in Ω_p, ρ is the density of the fluid, g is the gravitational acceleration. (The usage of g as gravitational vector or source term will be clear from the context in which it occurs.). Further, z is the elevation from a reference level. The presentation of the coupled problem with separate discretizations and differing time steps involves substantial notation. We therefore make some simplifying assumptions to reduce the notational complexity. In particular, we assume $z = 0$ and that $\mathbf{K} = diag(K,\cdots,K)$ with $K \in L^\infty(\Omega_p), K > 0$, which implies that the porous media is homogeneous. By using Darcy's law, (1.5) can be rewritten in the parabolic form

$$S_0\phi_t - \nabla\cdot(K\nabla\phi) = g \quad \texttt{in } \Omega_p\times(0,T], \tag{1.10}$$
$$\phi(x,0) = \phi_0 \quad \texttt{in } \Omega_p. \tag{1.11}$$

For the Stokes-Darcy model, the interface coupling conditions is a key part, the following interface conditions have been extensively considered and studied:

$$u\cdot\mathbf{n}_f + u_p\cdot\mathbf{n}_p = 0 \quad \texttt{on } \Gamma\times(0,T], \tag{1.12}$$
$$p - \nu\mathbf{n}_f\frac{\partial u}{\partial\mathbf{n}_f} = \rho g\phi \quad \texttt{on } \Gamma\times(0,T], \tag{1.13}$$
$$-\nu\tau_i\frac{\partial u}{\partial n_f} = \frac{\alpha}{\sqrt{\tau_i\cdot K\tau_i}}u\cdot\tau_i,\ i = 1,\cdots,d-1 \quad \texttt{on } \Gamma\times(0,T], \tag{1.14}$$

where α is a positive parameter depending on the properties of the porous medium and must be experimentally determined. The first interface condition (1.12) ensures the mass conservation across the interface Γ, and using (1.6) and (1.7), it can be rewritten as

$$u\cdot\mathbf{n}_f = \frac{\mathbf{K}}{n}\frac{\partial\phi}{\partial\mathbf{n}_p} \quad on \ \Gamma\times(0,T]. \tag{1.15}$$

The second condition (1.13) is the balance of the normal forces across the interface.

In the last ten years there has been an explosion of work on numerical analysis of coupling surface water to ground water. For a comprehensive overview of other work on this important problem, see [9] and the 125 references therein. Much of the work has studied the equilibrium problem, e.g., [8, 9, 14]. Various quasi-static models (not considered herein) have also been proposed with time dependence in one region and in the other at equilibrium. To our knowledge, justification of the quasi-static assumption based on the rates of return to equilibrium in either sub problem in the context of the fully evolutionary setting is still open. Among the many fewer papers (so far) on the numerical analysis of the *fully evolutionary* Stokes-Darcy problem (considered herein), Mu and Zhu [16] study a partitioned method which we build upon herein. Cao, Gunzburger, Hu, Hua, Wang and Zhao [3, 4] study a fully, monolithically coupled implicit method for the much harder and physically more accurate case of Beavers-Joseph coupling conditions (without Saffman's simplification).

2 Variational formulation of the continuous problem

Denote $W = H_f \times H_p$ and $Q = L^2(\Omega_f)$, where

$$H_f = \{v \in (H^1(\Omega_f))^d : v = 0 \text{ on } \partial\Omega_f \setminus \Gamma\}, H_p = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p \setminus \Gamma\}.$$

The space $L^2(D)$, where $D = \Omega_f$ or Ω_p, is equipped with the usual L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $||\cdot||_{L^2} \triangleq ||\cdot||_0$. The spaces H_f and H_p are equipped with the following norms:

$$||u||_{H_f} = || \bigtriangledown u||_0 = \sqrt{(\nabla u, \nabla u)} \qquad \forall\ u \in H_f, \tag{2.1}$$

$$||\phi||_{H_p} = || \bigtriangledown \phi||_0 = \sqrt{(\nabla \phi, \nabla \phi)} \qquad \forall\ u \in H_p. \tag{2.2}$$

We equip the space W with the following norms: $\forall\ \mathbf{u} = (u, \phi) \in W$,

$$||\mathbf{u}||_0 = \sqrt{n(\mathbf{u},\mathbf{u})_{\Omega_f} + \rho g S_0(\phi,\phi)_{\Omega_p}}, \tag{2.3}$$

$$||\mathbf{u}||_W = \sqrt{n\nu(\mathbf{u},\mathbf{u})_{\Omega_f} + \rho g \mathbf{K}(\phi,\phi)_{\Omega_p}} \approx ||\nabla u||_0, \tag{2.4}$$

where $(\cdot,\cdot)_D$ refers to the scalar product $(\cdot,\cdot)$ in the corresponding domain D for $D = \Omega_f$ or Ω_p, and $\approx$ refers to equivalent norms.

For simplicity, we assume n, ρ, g, S_0, ν and K are constants, without loss of generality we assume n, ρ, g and S_0 are positive and $O(1)$, in particular that

$$K(x) \geq k_{min} > 0.$$

The given data u_0, ϕ_0, f and g are assumed to be smooth enough.

The weak formulation of the time-dependent Stokes-Darcy model reads as follows: find $\mathbf{u} = (u, \phi) \in W$ and $p \in Q$, such that , $\forall\ t \in (0, T]$,

$$\begin{aligned} (\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \ \ \texttt{in}\ \Omega, \\ b(\mathbf{u}, q) &= 0 \ \ \texttt{in}\ \Omega, \\ \mathbf{u}(0) &= \mathbf{u}_0 \ \ \texttt{on}\ \partial\Omega, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= a_f(u, v) + a_p(\phi, \psi) + a_\Gamma(\mathbf{u}, \mathbf{v}), \\ a_f(u, v) &= (\nu\nabla u, \nabla v)_{\Omega_f} + \sum_{i=1}^{d-1} \int_\Gamma \frac{\alpha}{\sqrt{\tau_i \cdot K\tau_i}} (u \cdot \tau_i)(v \cdot \tau_i), \\ a_p(\phi, \psi) &= (K\nabla\phi, \nabla\psi)_{\Omega_p}, \\ a_\Gamma(\mathbf{u}, \mathbf{v}) &= \int_\Gamma \phi v \cdot \mathbf{n}_f - \psi u \cdot \mathbf{n}_f, \\ b(\mathbf{v}, p) &= -(p, divv)_{\Omega_f}, \\ (\mathbf{f}, \mathbf{v}) &= (f, v)_{\Omega_f} + (g, \psi)_{\Omega_p}. \end{aligned}$$

The well-posedness of the mixed Stokes-Darcy model(2.5) can be found in [7, 8, 14] for the stationary case and is assumed to hold similarly for the non-stationary case. In the paper, we focus on its numerical solution.

There is one known partitioned method of Mu and Zhu [16] for uncoupling the Stokes-Darcy problem in which subdomain terms are discretized by the implicit method in time and the coupling terms by the explicit method. Herein we extend the partitioned method to allow for different size time steps for the decoupled subproblems, say $\triangle t$ on Ω_f and $\triangle s$ on Ω_p, with any integer ratio $n = \triangle s/\triangle t$ between them. The reason for using different time step size is that physical processes happen at different rate, e.g., [10] whose analysis is consistent with the intuition that fluid flow is faster than that in the porous medium. The methods extend immediately to the case where the regions of small and large time steps are reversed. For the other point of view, the natural CFL condition demands $\frac{v\triangle t}{h} \le 1$ where v denote the velocity in the sub-domain. Since different domain have different flow velocities, practical computing often will require different time steps and even possibly adapting $\triangle t$ separately in each sub-region..

It is known [7, 16] that $a_f(\cdot, \cdot)$, $a_p(\cdot, \cdot)$, and $a_\Gamma(\cdot, \cdot)$ are continuous and coercive and $a(\cdot, \cdot)$ is continuous. coercive

$$a_f(u, v) \le C||u||_{H_f}||v||_{H_f}, \ \ a_f(v, v) \ge \nu||v||^2_{H_f}, \quad \forall\ u, v \in H_f, \tag{2.6}$$

$$a_p(\phi, \psi) \le C||\phi||_{H_p}||\psi||_{H_p}, \ \ a_p(\psi, \psi) \ge k_{min}||\psi||^2_{H_p}, \quad \forall\ \phi, \psi \in H_p, \tag{2.7}$$

where α_0 is a positive constant. Furthermore, $a_\Gamma(\cdot, \cdot)$ satisfy the following properties:

$$a_\Gamma(\mathbf{u}, \mathbf{v}) = -a_\Gamma(\mathbf{v}, \mathbf{u}), \qquad a_\Gamma(\mathbf{u}, \mathbf{u}) = 0, \qquad \forall\ \mathbf{u}, \mathbf{v} \in W. \tag{2.8}$$

There are many appealing reasons as discussed in [15] that have led to active research models so that existing single-model solvers can be applied locally with little extra computational and software overhead. In this paper, a decoupling approach with different time step in each domain is proposed for mixed Stokes-Darcy problem. The rest of the paper is organized as follows. Both coupled and decoupled algorithms are presented in Section 3. The zero-stability of the decoupled algorithm is given in Section 4. In Section 5, we analyze the error estimation. Numerical tests are reported in Section 6, followed by conclusions in Section 7.

3 Numerical algorithms

We consider a triangulation $\mathcal{T}_h$ of the domain $\overline{\Omega}_f \cup \overline{\Omega}_p$, depending on a positive parameter $h > 0$, made up of triangles if $d = 2$, or tetrahedra if $d = 3$.

Let $W_h = H_{fh} \times H_{ph} \subset W$ and $Q_h \subset Q$ denote the finite element subspaces. The finite element spaces H_{fh} and Q_h approximating velocity and pressure in the fluid flow region are assumed to satisfy the well known discrete inf-sup condition: there exists a positive constant β, independent of h, such that $\forall\ q_h \in Q_h, \exists\ v_h \in W_h, v_h \neq 0$,

$$b(v_h, q_h) \geq \beta ||v_h||_W ||q_h||_Q.$$

The following estimates on the coupling term are useful in our analysis.

Lemma 3.1. $\forall \mathbf{u}, \mathbf{v} \in W$, there exists $C \geq 0$, such that $\forall \varepsilon \geq 0$,

$$|a_\Gamma(\mathbf{u}, \mathbf{v})| \leq \frac{1}{4\varepsilon} ||\mathbf{u}||_W^2 + C\varepsilon ||\mathbf{v}||_W^2. \tag{3.1}$$

Further, we have $\forall\ \mathbf{u}, \mathbf{v} \in W$, there exists $C \geq 0$ such that .

$$|a_\Gamma(\mathbf{u}, \mathbf{v})| \leq \frac{1}{4\varepsilon}(||\mathbf{u}||_W^2 + ||\mathbf{v}||_W^2) + C\varepsilon(||\mathbf{u}||_0^2 + ||\mathbf{v}||_0^2). \tag{3.2}$$

In addition, if the finite element spaces satisfy the inverse inequality, then $\forall\ u_h, v_h \in W_h$, there exists $C \geq 0$ such that .

$$|a_\Gamma(\mathbf{u}_h, \mathbf{v}_h)| \leq \frac{1}{4\varepsilon} ||\mathbf{u}_h||_W^2 + C\varepsilon h^{-1} ||\mathbf{v}_h||_0^2. \tag{3.3}$$

Proof. (3.1) is proven in [16]. For (3.2), the proof is the same but uses a bit more care in applying trace + embedding + Poincaré inequality (in both regions). (3.3) follows immediately from (3.1) and is also proven in [16]. □

We also introduce a subspace V_h of W_h defined by

$$V_h = \{v_h \in W_h : b(v_h, q_h) = 0 \ \ \forall\ q_h \in Q_h\},$$

and correspondingly, as shown in [16], define a projection operator $P_h : (\mathbf{w}(t), p(t)) \in (W, Q) \mapsto (P_h\mathbf{w}(t), P_hp(t)) \in (W_h, Q_h)$, $\forall t \in [0, T]$ by

$$a(P_h\mathbf{w}(t), \mathbf{v}_h) + b(\mathbf{v}_h, P_hp(t)) = a(\mathbf{w}(t), \mathbf{v}_h) + b(\mathbf{v}_h, p(t)) \quad \forall\ \mathbf{v}_h \in W_h, \tag{3.4}$$

$$b(P_h\mathbf{w}(t), q_h) = 0 \quad \forall\ q_h \in Q_h. \tag{3.5}$$

Apparently, P_h is linear operator. Furthermore, under a certain smoothness assumption on $(\mathbf{w}(t), p(t))$, the following approximation properties hold:

$$||P_h\mathbf{w}(t) - \mathbf{w}(t)||_0 \le Ch^2,$$

$$||P_h\mathbf{w}(t) - \mathbf{w}(t)||_W \le Ch,$$

$$||P_hp(t) - p(t)||_0 \le Ch.$$

From now on, we always assume that $(u(t), \phi(t)) \in (H^2(\Omega_f)^d, H^2(\Omega_p))$, $(u_t(t), \phi_t(t)) \in (H^1(\Omega_f)^d, H^1(\Omega_p))$ and $(u_{tt}(t), \phi_{tt}(t)) \in (L^2(\Omega_f)^d, L^2(\Omega_p))$ for the solutions of (2.5).

3.1 The monolithically coupled, implicit method

In this section, we provide a monolithically coupled scheme which is used for comparison. Choose a uniform distribution of discrete time level,

$$\mathcal{Q} = \{0 = t^0, t^1, t^2, \cdots, t^N = T\},$$

where $t^m = m\triangle t, m = 0, 1, 2, \cdots, N$ for $\triangle t = \frac{T}{N}$. Here $(u^{h,m}, p^{h,m}, \phi^{h,m})$ denotes the discrete approximation to $(u(t^m), p(t^m), \phi(t^m))$.

Algorithm 3.1(Coupled scheme) Find $\mathbf{u}^{h,m+1} = (u^{h,m+1}, \phi^{h,m+1}) \in W_h$ and $p^{h,m+1} \in Q_h, m = 0, \cdots, N-1$, such that $\forall\ \mathbf{v} = (v_h, \psi_h) \in W_h$ and $\forall\ q_h \in Q_h$,

$$(\frac{\mathbf{u}^{h,m+1} - \mathbf{u}^{h,m}}{\triangle t}, \mathbf{v}) + a(\mathbf{u}^{h,m+1}, \mathbf{v}) + b(\mathbf{v}, p^{h,m+1}) = \mathbf{f}^{m+1}(\mathbf{v}), \tag{3.6}$$

$$b(\mathbf{u}^{h,m+1}, q_h) = 0, \tag{3.7}$$

$$\mathbf{u}^{h,0} = \mathbf{u}_0. \tag{3.8}$$

At each time step, the discrete model (3.6)-(3.8) is equivalent to two coupled problem that correspond to a Stokes problem in Ω_f and a Darcy problem in Ω_p, respectively, with associated the common boundary conditions on Γ. More specifically, the discrete Stokes problem in the fluid region Ω_f reads as follows: Find $u^{h,m+1} \in H_{fh}$ and $p^{h,m+1} \in Q_h, m = 0, \cdots, N-1$, such that $\forall\ v_h \in H_{fh}$ and $q_h \in Q_h$,

$$(\frac{u^{h,m+1} - u^{h,m}}{\triangle t}, v_h) + a_f(u^{h,m+1}, v_h) + b(v_h, p^{h,m+1}) + \int_\Gamma \phi^{h,m+1} v_h \cdot \mathbf{n}_f = f^{m+1}(v), \tag{3.9}$$

$$b(u^{h,m+1}, q_h) = 0, \tag{3.10}$$

$$u^{h,0} = u_0, \tag{3.11}$$

and the discrete Darcy problem in the porous media region Ω_p reads as follows: Find $\phi^{h,m+1} \in H_{ph}, m = 0, \cdots, N-1$, such that $\forall\ \psi \in H_{ph}$,

$$(\frac{\phi^{h,m+1} - \phi^{h,m}}{\triangle t}, \psi_h) + a_p(\phi^{h,m+1}, \psi_h) - \int_\Gamma \psi_h u^{h,m+1} \cdot \mathbf{n}_f = g^{m+1}(\psi_h), \tag{3.12}$$

$$\phi^{h,0} = \phi_0. \tag{3.13}$$

3.2 A Partitioned, Decoupled Scheme with different time step size

To streamline our notation further, we shall suppress the subscript "h" and replace $u_h^m,\ \phi_h^m,\ p_h^m$ by $u^m,\ \phi^m,\ p^m$, respectively. First, we choose discrete time levels

$$\mathcal{P} = \{0 = t^0, t^1, t^2, \cdots, t^N = T\},$$

where $t^m = m\triangle t,\ m = 0, 1, 2, \cdots, N$ for $\triangle t = \frac{T}{N}$. Denote by

$$\mathcal{S} = \{t^{m_0}, t^{m_1}, \cdots, t^{m_M}\} \subset \mathcal{P},$$

a subset satisfying $t^{m_k} = kn\triangle t$ such that $n \in \mathbb{N}$ is fixed and $Mn = N$. The time step size on Ω_p is given a separate notations hereafter, $\triangle s = n\triangle t$. For $t^m, t^{m_k} \in [0, T]$, (u^m, p^m, ϕ^{m_k}) will denote the discrete approximation to $(u(t^m), p(t^m), \phi(t^{m_k}))$. The approximations $(u^{m+1}, p^{m+1}) \in (H_{fh}, Q_h)$, for $m = m_0, m_0 + 1, \cdots, N-1$ and $\phi^{m_k+1} \in H_{ph}$ for $k = 0, 1, \cdots, M-1$ are calculated using Algorithm 3.2. In practice only the data at time t^0 would need to be provided. One important feature of Algorithm 3.2 is that (u^{m+1}, p^{m+1}) can be calculated for $m = m_k, m_k + 1, \cdots, m_{k+1} - 1$ in parallel with $\phi^{m_{k+1}}$.

Algorithm 3.2(Decoupled scheme)

- Find $(u^{m+1}, p^{m+1}) \in (H_{fh}, Q_h)$, with $m = m_k, m_k+1, \cdots, m_{k+1}-1$, such that $\forall (v, q) \in (H_{fh}, Q_h)$:

$$(\frac{u^{m+1} - u^m}{\triangle t}, v) + a_f(u^{m+1}, v) + b(v, p^{m+1}) = (f^{m+1}, v) - \int_\Gamma \phi^{m_k} v \cdot \mathbf{n}_f, \tag{3.14}$$

$$b(u^{m+1}, q) = 0, \tag{3.15}$$

$$u^0 = u_0, \tag{3.16}$$

  with the small time step size $\triangle t$.

- Set $S^{m_k} = \frac{1}{n} \sum_{i=m_k}^{m_{k+1}-1} u^i$,

- find $\phi^{m_{k+1}} \in H_{ph}$, such that $\psi \in H_{ph}$:

$$(\frac{\phi^{m_{k+1}} - \phi^{m_k}}{\triangle s}, \psi) + a_p(\phi^{m_{k+1}}, \psi) = (g^{m_{k+1}}, \psi) + \int_\Gamma \psi S^{m_k} \mathbf{n}_f, \tag{3.17}$$

$$\phi^{m_0} = \phi_0, \tag{3.18}$$

with the large time step size $\triangle s = n\triangle t$.

- Set $k = k + 1$ and repeat until $k = M - 1$.

4 Zero-Stability of the method

In this section, under a modest time step restriction of the form $\triangle t \leq C$ where $C = C(physical\ parameters)$ we prove the 0-stability (possibly including terms like $exp(aT)$) over bounded time intervals $[0, T]$ of the partitioned method Algorithm 3.2.

Theorem 4.1 (0-Stability) Choose the initial data $\phi^{m_0} = \phi^0$, $u^{m_0} = u^0$, and $\phi^{m_{k+1}+J+1} = \phi^{m_{k+1}}$, $g^{m_{k+1}+J+1} = g^{m_{k+1}}$, $(-1 \leq J \leq n-2, 0 \leq k \leq l)$. There is $C(\Omega) < \infty$ such that if

$$\frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}} < 1,$$

for $-1 \leq l \leq M - 1$, we have

$$\begin{aligned} ||u^{m_{l+1}+J+1}||_0^2 &+ \frac{\nu\triangle t}{2}\sum_{i=0}^{m_{l+1}+J} ||u^{i+1}||_{H_f}^2 + ||\phi^{m_{l+1}+J+1}||_0^2 + \frac{k_{min}\triangle t}{4n}\sum_{i=0}^{m_{l+1}+J} ||\phi^{i+1}||_{H_p}^2 \\ &\leq C(T)\left[\frac{\triangle t}{\nu}\sum_{i=0}^{m_{l+1}+J} ||f^{i+1}||_{H_f'}^2 + \frac{\triangle t}{k_{min}}\sum_{i=0}^{m_{l+1}+J} ||g^{i+1}||_{H_p'}^2\right] \\ &+ \frac{\triangle t}{2}(\nu||u^0||_{H_f}^2 + k_{min}||\phi^0||_{H_p}^2) + ||u^0||_0^2 + ||\phi^0||_0^2. \end{aligned} \tag{4.1}$$

Proof. Taking $v = 2\triangle t u^{m+1}$ in (3.14), using the divergence-free property, sum over $m = m_k, m_k + 1, \cdots, m_{k+1} - 1$,

$$\begin{aligned} ||u^{m_{k+1}}||_0^2 &+ \sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1} - u^i||_0^2 - ||u^{m_k}||_0^2 + 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_f(u^{i+1}, u^{i+1}) \\ &= 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} (f^{i+1}, u^{i+1}) - 2\triangle t \int_\Gamma \phi^{m_k}(\sum_{i=m_k}^{m_{k+1}-1} u^{i+1}) \cdot \mathbf{n}_f. \end{aligned} \tag{4.2}$$

Taking $\psi = 2\triangle s\phi^{m_{k+1}} = 2\triangle tn\phi^{m_{k+1}} = 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}\phi^{m_{k+1}}$ in (3.17),

$$\begin{aligned}||\phi^{m_{k+1}}||_0^2 + ||\phi^{m_{k+1}} - \phi^{m_k}||_0^2 - ||\phi^{m_{k+1}}||_0^2 + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1} a_p(\phi^{m_{k+1}}, \phi^{m_{k+1}}) \\ = 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(g^{m_{k+1}}, \phi^{m_{k+1}}) + 2\triangle t\int_\Gamma \phi^{m_{k+1}}(\sum_{i=m_k}^{m_{k+1}-1} u^i)\cdot \mathbf{n}_f.\end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$\begin{aligned}&||u^{m_{k+1}}||_0^2 + \sum_{i=m_k}^{m_{k+1}-1}||u^{i+1} - u^i||_0^2 - ||u^{m_k}||_0^2 + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1} a_f(u^{i+1}, u^{i+1}) \\ &+ ||\phi^{m_{k+1}}||_0^2 + ||\phi^{m_{k+1}} - \phi^{m_k}||_0^2 - ||\phi^{m_k}||_0^2 + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1} a_p(\phi^{m_{k+1}}, \phi^{m_{k+1}}) \\ &= 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(f^{i+1}, u^{i+1}) + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(g^{m_{k+1}}, \phi^{m_{k+1}}) \\ &\quad - 2\triangle t a_\Gamma(\phi^{m_k}, \sum_{i=m_k}^{m_{k+1}-1} u^i; \phi^{m_{k+1}}, \sum_{i=m_k}^{m_{k+1}-1} u^{i+1}),\end{aligned} \tag{4.4}$$

here and the following, we define $a_\Gamma(\phi, u; \psi, v) = \int_\Gamma \phi v\cdot \mathbf{n}_f - \psi u\cdot \mathbf{n}_f$.

The first two terms of RHS (right hand side) in (4.4) is bound by Young and Hölder inequalities,

$$\begin{aligned}&2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(f^{i+1}, u^{i+1}) + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(g^{m_{k+1}}, \phi^{m_{k+1}}) \\ &\le \frac{C\triangle t}{\nu}\sum_{i=m_k}^{m_{k+1}-1}||f^{i+1}||_{H_f'}^2 + \frac{C\triangle t}{k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||g^{m_{k+1}}||_{H_p'}^2 \\ &\quad + \frac{\nu\triangle t}{2}\sum_{i=m_k}^{m_{k+1}-1}||u^{i+1}||_{H_f}^2 + \frac{k_{min}\triangle t}{2}\sum_{i=m_k}^{m_{k+1}-1}||\phi^{m_{k+1}}||_{H_p}^2.\end{aligned}$$

The remains of RHS in (4.4) have the following bound by (3.2)

$$
\begin{aligned}
&-2\triangle t a_{\Gamma}(\phi^{m_k}, \sum_{i=m_k}^{m_{k+1}-1} u^i; \phi^{m_{k+1}}, \sum_{i=m_k}^{m_{k+1}-1} u^{i+1}) \\
&\leq \frac{\triangle t}{4}(\nu|| \sum_{i=m_k}^{m_{k+1}-1} u^{i+1}||_{H_f}^2 + k_{min}||\phi^{m_{k+1}}||_{H_p}^2 + \nu|| \sum_{i=m_k}^{m_{k+1}-1} u^i||_{H_f}^2 + k_{min}||\phi^{m_k}||_{H_p}^2) \\
&\quad + \frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}}(|| \sum_{i=m_k}^{m_{k+1}-1} u^{i+1}||_0^2 + ||\phi^{m_{k+1}}||_0^2 + || \sum_{i=m_k}^{m_{k+1}-1} u^i||_0^2 + ||\phi^{m_k}||_0^2) \\
&\leq \frac{\triangle t}{2}(\sum_{i=m_k}^{m_{k+1}-1} \nu||u^{i+1}||_{H_f}^2 + \nu||u^{m_k}||_{H_f}^2 + k_{min}||\phi^{m_{k+1}}||_{H_p}^2 + k_{min}||\phi^{m_k}||_{H_p}^2) \\
&\quad + \frac{2C(\Omega)\triangle t}{\sqrt{\nu k_{min}}}(\sum_{i=m_k}^{m_{k+1}} ||u^i||_0^2 + ||\phi^{m_{k+1}}||_0^2 + ||\phi^{m_k}||_0^2).
\end{aligned}
$$

Combining the above inequalities, using Holder's and Young's inequality, we obtain

$$
\begin{aligned}
&||u^{m_{k+1}}||_0^2 + \sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1} - u^i||_0^2 - ||u^{m_k}||_0^2 + \nu\triangle t \sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1}||_{H_f}^2 \\
&\quad - \frac{\nu\triangle t}{2}||u^{m_k}||_{H_f}^2 + ||\phi^{m_{k+1}}||_0^2 + ||\phi^{m_{k+1}} - \phi^{m_k}||_0^2 - ||\phi^{m_k}||_0^2 \\
&\quad + k_{min}\triangle t \sum_{i=m_k}^{m_{k+1}-1} ||\phi^{m_{k+1}}||_{H_p}^2 - \frac{k_{min}\triangle t}{2}||\phi^{m_k}||_{H_p}^2 \\
&\quad \leq \frac{C\triangle t}{\nu} \sum_{i=m_k}^{m_{k+1}-1} ||f^{i+1}||_{H_f{}'}^2 + \frac{C\triangle t}{k_{min}}||g^{m_{k+1}}||_{H_p{}'}^2 \\
&\quad\quad + \frac{2C(\Omega)\triangle t}{\sqrt{\nu k_{min}}}(\sum_{i=m_k}^{m_{k+1}} ||u^i||_0^2 + ||\phi^{m_{k+1}}||_0^2 + ||\phi^{m_k}||_0^2).
\end{aligned}
\tag{4.5}
$$

Sum over $k = 0, 1, \cdots, l$, with $0 \leq l \leq M-1$ we have

$$
\begin{aligned}
&||u^{m_{l+1}}||_0^2 + \frac{\nu\triangle t}{2} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1}||_{H_f}^2 + ||\phi^{m_{l+1}}||_0^2 + \frac{k_{min}\triangle t}{2} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||\phi^{m_{k+1}}||_{H_p}^2 \\
&\quad \leq \frac{2C(\Omega)\triangle t}{\sqrt{\nu k_{min}}} \sum_{k=0}^{l}(\sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1}||_0^2 + ||\phi^{m_{k+1}}||_0^2) \\
&\quad\quad + \frac{C\triangle t}{\nu} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||f^{i+1}||_{H_f{}'}^2 + \frac{C\triangle t}{k_{min}} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||g^{m_{k+1}}||_{H_p{}'}^2 \\
&\quad\quad + \frac{\triangle t}{2}(\nu||u^{m_0}||_{H_f}^2 + k_{min}||\phi^{m_0}||_{H_p}^2) + ||u^{m_0}||_0^2 + ||\phi^{m_0}||_0^2.
\end{aligned}
\tag{4.6}
$$

Taking $v = 2\triangle t u^{m+1}$ in (3.14), using the divergence-free property again, sum over $m = m_{l+1}, m_{l+1}+1, \cdots, m_{l+1}+J$, $(0 \leq J \leq n-2)$

$$\begin{aligned} &||u^{m_{l+1}+J+1}||_0^2 + \sum_{i=m_{l+1}}^{m_{l+1}+J} ||u^{i+1}-u^i||_0^2 - ||u^{m_{l+1}}||_0^2 + 2\triangle t \sum_{i=m_{l+1}}^{m_{l+1}+J} a_f(u^{i+1}, u^{i+1}) \\ &= 2\triangle t \sum_{i=m_{l+1}}^{m_{l+1}+J} (f^{i+1}, u^{i+1}) - 2\triangle t \int_\Gamma \phi^{m_{l+1}} (\sum_{i=m_{l+1}}^{m_{l+1}+J} u^{i+1}) \cdot \mathbf{n}_f \\ &\leq \frac{C\triangle t}{\nu} \sum_{i=m_{l+1}}^{m_{l+1}+J} ||f^{i+1}||_{H_f{}'}^2 + \frac{\nu \triangle t}{2} \sum_{i=m_{l+1}}^{m_{l+1}+J} ||u^{i+1}||_{H_f}^2 \\ &+ \frac{\triangle t}{4} (\sum_{i=m_{l+1}}^{m_{l+1}+J} \nu ||u^{i+1}||_{H_f}^2 + k_{min} ||\phi^{m_{l+1}}||_{H_p}^2) \\ &+ \frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}} (\sum_{i=m_{l+1}}^{m_{l+1}+J} ||u^{i+1}||_0^2 + ||\phi^{m_{l+1}}||_0^2). \end{aligned} \tag{4.7}$$

Rearrange the inequality, yield

$$\begin{aligned} &||u^{m_{l+1}+J+1}||_0^2 + \sum_{i=m_{l+1}}^{m_{l+1}+J} ||u^{i+1}-u^i||_0^2 - ||u^{m_{l+1}}||_0^2 + \frac{\nu\triangle t}{2} \sum_{i=m_{l+1}}^{m_{l+1}+J} ||u^{i+1}||_{H_f}^2 \\ &\leq \frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}} \sum_{i=m_{l+1}}^{m_{l+1}+J} ||u^{i+1}||_0^2 + \frac{C\triangle t}{\nu} \sum_{i=m_{l+1}}^{m_{l+1}+J} ||f^{i+1}||_{H_f{}'}^2 \\ &+ \frac{k_{min}\triangle t}{4} ||\phi^{m_{l+1}}||_{H_f}^2 + \frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}} ||\phi^{m_{l+1}}||_0^2. \end{aligned} \tag{4.8}$$

Considering the special case, when $l = -1$, then $\phi^{m_{l+1}} = \phi^0$, $u^{m_{l+1}} = u^0$, the last equation can be written as follows:

$$\begin{aligned} &||u^{J+1}||_0^2 + \sum_{i=0}^{J} ||u^{i+1}-u^i||_0^2 + \frac{\nu\triangle t}{2} \sum_{i=0}^{J} ||u^{i+1}||_{H_f}^2 \\ &\leq \frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}} \sum_{i=0}^{J} ||u^{i+1}||_0^2 + \frac{C\triangle t}{\nu} \sum_{i=0}^{J} ||f^{i+1}||_{H_f{}'}^2 \\ &+ \frac{k_{min}\triangle t}{4} ||\phi^0||_{H_f}^2 + \frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}} ||\phi^0||_0^2 + ||u^0||_0^2. \end{aligned} \tag{4.9}$$

Add both sides by $\frac{k_{min}\triangle t}{4} ||\phi^0||_{H_p}^2 + ||\phi^0||_0^2$, and set $\phi^{J+1} = \phi^0,\ g^{J+1} = g^0,\ (0 \leq J \leq n-2)$

since $\frac{\triangle t}{4n}\sum_{i=0}^{J}||\phi^{i+1}||^2_{H_f} \leq \frac{\triangle t}{4}||\phi^0||^2_{H_f}$, then,

$$\begin{aligned}||u^{J+1}||_0^2 + \sum_{i=0}^{J}||u^{i+1}-u^i||_0^2 + \frac{\nu\triangle t}{2}\sum_{i=0}^{J}||u^{i+1}||^2_{H_f} + ||\phi^{J+1}||_0^2 + \frac{k_{min}\triangle t}{4n}\sum_{i=0}^{J}||\phi^{i+1}||^2_{H_f}\\ \leq \frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}}\sum_{i=0}^{J}(||u^{i+1}||_0^2 + ||\phi^{i+1}||_0^2) + \frac{C\triangle t}{\nu}\sum_{i=0}^{J}||f^{i+1}||^2_{H_f'} + \frac{C\triangle t}{k_{min}}\sum_{i=0}^{J}||g^{i+1}||^2_{H_p'}\\ + \frac{\triangle t}{2}(\nu||u^0||^2_{H_f} + k_{min}||\phi^0||^2_{H_p}) + ||u^0||_0^2 + ||\phi^0||_0^2. \qquad (4.10)\end{aligned}$$

Combine (4.6) and (4.8), and set $\phi^{m_{k+1}+J+1} = \phi^{m_{k+1}},\ g^{m_{k+1}+J+1} = g^{m_{k+1}},\ (-1 \leq J \leq n-2,\ \forall\ l \geq -1)$, we arrive at

$$\begin{aligned}||u^{m_{l+1}+J+1}||_0^2 + \frac{\nu\triangle t}{2}\sum_{i=0}^{m_{l+1}+J}||u^{i+1}||^2_{H_f} + ||\phi^{m_{l+1}+J+1}||_0^2 + \frac{k_{min}\triangle t}{4n}\sum_{i=0}^{m_{l+1}+J}||\phi^{i+1}||^2_{H_p}\\ \leq \frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}}\sum_{i=0}^{m_{l+1}+J}(||u^{i+1}||_0^2 + ||\phi^{i+1}||_0^2) + \frac{C\triangle t}{\nu}\sum_{i=0}^{m_{l+1}+J}||f^{i+1}||^2_{H_f'}\\ + \frac{C\triangle t}{k_{min}}\sum_{i=0}^{m_{l+1}+J}||g^{i+1}||^2_{H_p'} + \frac{\triangle t}{2}(\nu||u^{m_0}||^2_{H_f} + k_{min}||\phi^{m_0}||^2_{H_p})\\ + ||u^{m_0}||_0^2 + ||\phi^{m_0}||_0^2. \qquad (4.11)\end{aligned}$$

Finally, choosing $\triangle t$, such that $\frac{C(\Omega)\triangle t}{\sqrt{\nu k_{min}}} < 1$, which is required to apply th e discrete Gronwall inequality to (4.11), (which contributes a C(T) term). □

5 Convergence Analysis

In this section, we analyze the error in Algorithm 3.2. We will use the following notations. Define $u_c^m = u(t^m)$, $\phi_c^m = \phi(t^m)$, $p_c^m = p(t^m)$. Following (3.4)-(3.5), we define $u_m = P_h u(t^m)$, $\phi_m = P_h\phi(t^m)$, $p_m = P_h p(t^m)$, then we set $e_c^m = u_c^m - u_m$, $\epsilon_c^m = \phi_c^m - \phi_m$, $\eta_c^m = p_c^m - p_m$, and $e^m = u_m - u^m$, $\epsilon^m = \phi_m - \phi^m$, $\eta^m = p_m - p^m$. Obviously, we observe that $u(t^m) - u^m = e_c^m + e^m$ and $\phi(t^m) - \phi^m = \epsilon_c^m + \epsilon^m$, from approximation properties, we have $||e_c^m||_0 + ||\epsilon_c^m||_0 \leq Ch^2$, $||e_c^m||_1 + ||\epsilon_c^m||_1 \leq Ch$.

Then, by the model (2.5) and (3.4)-(3.5), for $(\mathbf{v}, q) \in (W_h, Q_h)$, we have the following equations:

$$(\frac{u_{m+1}-u_m}{\triangle t}, v) + a_f(u_{m+1}, v) + b(v, p_{m+1}) = -(w_{f,t}^{m+1}, v) + (f^{m+1}, v) - \int_\Gamma \phi_{m+1} v\cdot\mathbf{n}_f \qquad (5.1)$$

$$b(u_{m+1}, q) = 0. \qquad (5.2)$$

$$(\frac{\phi_{m+1}-\phi_m}{\triangle t},\psi)+a_p(\phi_{m+1},\psi)=-(w_{p,t}^{m+1},\psi)+(g^{m+1},\psi)+\int_\Gamma \psi u_{m+1}\cdot \mathbf{n}_f, \tag{5.3}$$

where

$$\begin{aligned} w_{f,t}^{m+1} &= \frac{u_{m+1}-u_m}{\triangle t}-u_t(t^{m+1}) \\ &= [\frac{u_{m+1}-u_m}{\triangle t}-\frac{u(t^{m+1})-u(t^m)}{\triangle t}]+[\frac{u(t^{m+1})-u(t^m)}{\triangle t}-u_t(t^{m+1})] \\ &= w_{f,t,1}^{m+1}+w_{f,t,2}^{m+1}, \end{aligned}$$

and

$$\begin{aligned} w_{p,t}^{m+1} &= \frac{\phi_{m+1}-\phi_m}{\triangle t}-\phi_t(t^{m+1}) \\ &= [\frac{\phi_{m+1}-\phi_m}{\triangle t}-\frac{\phi(t^{m+1})-\phi(t^m)}{\triangle t}]+[\frac{\phi(t^{m+1})-\phi(t^m)}{\triangle t}-\phi_t(t^{m+1})] \\ &= w_{p,t,1}^{m+1}+w_{p,t,2}^{m+1}. \end{aligned}$$

It is easy to verify that the following properties of $w_{f,t,1}^{m+1}, w_{f,t,2}^{m+1}, w_{p,s,1}^{m+1}$ and $w_{p,s,2}^{m+1}$ hold: from the definition

$$w_{f,t,1}^{m+1}=(P_h-I)\frac{u(t^{m+1})-u(t^m)}{\triangle t}=\frac{1}{\triangle t}\int_{t^m}^{t^{m+1}}(P_h-I)u_t(t)dt,$$

we have

$$\begin{aligned} ||w_{f,t,1}^{m+1}||_0^2 &= \frac{1}{\triangle t^2}\int_\Omega (\int_{t^m}^{t^{m+1}}(P_h-I)u_t(t)dt)^2dx \\ &\le \frac{1}{\triangle t^2}\int_\Omega \int_{t^m}^{t^{m+1}}((P_h-I)u_t(t))^2dt\int_{t^m}^{t^{m+1}}1^2dtdx \\ &\le \frac{1}{\triangle t}\int_{t^m}^{t^{m+1}}||(P_h-I)u_t(t)||_0^2dt. \end{aligned} \tag{5.4}$$

Similarly,

$$\triangle t w_{f,t,2}^{m+1}=u(t^{m+1})-u(t^m)-\triangle t u_t(t^{m+1})=-\int_{t^m}^{t^{m+1}}(t-t^m)u_{tt}(t)dt,$$

which means

$$\begin{aligned} ||w_{f,t,2}^{m+1}||_0^2 &= \frac{1}{\triangle t^2}\int_\Omega (\int_{t^m}^{t^{m+1}}(t-t^m)u_{tt}(t)dt)^2dx \\ &\le \frac{1}{\triangle t^2}\int_\Omega \int_{t^m}^{t^{m+1}}(u_{tt}(t))^2dt\int_{t^m}^{t^{m+1}}(t-t^m)^2dtdx \le \triangle t\int_{t^m}^{t^{m+1}}||u_{tt}||_0^2dt. \end{aligned} \tag{5.5}$$

The same as $w_{p,t,1}^{m+1}$, $w_{p,t,2}^{m+1}$, while consider the large time step size $\triangle s$, then,

$$||w_{p,s,1}^{m_{k+1}}||_0^2 \leq \frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}} ||(P_h - I)\phi_s(s)||_0^2 ds, \tag{5.6}$$

and

$$||w_{p,s,2}^{m_{k+1}}||_0^2 \leq \triangle s \int_{t^{m_k}}^{t^{m_{k+1}}} ||\phi_{ss}||_0^2 ds. \tag{5.7}$$

By the equivalence between $||u||_{H_f}$ and $||\nabla u||_0$, $||\phi||_{H_p}$ and $||\nabla\phi||_0$,

$$\begin{aligned}||u_{m+1} - u_m||_{H_f}^2 &= ||P_h(u(t^{m+1}) - u(t^m))||_{H_f}^2 \leq C||u(t^{m+1}) - u(t^m)||_{H_f}^2\\ &\leq C\int_{\Omega_f}(\nabla(u(t^{m+1}) - u(t^m)))^2 dx \leq C\int_{\Omega_f}(\int_{t^m}^{t^{m+1}} \nabla u_t dt)^2 dx\\ &\leq C\int_{\Omega_f}\int_{t^m}^{t^{m+1}} \nabla u_t^2 dt * \int_{t^m}^{t^{m+1}} 1 dt dx \leq C\triangle t\int_{t^m}^{t^{m+1}} ||u_t||_{H_f}^2 dt. \end{aligned}\tag{5.8}$$

Do the same as (5.8), we have

$$||\phi_{m+1} - \phi_m||_{H_p}^2 \leq C\triangle t\int_{t^m}^{t^{m+1}} ||\phi_t||_{H_p}^2 dt, \tag{5.9}$$

$$||u_{m_{k+1}} - u_{m_k}||_{H_f}^2 \leq C\triangle s\int_{t^{m_k}}^{t^{m_{k+1}}} ||u_s||_{H_f}^2 ds, \tag{5.10}$$

$$||\phi_{m_{k+1}} - \phi_{m_k}||_{H_p}^2 \leq C\triangle s\int_{t^{m_k}}^{t^{m_{k+1}}} ||\phi_s||_{H_p}^2 ds. \tag{5.11}$$

Consider small time step size $\triangle t$, subtract (5.1) from (3.14), we obtain

$$\begin{aligned}&(\frac{e^{m+1} - e^m}{\triangle t}, v) + a_f(e^{m+1}, v) + b(v, \eta^{m+1})\\ &\qquad = -(w_{f,t}^{m+1}, v) - \int_\Gamma(\phi_{m+1} - \phi_m)v\cdot\mathbf{n}_f - \int_\Gamma(\phi_m - \phi^{m_k})v\cdot\mathbf{n}_f, \\ &\quad b(e^{m+1}, q) = 0.\end{aligned}\tag{5.12}$$

Consider larger time step size $\triangle s = n\triangle t$, subtract (5.3) from (3.17), we obtain

$$\begin{aligned}&(\frac{\epsilon^{m_{k+1}} - \epsilon^{m_k}}{\triangle s}, \psi) + a_p(\epsilon^{m_{k+1}}, \psi)\\ &\qquad = -(w_{p,s}^{m_{k+1}}, \psi) + \int_\Gamma \psi(u_{m_{k+1}} - u_{m_k})\cdot\mathbf{n}_f + \int_\Gamma \psi(u_{m_k} - S^{m_k})\cdot\mathbf{n}_f.\end{aligned}\tag{5.13}$$

Theorem 5.1. Suppose the true solution is smooth, the initial approximations are sufficiently accurate and that the time step and mesh width $\triangle t, h$ satisfy $\tilde{C}\triangle t h^{-1} \leq 1$, where $\tilde{C}$

depends on the parameters ν, k_{min} and the domain Ω. Then, the following estimate for the error at the larger time steps (the synchronization points) holds:

$$||e^{m_{l+1}}||_0^2 + \nu\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}||_{H_f}^2 + ||\epsilon^{m_{l+1}}||_0^2 + k_{min}\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\epsilon^{m_{k+1}}||_{H_p}^2$$
$$\leq C(T)(\triangle t^2 + h^4). \tag{5.14}$$

Proof. Taking $v = 2\triangle te^{m+1}$ in (5.12), using the divergence-free property, sum over $m = m_k, m_k+1, \cdots, m_{k+1}-1$, yield

$$||e^{m_{k+1}}||_0^2 + \sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}-e^i||_0^2 - ||e^{m_k}||_0^2 + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_f(e^{i+1}, e^{i+1})$$
$$= -2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1}, e^{i+1}) - 2\triangle t\int_\Gamma\sum_{i=m_k}^{m_{k+1}-1}(\phi_{i+1}-\phi_i)e^{i+1}\cdot\mathbf{n}_f \tag{5.15}$$
$$- 2\triangle t\int_\Gamma\sum_{i=m_k}^{m_{k+1}-1}(\phi_i - \phi^{m_k})e^{i+1}\cdot\mathbf{n}_f.$$

Taking $\psi = 2\triangle tn\epsilon^{m_{k+1}} = 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}\epsilon^{m_{k+1}}$ in (5.13),

$$||\epsilon^{m_{k+1}}||_0^2 + ||\epsilon^{m_{k+1}} - \epsilon^{m_k}||_0^2 - ||\epsilon^{m_{k+1}}||_0^2 + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_p(\epsilon^{m_{k+1}}, \epsilon^{m_{k+1}})$$
$$= -2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{p,s}^{m_{k+1}}, \epsilon^{m_{k+1}}) + 2\triangle t\int_\Gamma\sum_{i=m_k}^{m_{k+1}-1}\epsilon^{m_{k+1}}(u_{m_{k+1}} - u_{m_k})\cdot\mathbf{n}_f \tag{5.16}$$
$$+ 2\triangle t\int_\Gamma\sum_{i=m_k}^{m_{k+1}-1}\epsilon^{m_{k+1}}(u_{m_k} - u^i)\cdot\mathbf{n}_f.$$

Combining the above equalities (5.15) and (5.16), we obtain

$$
\begin{aligned}
&||e^{m_{k+1}}||_0^2 + \sum_{i=m_k}^{m_{k+1}-1} ||e^{i+1} - e^i||_0^2 - ||e^{m_k}||_0^2 + 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_f(e^{i+1}, e^{i+1}) \\
&\quad + ||\epsilon^{m_{k+1}}||_0^2 + ||\epsilon^{m_{k+1}} - \epsilon^{m_k}||_0^2 - ||\epsilon^{m_k}||_0^2 + 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_p(\epsilon^{m_{k+1}}, \epsilon^{m_{k+1}}) \\
&\quad = -2\triangle t \sum_{i=m_k}^{m_{k+1}-1} (w_{f,t}^{i+1}, e^{i+1}) - 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} (w_{p,s}^{m_{k+1}}, \epsilon^{m_{k+1}}) \\
&\quad - 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\phi_{i+1} - \phi_i, u_{m_{k+1}} - u_{m_k}; \epsilon^{m_{k+1}}, e^{i+1}) \\
&\quad - 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\phi_i - \phi^{m_k}, u_{m_k} - u^i; \epsilon^{m_{k+1}}, e^{i+1}). \qquad (5.17)
\end{aligned}
$$

The first term of RHS in (5.17) is bound by Young, Poincar*é* and H*ö*lder inequalities

$$
\begin{aligned}
&- 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} (w_{f,t}^{i+1}, e^{i+1}) - 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} (w_{p,s}^{m_{k+1}}, \epsilon^{m_{k+1}}) \\
&\le \frac{\triangle t}{4} \sum_{i=m_k}^{m_{k+1}-1} (\nu||e^{i+1}||_{H_f}^2 + k_{min}||\epsilon^{m_{k+1}}||_{H_p}^2) + C(\Omega)\triangle t \sum_{i=m_k}^{m_{k+1}-1} (\frac{1}{\nu}||w_{f,t}^{i+1}||_0^2 + \frac{1}{k_{min}}||w_{p,s}^{m_{k+1}}||_0^2) \\
&\le \frac{\triangle t}{4} \sum_{i=m_k}^{m_{k+1}-1} (\nu||e^{i+1}||_{H_f}^2 + k_{min}||\epsilon^{m_{k+1}}||_{H_p}^2) \\
&\quad + C(\Omega)\triangle t \sum_{i=m_k}^{m_{k+1}-1} (\frac{1}{\nu}||w_{f,t,1}^{i+1}||_0^2 + \frac{1}{\nu}||w_{f,t,2}^{i+1}||_0^2 + \frac{1}{k_{min}}||w_{p,s,1}^{m_{k+1}}||_0^2 + \frac{1}{k_{min}}||w_{p,s,2}^{m_{k+1}}||_0^2), \qquad (5.18)
\end{aligned}
$$

where $C(\Omega)$ is a constant which depends on the domain Ω. The second term of RHS is bound by

$$
\begin{aligned}
&- 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\phi_{i+1} - \phi_i, u_{m_{k+1}} - u_{m_k}; \epsilon^{m_{k+1}}, e^{i+1}) \\
&\quad \le \frac{\triangle t}{4} \sum_{i=m_k}^{m_{k+1}-1} (\nu||e^{i+1}||_{H_f}^2 + k_{min}||\epsilon^{m_{k+1}}||_{H_p}^2) \\
&\quad\quad + \frac{C\triangle t}{\nu} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_{i+1} - \phi_i||_{H_p}^2 + \frac{C\triangle t}{k_{min}}||u_{m_{k+1}} - u_{m_k}||_{H_f}^2. \qquad (5.19)
\end{aligned}
$$

The third term of RHS is bound by

$$
\begin{aligned}
& - 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\phi_i - \phi^{m_k}, u_{m_k} - u^i; \epsilon^{m_{k+1}}, e^{i+1}) \\
& = -2\triangle t \sum_{i=m_k}^{m_{k+1}-1} \{a_\Gamma(\phi_i - \phi_{m_k}, u_{m_k} - u_i; \epsilon^{m_{k+1}}, e^{i+1}) + a_\Gamma(\epsilon^{m_k}, e^i; \epsilon^{m_{k+1}}, e^{i+1})\} \\
& = 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} \{a_\Gamma(\epsilon^{m_{k+1}} - \epsilon^{m_k}, e^{i+1} - e^i; \epsilon^{m_{k+1}}, e^{i+1}) - a_\Gamma(\phi_i - \phi_{m_k}, u_{m_k} - u_i; \epsilon^{m_{k+1}}, e^{i+1})\} \\
& \leq \frac{\triangle t}{2} \sum_{i=m_k}^{m_{k+1}-1} (\nu ||e^{i+1}||^2_{H_f} + k_{min} ||\epsilon^{m_{k+1}}||^2_{H_p}) + \tilde{C}\triangle t h^{-1} (\sum_{i=m_k}^{m_{k+1}-1} ||e^{i+1} - e^i||_0^2 + ||\epsilon^{m_{k+1}} - \epsilon^{m_k}||_0^2) \\
& \quad + C\triangle t \sum_{i=m_k}^{m_{k+1}-1} (\frac{1}{\nu} ||u_{m_k} - u_i||^2_{H_f} + \frac{1}{k_{min}} ||\phi_i - \phi_{m_k}||^2_{H_p}). \qquad (5.20)
\end{aligned}
$$

where $\tilde{C} = max\{\frac{1}{\nu}, \frac{1}{k_{min}}\}$ and also depends on the domain Ω, by choosing $\triangle t, h$, such that . $\tilde{C}\triangle t h^{-1} \leq 1$, combine the above inequalities, sum over $k = 0, 1, \cdots, l$, we arrive at

$$
\begin{aligned}
& ||e^{m_{l+1}}||_0^2 - ||e^{m_0}||_0^2 + \nu \triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||e^{i+1}||^2_{H_f} \\
& \qquad + ||\epsilon^{m_{l+1}}||_0^2 - ||\epsilon^{m_0}||_0^2 + k_{min}\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||\epsilon^{m_{k+1}}||^2_{H_p} \\
& \leq \tilde{C}\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} (||w^{i+1}_{f,t,1}||_0^2 + ||w^{i+1}_{f,t,2}||_0^2 + ||w^{m_{k+1}}_{p,s,1}||_0^2 + ||w^{m_{k+1}}_{p,s,2}||_0^2) \\
& \quad + \tilde{C}\triangle t \sum_{k=0}^{l} ||u_{m_{k+1}} - u_{m_k}||^2_{H_f} \\
& \quad + \tilde{C}\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} (||u_{m_k} - u_i||^2_{H_f} + ||\phi_i - \phi_{m_k}||^2_{H_p} + ||\phi_{i+1} - \phi_i||^2_{H_p}). \qquad (5.21)
\end{aligned}
$$

By (5.4)-(5.11) and the approximate properties of P_h, the first term of RHS in (5.21) bound by

$$
\begin{aligned}
&\tilde{C}\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}(||w_{f,t,1}^{i+1}||_0^2+||w_{f,t,2}^{i+1}||_0^2+||w_{p,s,1}^{m_{k+1}}||_0^2+||w_{p,s,2}^{m_{k+1}}||_0^2)\\
&\leq \tilde{C}\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}(\frac{1}{\triangle t}\int_{t^i}^{t^{i+1}}||(P_h-I)u_t(t)||_0^2dt+\triangle t\int_{t^i}^{t^{i+1}}||u_{tt}||_0^2dt)\\
&\quad+\tilde{C}n\triangle t\sum_{k=0}^{l}(\frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}}||(P_h-I)\phi_s(s)||_0^2ds+\triangle s\int_{t^{m_k}}^{t^{m_{k+1}}}||\phi_{ss}||_0^2ds)\\
&\leq \tilde{C}(\int_0^T||(P_h-I)u_t(t)||_0^2dt+\triangle t^2\int_0^T||u_{tt}||_0^2dt\\
&\quad+\int_0^T||(P_h-I)\phi_s(s)||_0^2ds+\triangle s^2\int_0^T||\phi_{ss}||_0^2ds)\\
&\leq C(T)(\triangle t^2+h^4),
\end{aligned}
\tag{5.22}
$$

here and afterwards $C(T)$ denotes a constant depending on ν, k_{min}, T and the domain Ω. The second term of RHS in (5.21) bound by

$$
\tilde{C}\triangle t\sum_{k=0}^{l}||u_{m_{k+1}}-u_{m_k}||_{H_f}^2\leq\tilde{C}\triangle t\triangle s\int_0^T||u_s(s)||_{H_f}^2ds\leq C(T)\triangle t^2.
$$

The third term of RHS in (5.21) bound by

$$
\begin{aligned}
&\tilde{C}\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}(||u_{m_k}-u_i||_{H_f}^2+||\phi_i-\phi_{m_k}||_{H_p}^2+||\phi_{i+1}-\phi_i||_{H_p}^2)\\
&\leq\tilde{C}n\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}(||u_{i+1}-u_i||_{H_f}^2+||\phi_i-\phi_{i+1}||_{H_p}^2)+\tilde{C}\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\phi_{i+1}-\phi_i||_{H_p}^2\\
&\leq\tilde{C}n\triangle t^2(\int_0^T||u_t(t)||_{H_f}^2dt+\int_0^T||\phi_t(t)||_{H_p}^2dt)+\tilde{C}\triangle t^2\int_0^T||\phi_t(t)||_{H_p}^2dt\\
&\leq C(T)\triangle t^2.
\end{aligned}
\tag{5.23}
$$

Combine the above bounds, add the initial data and yields the final result,

$$
\begin{aligned}
||e^{m_{l+1}}||_0^2+\nu\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}||_{H_f}^2+||\epsilon^{m_{l+1}}||_0^2+k_{min}\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\epsilon^{m_{k+1}}||_{H_p}^2&\\
\leq C(T)(\triangle t^2+h^4).&
\end{aligned}
\tag{5.24}
$$

□

For the error in time derivatives, we have the following error estimate.

Theorem 5.2. Under the assumptions of the previous theorem, including, $\tilde{C}\triangle t h^{-1} \leq 1$, where $\tilde{C}$ is defined as above, the following error estimate holds:

$$\begin{aligned}
\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||d_t e^{i+1}||_0^2 + \nu ||e^{m_{l+1}}||_{H_f}^2 + \lambda(e^{m_{l+1}}, e^{m_{l+1}}) \\
+ \triangle t \sum_{k=0}^{l} ||d_t \epsilon^{m_{k+1}}||_0^2 + k_{min} \sum_{i=m_k}^{m_{k+1}-1} ||\epsilon^{m_{k+1}}||_{H_p}^2 \\
\leq C(T)(\triangle t + h^4 + \triangle t^{-1} h^4).
\end{aligned} \tag{5.25}$$

Proof. Taking $v = 2\triangle t d_t e^{m+1} = 2(e^{m+1} - e^m)$ in (5.12), using the divergence-free property, sum over $m = m_k, m_k + 1, \cdots, m_{k+1} - 1$, we get

$$\begin{aligned}
2\triangle t \sum_{i=m_k}^{m_{k+1}-1} ||d_t e^{i+1}||_0^2 + a_f(e^{m_{k+1}}, e^{m_{k+1}}) - a_f(e^{m_k}, e^{m_k}) + \triangle t^2 \sum_{i=m_k}^{m_{k+1}-1} a_f(d_t e^{i+1}, d_t e^{i+1}) \\
= -2\triangle t \sum_{i=m_k}^{m_{k+1}-1} (w_{f,t}^{i+1}, d_t e^{i+1}) - 2\triangle t \int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} (\phi_{i+1} - \phi_i) d_t e^{i+1} \cdot \mathbf{n}_f \\
- 2\triangle t \int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} (\phi_i - \phi^{m_k}) d_t e^{i+1} \cdot \mathbf{n}_f.
\end{aligned} \tag{5.26}$$

Taking $\psi = 2n\triangle t d_t \epsilon^{m_{k+1}} = 2n(\epsilon^{m_{k+1}} - \epsilon^{m_k}) = 2\sum_{i=m_k}^{m_{k+1}-1}(\epsilon^{m_{k+1}} - \epsilon^{m_k})$ in (5.13) yield

$$\begin{aligned}
2\triangle t ||d_t \epsilon^{m_{k+1}}||_0^2 + \sum_{i=m_k}^{m_{k+1}-1} \left\{ a_p(\epsilon^{m_{k+1}}, \epsilon^{m_{k+1}}) - a_p(\epsilon^{m_k}, \epsilon^{m_k}) + \triangle t^2 a_p(d_t \epsilon^{m_{k+1}}, d_t \epsilon^{m_{k+1}}) \right\} \\
= 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} (w_{p,s}^{m_{k+1}}, d_t \epsilon^{m_{k+1}}) + 2\triangle t \int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} d_t \epsilon^{m_{k+1}} (u_{m_{k+1}} - u_{m_k}) \cdot \mathbf{n}_f \\
+ 2\triangle t \int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} d_t \epsilon^{m_{k+1}} (u_{m_k} - u^i) \cdot \mathbf{n}_f.
\end{aligned} \tag{5.27}$$

Combining the above two equalities (5.26) and (5.27), we have

$$
\begin{aligned}
&2\triangle t\sum_{i=m_k}^{m_{k+1}-1}||d_te^{i+1}||_0^2+a_f(e^{m_{k+1}},e^{m_{k+1}})-a_f(e^{m_k},e^{m_k})+\triangle t^2\sum_{i=m_k}^{m_{k+1}-1}a_f(d_te^{i+1},d_te^{i+1})\\
&+2\triangle t||d_t\epsilon^{m_{k+1}}||_0^2+\sum_{i=m_k}^{m_{k+1}-1}\left\{a_p(\epsilon^{m_{k+1}},\epsilon^{m_{k+1}})-a_p(\epsilon^{m_k},\epsilon^{m_k})+\triangle t^2a_p(d_t\epsilon^{m_{k+1}},d_t\epsilon^{m_{k+1}})\right\}\\
&\quad=-2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1},d_te^{i+1})-2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{p,s}^{m_{k+1}},d_t\epsilon^{m_{k+1}})\\
&\quad-2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(\phi_{i+1}-\phi_i,u_{m_{k+1}}-u_{m_k};d_t\epsilon^{m_{k+1}},d_te^{i+1})\\
&\quad-2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(\phi_i-\phi^{m_k},u_{m_k}-u^i;d_t\epsilon^{m_{k+1}},d_te^{i+1}).
\end{aligned}\tag{5.28}
$$

The first term of RHS in (5.28) is bound by Young and Hölder inequalities

$$
\begin{aligned}
&-2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1},d_te^{i+1})-2\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{p,s}^{m_{k+1}},d_t\epsilon^{m_{k+1}})\\
&\le\triangle t\sum_{i=m_k}^{m_{k+1}-1}||d_te^{i+1}||_0^2+\triangle t||d_t\epsilon^{m_{k+1}}||_0^2+C\triangle t\sum_{i=m_k}^{m_{k+1}-1}(||w_{f,t}^{i+1}||_0^2+||w_{p,s}^{m_{k+1}}||_0^2)\\
&\le C\triangle t\sum_{i=m_k}^{m_{k+1}-1}(||w_{f,t,1}^{i+1}||_0^2+||w_{f,t,2}^{i+1}||_0^2+||w_{p,s,1}^{m_{k+1}}||_0^2+||w_{p,s,2}^{m_{k+1}}||_0^2)\\
&\quad+\triangle t\sum_{i=m_k}^{m_{k+1}-1}||d_te^{i+1}||_0^2+\triangle t||d_t\epsilon^{m_{k+1}}||_0^2.
\end{aligned}\tag{5.29}
$$

The second term of the RHS is bound by

$$
\begin{aligned}
-2\triangle t&\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(\phi_{i+1}-\phi_i,u_{m_{k+1}}-u_{m_k};d_t\epsilon^{m_{k+1}},d_te^{i+1})\\
&\le\frac{2\triangle t^2}{3}\sum_{i=m_k}^{m_{k+1}-1}(\nu||d_te^{i+1}||_{H_f}^2+k_{min}||\epsilon^{m_{k+1}}||_{H_p}^2)\\
&\quad+\tilde C\sum_{i=m_k}^{m_{k+1}-1}||\phi_{i+1}-\phi_i||_{H_p}^2+\tilde C||u_{m_{k+1}}-u_{m_k}||_{H_f}^2.
\end{aligned}\tag{5.30}
$$

The third term of the RHS is bound by

$$
\begin{aligned}
&-2\triangle t\sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\phi_i-\phi^{m_k},u_{m_k}-u^i;d_t\epsilon^{m_{k+1}},d_te^{i+1})\\
&=-2\triangle t\sum_{i=m_k}^{m_{k+1}-1}\{a_\Gamma(\phi_i-\phi_{m_k},u_{m_k}-u_i;d_t\epsilon^{m_{k+1}},d_te^{i+1})+a_\Gamma(\epsilon^{m_k},e^i;d_t\epsilon^{m_{k+1}},d_te^{i+1})\}\\
&\le\frac{4\triangle t^2}{3}\sum_{i=m_k}^{m_{k+1}-1}(\nu||d_te^{i+1}||_{H_f}^2+k_{min}||d_t\epsilon^{m_{k+1}}||_{H_p}^2)+\tilde C\sum_{i=m_k}^{m_{k+1}-1}(||e^i||_{H_f}^2+||\epsilon^{m_k}||_{H_p}^2)\\
&\quad+\tilde C\sum_{i=m_k}^{m_{k+1}-1}(||u_{m_k}-u_i||_{H_f}^2+||\phi_i-\phi_{m_k}||_{H_p}^2)\\
&\le\frac{4\triangle t^2}{3}\sum_{i=m_k}^{m_{k+1}-1}(\nu||d_te^{i+1}||_{H_f}^2+k_{min}||d_t\epsilon^{m_{k+1}}||_{H_p}^2)+\tilde C\sum_{i=m_k}^{m_{k+1}-1}(||e^i||_{H_f}^2+||\epsilon^{m_k}||_{H_p}^2)\\
&\quad+\tilde Cn\sum_{i=m_k}^{m_{k+1}-1}(||u_{i+1}-u_i||_{H_f}^2+||\phi_i-\phi_{i+1}||_{H_p}^2). \qquad (5.31)
\end{aligned}
$$

For simplicity, we define

$$\lambda(u,v)=\sum_{i=1}^{d-1}\int_\Gamma\frac{\alpha}{\sqrt{\tau_i\cdot K\tau_i}}(u\cdot\tau_i)(v\cdot\tau_i).$$

Then, by using (5.28)-(5.31), we have

$$
\begin{aligned}
&\triangle t\sum_{i=m_k}^{m_{k+1}-1}||d_te^{i+1}||_0^2+\nu||e^{m_{k+1}}||_{H_f}^2-\nu||e^{m_k}||_{H_f}^2+\lambda(e^{m_{k+1}},e^{m_{k+1}})-\lambda(e^{m_k},e^{m_k})\\
&+\triangle t^2\sum_{i=m_k}^{m_{k+1}-1}\lambda(d_te^{i+1},d_te^{i+1})+\triangle t||d_t\epsilon^{m_{k+1}}||_0^2+k_{min}\sum_{i=m_k}^{m_{k+1}-1}\Big\{||\epsilon^{m_{k+1}}||_{H_p}^2-||\epsilon^{m_k}||_{H_p}^2\Big\}\\
&\le C\triangle t\sum_{i=m_k}^{m_{k+1}-1}(||w_{f,t,1}^{i+1}||_0^2+||w_{f,t,2}^{i+1}||_0^2+||w_{p,s,1}^{m_{k+1}}||_0^2+||w_{p,s,2}^{m_{k+1}}||_0^2)+\tilde C\sum_{i=m_k}^{m_{k+1}-1}||\phi_{i+1}-\phi_i||_{H_p}^2\\
&\quad+\tilde C||u_{m_{k+1}}-u_{m_k}||_{H_f}^2+\tilde Cn\sum_{i=m_k}^{m_{k+1}-1}(||u_{i+1}-u_i||_{H_f}^2+||\phi_i-\phi_{i+1}||_{H_p}^2)\\
&\quad+\tilde C\sum_{i=m_k}^{m_{k+1}-1}(||e^i||_{H_f}^2+||\epsilon^{m_k}||_{H_p}^2). \qquad (5.32)
\end{aligned}
$$

Sum over $k = 0, 1, \cdots, l$, since $\lambda(u, u) \geq 0$, we arrive at

$$\begin{aligned}
&\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||d_t e^{i+1}||_0^2 + \nu||e^{m_{l+1}}||_{H_f}^2 + \lambda(e^{m_{l+1}}, e^{m_{l+1}}) \\
&\quad + \triangle t \sum_{k=0}^{l} ||d_t \epsilon^{m_{k+1}}||_0^2 + k_{min} \sum_{i=m_k}^{m_{k+1}-1} ||\epsilon^{m_{l+1}}||_{H_p}^2 \\
&\leq C\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} (||w_{f,t,1}^{i+1}||_0^2 + ||w_{f,t,2}^{i+1}||_0^2 + ||w_{p,s,1}^{m_{k+1}}||_0^2 + ||w_{p,s,2}^{m_{k+1}1}||_0^2) \\
&\quad + \tilde{C} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_{i+1} - \phi_i||_{H_p}^2 + \tilde{C} \sum_{k=0}^{l} ||u_{m_{k+1}} - u_{m_k}||_{H_f}^2 \\
&\quad + \tilde{C}n \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} (||u_{i+1} - u_i||_{H_f}^2 + ||\phi_i - \phi_{i+1}||_{H_p}^2) + \tilde{C} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} (||e^i||_{H_f}^2 + ||\epsilon^{m_k}||_{H_p}^2) \\
&\quad + \nu||e^{m_0}||_{H_f}^2 + \lambda(e^{m_0}, e^{m_0}) + k_{min} \sum_{i=m_k}^{m_{k+1}-1} ||\epsilon^{m_0}||_{H_p}^2. \qquad (5.33)
\end{aligned}$$

Do similarly as (5.22)- (5.23), we have

$$C\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} (||w_{f,t,1}^{i+1}||_0^2 + ||w_{f,t,2}^{i+1}||_0^2 + ||w_{p,s,1}^{m_{k+1}}||_0^2 + ||w_{p,s,2}^{m_{k+1}}||_0^2) \leq C(T)(\triangle t^2 + h^4),$$

$$\tilde{C} \sum_{k=0}^{l} ||u_{m_{k+1}} - u_{m_k}||_{H_f}^2 \leq C(T)\triangle t,$$

$$\tilde{C} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_{i+1} - \phi_i||_{H_p}^2 + \tilde{C}n \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} (||u_{i+1} - u_i||_{H_f}^2 + ||\phi_i - \phi_{i+1}||_{H_p}^2) \leq C(T)\triangle t.$$

From Theorem 5.1, we have,

$$\tilde{C} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} (||e^i||_{H_f}^2 + ||\epsilon^{m_k}||_{H_p}^2) \leq C(T)(\triangle t + \triangle t^{-1} h^4).$$

Combine the above bounds, add the initial data and yields the final result,

$$\begin{aligned}
\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||d_t e^{i+1}||_0^2 + \nu||e^{m_{l+1}}||_{H_f}^2 + \lambda(e^{m_{l+1}}, e^{m_{l+1}}) \\
+ \triangle t \sum_{k=0}^{l} ||d_t \epsilon^{m_{k+1}}||_0^2 + k_{min} \sum_{i=m_k}^{m_{k+1}-1} ||\epsilon^{m_{l+1}}||_{H_p}^2 \\
\leq C(T)(\triangle t + h^4 + \triangle t^{-1} h^4).
\end{aligned}$$

□

At the smaller time steps used for the faster problem we have the following error estimate.
Theorem 5.3. Under the assumptions of the previous theorem, including, $\tilde{C}\triangle t h^{-1} \le 1$, where $\tilde{C}$ is defined as above, the following error estimate holds: for $J = 1, 2, \cdots, n-1$, and $k = 0, 1, \cdots, l$,

$$||e^{m_k+J+1}||_0^2 + \nu\triangle t \sum_{i=m_k}^{m_k+J} ||e^{i+1}||_{H_f}^2 \le C(T)(\triangle t^2 + h^4). \tag{5.34}$$

Proof. Taking $v = 2\triangle t e^{m+1}$ in (5.12), using the divergence-free property, sum over $m = m_k, m_k+1, \cdots, m_k+J$, yield

$$\begin{aligned}
&||e^{m_k+J+1}||_0^2 + \sum_{i=m_k}^{m_k+J} ||e^{i+1} - e^i||_0^2 - ||e^{m_k}||_0^2 + 2\triangle t \sum_{i=m_k}^{m_k+J} a_f(e^{i+1}, e^{i+1}) \\
&= -2\triangle t \sum_{i=m_k}^{m_k+J} (w_{f,t}^{i+1}, e^{i+1}) - 2\triangle t \int_\Gamma \sum_{i=m_k}^{m_k+J} (\phi_{i+1} - \phi_i) e^{i+1} \cdot \mathbf{n}_f \\
&\quad - 2\triangle t \int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} (\phi_i - \phi^{m_k}) e^{i+1} \cdot \mathbf{n}_f \\
&\le \frac{\triangle t}{3} \sum_{i=m_k}^{m_k+J} \nu ||e^{i+1}||_{H_f}^2 + \tilde{C}\triangle t \sum_{i=m_k}^{m_k+J} ||w_{f,t}^{i+1}||_0^2 + \frac{\triangle t}{3} \sum_{i=m_k}^{m_k+J} \nu ||e^{i+1}||_{H_f}^2 \\
&\quad + \tilde{C}\triangle t \sum_{i=m_k}^{m_k+J} ||\phi_{i+1} - \phi_i||_{H_p}^2 + \frac{\triangle t}{3} \sum_{i=m_k}^{m_k+J} \nu ||e^{i+1}||_{H_f}^2 + \tilde{C}\triangle t \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi^{m_k}||_{H_p}^2 \\
&\le \triangle t \sum_{i=m_k}^{m_k+J} \nu ||e^{i+1}||_{H_f}^2 + \tilde{C}\triangle t \sum_{i=m_k}^{m_k+J} (||w_{f,t,1}^{i+1}||_0^2 + ||w_{f,t,2}^{i+1}||_0^2) \\
&\quad + \tilde{C}\triangle t \sum_{i=m_k}^{m_k+J} ||\phi_{i+1} - \phi_i||_{H_p}^2 + \tilde{C}\triangle t \sum_{i=m_k}^{m_k+J} (||\phi_i - \phi_{m_k}||_{H_p}^2 + ||\phi_{m_k} - \phi^{m_k}||_{H_p}^2)
\end{aligned} \tag{5.35}$$

By (5.4), (5.5), (5.9), (5.11), and Theorem 5.1, we have

$$C\triangle t\sum_{i=m_k}^{m_k+J}(||w_{f,t,1}^{i+1}||_0^2+||w_{f,t,2}^{i+1}||_0^2)\le C(T)(\triangle t^2+h^4),$$

$$\tilde{C}\triangle t\sum_{i=m_k}^{m_k+J}||\phi_{i+1}-\phi_i||_{H_p}^2\le C(T)\triangle t^2,$$

$$\tilde{C}\triangle t\sum_{i=m_k}^{m_k+J}||\phi_i-\phi_{m_k}||_{H_p}^2\le \tilde{C}n\triangle t\sum_{i=m_k}^{m_k+J}||\phi_i-\phi_{i+1}||_{H_p}^2\le C(T)\triangle t^2,$$

$$\tilde{C}\triangle t\sum_{i=m_k}^{m_k+J}||\phi_{m_k}-\phi^{m_k}||_{H_p}^2\le \tilde{C}n\triangle t||\phi_{m_k}-\phi^{m_k}||_{H_p}^2\le C(T)(\triangle t^2+h^4).$$

Note that the last two inequalities above, we used the fact $m_k+J-m_k\le n$ for $J=1,2,\cdots,n-1$ and the general triangle inequality. Combine the above bounds, the final result follows by Theorem 5.1,

$$\begin{aligned}&||e^{m_k+J+1}||_0^2+\sum_{i=m_k}^{m_k+J}\nu||e^{i+1}-e^i||_0^2+\triangle t\sum_{i=m_k}^{m_k+J}||e^{i+1}||_{H_f}^2\\&\le C(T)(\triangle t^2+h^4)+||e^{m_k}||_0^2\le C(T)(\triangle t^2+h^4).\end{aligned}$$

□

For the error in time derivatives on smaller time steps, we have the following error estimate.

Theorem 5.4. Based on the smoothness assumption on the true solution, $J=1,2,\cdots,n-1$, and k can be $0,1,\cdots,l$, the following estimate holds:

$$\begin{aligned}&\triangle t\sum_{i=m_k}^{m_k+J}||d_te^{i+1}||_0^2+\nu||e^{m_k+J+1}||_{H_f}^2+\lambda(e^{m_k+J+1},e^{m_k+J+1})\\&\le C(T)(\triangle t+h^4+\triangle t^{-1}h^4).\end{aligned}\tag{5.36}$$

Proof. Taking $2\triangle td_te^{m+1}=2(e^{m+1}-e^m)$ in (5.12), using the divergence-free property, sum over $m = m_k, m_k+1, \cdots, m_k+J$, we obtain

$$\begin{aligned}
&2\triangle t \sum_{i=m_k}^{m_k+J} ||d_t e^{i+1}||_0^2 + a_f(e^{m_k+J+1}, e^{m_k+J+1}) - a_f(e^{m_k}, e^{m_k}) + \triangle t^2 \sum_{i=m_k}^{m_k+J} a_f(d_t e^{i+1}, d_t e^{i+1}) \\
&= -2\triangle t \sum_{i=m_k}^{m_k+J} (w_{f,t}^{i+1}, d_t e^{i+1}) - 2\triangle t \int_\Gamma \sum_{i=m_k}^{m_k+J} (\phi_{i+1} - \phi_i) d_t e^{i+1} \cdot \mathbf{n}_f \\
&\quad - 2\triangle t \int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} (\phi_i - \phi^{m_k}) d_t e^{i+1} \cdot \mathbf{n}_f \\
&\leq \triangle t \sum_{i=m_k}^{m_k+J} ||d_t e^{i+1}||_0^2 + C\triangle t \sum_{i=m_k}^{m_k+J} ||w_{f,t}^{i+1}||_0^2 + \frac{\triangle t^2}{2} \sum_{i=m_k}^{m_k+J} \nu ||d_t e^{i+1}||_{H_f}^2 \\
&\quad + \tilde{C} \sum_{i=m_k}^{m_k+J} ||\phi_{i+1} - \phi_i||_{H_p}^2 + \frac{\triangle t^2}{2} \sum_{i=m_k}^{m_k+J} \nu ||d_t e^{i+1}||_{H_f}^2 + \tilde{C} \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi^{m_k}||_{H_p}^2 \\
&\leq \triangle t \sum_{i=m_k}^{m_k+J} ||d_t e^{i+1}||_0^2 + \triangle t^2 \sum_{i=m_k}^{m_k+J} \nu ||d_t e^{i+1}||_{H_f}^2 + C\triangle t \sum_{i=m_k}^{m_k+J} (||w_{f,t,1}^{i+1}||_0^2 + ||w_{f,t,2}^{i+1}||_0^2) \\
&\quad + \tilde{C} \sum_{i=m_k}^{m_k+J} ||\phi_{i+1} - \phi_i||_{H_p}^2 + \tilde{C} \sum_{i=m_k}^{m_k+J} (||\phi_i - \phi_{m_k}||_{H_p}^2 + ||\phi_{m_k} - \phi^{m_k}||_{H_p}^2). \qquad (5.37)
\end{aligned}$$

Just as the proof of the last theorem, using (5.4), (5.5), (5.9), (5.11) and Theorem 5.1, as well as the general triangle inequality and the fact $m_k + J - m_k \leq n$ for $J = 1, 2, \cdots, n-1$, we obtain

$$\begin{aligned}
C\triangle t \sum_{i=m_k}^{m_k+J} (||w_{f,t,1}^{i+1}||_0^2 + ||w_{f,t,2}^{i+1}||_0^2) &\leq C(T)(\triangle t^2 + h^4), \\
\tilde{C} \sum_{i=m_k}^{m_k+J} ||\phi_{i+1} - \phi_i||_{H_p}^2 &\leq C(T)\triangle t, \\
\tilde{C} \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi_{m_k}||_{H_p}^2 \leq \tilde{C} n \triangle t \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi_{i+1}||_{H_p}^2 &\leq C(T)\triangle t, \\
\tilde{C} \sum_{i=m_k}^{m_k+J} ||\phi_{m_k} - \phi^{m_k}||_{H_p}^2 \leq \tilde{C} n ||\phi_{m_k} - \phi^{m_k}||_{H_p}^2 &\leq C(T)(\triangle t + \triangle t^{-1} h^4).
\end{aligned}$$

Combine the above bounds, by Theorem 5, yield the final result,

$$\begin{aligned}\triangle t \sum_{i=m_k}^{m_k+J} ||d_t e^{i+1}||_0^2 + \nu||e^{m_k+J+1}||_{H_f}^2 + \lambda(e^{m_k+J+1}, e^{m_k+J+1}) \\ \leq C(T)(\triangle t + h^4 + \triangle t^{-1}h^4) + \nu||e^{m_k}||_{H_f}^2 + \lambda(e^{m_k}, e^{m_k}) \\ \leq C(T)(\triangle t + h^4 + \triangle t^{-1}h^4).\end{aligned}$$

□

Corollary 5.5. Under the assumptions of the previous theorem, including, $\tilde{C}\triangle t h^{-1} \leq 1$, where $\tilde{C}$ depends on ν, k_{min} and the domain Ω, such that when $\tilde{C}\triangle t h^{-1} \leq 1$, for $k = 0, 1, \cdots, M-1$, and $m = 1, 2, \cdots, N$, the following estimates hold:

$$||u_h^m - u(t^m)||_0 \leq C(T)(\triangle t + h^2), \tag{5.38}$$

$$||\phi_h^{m_{k+1}} - \phi(t^{m_{k+1}})||_0 \leq C(T)(\triangle t + h^2), \tag{5.39}$$

$$||u_h^m - u(t^m)||_1 \leq C(T)(\triangle t^{1/2} + h + \triangle t^{-1/2}h^2), \tag{5.40}$$

$$||\phi_h^{m_{k+1}} - \phi(t^{m_{k+1}})||_1 \leq C(T)(\triangle t^{1/2} + h + \triangle t^{-1/2}h^2). \tag{5.41}$$

Proof. By using the triangle inequality, combine the approximation properties and Theorem 5.1-5.4, it is easy to obtain the claim of this theorem. □

Remark: In this paper, different conditions are needed for stability and error estimation, for stability, we need $\triangle t$ satisfies that $\triangle t < C$ with C is a constant depends on domain Ω. For the error estimation, we assume that $\triangle t$ satisfies that $\tilde{C}\triangle t h^{-1}\triangle t < 1$ with $\tilde{C}$ is a constant depends on ν, k_{min} as well as the domain Ω. Which condition is better is still an open question, it depends on the problem and many other factors.

6 Numerical tests

In this section, we present some results of numerical tests which confirm the theoretical analysis. Assume $\Omega_f = [0, 1] \times [1, 2]$ and $\Omega_p = [0, 1] \times [0, 1]$ with interface $\Gamma = (0, 1) \times \{1\}$. The exact solution is given by

$$\begin{aligned}&(u1, u2) = ([x^2(y-1)^2 + y]cos(\omega t), [-\frac{2}{3}x(y-1)^3]cos(\omega t) + [2 - \pi sin(\pi x)]cos(t)),\\ &p = [2 - \pi sin(\pi x)]sin(0.5\pi y)cos(t),\\ &\phi = [2 - \pi sin(\pi x)][1 - y - cos(\pi y)]cos(t).\end{aligned}$$

Here $\omega = 5$, and the initial conditions, boundary conditions, and the forcing terms follows the solution.

The finite element spaces are constructed by using the well-known MINI elements ($P1b-P1$) for the Stokes problem and the linear Lagrangian elements ($P1$) for the Darcy flow. The code was implemented using the software package FreeFEM++[11]. For the monolithically coupled scheme, the GMRES routine is used to solve the (non-symmetric) coupled system. For the uncoupled scheme, a multi-frontal Gauss LU factorization implemented to solve the SPD sub-systems.

We define some notations first, for coupled scheme, we denote

$$e_u^{h,m} = u^{h,m} - u(t^m), e_p^{h,m} = p^{h,m} - p(t^m), e_\phi^{h,m} = \phi^{h,m} - \phi(t^m).$$

For the decoupled scheme, we denote

$$e_{h,u}^m = u_h^m - u(t^m), e_{h,p}^m = p_h^m - p(t^m), e_{h,\phi}^m = \phi_h^m - \phi(t^m).$$

First, we compare the convergence performance and CPU time for both the coupled scheme and the decouple scheme. In Table 1-2, we consider both schemes at time $t^m = 1.0$, with varying mesh h but fixed time step $\triangle t$ and $\triangle s = \omega \triangle t$. The two schemes achieve similar precision, although the monolithically coupled scheme is slightly more accurate than the decoupled scheme. However, the coupled scheme required much more CPU time than the decoupled scheme. The relative advantage of the decoupled scheme increased as the mesh was decreased. In Table 3-4, at the same time $t^m = 1.0$, with varying time step $\triangle t$ and $\triangle s = \omega \triangle t$ but fixed mesh $h = \frac{1}{8}$ are tested for both schemes. The two schemes almost get the same accuracy, but the coupled scheme needs much more CPU time than the decoupled scheme. In all, the decoupled scheme is comparable with the coupled scheme, and cheaper and more efficient.

Next, we will focus on the decoupled scheme, and examine the orders of convergence with respect to the spacing h or the time step $\triangle t$. Following [16], we introduce a more accurate approach to examine the orders of convergence with respect to the time step $\triangle t$ or the mesh size h due to the approximation errors $O(\triangle t^\gamma) + O(h^\mu)$. For example, assuming

$$v_h^{\triangle t}(x, t^m) \approx v(x, t^m) + C_1(x, t^m)\triangle t^\gamma + C_2(x, t^m)h^\mu.$$

Thus,

$$\rho_{v,h,i} = \frac{||v_h^{\triangle t}(x, t^m) - v_{\frac{h}{2}}^{\triangle t}(x, t^m)||_i}{||v_{\frac{h}{2}}^{\triangle t}(x, t^m) - v_{\frac{h}{4}}^{\triangle t}(x, t^m)||_i} \approx \frac{4^\mu - 2^\mu}{2^\mu - 1}.$$

$$\rho_{v,\triangle t,i} = \frac{||v_h^{\triangle t}(x, t^m) - v_h^{\frac{\triangle t}{2}}(x, t^m)||_i}{||v_h^{\frac{\triangle t}{2}}(x, t^m) - v_h^{\frac{\triangle t}{4}}(x, t^m)||_i} \approx \frac{4^\gamma - 2^\gamma}{2^\gamma - 1}.$$

Here, v can be u, p, ϕ and i can be 0, 1. While $\rho_{v,h,i}$, $\rho_{v,\triangle t,i}$ approach 4.0 or 2.0, the convergence order will be 2.0 or 1.0, respectively.

In Table 5, we study the convergence order with a fixed time step $\triangle t = 0.01$ and $\triangle s = \omega \triangle t$ and varying spacing $h = 1/2, 1/4, 1/8, 1/16, 1/32$. Observe that, $\rho_{u,h,0}$, $\rho_{\phi,h,0}$

is a little larger than 4.0, and $\rho_{u,h,1}$, $\rho_{p,h,0}$, $\rho_{\phi,h,1}$ approach 2.0, which suggest that the error estimates $O(h^2)$ for the L^2-norm of u and ϕ, $O(h)$ for the H^1-norm of u and ϕ and the L^2-norm of p is optimal in space for the decoupled scheme. However, in Table 5, we study the convergence order with a fixed spacing $h = 1/8$ and varying time step $\triangle t = 0.02,\ 0.01,\ 0.005,\ 0.0025,\ 0.00125$ and $\triangle s = \omega\triangle t$. The numerical experiments strongly suggest that the orders of convergence in time for all should be $O(\triangle t)$, which implies that the error estimates for the L^2-norm of u and ϕ is optimal, however, the error estimates for the H^1-norm of u and ϕ might not be optimal for the decoupled scheme, and may be further improved from $O(\triangle t^{1/2})$ to $O(\triangle t)$ by a finer analysis- an open problem for further work.

Table 1: The convergence performance and CPU time of coupled scheme at time $t^m = 1.0$, with varying mesh h but fixed time step $\triangle t = 0.01$.

h	$\|\|e_u^{h,m}\|\|_0$	$\|\|e_u^{h,m}\|\|_1$	$\|\|e_p^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	CPU
$\frac{1}{2}$	0.260588	1.50020	0.84932	0.154474	1.37573	4.428
$\frac{1}{4}$	0.073905	1.03481	0.82981	0.058474	0.86908	8.741
$\frac{1}{8}$	0.017644	0.40179	0.20873	0.010962	0.38724	32.081
$\frac{1}{16}$	0.004265	0.19129	0.07193	0.002688	0.19679	149.358
$\frac{1}{32}$	0.001120	0.09931	0.03493	0.000756	0.10059	698.809

Table 2: The convergence performance and CPU time of decoupled scheme at time $t^m = 1.0$, with varying mesh h but fixed small time step $\triangle t = 0.01$ and fixed large time step $\triangle s = \omega\triangle t$.

h	$\|\|e_{h,u}^m\|\|_0$	$\|\|e_{h,u}^m\|\|_1$	$\|\|e_{h,p}^m\|\|_0$	$\|\|e_{h,\phi}^m\|\|_0$	$\|\|e_{h,\phi}^m\|\|_0$	CPU
$\frac{1}{2}$	0.260588	1.50020	0.85337	0.154915	1.37554	0.856
$\frac{1}{4}$	0.070324	0.80750	0.47382	0.047873	0.79309	3.020
$\frac{1}{8}$	0.017953	0.41543	0.224210	0.013647	0.40958	10.038
$\frac{1}{16}$	0.004287	0.18950	0.07584	0.003879	0.19556	38.423
$\frac{1}{32}$	0.001185	0.09608	0.03781	0.002168	0.10105	143.963

7 Conclusions

A decoupled method with different time step in each domain for mixed Stokes-Darcy problem is proposed and analyzed in this paper. Under a modest time step restriction we

Table 3: The convergence performance and CPU time of coupled scheme at time $t^m = 1.0$, with varying time step $\triangle t$ but fixed mesh $h = \frac{1}{8}$.

$\triangle t$	$\|\|e_u^{h,m}\|\|_0$	$\|\|e_u^{h,m}\|\|_1$	$\|\|e_p^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	CPU
0.02	0.017658	0.401804	0.209185	0.010998	0.387225	19.656
0.01	0.017644	0.401971	0.208733	0.010962	0.387235	31.839
0.005	0.017638	0.401786	0.208770	0.010944	0.387240	55.723
0.0025	0.017639	0.401786	0.208897	0.010935	0.387242	103.725
0.00125	0.017639	0.401786	0.208942	0.010930	0.387242	215.046

Table 4: The convergence performance and CPU of decoupled scheme at time $t^m = 1.0$, with varying small time step $\triangle t$ and large time step $\triangle s = \omega\triangle t$ and but fixed mesh $h = \frac{1}{8}$.

$\triangle t$	$\|\|e_{h,u}^m\|\|_0$	$\|\|e_{h,u}^m)\|\|_1$	$\|\|e_{h,p}^m\|\|_0$	$\|\|e_{h,\phi}^m\|\|_0$	$\|\|e_{h,\phi}^m\|\|_0$	CPU
0.02	0.017849	0.415564	0.226369	0.014735	0.409701	5.429
0.01	0.017953	0.415431	0.224210	0.013647	0.409579	10.639
0.005	0.018010	0.415403	0.223604	0.013128	0.409577	21.435
0.0025	0.018038	0.415398	0.223444	0.012877	0.409592	41.262
0.00125	0.018050	0.415397	0.223404	0.012753	0.409603	76.190

Table 5: Convergence orders of $O(h^\mu)$ of Uncouple scheme at time $t^m = 1.0$, with varying mesh h but fixed small time step $\triangle t = 0.01$ and fixed large time step $\triangle s = \omega\triangle t$.

h	$\|\|u_h^m - u_{\frac{h}{2}}^m\|\|_0$	$\rho_{u,h,0}$	$\|\|u_h^m - u_{\frac{h}{2}}^m\|\|_1$	$\rho_{u,h,1}$	$\|\|p_h^m - p_{\frac{h}{2}}^m\|\|_0$	$\rho_{p,h,0}$
$\frac{1}{2}$	0.210264	3.74520	1.60993	1.94293	0.71638	1.48895
$\frac{1}{4}$	0.056142	3.83200	0.82861	1.93881	0.48113	2.15270
$\frac{1}{8}$	0.014651	4.23579	0.42738	2.14606	0.22350	2.89976
$\frac{1}{16}$	0.003458		0.19915		0.07708	
h	$\|\|\phi_h^m - \phi_{\frac{h}{2}}^m\|\|_0$	$\rho_{\phi,h,0}$	$\|\|\phi_h^m - \phi_{\frac{h}{2}}^m\|\|_0$	$\rho_{\phi,h,1}$		
$\frac{1}{2}$	0.134538	3.38510	1.30491	1.67120		
$\frac{1}{4}$	0.039744	3.56065	0.78083	1.87755		
$\frac{1}{8}$	0.011162	4.81406	0.41587	2.05836		
$\frac{1}{16}$	0.002319		0.20204			

Table 6: Convergence orders of $O(\triangle t^{\gamma})$ of Uncouple at time $t^m = 1.0$, with varying small time step $\triangle t$ and large time step $\triangle s = \omega \triangle t$ and but fixed mesh $h = \frac{1}{8}$.

$\triangle t$	$\|\|u^m_{\triangle t} - u^m_{\frac{\triangle t}{2}}\|\|_0$	$\rho_{u,\triangle t,0}$	$\|\|u^m_{\triangle t} - u^m_{\frac{\triangle t}{2}}\|\|_1$	$\rho_{u,\triangle t,1}$	$\|\|p^m_{\triangle t} - p^m_{\frac{\triangle t}{2}}\|\|_0$	$\rho_{p,\triangle t,0}$
0.02	6.49961e-4	2.03698	6.52832e-3	2.05855	1.68035e-2	1.91948
0.01	3.19081e-4	2.15518	3.17132e-3	2.18070	8.75420e-3	1.99190
0.005	1.48053e-4	2.17448	1.45427e-3	2.21398	4.39490e-3	2.01493
0.0025	6.80866e-5		6.656858e-4		2.18117e-3	
$\triangle s$	$\|\|\phi^m_{\triangle s} - \phi^m_{\frac{\triangle s}{2}}\|\|_0$	$\rho_{\phi,\triangle s,0}$	$\|\|\phi^m_{\triangle s} - \phi^m_{\frac{\triangle s}{2}}\|\|_0$	$\rho_{\phi,\triangle s,1}$		
0.1	1.51669e-3	1.96730	7.98752e-3	1.96671		
0.05	7.70949e-4	1.98387	4.06136e-3	1.98326		
0.025	3.88608e-4	1.99199	2.04781e-4	1.99160		
0.0125	1.95085e-4		1.02822e-4			

prove the zero-stability over bounded time intervals of the method. An analysis of the asymptotic stability over infinite time intervals and the possibly uniformity of the error in time is an important open problem. An error estimation is presented and numerical experiments are conducted to demonstrate the computational effectiveness of the decoupling approach.

No one paper can analyze every algorithmic option. We have made several choices to simplify the method (to offset the notational complexity of asynchronous time stepping methods). In particular we have studied a formulation of the porous media problem as one second order problem for the Darcy pressure instead of as a mixed system for the pressure and Darcy velocity. Extension to a mixed discretization in the porous media region is also an important open problem. At this early stage of development, it does seem like uncoupled, partitioned methods are very promising for solving coupled surface water-ground water flow problems. They are very efficient, can be accurate and do not require reference to any monolithically coupled system of even iteration between sub-problems.

References

[1] M. Anitescu, W. J. Layton and F. Pahlevani, Implicit for local effects and explicit for nonlocal effects is unconditionally stable, ETNA 18 (2004) 174–187.

[2] G. Beavers and D. Joseph, Boundary conditions at a naturally impermeable wall, J. Fluid. Mech, 30 (1967), 197-207.

[3] Y. Cao, M. Gunzburger, F. Hua and X. Wang, Coupled Stokes-Darcy Model with Beavers-Joseph Interface Boundary Condition, Comm. Math. Sci., 8, (2010), 1-25.

[4] Y. Cao, M. Gunzburger, X. Hu, F. Hua, X. Wang and W. Zhao, Finite element approximations for Stokes-Darcy flow with Beavers-Joseph interface conditions, SINUM 47(2010) 4239-4256.

[5] J. M. Connors, Partitioned time discretization for atmosphere-ocean interaction, PhD dissertation, University of Pittsburgh, 2010.

[6] J. M. Connors and J. S. Howell, Variable time stepping for decoupled computation of fluid-fluid interaction, technical report 2010.

[7] M. Discacciati, Domain decomposition methods for the coupling of surface and groundwater flows, Ph.D. dissertation, École Polytechnique Fédérale de Lausanne, 2004.

[8] M. Discacciati, E. Miglio and A. Quarteroni, Mathematical and numerical models for coupling surface and groundwater flows, Appl. Numer. Math., 43 (2002), 57-74.

[9] M. Discacciati and A. Quarteroni, Navier-Stokes/Darcy Coupling: Modeling, Analysis, and Numerical Approximation, Revista Matemática Complutense, 2009

[10] H. I. Ene and E. Sanchez-Palencia, Equations et ph´enomen‘es de surface pour l’´ecoulement dans un mod‘ele de milieu poreux, J. M´ecanique, 14(1):73–108, 1975.

[11] F. Hecht, O. Pironneau, and K. Ohtsuka, FreeFEM++, http://www.freefem.org/ff++/ftp/ (2010).

[12] W. Jaeger and A. Mikelic, On the interface boundary conditions of Beavers, Joseph and Saffman, SIAM J Applied Math. 60(2000) 1111-1127.

[13] V. John, W. J. Layton and C. Manica, Time Averaged convergence of algorithms for flow problems, SINUM, 46, 2007, 151-179.

[14] W. J. Layton, F. Schieweck and I. Yotov, Coupling fluid flow with porous media flow, SIAM J. Numer. Anal. 40(2003) 2195-2218.

[15] M. Mu and J. Xu, A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow, SIAM J. Numer. Anal., 45 (2007), 1801-1813.

[16] M. Mu and X. H. Zhu, Decoupled schemes for a non-stationary mixed Stokes-Darcy model,Mathematics of Computation, 79 (2010), 707-731.

[17] L. E. Payne, J .C. Song and B. Straughan, Continuous dependence and convergence results for Brinkman and Forcheimer models with variable viscosity, Proc. Royal Soc. London, A 455(1999) 2173-2190.

[18] L. E. Payne and B. Straughan, Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modeling questions, J Math. Pure Appl. 77(1998) 1959-1977.

[19] P. G. Saffman, On the boundary condition at the interface of a porous medium, Studies in Appl. Math. 1(1971) 93-101.