NUMERICAL ANALYSIS OF THE FULLY DISCRETE FINITE ELEMENT SCHEME FOR THE LIGHTHILL ACOUSTIC ANALOGY AND ESTIMATING THE ERROR IN THE SOUND POWER

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ABSTRACT. This paper gives rigorous numerical analysis of the error in prediction of aeroacoustic noise via Lighthill analogy. The first fundamental and intractable problem is to predict the sound power on surfaces. We give a full analysis of three methods of prediction. The second fundamental problem is the limited regularity of the underlying turbulent flow. This is handled herein by giving a negative norm error analysis which reduces the required regularity. We also give a comprehensive analysis of a fully discrete scheme including effects of the error coming into acoustic equation from the turbulent flow simulation.

1. INTRODUCTION

This paper presents and studies the fully discrete Finite Element Method for the Lighthill analogy [19] used for computing the acoustic pressure of the noise generated by turbulent flows. Here we present the Lighthill analogy without a derivation which was reviewed in [21]. The general result is presented in Theorem 2. Next, we refer to the semidiscrete scheme built and analyzed in [21]. In this paper we continue the analysis using the negative norms for the error and present the results in Theorem 3. Finally, the ways for computing the acoustic power are suggested and for each one the error estimate is presented.

Prediction of the acoustic noise generated by a turbulent flow has been an important fundamental problem in various engineering applications. First of all, the applications lie in all types of transport. The most noisy ones are trains and, specifically, jet airplanes. In these cases for high velocities the aerodynamic noise tends to dominate compared to other sources of noise, [29]. The engines of the new generation fighter jets are expected to produce more than 140 decibels of noise while 150 already damage internal organs. One of the other sources of annoying aerodynamic noise can be an everyday home technology, such as coffee makers or climate systems. Other important applications lie in ocean acoustics, for example, in submarine detection. There's also an interest in medicine. Measuring characteristics of the sound emitted from a blood flow in a valve of a heart would help diagnose heart murmurs. There are also many engineering devices such as, for example, wind turbines and helicopter rotors that produce significant amount of noise that is needed to be reduced.

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The research of aeroacoustics was pioneered by Lighthill in 1951. He proposed the fundamental model of noise generated by turbulence. Given the turbulent flow's velocity $\mathbf{u}$ and density ρ_0, the Lighthill's model for the small acoustic pressure fluctuations q is a wave equation with a nonlinear source term :

$$\frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta q = \nabla\cdot(\nabla\cdot(\rho_0 \mathbf{u}\otimes\mathbf{u}) - \nabla\cdot\mathbb{S} - \rho_0\mathbf{f}), \tag{1.1}$$

with the *deviatoric* stress tensor $\mathbb{S}$, the sound speed $a_0 = \sqrt{\frac{\partial p}{\partial\rho}|_{\rho=\rho_0}}$, the external body force $\mathbf{f}$ and the density ρ_0.

So far the only known paper with rigorous mathematical derivation of the Lighthill model is by Novotny and Layton [23]. For low Mach numbers the generated noise itself plays little role in changing the flow and thus the model describes a one-sided process, i.e. the noise is generated by the flow whose motion is dependent solely on the known external forces, and no feedback from the noise to the turbulent flow is considered, [19]. Also, for small Mach numbers the compressibility of the flow has negligible impact on the sound generation, see, for example, [29]. Therefore, the noise can be predicted by solving the incompressible Navier-Stokes equations (NSE) for $\mathbf{u}$ and inserting the incompressible velocity and density ρ_0 into the right-hand side (RHS) of (1.1) and then solving (1.1) for the acoustic pressure q. For the incompressible case $\nabla\cdot\nabla\cdot\mathbb{S} = 0$, [21]. More on computational practice with Lighthill analogy may be found, for example, in [8] and [25] as well as [31].

The whole acoustic domain of our model equation (1.1) is divided in two parts. These are the turbulent region Ω_1 with the flow where the generation of sound occurs and the far field Ω_2 where the acoustic waves propagate. In this paper, as in [21], Ω_1 is surrounded by Ω_2. The whole domain is $\Omega = \Omega_1 \cup \Omega_2$. This is shown on figure 1.

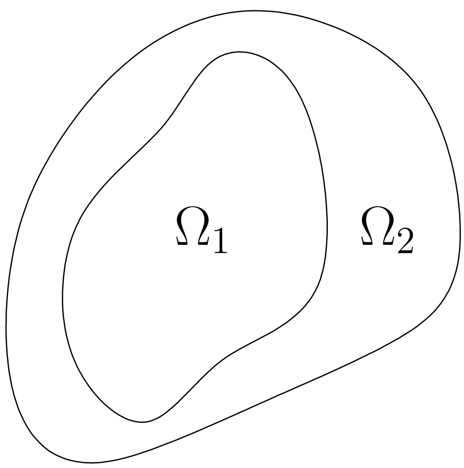

FIGURE 1. One domain inside the other

The Initial Boundary Value Problem is the following :

$$
\begin{aligned}
(1.2) \qquad & \frac{1}{a_0^2}\frac{\partial^2 q}{\partial t^2} - \Delta q = R(t,x) + \frac{1}{a_0^2}G(t,x)\ \forall (t,x) \in (0,T)\times\Omega,\\
& q(0,x) = q_1(x),\ \frac{\partial q}{\partial t}(0,x) = q_2(x)\ \forall x \in \Omega,\\
& \nabla q\cdot \mathbf{n} + \frac{1}{a_0}\frac{\partial q}{\partial t} = g(t,x)\ \forall (t,x)\in (0,T)\times\partial\Omega,
\end{aligned}
$$

where $R(t,x) = \nabla\cdot(\nabla\cdot(\rho_0\mathbf{u}\otimes\mathbf{u}) - \rho_0\mathbf{f})$ inside Ω_1 and 0 around it in Ω_2, assumed $\mathbf{u}$ is the solution of the incompressible NSE in Ω_1. The function $G(t,x)$ and $g(t,x)$ are arbitrary control functions that we add according to the problem's physics and goals. The case $g\equiv 0$ in (1.2) gives the non-reflecting boundary conditions of the first order.

The semidiscrete scheme for (1.2) with approximate R_{h_1} was fully presented and studied in [21] and was relying on Dupont's analysis [12]. The main result was that for the approximating space of continuous piecewise polynomials of degree no more than $k-1$ and 'sufficiently good' initial data on the mesh of size $h<1$, the error satisfies

$$
(1.3) \quad \left\| \frac{\partial}{\partial t}(q-q_h)\right\|_{L^\infty(L^2(\Omega))} + \|q-q_h\|_{L^\infty(L^2(\Omega))} \leqslant C(h^k + \|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}),
$$

if the exact solution q is regular enough, more specifically, $q, q_t \in L^\infty(H^k(\Omega))$, $q_{tt}\in L^2(H^k(\Omega))$. Here $Q=\rho_0\nabla\cdot\nabla\cdot(\mathbf{u}\otimes\mathbf{u})$ and $Q_{h_1} = \rho_0\nabla\cdot\nabla\cdot(\mathbf{u}_{h_1}\otimes\mathbf{u}_{h_1})$, where velocity $\mathbf{u}_{h_1}$ is computed on another independent grid of mesh size h_1 inside Ω_1.

The fully discrete scheme for (1.2) is studied in section 3. The analysis of the scheme is based on Dupont's work [12] where the basic FEM scheme, both continuous and discrete in time, for the wave equation with RHS known exactly was analyzed. Our analysis differs by the presence of the computational error in the RHS of the wave equation in (1.2), more specifically, in term Q. Since the optimal estimate for the error $\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}$ from (1.3) is not known, it's worth estimating negative norms of the error $q-q_h$. This results in increase of the accuracy by multiplying the error $\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}$ by a power of h. Section 4 is devoted to the negative norm analysis. The main result was obtained for a particular case of Neumann boundary conditions without the time-derivative term and is presented in Theorem 3.

One of the most important indicators of the sound emission is the acoustic intensity (see [20])

$$
I = q\cdot\mathbf{v},
$$

where $\mathbf{v}$ is the velocity of the fluid, and in the case of the far field it's a small velocity fluctuation about the zero state. The flux integral of the intesity along a surface S gives a sound power

$$
A = \int_S I\cdot\mathbf{n}dS.
$$

Section 5 studies three different approaches on calculating the sound power on a given surface S and estimating the numerical error for it. The first method uses the linearized continuity and momentum equations as a starting point in order to obtain an exact analytical formula for computing velocity $\mathbf{v}$ in the far field. This is the cheapest method computationally, but is the least accurate. The improvement in the rate of covergence may be made in case when $S \subset \partial\Omega$. The other approach suggests to obtain an upper bound for the sound power so that this bound was computed via fluctuating pressure q_h only, and then the numerical error for the bound is analyzed. The last method is based on the duality analysis and is only used for the case $S \subset \partial\Omega$. This method breaks the problem in two computational subproblems, one is for finding q_h and the other is for finding $\mathbf{v}_{h_2}$ on the other grid of mesh size $h_2 < 1$. Although duality method gives the highest possible rate of convergence for the term containing q_h, the scheme for $\mathbf{v}_{h_2}$ still requires more research since the rate of convergence it provides is one power less than that for the term with q_h. From this point of view, we can only say that duality method is less preferable compared to the exact formula approach, since they both give the same rate of convergence in case $h = O(h_2)$ and computationally the duality method is much more expensive.

2. Notation and preliminaries

In this paper we assume that both Ω and Ω_1 are open bounded connected domains in $\mathbb{R}^d$, $d = 2, 3$, having smooth enough boundaries $\partial\Omega$ and $\partial\Omega_1$ respectively. $(\cdot,\cdot)$ and $\|\cdot\|$ without a subscript denote the $L^2(\Omega)$ or $L^2(\Omega_1)$ inner product and norm depending on which domain is considered at the moment. The norms $\|\cdot\|_{(L^p(\Omega))^d}$ are used for vector functions $\mathbf{u}$ with two or three components. If $1 \leqslant p < \infty$, they should be understood as

$$\|\mathbf{u}\|_{(L^p(\Omega))^d} = \left(\sum_{i=1}^{d} \|u_i\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}},$$

where u_i denotes i-th component of $\mathbf{u}$ and d is the number of components. The inner product should be understood as

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{d} (u_i, v_i).$$

$L^2(\partial\Omega)$ denotes the space of the real-valued square-integrable functions on the boundary $\partial\Omega$ of the domain Ω. The inner product in this space is denoted as $< \cdot, \cdot >$:

$$< u, v >= \int_{\partial\Omega} u \cdot v dS \text{ for } u, v \in L^2(\partial\Omega).$$

The norm induced by this inner product is denoted as $|\cdot|$:

$$|v| = \|v\|_{L^2(\partial\Omega)} = \sqrt{< v, v >} \text{ for } v \in L^2(\partial\Omega).$$

For any integer $s \geqslant 0$ let $H^s(\Omega)$ denote a Sobolev space $W^{s,2}(\Omega)$ of real-valued functions on a domain Ω. The inner product and norm in the space $H^s(\Omega)$ are defined by

$$(u, v)_{H^s(\Omega)} = (u, v)_s = \sum_{|\alpha|=0}^{s} (\partial^\alpha u, \partial^\alpha v), \ \|u\|_{H^s(\Omega)} = \|u\|_s = \sqrt{(u, u)_{H^s(\Omega)}},$$

where α is a multiindex and $\partial^\alpha u$ denotes a weak partial derivative of the order $|\alpha|$ of the function u. The space $H_{div}(\Omega)$ denotes all such vector square-integrable on Ω functions that their divergence is also square-integrable on Ω, [24]. Next, if B

denotes a Banach space with norm $\|\cdot\|_B$ and $u : [0,T] \to B$ is Lebesgue measurable, then we define

$$\|u\|_{L^p(0,T;B)} = \left(\int_0^T \|u\|_B^p dt \right)^{\frac{1}{p}}, \|u\|_{L^\infty(0,T;B)} = esssup_{0 \leqslant t \leqslant T} \|u(t,\cdot)\|_B,$$

and the space

$$L^p(0,T;B) = L^p(B) = \{u : [0,T] \to B | \|u\|_{L^p(0,T;B)} < \infty\} \text{ for } 1 \leqslant p \leqslant \infty.$$

Theorem 1. *(Trace theorem) Let $v \in H^1(\Omega)$. Then $v \in H^{\frac{1}{2}}(\partial\Omega)$ and the following inequality holds*

$$\|v\|_{L^2(\partial\Omega)} \leqslant C_{tr} \|v\|_1,$$

where C_{tr} is a constant that depends only on the geometry of the domain Ω.

2.1. **Finite Element Space.** Let us build non-degenerate, edge-to-edge, shape regular triangular mesh by introducing the partition $\Pi = \{T_1, T_2, ..., T_M\}$ of Ω into the finite triangles. The characteristic size of the mesh $h < 1$ is defined by

$$h = max_{1 \leqslant i \leqslant M} diam(T_i).$$

Define

$$M^m(\Omega) = \{u \in L^2(\Omega) \mid u|_T \in P_{m-1} \ \forall T \in \Pi\} \text{ and } M_0^m(\Omega) = M^m(\Omega) \cap C^0(\Omega),$$

where P_m is the space of polynomials of degree no more than m and $C^0(\Omega)$ is the space of continuous on Ω functions. Therefore, by $M_0^m(\Omega)$ we mean the space of continuous piecewise polynomials of degree no more than $m-1$ on Ω. Obviously, $M_0^m(\Omega)$ is defined for $m \geqslant 2$.

From now on, C will denote a generic constant, not necessarily the same in two places. As in [12], we suppose there exist a positive constant C and integer $k \geqslant 2$ such that the spaces $M_0^m(\Omega)$ have the property that for $0 \leqslant s \leqslant 1$, $2 \leqslant m \leqslant k$ and $V \in H^m(\Omega)$

$$inf_{\chi \in M_0^m(\Omega)} \|V - \chi\|_{H^s(\Omega)} \leqslant C h^{m-s} \|V\|_{H^m(\Omega)}.$$

Following [12], we define the H^1-projection $\hat{u} \in M_0^m(\Omega)$ for $u \in H^1(\Omega)$ by the formula :

$$a_0^2(\nabla u, \nabla u_h) + (u, u_h) = a_0^2(\nabla \hat{u}, \nabla u_h) + (\hat{u}, u_h) \ \forall u_h \in M_0^m(\Omega).$$

Below is the lemma that will be used in the proof of the main theorem about the error estimate for the fully discrete scheme.

Lemma 1. *(Dupont [12], Lemma 5) Let $u, \frac{\partial u}{\partial t} \in L^\infty(H^k(\Omega))$ and $\frac{\partial^2 u}{\partial t^2} \in L^2(H^k(\Omega))$ for some positive integer k, $m \geqslant k \geqslant 2$. Then for some constant C independent of h the error in the H^1-projection $\hat{u}$ satisfies*

$$\left\| \frac{\partial^r (u - \hat{u})}{\partial t^r} \right\|_{L^s(L^2(\Omega))} + \left\| \frac{\partial^r (u - \hat{u})}{\partial t^r} \right\|_{L^s(H^{-\frac{1}{2}}(\partial\Omega))} \leqslant C h^k,$$

where $s = \infty, \infty, 2$ for $r = 0, 1, 2$ respectively.

A mesh with above properties is called quasi-uniform, if there exist constants C_1 and C_2 independent of h, such that

$$C_1 \cdot diam(T_i) \leqslant diam(T_j) \leqslant C_2 \cdot diam(T_i)$$

for any distinct triangular elements T_i and T_j of the mesh.

For a given FEM space $M_0^m(\Omega)$, $m \geqslant 2$, consider the nodal basis consisting of functions ϕ_j. An arbitrary function $u \in H^m(\Omega)$ has a unique continuous representation on Ω, and therefore it's possible to define a piecewise polynomial interpolant $I_h(u)$ for this function by the formula

$$I_h(u) = \sum_j u(N_j)\phi_j,$$

where N_j denote the nodal points.

For the discrete in time numerical analysis the discrete Gronwall lemma will be used. It formulates as follows.

Lemma 2. *(Discrete Gronwall lemma) Assume a_n, b_n are two non-negative sequences, and b_n is non-decreasing, such that $a_0 \leqslant b_0$ and $\forall n$ $a_n \leqslant b_n + \sum_{i=0}^{n} \lambda a_i$, where $0 < \lambda < 1$ is independent of n. Then $\forall n$:*

$$a_n \leqslant \frac{b_n}{1-\lambda} e^{\frac{n\lambda}{1-\lambda}}$$

3. Fully discrete scheme

Consider the initial boundary-value problem

$$\frac{\partial^2 q}{\partial t^2} - a_0^2 \Delta q = a_0^2 (Q(\mathbf{u},\mathbf{u}) - \rho_0 \nabla \cdot \mathbf{f}) + G(t,x) \ \forall (t,x) \in (0,T) \times \Omega_1, \tag{3.1}$$

$$\frac{\partial^2 q}{\partial t^2} - a_0^2 \Delta q = 0 \ \forall (t,x) \in (0,T) \times \Omega / \Omega_1,$$

$$q(0,x) = q_1(x), \ \frac{\partial q}{\partial t}(0,x) = q_2(x) \ \forall x \in \Omega,$$

$$\nabla q \cdot \mathbf{n} + \frac{1}{a_0} \frac{\partial q}{\partial t} = g(t,x) \ \forall (t,x) \in (0,T) \times \partial\Omega,$$

where all functions on the RHS are known and $\mathbf{n}$ being the outward normal on the boundary $\partial\Omega$.

The exact variational formulation is as follows (see [21]): assume that

$$Q(\mathbf{u},\mathbf{u}) - \rho_0 \nabla \cdot \mathbf{f} + \frac{1}{a_0^2} G \in L^2(0,T;L^2(\Omega_1)), q(0,\cdot) \in H^1(\Omega),$$

$$\frac{\partial q}{\partial t}(0,\cdot) \in L^2(\Omega), g \in L^2(0,T;L^2(\partial\Omega)).$$

Find $q \in L^2(0,T;H^1(\Omega))$ such that $\frac{\partial q}{\partial t} \in L^2(0,T;H^1(\Omega))$, $\frac{\partial^2 q}{\partial t^2} \in L^2(0,T;L^2(\Omega))$ and

$$\left(\frac{\partial^2 q}{\partial t^2}, v\right) + a_0^2 (\nabla q, \nabla v) + a_0 \left\langle \frac{\partial q}{\partial t}, v \right\rangle = \tag{3.2}$$

$$= a_0^2 \left(Q(\mathbf{u},\mathbf{u}) - \rho_0 \nabla \cdot \mathbf{f} + \frac{1}{a_0^2} G, v \right)_{\Omega_1} + a_0^2 < g, v >$$

$$\forall v \in H^1(\Omega), 0 < t < T,$$

$$(q(0,\cdot), v) = (q_1(\cdot), v) \ \forall v \in H^1(\Omega), \tag{3.3}$$

$$\left(\frac{\partial q}{\partial t}(0,\cdot), v\right) = (q_2(\cdot), v) \ \forall v \in H^1(\Omega). \tag{3.4}$$

Next, we construct the fully discrete Finite Element approximation. It will be based on finite-dimensional spaces $\{M_0^m(\Omega)\} \subset H^1(\Omega)$ of continuous piecewise polynomials of degree no more than $m-1$, section 2. The approximation in time uses the second order scheme. The total error between the exact solution q of (3.2) and the approximate q_h will consist of the scheme error and the perturbation of the RHS caused by replacing $R = Q(\mathbf{u},\mathbf{u}) - \rho_0 \nabla \cdot \mathbf{f}$ with $R^{'} = Q^{'} - \rho_0 \nabla \cdot \mathbf{f}$, where $Q^{'} = Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1})$. We also implicitly assume that both R and $R^{'}$ are defined outside Ω_1 as zero functions.

Below we will follow Dupont's notations from [12]. Suppose the time step $\Delta t = T/N$ for some fixed positive integer N. If some function f is defined for time levels $i\Delta t$ with all integers i, $0 \leqslant i \leqslant N$, then denote by f_n the function f at the time level $t_n = n\Delta t$. Other notations are

$$f_{n+\frac{1}{2}} = \frac{1}{2}(f_{n+1} + f_n),\ f_{n,\frac{1}{4}} = \frac{1}{4}f_{n-1} + \frac{1}{2}f_n + \frac{1}{4}f_{n+1},$$

$$\partial_t f_{n+\frac{1}{2}} = \frac{f_{n+1} - f_n}{\Delta t},\ \partial_t^2 f_n = \frac{f_{n+1} - 2f_n + f_{n-1}}{(\Delta t)^2},\ \delta_t f_n = \frac{f_{n+1} - f_{n-1}}{2\Delta t}$$

and for any norm $\|\cdot\|_X$

$$\|f\|_{\tilde{L}^\infty(X)} = max_{0<n<N}\|f_n\|_X,\ \|f\|_{\hat{L}^\infty(X)} = max_{0\leqslant n<N}\|f_{n+\frac{1}{2}}\|_X.$$

We assume that the term $Q(\mathbf{u}_{h_1}, \mathbf{u}_{h_1})$ is given either continuously or discretely in time. In the second case we additionaly impose that this term is defined for all the time levels t_n used for the wave equation. Consider the discrete scheme

$$(3.5)\qquad \begin{aligned} &(\partial_t^2 q_{h,n}, v_h) + a_0^2(\nabla q_{h,n,\frac{1}{4}}, \nabla v_h) + a_0 < \delta_t q_{h,n}, v_h >= \\ &\qquad = a_0^2(R^{'}_{n,\frac{1}{4}} + \frac{1}{a_0^2} G_{n,\frac{1}{4}}, v_h) + a_0^2 < g_{n,\frac{1}{4}}, v_h > \\ &\forall v_h \in M_0^m(\Omega),\ \text{for } n = 1, ..., N-1, \\ &q_{h,0} \text{ and } q_{h,1} \text{ are the initial data.} \end{aligned}$$

Theorem 2. *Let q be the solution of (3.2) and $q, q_t \in L^\infty(H^k(\Omega))$ and $q_{tt} \in L^2(H^k(\Omega))$ for some integer k, $m \geqslant k \geqslant 2$. Also let $\frac{\partial^4 q}{\partial t^4} \in L^2(L^2(\Omega))$, $\frac{\partial^3 q}{\partial t^3} \in L^2(L^2(\partial\Omega))$. Finally, assume that the initial data satisfies conditions*

$$\|q_{h,0} - \hat{q}_0\|_{H^1(\Omega)} + \|q_{h,1} - \hat{q}_1\|_{H^1(\Omega)} + \left\| \frac{q_{h,1} - q_{h,0}}{\Delta t} - \frac{\hat{q}_1 - \hat{q}_0}{\Delta t} \right\| \leqslant Ch^k$$

with constant C independent of h. Then the solution q_h of (3.5) satisfies

$$\begin{aligned} &\|\partial_t(q - q_h)\|_{\hat{L}^\infty(L^2(\Omega))} + \|q - q_h\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant \\ &\qquad C\left(h^k + \sqrt{\sum_{n=1}^{N-1} \Delta t \|Q^{'}_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2} + (\Delta t)^2\right) \end{aligned}$$

with constant C independent of h.

Proof. The exact solution q satisfies

$$(\partial_t^2 q_n, v_h) + a_0^2(\nabla q_{n,\frac{1}{4}}, \nabla v_h) + a_0 < \delta_t q_n, v_h >= a_0^2(R_{n,\frac{1}{4}} + \frac{1}{a_0^2}(G_{n,\frac{1}{4}} + r_n), v_h) +$$

$$+ a_0 < r^{'}_n, v_h > + a_0^2 < g_{n,\frac{1}{4}}, v_h > .$$

Here r_n and $r_n^{'}$ are the approximation errors and

$$\|r_n\|^2 \leqslant C(\Delta t)^3 \int_{t_{n-1}}^{t_{n+1}} \left\| \frac{\partial^4 q}{\partial t^4} \right\|^2 d\tau \text{ and } |r_n^{'}|^2 \leqslant C(\Delta t)^3 \int_{t_{n-1}}^{t_{n+1}} \left| \frac{\partial^3 q}{\partial t^3} \right|^2 d\tau.$$

Let $\eta = \hat{q} - q$, $\psi = q_h - \hat{q}$. Then

$$(\partial_t^2 \psi_n, v_h) + a_0^2 (\nabla \psi_{n,\frac{1}{4}}, \nabla v_h) + a_0 < \delta_t \psi_n, v_h >=$$

$$= a_0^2 (Q_{n,\frac{1}{4}}^{'} - Q_{n,\frac{1}{4}}, v_h) + (\eta_{n,\frac{1}{4}} - \partial_t^2 \eta_n, v_h) - a_0 < \delta_t \eta_n, v_h > - (r_n, v_h) - a_0 < r_n^{'}, v_h > .$$

Set $v_h = \delta_t \psi_n$. Then we will have

$$\frac{1}{2} \frac{\|\partial_t \psi_{n+\frac{1}{2}}\|^2 - \|\partial_t \psi_{n-\frac{1}{2}}\|^2}{\Delta t} + \frac{a_0^2}{2} \frac{\|\nabla \psi_{n+\frac{1}{2}}\|^2 - \|\nabla \psi_{n-\frac{1}{2}}\|^2}{\Delta t} + a_0 |\delta_t \psi_n|^2 =$$

$$= a_0^2 (Q_{n,\frac{1}{4}}^{'} - Q_{n,\frac{1}{4}}, \delta_t \psi_n) + (\eta_{n,\frac{1}{4}} - \partial_t^2 \eta_n, \delta_t \psi_n) - a_0 < \delta_t \eta_n, \delta_t \psi_n > -$$

$$- (r_n, \delta_t \psi_n) - a_0 < r_n^{'}, \delta_t \psi_n > .$$

Next add inequality

$$\frac{1}{2\Delta t} \left(\|\psi_{n+\frac{1}{2}}\|^2 - \|\psi_{n-\frac{1}{2}}\|^2 \right) \leqslant \frac{1}{2} \left(\|\delta_t \psi_n\|^2 + \|\psi_{n,\frac{1}{4}}\|^2 \right)$$

and use Young's inequalities on the RHS to get

$$\frac{1}{2} \frac{\|\partial_t \psi_{n+\frac{1}{2}}\|^2 - \|\partial_t \psi_{n-\frac{1}{2}}\|^2}{\Delta t} + \frac{a_0^2}{2} \frac{\|\nabla \psi_{n+\frac{1}{2}}\|^2 - \|\nabla \psi_{n-\frac{1}{2}}\|^2}{\Delta t} + a_0 |\delta_t \psi_n|^2 +$$

$$+ \frac{1}{2\Delta t} \left(\|\psi_{n+\frac{1}{2}}\|^2 - \|\psi_{n-\frac{1}{2}}\|^2 \right) \leqslant C(\|\delta_t \psi_n\|^2 + \|\psi_{n,\frac{1}{4}}\|^2 + \|\eta_{n,\frac{1}{4}}\|^2 + \|\partial_t^2 \eta_n\|^2 +$$

$$+ \|Q_{n,\frac{1}{4}}^{'} - Q_{n,\frac{1}{4}}\|^2 + \|r_n\|^2) - a_0 < \delta_t \eta_n, \delta_t \psi_n > - a_0 < r_n^{'}, \delta_t \psi_n > .$$

Let $1 \leqslant i \leqslant N$ be an integer. Note that for $i \geqslant 4$

$$\sum_{n=1}^{i-1} \Delta t < \delta_t \eta_n, \delta_t \psi_n >=< \delta_t \eta_{i-1}, \psi_{i-\frac{1}{2}} > - \frac{1}{2} < \delta_t \eta_1, \psi_0 > - \frac{1}{2} < \delta_t \eta_2, \psi_1 > +$$

$$- \frac{\Delta t}{2} \left\langle \frac{\delta_t \eta_{i-1} - \delta_t \eta_{i-2}}{\Delta t}, \psi_{i-1} \right\rangle - \sum_{n=2}^{i-2} \Delta t \left\langle \frac{\eta_{n+2} - 2\eta_n + \eta_{n-2}}{4(\Delta t)^2}, \psi_n \right\rangle .$$

Then we have

$$-a_0 \sum_{n=1}^{i-1} \Delta t < \delta_t \eta_n, \delta_t \psi_n >\leqslant C(\|\delta_t \eta_{i-1}\|_{H^{-\frac{1}{2}}(\partial\Omega)} \cdot \|\psi_{i-\frac{1}{2}}\|_{H^1(\Omega)} +$$

$$+ \|\delta_t \eta_1\|_{H^{-\frac{1}{2}}(\partial\Omega)} \cdot \|\psi_0\|_{H^1(\Omega)} + \sum_{n=2}^{i-2} \Delta t \left\| \frac{\eta_{n+2} - 2\eta_n + \eta_{n-2}}{4(\Delta t)^2} \right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \cdot \|\psi_n\|_{H^1(\Omega)}$$

$$+ \|\delta_t \eta_2\|_{H^{-\frac{1}{2}}(\partial\Omega)} \cdot \|\psi_1\|_{H^1(\Omega)} + \Delta t \left\| \frac{\delta_t \eta_{i-1} - \delta_t \eta_{i-2}}{\Delta t} \right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \cdot \|\psi_{i-1}\|_{H^1(\Omega)}).$$

The last expression may be bounded by

$$C(\|\delta_t \eta\|^2_{\tilde{L}^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \|\psi_0\|^2_{H^1(\Omega)} + \|\psi_1\|^2_{H^1(\Omega)} +$$

$$+ \sum_{n=2}^{i-2} \Delta t \left\| \frac{\eta_{n+2} - 2\eta_n + \eta_{n-2}}{4(\Delta t)^2} \right\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \sum_{n=2}^{i-1} \Delta t \|\psi_n\|^2_{H^1(\Omega)} +$$

$$+\Delta t \left\| \frac{\delta_t \eta_{i-1} - \delta_t \eta_{i-2}}{\Delta t} \right\|^2_{H^{-\frac{1}{2}}(\partial\Omega)}) + \epsilon \|\psi_{i-\frac{1}{2}}\|^2_{H^1(\Omega)},$$

where positive ϵ is of our choice. Here $C = C(\epsilon)$. Also

$$-a_0 \sum_{n=1}^{i-1} \Delta t < r_n', \delta_t \psi_n > \leqslant \frac{a_0}{2} \left(\sum_{n=1}^{i-1} \Delta t |r_n'|^2 + \sum_{n=1}^{i-1} \Delta t |\delta_t \psi_n|^2 \right).$$

Summation over n from 1 to $i-1$ gives

$$\|\partial_t \psi_{i-\frac{1}{2}}\|^2 + \|\psi_{i-\frac{1}{2}}\|^2_{H^1(\Omega)} + \sum_{n=1}^{i-1} \Delta t |\delta_t \psi_n|^2 \leqslant$$

$$\leqslant C(\sum_{n=1}^{i} \Delta t \|\partial_t \psi_{n-\frac{1}{2}}\|^2 + \sum_{n=1}^{i} \Delta t \|\psi_{n-\frac{1}{2}}\|^2 + \sum_{n=1}^{i-1} \Delta t \|\eta_{n,\frac{1}{4}}\|^2 + \sum_{n=1}^{i-1} \Delta t \|\partial_t^2 \eta_n\|^2 +$$
$$+ \|\partial_t \psi_{\frac{1}{2}}\|^2 + \|\psi_{\frac{1}{2}}\|^2_{H^1(\Omega)} + \|\delta_t \eta\|^2_{\tilde{L}^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \|\psi_0\|^2_{H^1(\Omega)} + \|\psi_1\|^2_{H^1(\Omega)} +$$
$$+ \sum_{n=1}^{i-1} \Delta t \|Q_{n,\frac{1}{4}}' - Q_{n,\frac{1}{4}}\|^2 + \sum_{n=2}^{i-2} \Delta t \left\| \frac{\eta_{n+2} - 2\eta_n + \eta_{n-2}}{4(\Delta t)^2} \right\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \sum_{n=1}^{i-1} \Delta t \|r_n\|^2 +$$
$$+ \sum_{n=2}^{i-1} \Delta t \|\psi_n\|^2_{H^1(\Omega)} + \sum_{n=1}^{i-1} \Delta t |r_n'|^2 + \Delta t \left\| \frac{\delta_t \eta_{i-1} - \delta_t \eta_{i-2}}{\Delta t} \right\|^2_{H^{-\frac{1}{2}}(\partial\Omega)}).$$

Note that

$$\sum_{n=1}^{i} \Delta t \|\psi_{n-\frac{1}{2}}\|^2 + \sum_{n=2}^{i-1} \Delta t \|\psi_n\|^2_{H^1(\Omega)} \leqslant \sum_{n=1}^{2i-1} \Delta t \|\psi_{\frac{n}{2}}\|^2_{H^1(\Omega)}$$

and

$$\sum_{n=1}^{i} \Delta t \|\partial_t \psi_{n-\frac{1}{2}}\|^2 \leqslant \sum_{n=1}^{i} \Delta t \|\partial_t \psi_{n-\frac{1}{2}}\|^2 + \sum_{n=1}^{i-1} \Delta t \|\delta_t \psi_n\|^2.$$

Also, for N large enough, i.e. small Δt,

$$\Delta t \left\| \frac{\delta_t \eta_{i-1} - \delta_t \eta_{i-2}}{\Delta t} \right\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} \leqslant C \left(\left\| \frac{\partial^2 \eta}{\partial t^2} \right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} + (\Delta t)^4 \left\| \frac{\partial^4 \eta}{\partial t^4} \right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} \right).$$

Thus we can apply discrete Gronwall's inequality. We will have

$$\|\partial_t \psi_{i-\frac{1}{2}}\|^2 + \|\psi_{i-\frac{1}{2}}\|^2_{H^1(\Omega)} \leqslant$$

$$\leqslant C(\sum_{n=1}^{i-1} \Delta t \|\eta_{n,\frac{1}{4}}\|^2 + \sum_{n=1}^{i-1} \Delta t \|\partial_t^2 \eta_n\|^2 + \|\partial_t \psi_{\frac{1}{2}}\|^2 + \|\delta_t \eta\|^2_{\tilde{L}^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \|\psi_0\|^2_{H^1(\Omega)} +$$
$$+ \|\psi_1\|^2_{H^1(\Omega)} + \sum_{n=1}^{i-1} \Delta t \|Q_{n,\frac{1}{4}}' - Q_{n,\frac{1}{4}}\|^2 + \left\| \frac{\partial^2 \eta}{\partial t^2} \right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} + (\Delta t)^4 +$$
$$+ \sum_{n=2}^{i-2} \Delta t \left\| \frac{\eta_{n+2} - 2\eta_n + \eta_{n-2}}{4(\Delta t)^2} \right\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \sum_{n=1}^{i-1} \Delta t \|r_n\|^2 + \sum_{n=1}^{i-1} \Delta t |r_n'|^2).$$

Here C is proportional to $e^{C_1 i \Delta t}$ with some positive constant C_1. Obviously, this exponent is no larger than $e^{C_1 T}$.

Next, for N large enough, i.e. small Δt,

$$\sum_{n=1}^{i-1}\Delta t\|\eta_{n,\frac{1}{4}}\|^2 \leqslant C\left(\|\eta\|^2_{L^2(L^2(\Omega))} + (\Delta t)^4\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))}\right),$$

$$\sum_{n=1}^{i-1}\Delta t\|\partial_t^2\eta_n\|^2 \leqslant C\left(\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))} + (\Delta t)^4\left\|\frac{\partial^4\eta}{\partial t^4}\right\|^2_{L^2(L^2(\Omega))}\right),$$

$$\|\delta_t\eta\|^2_{\tilde{L}^\infty(H^{-\frac{1}{2}}(\partial\Omega))} \leqslant C\left(\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + (\Delta t)^4\left\|\frac{\partial^3\eta}{\partial t^3}\right\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}\right),$$

$$\sum_{n=2}^{i-2}\Delta t\left\|\frac{\eta_{n+2}-2\eta_n+\eta_{n-2}}{4(\Delta t)^2}\right\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} \leqslant$$

$$\leqslant C\left(\left\|\frac{\partial^2\eta}{\partial t^2}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} + (\Delta t)^4\left\|\frac{\partial^4\eta}{\partial t^4}\right\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega))}\right).$$

The constants are chosen so that the above inequalities held uniformly with respect to i. Finally,

$$\sum_{n=1}^{i-1}\Delta t\|Q'_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2 \leqslant \sum_{n=1}^{N-1}\Delta t\|Q'_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2.$$

So we obtain

$$\|\partial_t\psi\|_{\hat{L}^\infty(L^2(\Omega))} + \|\psi\|_{\hat{L}^\infty(H^1(\Omega))} \leqslant$$

$$\leqslant C(\|\eta\|_{L^2(L^2(\Omega))} + \left\|\frac{\partial^2\eta}{\partial t^2}\right\|_{L^2(L^2(\Omega))} + \left\|\frac{\partial\eta}{\partial t}\right\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \left\|\frac{\partial^2\eta}{\partial t^2}\right\|_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} +$$

$$+\|\partial_t\psi_{\frac{1}{2}}\| + \|\psi_0\|_{H^1(\Omega)} + \|\psi_1\|_{H^1(\Omega)} + \sqrt{\sum_{n=1}^{N-1}\Delta t\|Q'_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2} + (\Delta t)^2).$$

The last step is to use the triangle inequality :

$$\|\partial_t e\|_{\hat{L}^\infty(L^2(\Omega))} + \|e\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant$$

$$\leqslant \|\partial_t\psi\|_{\hat{L}^\infty(L^2(\Omega))} + \|\psi\|_{\hat{L}^\infty(L^2(\Omega))} + \|\partial_t\eta\|_{\hat{L}^\infty(L^2(\Omega))} + \|\eta\|_{\hat{L}^\infty(L^2(\Omega))}.$$

For the last two terms we have

$$\|\partial_t\eta\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant C\left(\left\|\frac{\partial\eta}{\partial t}\right\|_{L^\infty(L^2(\Omega))} + (\Delta t)^2\left\|\frac{\partial^3\eta}{\partial t^3}\right\|_{L^\infty(L^2(\Omega))}\right),$$

$$\|\eta\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant C\left(\|\eta\|_{L^\infty(L^2(\Omega))} + (\Delta t)^2\left\|\frac{\partial^2\eta}{\partial t^2}\right\|_{L^\infty(L^2(\Omega))}\right).$$

Therefore, the final result will be

$$\|\partial_t e\|_{\hat{L}^\infty(L^2(\Omega))} + \|e\|_{\hat{L}^\infty(L^2(\Omega))} \leqslant$$

$$\leqslant C(\|\eta\|_{L^\infty(L^2(\Omega))} + \left\|\frac{\partial^2\eta}{\partial t^2}\right\|_{L^2(L^2(\Omega))} + \left\|\frac{\partial\eta}{\partial t}\right\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))} + \left\|\frac{\partial^2\eta}{\partial t^2}\right\|_{L^2(H^{-\frac{1}{2}}(\partial\Omega))} +$$

$$+\left\|\frac{\partial\eta}{\partial t}\right\|_{L^\infty(L^2(\Omega))} + \|\partial_t\psi_{\frac{1}{2}}\| + \|\psi_0\|_{H^1(\Omega)} + \|\psi_1\|_{H^1(\Omega)} + \sqrt{\sum_{n=1}^{N-1}\Delta t\|Q'_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2} + (\Delta t)^2).$$

Use Lemma 1 and obtain the theorem. □

Remark 1. *The term* $\sqrt{\sum_{n=1}^{N-1}\Delta t\|Q'_{n,\frac{1}{4}} - Q_{n,\frac{1}{4}}\|^2}$ *is a discrete analogue of the term* $\|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))}$ *from* (1.3).

4. Negative norm analysis

Consider the problem

$$\begin{cases} q_{tt} - a_0^2\Delta q = a_0^2 R + G, \text{ in } (0,T)\times\Omega \\ \nabla q\cdot\mathbf{n} + \frac{1}{a_0}q_t = g, \text{ in } (0,T)\times\partial\Omega \end{cases}$$

with some initial conditions on $q(0,\cdot)$ and $q_t(0,\cdot)$. Here

$$R = \begin{cases} Q - \rho_0\nabla\cdot\mathbf{f}, \text{ if } x\in\Omega_1 \\ 0, \text{ if } x\in\Omega/\Omega_1 \end{cases}$$

and G and g are control functions. $G = 0$ outside Ω_1. Also extend Q to the whole Ω by setting it to zero outside Ω_1. Introduce operators T and T_1 as shown below. For T, consider the elliptic problem

$$\begin{cases} -a_0^2\Delta p + p = f, \text{ in } \Omega \\ \nabla p\cdot\mathbf{n} = 0, \text{ in } \partial\Omega. \end{cases}$$

$T : L^2(\Omega) \to H^1(\Omega)$ is a solution operator to this problem and is given by the formula $Tf = p$, for f being a given data. This operator is well-defined on the whole $L^2(\Omega)$, which follows from the Lax-Milgram theorem. Clearly, T is self-adjoint and positive definite.

For T_1 consider another elliptic problem :

$$\begin{cases} -a_0^2\Delta p + p = 0, \text{ in } \Omega \\ \nabla p\cdot\mathbf{n} = g, \text{ in } \partial\Omega. \end{cases}$$

$T_1 : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is a solution operator to this problem and is given by the formula $T_1 g = p$. The existence of this operator again follows from the Lax-Milgram theorem.

Also we'll use the trace operator $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$.

Rewrite the given hyperbolic problem in the form

$$\begin{cases} q_{tt} - a_0^2\Delta q + q - q = a_0^2 R + G, \text{ in } (0,T)\times\Omega \\ \nabla q\cdot\mathbf{n} = -\frac{1}{a_0}q_t + g, \text{ in } (0,T)\times\partial\Omega. \end{cases}$$

Now apply operator T to both sides of the wave equation and take into account the non-homogeneous boundary condition.

$$Tq_{tt} + q - Tq + \frac{1}{a_0}T_1(\gamma q_t - a_0 g) = T(a_0^2 R + G), \text{ in } (0,T)\times\Omega. \tag{4.1}$$

This is the main equation in the negative norm analysis to start from. Next define its semidiscrete analogue with approximate operators T_h, $T_{1,h}$ and γ_h(see [28] for details).

$$T_h q_{h,tt} + q_h - T_h q_h + \frac{1}{a_0}T_{1,h}(\gamma_h q_{h,t} - a_0 g) = T_h(a_0^2 R_{h_1} + G), \text{ in } (0,T)\times\Omega. \tag{4.2}$$

The last term contains R_{h_1} which comes from the DNS of the incompressible flow on the different grid of size h_1 in Ω_1.

Introduce the inner product and the norm

$$(u,v)_{-1} = (Tu,v),\ \|u\|_{-1} = \sqrt{(u,u)_{-1}}$$

and the semi-inner product and the semi-norm

$$(u,v)_{-1,h} = (T_h u,v),\ \|u\|_{-1,h} = \sqrt{(u,u)_{-1,h}},$$

defined on all functions $u,v \in L^2(\Omega)$. The error equation comes from subtracting the exact and discrete ones, i.e. if $e = q - q_h$, then

$$\begin{aligned} T_h e_{tt} + e - T_h e + (T - T_h)q_{tt} - (T - T_h)q + \frac{1}{a_0}(T_1\gamma - T_{1,h}\gamma_h)q_t - \\ - (T_1 - T_{1,h})g + \frac{1}{a_0}T_{1,h}\gamma_h e_t = (T - T_h)(a_0^2 R + G) + a_0^2 T_h(Q - Q_{h_1}). \end{aligned} \tag{4.3}$$

Multiply by e_t and integrate in space :

(4.4)

$$\begin{aligned} (T_h e_{tt}, e_t) + (e, e_t) = -\frac{1}{a_0}((T_1\gamma - T_{1,h}\gamma_h)q_t, e_t) + ((T_1 - T_{1,h})g, e_t) + (T_h e, e_t) - \\ -\frac{1}{a_0}(T_{1,h}\gamma_h e_t, e_t) + ((T - T_h)(a_0^2 R + G + q - q_{tt}), e_t) + a_0^2(T_h(Q - Q_{h_1}), e_t). \end{aligned}$$

We moved the term $(T_{1,h}\gamma_h e_t, e_t)$ to the RHS because the operator $T_{1,h}\gamma_h$ is not positive definite and thus we cannot hide it in the LHS as a part of the global error. From this point, we'll only work with the Neumann boundary condition that has no time derivative, since the mentioned term $(T_{1,h}\gamma_h e_t, e_t)$ is not of high order with respect to the others and we can't increase the accuracy in this case. Therefore, we are now considering the problem

$$\left\{ \begin{array}{l} q_{tt} - a_0^2 \Delta q = a_0^2 R + G, \text{ in } (0,T)\times\Omega \\ \nabla q \cdot \mathbf{n} = g, \text{ in } (0,T)\times\partial\Omega, \end{array} \right. \tag{4.5}$$

and the equation (4.4) reduces to

$$\begin{aligned} (T_h e_{tt}, e_t) + (e, e_t) = (T_h e, e_t) + ((T - T_h)(a_0^2 R + G + q - q_{tt}), e_t) + \\ + a_0^2(T_h(Q - Q_{h_1}), e_t) + ((T_1 - T_{1,h})g, e_t) \end{aligned} \tag{4.6}$$

Theorem 3. *Let the exact solution q of the variational formulation of* (4.5) *satisfy conditions : $q, q_t \in L^\infty(H^k(\Omega))$, $q_{tt} \in L^2(H^k(\Omega))$ with integer k, $m \geqslant k \geqslant 2$. Also let the initial data satisfy conditions*

$$\|(q_h - \hat{q})(0,\cdot)\|_{H^1(\Omega)} + \left\| \frac{\partial}{\partial t}(q_h - \hat{q})(0,\cdot) \right\| \leqslant C_1 h^k$$

with the constant C_1 independent of h. Finally, let $a_0^2 R + G \in L^2(H^k(\Omega))$ and $g \in L^2(H^{\frac{1}{2}+k}(\partial\Omega))$. Then

$$\begin{aligned} \left\| \frac{\partial}{\partial t}(q - q_h) \right\|_{L^\infty(H^{-1}(\Omega))} + \|q - q_h\|_{L^\infty(L^2(\Omega))} \leqslant \\ C(h^{k+1} + h^{-1}\|Q - Q_{h_1}\|_{L^2(H^{-2}(\Omega))} + h\|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))} + \\ + \left\| \frac{\partial}{\partial t}(q - q_h)(0,\cdot) \right\|_{-1} + \|(q - q_h)(0,\cdot)\|) \end{aligned}$$

with constant C independent of h.

Proof. (4.6) is equivalent to

$$\frac{1}{2}\frac{d}{dt}\{\|e_t\|^2_{-1,h}+\|e\|^2\}=(T_he,e_t)+((T-T_h)(a_0^2R+G+q-q_{tt}),e_t)+$$
$$+a_0^2(T_h(Q-Q_{h_1}),e_t)+((T_1-T_{1,h})g,e_t).$$

It's obvious that

$$(T_he,e_t)=(e,T_he_t).$$

Integration of (4.6) yields

$$\|e_t\|^2_{-1,h}+\|e\|^2\leqslant\int_0^t(\|e\|^2+\|T_he_t\|^2)+$$
$$+2\int_0^t|((T-T_h)(a_0^2R+G+q-q_{tt}),e_t)|+2a_0^2\int_0^t|(T_h(Q-Q_{h_1}),e_t)|+$$
$$+2\int_0^t|((T_1-T_{1,h})g,e_t)|+\|e_t\|^2_{-1,h}(0)+\|e\|^2(0).$$

The term $\|T_he_t\|^2=\|e_t\|^2_{-2,h}\leqslant\|e_t\|^2_{-1,h}$.

Using Gronwall's lemma, we obtain

$$\|e_t\|^2_{-1,h}+\|e\|^2\leqslant$$
$$C(\int_0^t|((T-T_h)(a_0^2R+G+q-q_{tt}),e_t)|+\int_0^t|(T_h(Q-Q_{h_1}),e_t)|+$$
$$+\int_0^t|((T_1-T_{1,h})g,e_t)|+\|e_t\|^2_{-1,h}(0)+\|e\|^2(0)).$$

Next,

$$|((T-T_h)(a_0^2R+G+q-q_{tt}),e_t)|\leqslant\|(T-T_h)(a_0^2R+G+q-q_{tt})\|\cdot\|e_t\|\leqslant$$
$$\leqslant\frac{1}{4}h^{2s+2}\|a_0^2R+G+q-q_{tt}\|^2_{H^s(\Omega)}+h^2\|e_t\|^2$$

with integer $s\geqslant 0$, and

$$|(T_h(Q-Q_{h_1}),e_t)|\leqslant C\left(h^{-2}\|T_h(Q-Q_{h_1})\|^2+h^2\|e_t\|^2\right)=$$
$$=C\left(h^{-2}\|Q-Q_{h_1}\|^2_{-2,h}+h^2\|e_t\|^2\right),$$
$$|((T_1-T_{1,h})g,e_t)|\leqslant\|(T_1-T_{1,h})g\|\cdot\|e_t\|\leqslant\frac{1}{4}h^{2s+2}\|g\|^2_{H^{\frac{1}{2}+s}(\partial\Omega)}+h^2\|e_t\|^2.$$

Thus

$$\|e_t\|^2_{L^\infty(H^{-1,h}(\Omega))}+\|e\|^2_{L^\infty(L^2(\Omega))}\leqslant$$
$$C(h^{2k+2}\|a_0^2R+G+q-q_{tt}\|^2_{L^2(H^k(\Omega))}+h^{-2}\|(Q-Q_{h_1})\|^2_{L^2(H^{-2,h}(\Omega))}+$$
$$+h^{2k+2}\|g\|^2_{L^2(H^{\frac{1}{2}+k}(\partial\Omega))}+h^2\|e_t\|^2_{L^2(L^2(\Omega))}+\|e_t\|^2_{-1,h}(0)+\|e\|^2(0)),$$

or

$$\|e_t\|_{L^\infty(H^{-1,h}(\Omega))}+\|e\|_{L^\infty(L^2(\Omega))}\leqslant$$
$$C(h^{k+1}\|a_0^2R+G+q-q_{tt}\|_{L^2(H^k(\Omega))}+h^{-1}\|(Q-Q_{h_1})\|_{L^2(H^{-2,h}(\Omega))}+$$
$$+h^{k+1}\|g\|_{L^2(H^{\frac{1}{2}+k}(\partial\Omega))}+h\|e_t\|_{L^2(L^2(\Omega))}+\|e_t\|_{-1,h}(0)+\|e\|(0)).$$

According to V. Thomee's results([28]),

$$\|e_t\|_{-1}\leqslant C(\|e_t\|_{-1,h}+h\|e_t\|),$$

and therefore

$$\|e_t\|_{L^\infty(H^{-1}(\Omega))} + \|e\|_{L^\infty(L^2(\Omega))} \leqslant$$
$$C(h^{k+1}\|a_0^2 R + G + q - q_{tt}\|_{L^2(H^k(\Omega))} + h^{-1}\|Q - Q_{h_1}\|_{L^2(H^{-2,h}(\Omega))} +$$
$$+h^{k+1}\|g\|_{L^2(H^{\frac{1}{2}+k}(\partial\Omega))} + h\|e_t\|_{L^\infty(L^2(\Omega))} + \|e_t\|_{-1,h}(0) + \|e\|(0)).$$

For the initial data

$$\|e_t\|_{-1,h}(0) \leqslant C(\|e_t\|_{-1}(0) + h\|e_t\|(0)).$$

For the term $Q - Q_{h_1}$ we have

$$h^{-1}\|(Q - Q_{h_1})\|_{-2,h} \leqslant C\left(h^{-1}\|Q - Q_{h_1}\|_{-2} + h\|Q - Q_{h_1}\|\right).$$

The final result is, due to (1.3),

$$\|e_t\|_{L^\infty(H^{-1}(\Omega))} + \|e\|_{L^\infty(L^2(\Omega))} \leqslant$$
$$C(h^{k+1} + h^{-1}\|Q - Q_{h_1}\|_{L^2(H^{-2}(\Omega))} + h\|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))} + \|e_t\|_{-1}(0) + \|e\|(0)).$$

□

Theorem 4. *Suppose the exact solution* $\mathbf{u}$ *of the incompressible NSE with the boundary condition '*$\mathbf{u} = 0$ *on* $\partial\Omega_1$*' satisfies condition*

$$\mathbf{u} \in L^\infty((H^1(\Omega_1))^d)$$

and also has a continuous representation on Ω_1 *for almost all* $0 < t < T$*. Assume the mesh on* Ω_1 *used for the DNS of the incompressible NSE is quasi-uniform. Then the following estimate holds :*

$$\|Q - Q_{h_1}\|_{L^2(H^{-2}(\Omega))} \leqslant C(\mathbf{u}) \cdot \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))}$$

with constant $C(\mathbf{u})$ *independent of* h_1*.*

Proof. The norm $\|\cdot\|_{-2}$ is equivalent to the norm

$$sup_{v\in H^2(\Omega)} \frac{(\cdot, v)}{\|v\|_2}.$$

Using this, we obtain

$$\|Q - Q_{h_1}\|_{-2} \leqslant C \cdot sup_{v\in H^2(\Omega)} \frac{(Q - Q_{h_1}, v)}{\|v\|_2}.$$

Since $Q - Q_{h_1}$ is zero outside the smaller domain Ω_1, it's obvious that

$$\|Q - Q_{h_1}\|_{-2} \leqslant C \cdot sup_{v\in H^2(\Omega_1)} \frac{(Q - Q_{h_1}, v)}{\|v\|_2}.$$

We know that

$$(Q - Q_{h_1}, v) \leqslant \rho_0|(\nabla\mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t, v)| + \rho_0|(\nabla(\mathbf{u} - \mathbf{u}_{h_1}) : \nabla\mathbf{u}_{h_1}^t, v)|.$$

For both terms use Holder's inequality. For example, for the first term we get

$$\rho_0|(\nabla\mathbf{u} : \nabla(\mathbf{u} - \mathbf{u}_{h_1})^t, v)| \leqslant C\|\nabla\mathbf{u}\|_{L^r(\Omega_1)}\|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|_{L^p(\Omega_1)}\|v\|_{L^\infty(\Omega_1)},$$

where $\frac{1}{r} + \frac{1}{p} = 1$. Choose $p, r = 2$ and use Sobolev embedding $\|v\|_{L^\infty(\Omega_1)} \leqslant C\|v\|_2$. This gives

$$\|Q - Q_{h_1}\|_{-2} \leqslant C(\|\nabla\mathbf{u}\| + \|\nabla\mathbf{u}_{h_1}\|) \cdot \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|.$$

Next,

$$\|\nabla\mathbf{u}_{h_1}\| \leqslant \|\nabla\mathbf{u}\| + \|\nabla(\mathbf{u} - I_{h_1}\mathbf{u})\| + \|\nabla(\mathbf{u}_{h_1} - I_{h_1}\mathbf{u})\|,$$

where I_{h_1} is the piecewise polynomial interpolant, section 2. The first two terms on the RHS are bounded uniformly in time. For the last one we use the inverse estimate, [7] :

$$\|\nabla(\mathbf{u}_{h_1} - I_{h_1}\mathbf{u})\| \leqslant Ch_1^{-1}\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\|.$$

Using triangle inequality, we obtain

$$h_1^{-1}\|\mathbf{u}_{h_1} - I_{h_1}\mathbf{u}\| \leqslant h_1^{-1}\|\mathbf{u} - I_{h_1}\mathbf{u}\| + h_1^{-1}\|\mathbf{u} - \mathbf{u}_{h_1}\|$$

These two terms are bounded for any continuous piecewise polynomial element satisfying LBB-condition, [18], and converging to the exact solution. Thus we showed that

$$\|Q - Q_{h_1}\|_{-2} \leqslant C \cdot \|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|$$

with some positive constant $C = C(\mathbf{u})$ depending on the solution $\mathbf{u}$. □

Remark 2. *If $h = O(h_1)$, then in order to have convergence for the total error in Theorem 3, it's sufficient that $\|\nabla(\mathbf{u} - \mathbf{u}_{h_1})\|_{L^2(L^2(\Omega_1))}$ converge superlinearly. This means we have to use high-order FEM scheme for the NSE. For example, Taylor-Hood element will be sufficient (see* [18] *).*

5. Estimating the error in acoustic power

From now on we consider the semidiscrete FEM scheme for solving the problem (3.1), referring to [21]. The exact acoustic power on the surface S is given by the formula

$$A(t) = \int_S q(t,\cdot)\mathbf{v}(t,\cdot) \cdot \mathbf{n} dS.$$

Its approximate analogue is defined as

$$A_h(t) = \int_S q_h(t,\cdot)\mathbf{v}_{h_2}(t,\cdot) \cdot \mathbf{n} dS.$$

Decompose the error in power in two terms as follows :

$$A(t) - A_h(t) = \int_S (q - q_h)\mathbf{v} \cdot \mathbf{n} dS + \int_S q_h(\mathbf{v} - \mathbf{v}_{h_2}) \cdot \mathbf{n} dS. \tag{5.1}$$

Denote the terms on the RHS as $E_1(t)$ and $E_2(t)$ respectively. For computing q_h we use the semidiscrete FEM scheme.

Everywhere throughout the paper we're assuming that S is Lipschitz continuous. Estimating the error in sound power depends on how we compute the velocity $\mathbf{v}_{h_2}$ in the first place. The straightforward method is described below.

5.1. **The exact formula.**

In order to find the exact formula for $\mathbf{v}$, consider the compressible linearized NSE in the far field

$$\begin{cases} \frac{1}{a_0^2}\frac{\partial q}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0, \\ \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla q = 0. \end{cases}$$

The second equation gives

$$\mathbf{v}(t,\cdot) = -\frac{1}{\rho_0}\int_0^t \nabla q(\tau,\cdot)d\tau + \mathbf{v}(0,\cdot).$$

Thus define

$$\mathbf{v}_{h_2}(t,\cdot) = -\frac{1}{\rho_0}\int_0^t \nabla q_h(\tau,\cdot)d\tau + \mathbf{v}(0,\cdot).$$

The errors will be

$$E_1(t) = -\frac{1}{\rho_0}\int_S (q-q_h)(t,\cdot)\left(\int_0^t \nabla q(\tau,\cdot)\cdot \mathbf{n}d\tau\right)dS + \int_S (q-q_h)(t,\cdot)\mathbf{v}(0,\cdot)\cdot\mathbf{n}dS$$

and

$$E_2(t) = -\frac{1}{\rho_0}\int_S q_h(t,\cdot)\left(\int_0^t \nabla(q-q_h)(\tau,\cdot)\cdot\mathbf{n}d\tau\right)dS.$$

Using Fubini's theorem, write the first term in the form

$$E_1(t) = -\frac{1}{\rho_0}\int_0^t\int_S (q-q_h)(t,\cdot)\nabla q(\tau,\cdot)\cdot\mathbf{n}dSd\tau + \int_S (q-q_h)(t,\cdot)\mathbf{v}(0,\cdot)\cdot\mathbf{n}dS.$$

Next obtain the bound :

$$|E_1(t)| \leqslant C\int_0^t\left|\int_S (q-q_h)(t,\cdot)\nabla q(\tau,\cdot)\cdot\mathbf{n}dS\right|d\tau + \left|\int_S (q-q_h)(t,\cdot)\mathbf{v}(0,\cdot)\cdot\mathbf{n}dS\right| \leqslant$$

$$\leqslant C\|(q-q_h)(t,\cdot)\|_1\cdot\int_0^t \|\nabla q(\tau,\cdot)\cdot\mathbf{n}\|_{H^{-\frac{1}{2}}(S)}d\tau + \|(q-q_h)(t,\cdot)\|_1\cdot\|\mathbf{v}(0,\cdot)\cdot\mathbf{n}\|_{H^{-\frac{1}{2}}(S)} \leqslant$$

$$\leqslant C\|q-q_h\|_{L^\infty(0,T;H^1(\Omega))}\cdot\left(\|\nabla q\cdot\mathbf{n}\|_{L^1(0,T;H^{-\frac{1}{2}}(S))} + \|\mathbf{v}(0,\cdot)\cdot\mathbf{n}\|_{H^{-\frac{1}{2}}(S)}\right).$$

For the second term, again, using Fubini's theorem, we obtain

$$E_2(t) = -\frac{1}{\rho_0}\int_0^t\int_S q_h(t,\cdot)\nabla(q-q_h)(\tau,\cdot)\cdot\mathbf{n}dSd\tau.$$

Thus, in the same manner,

$$|E_2(t)| \leqslant C\|q_h(t,\cdot)\|_{H^1(\Omega)}\cdot\int_0^t \|\nabla(q-q_h)(\tau,\cdot)\cdot\mathbf{n}\|_{H^{-\frac{1}{2}}(S)} \leqslant$$

$$\leqslant C\|q_h\|_{L^\infty(0,T;H^1(\Omega))}\cdot\|\nabla(q-q_h)\cdot\mathbf{n}\|_{L^1(0,T;H^{-\frac{1}{2}}(S))}.$$

For a regular enough function q, the rate of convergence in the term E_1 is no slower than that of $\|q-q_h\|_{L^\infty(0,T;H^1(\Omega))}$, which is $O(h^{k-1}+\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))})$ for continuous piecewise polynomials of degree no more than $m-1$, $m\geqslant k\geqslant 2$. In the term E_2 the rate of convergence is defined by that of the term $\|\nabla(q-q_h)\cdot\mathbf{n}\|_{L^1(0,T;H^{-\frac{1}{2}}(S))}$ which is $O(h^{k-\frac{3}{2}}+h^{-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))})$. Thus the rate of convergence for the total error may be estimated as $O(h^{k-\frac{3}{2}}+h^{-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))})$. The only assumption we require is

$$\mathbf{v}(0,\cdot)\in H_{div}(\Omega),$$

since conditions $\|\nabla q\cdot\mathbf{n}\|_{L^1(0,T;H^{-\frac{1}{2}}(S))}<\infty$ and $\|q_h\|_{L^\infty(0,T;H^1(\Omega))}<\infty$ will be guranteed by the regularity assumption $q\in L^\infty(H^k(\Omega))$ for $k\geqslant 2$ and the stability theorem for q_h (see [21]) respectively.

The advantage of this exact approach is in its cheapness. The velocity and thus the sound power are computed quickly once q_h is known. The disadvantage is that we lose $\frac{3}{2}$ power of h compared to the L^2-norm of the error $q-q_h$. This is the least accurate method among those presented here.

In the particular case $S\subset\partial\Omega$ we can make an improvement. In the term $E_2(t)$, due to the boundary condition,

$$\nabla(q-q_h)\cdot\mathbf{n} = -\frac{1}{a_0}\left(\frac{\partial q}{\partial t}-\frac{\partial q_h}{\partial t}\right),$$

and so

$$\|\nabla(q-q_h)\cdot \mathbf{n}\|_{L^1(H^{-\frac{1}{2}}(\partial\Omega))} \leqslant C\left\|\frac{\partial q}{\partial t}-\frac{\partial q_h}{\partial t}\right\|_{L^1(H^{\frac{1}{2}}(\Omega))} \leqslant$$
$$\leqslant C(h^{k-\frac{1}{2}}+h^{-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}).$$

Then the total rate of convergence will be of order $O(h^{k-1}+h^{-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))})$, which comes from the term $E_1(t)$. In the case $S\subset\partial\Omega$ there's a loss of only one power of h compared to the L^2-error of $q-q_h$.

5.2. **The bound for the sound power.** Instead of finding the sound power, let us find some its upper bound. This may be used in applications where one does not necessarily need to know the exact sound power but rather needs to know whether the loudness surpasses a certain level. If the flow variable Q is given exactly as a function of space and time, then using this method only has meaning if S is not a part of $\partial\Omega$ since otherwise it has absolutely no advantage compared to the first approach. We have for some arbitrary fixed $0<t<T$

$$\int_S q\mathbf{v}\cdot\mathbf{n}dS \leqslant \|q\|_{H^{\frac{1}{2}}(S)}\cdot\|\mathbf{v}\cdot\mathbf{n}\|_{H^{-\frac{1}{2}}(S)} \leqslant C_1(\tilde\Omega)\cdot C_2(\tilde\Omega)\|q\|_{H^1(\tilde\Omega)}\cdot\|\mathbf{v}\|_{H_{div}(\tilde\Omega)}.$$

Here $\tilde\Omega$ is some domain of our choice that has S as a part of its boundary and that doesn't coincide with the turbulent region, figure 2. Constant C_1 is the norm of the trace operator from $H^1(\tilde\Omega)$ to $H^{\frac{1}{2}}(\partial\tilde\Omega)$ and C_2 is a norm of the continuous linear operator from $H_{div}(\tilde\Omega)$ to $H^{-\frac{1}{2}}(\partial\tilde\Omega)$, [24]. In fact, C_1 is constant C_{tr} from the trace theorem 1, section 2.

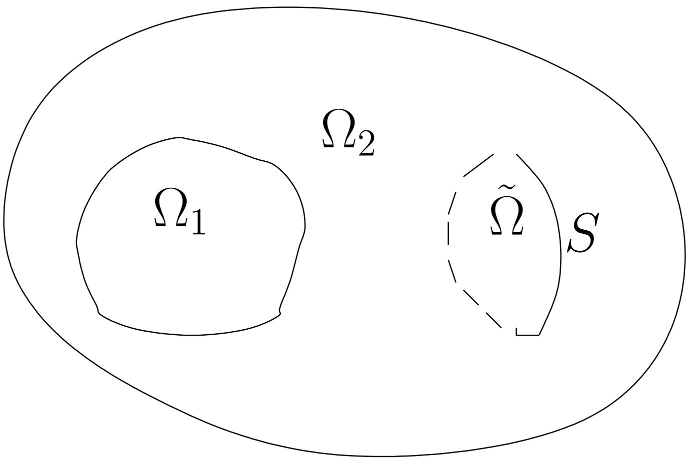

FIGURE 2. Domain $\tilde\Omega$

Next,

$$\|\mathbf{v}\|_{H_{div}(\tilde\Omega)} = \sqrt{\|\mathbf{v}\|^2_{L^2(\tilde\Omega)}+\|\nabla\cdot\mathbf{v}\|^2_{L^2(\tilde\Omega)}}.$$

From the continuity equation of the linearized compressible NSE we have

$$\nabla\cdot\mathbf{v} = -\frac{1}{a_0^2\rho_0}\frac{\partial q}{\partial t}$$

and also

$$\|\mathbf{v}(t,\cdot)\|_{L^2(\tilde{\Omega})} - \|\mathbf{v}(0,\cdot)\|_{L^2(\tilde{\Omega})} \leqslant \|\mathbf{v}(t,\cdot) - \mathbf{v}(0,\cdot)\|_{L^2(\tilde{\Omega})} = \left\| \int_0^t \frac{\partial \mathbf{v}}{\partial t} d\tau \right\|_{L^2(\tilde{\Omega})} =$$

$$= \frac{1}{\rho_0} \left\| \int_0^t \nabla q d\tau \right\|_{L^2(\tilde{\Omega})} \leqslant \frac{1}{\rho_0} \int_0^t \|\nabla q\|_{L^2(\tilde{\Omega})} d\tau.$$

Thus at time t

$$\|\mathbf{v}\|_{L^2(\tilde{\Omega})} \leqslant \frac{1}{\rho_0} \|\nabla q\|_{L^1(0,t;L^2(\tilde{\Omega}))} + \|\mathbf{v}(0,\cdot)\|_{L^2(\tilde{\Omega})}.$$

That's why we obtain

$$\|\mathbf{v}\|_{H_{div}(\tilde{\Omega})} \leqslant \sqrt{\frac{2}{\rho_0^2} \|\nabla q\|^2_{L^1(0,t;L^2(\tilde{\Omega}))} + \frac{1}{a_0^4 \rho_0^2} \left\| \frac{\partial q}{\partial t} \right\|^2_{L^2(\tilde{\Omega})} + 2\|\mathbf{v}(0,\cdot)\|^2_{L^2(\tilde{\Omega})}}.$$

Our bound will be

(5.2)

$$P(t) =$$

$$= C_1(\tilde{\Omega}) C_2(\tilde{\Omega}) \|q\|_{H^1(\tilde{\Omega})} \cdot \sqrt{\frac{2}{\rho_0^2} \|\nabla q\|^2_{L^1(0,t;L^2(\tilde{\Omega}))} + \frac{1}{a_0^4 \rho_0^2} \left\| \frac{\partial q}{\partial t} \right\|^2_{L^2(\tilde{\Omega})} + 2\|\mathbf{v}(0,\cdot)\|^2_{L^2(\tilde{\Omega})}}.$$

Introduce

(5.3)

$$P_h(t) =$$

$$= C_1(\tilde{\Omega}) C_2(\tilde{\Omega}) \|q_h\|_{H^1(\tilde{\Omega})} \cdot \sqrt{\frac{2}{\rho_0^2} \|\nabla q_h\|^2_{L^1(0,t;L^2(\tilde{\Omega}))} + \frac{1}{a_0^4 \rho_0^2} \left\| \frac{\partial q_h}{\partial t} \right\|^2_{L^2(\tilde{\Omega})} + 2\|\mathbf{v}(0,\cdot)\|^2_{L^2(\tilde{\Omega})}}.$$

Our purpose now is to get the rate of convergence for the error $P - P_h$. For simplicity, denote

$$SQ(t) = \sqrt{\frac{2}{\rho_0^2} \|\nabla q\|^2_{L^1(0,t;L^2(\tilde{\Omega}))} + \frac{1}{a_0^4 \rho_0^2} \left\| \frac{\partial q}{\partial t} \right\|^2_{L^2(\tilde{\Omega})} + 2\|\mathbf{v}(0,\cdot)\|^2_{L^2(\tilde{\Omega})}}$$

and

$$SQ_h(t) = \sqrt{\frac{2}{\rho_0^2} \|\nabla q_h\|^2_{L^1(0,t;L^2(\tilde{\Omega}))} + \frac{1}{a_0^4 \rho_0^2} \left\| \frac{\partial q_h}{\partial t} \right\|^2_{L^2(\tilde{\Omega})} + 2\|\mathbf{v}(0,\cdot)\|^2_{L^2(\tilde{\Omega})}}.$$

If we require that

$$\mathbf{v}(0,\cdot) \in (L^2(\Omega))^d,$$

then both $SQ(t)$ and $SQ_h(t)$ will be bounded due to earlier regularity assumptions and stability theorem from [21]. Obviously,

$$P(t) - P_h(t) = C_1(\tilde{\Omega}) \cdot C_2(\tilde{\Omega}) \cdot (\|q(t,\cdot)\|_{H^1(\tilde{\Omega})} - \|q_h(t,\cdot)\|_{H^1(\tilde{\Omega})}) \cdot SQ(t) +$$
$$+ C_1(\tilde{\Omega}) \cdot C_2(\tilde{\Omega}) \cdot \|q_h(t,\cdot)\|_{H^1(\tilde{\Omega})} \cdot (SQ(t) - SQ_h(t)),$$

The first term of the error may be bounded by

$$C_1(\tilde{\Omega}) \cdot C_2(\tilde{\Omega}) \cdot \|q - q_h\|_{H^1(\tilde{\Omega})} \cdot SQ,$$

and thus converges as $O(h^{k-1} + \|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))})$. The second term of the error may be bounded by

$$C_1(\tilde{\Omega}) \cdot C_2(\tilde{\Omega}) \cdot \|q_h\|_{H^1(\tilde{\Omega})} \cdot \frac{|SQ^2 - SQ_h^2|}{SQ + SQ_h}$$

Next,

$$|SQ^2 - SQ_h^2| =$$

$$= \frac{2}{\rho_0^2}\left(\|\nabla q\|^2_{L^1(0,t;L^2(\tilde{\Omega}))} - \|\nabla q_h\|^2_{L^1(0,t;L^2(\tilde{\Omega}))}\right) + \frac{1}{a_0^4\rho_0^2}\left(\left\|\frac{\partial q}{\partial t}\right\|^2_{L^2(\tilde{\Omega})} - \left\|\frac{\partial q_h}{\partial t}\right\|^2_{L^2(\tilde{\Omega})}\right)$$

The first and the second terms in this expression converge as $O(h^{k-1} + \|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))})$ and $O(h^k + \|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))})$ respectively. Therefore, we conclude that the rate of convergence for the total error $P - P_h$ is $O(h^{k-1} + \|Q - Q_{h_1}\|_{L^2(L^2(\Omega_1))})$. The advantage of this approach is obvious : it gives more accurate approximation in case when S is not a part of $\partial\Omega$. A big disadvantage is that we compute the upper bound for the sound power instead of itself. This approach also suffers from the necessity for the user to know constants $C_1(\tilde{\Omega})$ and $C_2(\tilde{\Omega})$ whose behavior depends on the geometry of the domain $\tilde{\Omega}$ chosen.

5.3. **Duality analysis.** The error in the sound power cannot converge to zero faster than the L^2-norm of the error $q - q_h$, i.e. the greatest rate of convergence may not be higher than $O(h^k)$. The way we may reach this rate is by using the duality approach. This method also allows to reduce the regularity of the exact solution needed to reach the desired rate of convergence. This advantage may be crucial if one works with turbulent irregular effects. In this case we work with time-averaged sound power

$$\bar{A} = \frac{1}{T}\int_0^T \int_S q\mathbf{v}\cdot\mathbf{n}dSd\tau,$$

and the error

$$T(\bar{A} - \bar{A}_h) = \int_0^T \int_S (q - q_h)\mathbf{v}\cdot\mathbf{n}dSd\tau + \int_0^T \int_S q_h(\mathbf{v} - \mathbf{v}_{h_2})\cdot\mathbf{n}dSd\tau. \tag{5.4}$$

Also we assume that $S \subset \partial\Omega$. Denote these error terms as $\bar{E}_1$ and $\bar{E}_2$ respectively.

Let us demonstrate the duality approach by estimating the error term $\bar{E}_1$ first. First, write the variational formulation for the wave equation, using integration both in space and time. If v denotes a test function, then

$$\int_0^T \left(\frac{\partial^2 q}{\partial t^2}, v\right) + a_0^2\int_0^T (\nabla q, \nabla v) + a_0\int_0^T \left\langle \frac{\partial q}{\partial t}, v\right\rangle =$$

$$= a_0^2 \int_0^T (R + \frac{1}{a_0^2}G, v)_{\Omega_1} + a_0^2\int_0^T < g, v > .$$

Integration by parts in time gives us

$$\left(\frac{\partial q}{\partial t}(T), v(T)\right) - \left(\frac{\partial q}{\partial t}(0), v(0)\right) - \left(\frac{\partial v}{\partial t}(T), q(T)\right) + \left(\frac{\partial v}{\partial t}(0), q(0)\right) + \int_0^T \left(\frac{\partial^2 v}{\partial t^2}, q\right) +$$

$$+ a_0^2\int_0^T (\nabla q, \nabla v) + a_0 < q(T), v(T) > - a_0 < q(0), v(0) > - a_0\int_0^T \left\langle q, \frac{\partial v}{\partial t}\right\rangle =$$

$$= a_0^2 \int_0^T (R + \frac{1}{a_0^2} G, v)_{\Omega_1} + a_0^2 \int_0^T < g, v > .$$

The initial data is given :

$$q(0,\cdot) = q_1(\cdot), \frac{\partial q}{\partial t}(0,\cdot) = q_2(\cdot),$$

and thus

$$\left(\frac{\partial q}{\partial t}(T), v(T)\right) - \left(\frac{\partial v}{\partial t}(T), q(T)\right) + \int_0^T \left(\frac{\partial^2 v}{\partial t^2}, q\right) + a_0^2 \int_0^T (\nabla q, \nabla v) - a_0 \int_0^T \left\langle q, \frac{\partial v}{\partial t}\right\rangle +$$

$$+ a_0 < q(T), v(T) >= a_0^2 \int_0^T (R + \frac{1}{a_0^2} G, v)_{\Omega_1} + (q_2, v(0)) - \left(\frac{\partial v}{\partial t}(0), q_1\right) +$$

$$+ a_0 < q_1, v(0) > + a_0^2 < g, v > .$$

Consider a function ψ by the formula

$$\psi(t,\mathbf{x}) = \begin{cases} \mathbf{v}\cdot\mathbf{n}, \text{ if } \mathbf{x} \in S, \\ 0, \text{ if } \mathbf{x} \in \partial\Omega/S. \end{cases}$$

The weak formulation for the dual problem with unknown function $\tilde{q}$ will be

$$\left(\frac{\partial v}{\partial t}(T), \tilde{q}(T)\right) - \left(\frac{\partial \tilde{q}}{\partial t}(T), v(T)\right) + \int_0^T \left(\frac{\partial^2 \tilde{q}}{\partial t^2}, v\right) + a_0^2 \int_0^T (\nabla \tilde{q}, \nabla v) -$$

$$- a_0 \int_0^T \left\langle v, \frac{\partial \tilde{q}}{\partial t}\right\rangle + a_0 < \tilde{q}(T), v(T) >= a_0^2 \int_0^T \langle \psi, v\rangle .$$

In order to get rid of the terms at the final time T, we may reduce this formulation to the following point-wise problem :

$$\begin{cases} \tilde{q}_{tt} - a_0^2 \Delta \tilde{q} = 0, \text{ on } \Omega \times (0,T) \\ \tilde{q}(T,\cdot) = 0, \text{ on } \Omega \\ \tilde{q}_t(T,\cdot) = 0, \text{ on } \Omega \\ \nabla\tilde{q}\cdot\mathbf{n} - \frac{1}{a_0}\tilde{q}_t = \psi, \text{ on } \partial\Omega \times (0,T) \end{cases} \tag{5.5}$$

and so

$$\int_0^T \left(\frac{\partial^2 \tilde{q}}{\partial t^2}, v\right) + a_0^2 \int_0^T (\nabla \tilde{q}, \nabla v) - a_0 \int_0^T \left\langle v, \frac{\partial \tilde{q}}{\partial t}\right\rangle = a_0^2 \int_0^T \langle \psi, v\rangle .$$

Next we present the stability lemma which will be needed for the error analysis.

Lemma 3. *Let* $\mathbf{v} \in L^2(0,T;(H^1(\Omega))^d)$*. Then the variational solution of the dual problem is stable in the following sense :*

$$\left\|\frac{\partial \tilde{q}}{\partial t}\right\|_{L^\infty(L^2(\Omega))} + a_0 \|\nabla \tilde{q}\|_{L^\infty(L^2(\Omega))} \leqslant a_0^{\frac{3}{2}} \|\mathbf{v}\cdot\mathbf{n}\|_{L^2(L^2(S))}.$$

Proof. The change of time variable $\tau = T - t$ will give us the problem

$$\int_0^T \left(\frac{\partial^2 \tilde{q}}{\partial \tau^2}, v\right) + a_0^2 \int_0^T (\nabla \tilde{q}, \nabla v) + a_0 \int_0^T \left\langle v, \frac{\partial \tilde{q}}{\partial \tau}\right\rangle = a_0^2 \int_0^T \langle \psi, v\rangle .$$

Set $v = \frac{\partial \tilde{q}}{\partial \tau}$.

$$\frac{1}{2}\left(\left\|\frac{\partial \tilde{q}}{\partial \tau}\right\|_{\tau=T}^2 - \left\|\frac{\partial \tilde{q}}{\partial \tau}\right\|_{\tau=0}^2\right) + \frac{1}{2} a_0^2 (\|\nabla \tilde{q}\|_{\tau=T}^2 - \|\nabla \tilde{q}\|_{\tau=0}^2) + a_0 \int_0^T \left|\frac{\partial \tilde{q}}{\partial \tau}\right|^2 =$$

$$= a_0^2 \int_0^T \left\langle \psi, \frac{\partial \tilde{q}}{\partial \tau} \right\rangle.$$

Since we have homogeneous conditions at time $t = T$, or $\tau = 0$, we can simplify the equation :

$$\left\| \frac{\partial \tilde{q}}{\partial \tau} \right\|^2_{\tau=T} + a_0^2 \|\nabla \tilde{q}\|^2_{\tau=T} + 2a_0 \int_0^T \left| \frac{\partial \tilde{q}}{\partial \tau} \right|^2 = 2a_0^2 \int_0^T \left\langle \psi, \frac{\partial \tilde{q}}{\partial \tau} \right\rangle.$$

Bound the RHS using Young's inequality as shown below :

$$\left\langle \psi, \frac{\partial \tilde{q}}{\partial \tau} \right\rangle \leqslant \frac{a_0}{4} |\psi|^2 + \frac{1}{a_0} \left| \frac{\partial \tilde{q}}{\partial \tau} \right|^2.$$

This results in cancelling the boundary term with the time derivative :

$$\left\| \frac{\partial \tilde{q}}{\partial \tau} \right\|^2_{\tau=T} + a_0^2 \|\nabla \tilde{q}\|^2_{\tau=T} \leqslant \frac{a_0^3}{2} \int_0^T |\psi|^2 = \frac{a_0^3}{2} \|\mathbf{v} \cdot \mathbf{n}\|^2_{L^2(0,T;L^2(S))}.$$

Extracting the square root out of both sides and using the fact that $|a| + |b| \leqslant \sqrt{2} \cdot \sqrt{a^2 + b^2}$, we obtain the formulation of the theorem. □

Remark 3. *The analogous stability result may be obtained for the FEM solution $\tilde{q}_h$ of the dual problem, if we use the same space $M_0^m(\Omega)$ of piecewise polynomials as for the original problem.*

We now may proceed with the error analysis. Set test function $v = q - q_h$ in the variational formulation of the dual problem to get

$$a_0^2 \int_0^T \langle \psi, q - q_h \rangle = \int_0^T \left(\frac{\partial^2 \tilde{q}}{\partial t^2}, q - q_h \right) + a_0^2 \int_0^T (\nabla \tilde{q}, \nabla (q - q_h)) - a_0 \int_0^T \left\langle q - q_h, \frac{\partial \tilde{q}}{\partial t} \right\rangle.$$

The LHS is exactly $a_0^2 \bar{E}_1$. Let us again integrate by parts :

$$a_0^2 \int_0^T \langle \psi, q - q_h \rangle =$$

$$= \int_0^T \left(\frac{\partial^2 (q - q_h)}{\partial t^2}, \tilde{q} \right) + \left(\tilde{q}(0), \frac{\partial (q - q_h)}{\partial t}(0) \right) + a_0^2 \int_0^T (\nabla \tilde{q}, \nabla (q - q_h)) -$$

$$- \left(\frac{\partial \tilde{q}}{\partial t}(0), (q - q_h)(0) \right) + a_0 \langle (q - q_h)(0), \tilde{q}(0) \rangle + a_0 \int_0^T \left\langle \frac{\partial (q - q_h)}{\partial t}, \tilde{q} \right\rangle.$$

Let v_h be some arbitrary test function from the approximating space $M_0^m(\Omega)$. Then Galerkin orthogonality gives

$$a_0^2 \int_0^T \langle \psi, q - q_h \rangle = \int_0^T \left(\frac{\partial^2 (q - q_h)}{\partial t^2}, \tilde{q} - v_h \right) + \left(\tilde{q}(0), \frac{\partial (q - q_h)}{\partial t}(0) \right) -$$

$$- \left(\frac{\partial \tilde{q}}{\partial t}(0), (q - q_h)(0) \right) + a_0^2 \int_0^T (\nabla (\tilde{q} - v_h), \nabla (q - q_h)) + a_0 \langle (q - q_h)(0), \tilde{q}(0) \rangle +$$

$$+ a_0 \int_0^T \left\langle \frac{\partial (q - q_h)}{\partial t}, \tilde{q} - v_h \right\rangle + a_0^2 \int_0^T (Q - Q_{h_1}, v_h)_{\Omega_1}.$$

The next step will be the integration by parts of the second derivative term once :

$$a_0^2 \int_0^T \langle \psi, q - q_h \rangle = \left(\frac{\partial (q - q_h)}{\partial t}(T), \tilde{q}(T) - v_h(T) \right) - \left(\frac{\partial (q - q_h)}{\partial t}(0), \tilde{q}(0) - v_h(0) \right) -$$

$$-\int_0^T \left(\frac{\partial(q-q_h)}{\partial t}, \frac{\partial(\tilde{q}-v_h)}{\partial t}\right) + \left(\tilde{q}(0), \frac{\partial(q-q_h)}{\partial t}(0)\right) - \left(\frac{\partial\tilde{q}}{\partial t}(0), (q-q_h)(0)\right) +$$

$$+a_0^2\int_0^T (\nabla(\tilde{q}-v_h), \nabla(q-q_h)) + a_0\left\langle (q-q_h)(0), \tilde{q}(0)\right\rangle + a_0\int_0^T \left\langle \frac{\partial(q-q_h)}{\partial t}, \tilde{q}-v_h\right\rangle +$$

$$+a_0^2\int_0^T (Q-Q_{h_1}, v_h)_{\Omega_1}.$$

Let $v_h = \tilde{q}_h$ be the FEM solution for $\tilde{q}$. We assume that at time $t = T$ $\tilde{q}_h$ and $\frac{\partial\tilde{q}_h}{\partial t}$ are chosen to be the H^1-ptojections of the corresponding functions, just as in case of q_h and $\frac{\partial q_h}{\partial t}$ being H^1-projections of exact functions at time $t = 0$. This implies that the first term in the RHS above is zero since $\tilde{q}(T,\cdot) = 0$ and H^1-projection of zero function is also zero. Finally, we have

$$a_0^2\int_0^T \langle\psi, q-q_h\rangle = \left(\frac{\partial(q-q_h)}{\partial t}(0), \tilde{q}_h(0)\right) - \int_0^T \left(\frac{\partial(q-q_h)}{\partial t}, \frac{\partial(\tilde{q}-\tilde{q}_h)}{\partial t}\right) - \tag{5.6}$$

$$-\left(\frac{\partial\tilde{q}}{\partial t}(0), (q-q_h)(0)\right) + a_0^2\int_0^T (\nabla(\tilde{q}-\tilde{q}_h), \nabla(q-q_h)) + a_0\left\langle (q-q_h)(0), \tilde{q}(0)\right\rangle +$$

$$+a_0\int_0^T \left\langle \frac{\partial(q-q_h)}{\partial t}, \tilde{q}-\tilde{q}_h\right\rangle + a_0^2\int_0^T (Q-Q_{h_1}, \tilde{q}_h)_{\Omega_1}.$$

Now we must bound optimally each term on the RHS.

Definition 1. *Let $r \in \mathbb{R}$ and $r > 0$. Then $]r[$ denotes the smallest possible integer $s \in \mathbb{N}$ with a property $s \geqslant r$.*

Theorem 5. *Assume the initial data satisfies*

$$q(0,\cdot) \in H^k(\Omega), \frac{\partial q}{\partial t}(0,\cdot) \in H^k(\Omega),$$

where integer k satisfies $2 \leqslant k \leqslant m$. Also let $q_h(0,\cdot)$, $\frac{\partial q_h}{\partial t}(0,\cdot)$ be H^1-projections of the initial data. If the exact solution q and the solution $\tilde{q}$ of the dual problem (5.5) *satisfy regularity conditions*

$$q, \tilde{q} \in L^\infty(0,T; H^{]\frac{k}{2}[+1}(\Omega)),$$
$$\frac{\partial q}{\partial t}, \frac{\partial\tilde{q}}{\partial t} \in L^\infty(0,T; H^{]\frac{k}{2}[+1}(\Omega)),$$
$$\frac{\partial^2 q}{\partial t^2}, \frac{\partial^2\tilde{q}}{\partial t^2} \in L^2(0,T; H^{]\frac{k}{2}[+1}(\Omega)),$$

then

$$\bar{E}_1 \leqslant C(h^k + h^{]\frac{k}{2}[-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))} + \|Q-Q_{h_1}\|_{L^1(H^{-1}(\Omega_1))})$$

with some positive constant C independent of h.

Proof. Using stability lemma, for the first term of (5.6) we obtain

$$\left|\left(\frac{\partial(q-q_h)}{\partial t}(0), \tilde{q}_h(0)\right)\right| \leqslant \left\|\frac{\partial(q-q_h)}{\partial t}(0)\right\|_{H^{-1}(\Omega)} \cdot \|\tilde{q}_h(0)\|_{H^1(\Omega)} \leqslant Ch^{k+1}.$$

Next, in the same manner,

$$\left|\left(\frac{\partial\tilde{q}}{\partial t}(0), (q-q_h)(0)\right)\right| \leqslant \left\|\frac{\partial\tilde{q}}{\partial t}(0)\right\| \cdot \|(q-q_h)(0)\| \leqslant Ch^k,$$

$$a_0\,|\langle (q-q_h)(0),\tilde q(0)\rangle| \leqslant C\|(q-q_h)(0)\|_{H^{-\frac12}(\partial\Omega)}\cdot\|\tilde q(0)\|_{H^1(\Omega)}\leqslant Ch^k.$$

For the integral terms, using (1.3), we obtain

$$\left|\int_0^T\left(\frac{\partial(q-q_h)}{\partial t},\frac{\partial(\tilde q-\tilde q_h)}{\partial t}\right)\right|\leqslant C\left\|\frac{\partial(q-q_h)}{\partial t}\right\|_{L^\infty(L^2(\Omega))}\cdot\left\|\frac{\partial(\tilde q-\tilde q_h)}{\partial t}\right\|_{L^\infty(L^2(\Omega))}\leqslant$$

$$\leqslant C(h^{2]\frac{k}{2}[+2}+h^{]\frac{k}{2}[+1}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}),$$

$$a_0^2\left|\int_0^T(\nabla(\tilde q-\tilde q_h),\nabla(q-q_h))\right|\leqslant C\|\nabla(\tilde q-\tilde q_h)\|_{L^\infty(L^2(\Omega))}\cdot\|\nabla(q-q_h)\|_{L^\infty(L^2(\Omega))}\leqslant$$

$$\leqslant C(h^{2]\frac{k}{2}[}+h^{]\frac{k}{2}[}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}),$$

$$a_0\left|\int_0^T\left\langle\frac{\partial(q-q_h)}{\partial t},\tilde q-\tilde q_h\right\rangle\right|\leqslant C\int_0^T\left\|\frac{\partial(q-q_h)}{\partial t}\right\|_{H^{-\frac12}(\partial\Omega)}\cdot\|\tilde q-\tilde q_h\|_{L^\infty(H^1(\Omega))}\leqslant$$

$$\leqslant Ch^{]\frac{k}{2}[}\left\|\frac{\partial(q-q_h)}{\partial t}\right\|_{L^1(H^{-\frac12}(\partial\Omega))}\leqslant C(h^{2]\frac{k}{2}[+\frac12}+h^{]\frac{k}{2}[-\frac12}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}),$$

and finally

$$\left|a_0^2\int_0^T(Q-Q_{h_1},\tilde q_h)_{\Omega_1}\right|\leqslant C\|Q-Q_{h_1}\|_{L^1(H^{-1}(\Omega_1))}\cdot\|\tilde q_h\|_{L^\infty(H^1(\Omega))}.$$

Therefore, the total rate of convergence for $\bar E_1$ is given by

$$\bar E_1\leqslant C(h^k+h^{]\frac{k}{2}[-\frac12}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))}+\|Q-Q_{h_1}\|_{L^1(H^{-1}(\Omega_1))})\tag{5.7}$$

□

To obtain the bound for $\bar E_2$ it's necessary to formulate and solve a variational problem for $\mathbf{v}$. We have the linearized continuity equation

$$\frac{1}{\rho_0}\frac{\partial q}{\partial t}+a_0^2\nabla\cdot\mathbf{v}=0,\tag{5.8}$$

and the linearized momentum equation

$$\frac{\partial\mathbf{v}}{\partial t}+\frac{1}{\rho_0}\nabla q=\frac{1}{\rho_0}\mathbf{F},\tag{5.9}$$

where $\mathbf{F}$ is zero in the far field and

$$\mathbf{F}=-\rho_0\nabla\cdot(\mathbf{u}\otimes\mathbf{u})+\rho_0\cdot\mathbf{f}+\nabla\cdot\mathbb{S}-\frac{1}{a_0^2}\mathbf{G}_1$$

in the turbulent region of the flow. Here $\mathbf{G}_1$ is such function that $\nabla\cdot\mathbf{G}_1=G$.

Take ∇ of the first equation and differentiate the second equation with respect to time t. The subtraction leads to one equation of variable $\mathbf{v}$ only :

$$\frac{\partial^2\mathbf{v}}{\partial t^2}-a_0^2\nabla(\nabla\cdot\mathbf{v})=\begin{cases}0, \text{ in the far field } \Omega/\Omega_1,\\ \frac{\partial}{\partial t}(-\nabla\cdot(\mathbf{u}\otimes\mathbf{u})+\mathbf{f}+\nu\Delta\mathbf{u})-\frac{1}{a_0^2\rho_0}\frac{\partial}{\partial t}\mathbf{G}_1, \text{ in } \Omega_1\end{cases}\tag{5.10}$$

with initial conditions

$$\mathbf{v}(0,x)=\mathbf{v}_1(x),\frac{\partial\mathbf{v}}{\partial t}(0,x)=\mathbf{v}_2(x).$$

The boundary condition is

$$\frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{n} + a_0 \nabla \cdot \mathbf{v} = -\frac{1}{\rho_0} g \text{ on } \partial\Omega \times (0,T).$$

The variational formulation for this problem will be as follows. Assume

$$\frac{\partial}{\partial t}(-\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \mathbf{f} + \nu\Delta\mathbf{u} - \frac{1}{a_0^2\rho_0}\mathbf{G}_1) \in L^2(0,T;(L^2(\Omega_1))^d), \mathbf{v}(0,\cdot) \in H_{div}(\Omega),$$

$$\frac{\partial \mathbf{v}}{\partial t}(0,\cdot) \in (L^2(\Omega))^d, g \in L^2(0,T;L^2(\partial\Omega)).$$

Find $\mathbf{v} \in L^2(0,T;H_{div}(\Omega))$ such that $\frac{\partial \mathbf{v}}{\partial t} \in L^2(0,T;H_{div}(\Omega))$ and $\frac{\partial^2 \mathbf{v}}{\partial t^2} \in L^2(0,T;(L^2(\Omega))^d)$ and which satisfies

$$\left(\frac{\partial^2 \mathbf{v}}{\partial t^2}, \mathbf{w}\right) + a_0^2(\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{w}) + a_0 \left\langle \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n} \right\rangle = \tag{5.11}$$

$$= \frac{1}{\rho_0}\left(\frac{\partial}{\partial t}\mathbf{F}, \mathbf{w}\right) - \frac{a_0}{\rho_0} < g, \mathbf{w} \cdot \mathbf{n} >$$

$$\forall \mathbf{w} \in H_{div}(\Omega), 0 < t < T,$$

$$(\mathbf{v}(0,\cdot), \mathbf{w}) = (\mathbf{v}_1(\cdot), \mathbf{w}) \ \forall \mathbf{w} \in H_{div}(\Omega),$$

$$\left(\frac{\partial \mathbf{v}}{\partial t}(0,\cdot), \mathbf{w}\right) = (\mathbf{v}_2(\cdot), \mathbf{w}) \ \forall \mathbf{w} \in H_{div}(\Omega).$$

Let $(M_0^{m_2}(\Omega))^d$ denote the space of vector continuous piecewise polynomials of degree no more than m_2-1, where $m_2 \geqslant 2$ is an integer. The mesh has a characteristic size $h_2 < 1$. The FEM semidiscrete approximation is as follows. Assume

$$\frac{\partial}{\partial t}(-\nabla\cdot(\mathbf{u}_{h_1}\otimes\mathbf{u}_{h_1})+\nu\Delta\mathbf{u}_{h_1}+\mathbf{f}-\frac{1}{a_0^2\rho_0}\mathbf{G}_1) \in L^2(0,T;(L^2(\Omega_1))^d), g \in L^2(0,T;L^2(\partial\Omega)).$$

Find a twice differentiable map $\mathbf{v}_{h_2} : [0,T] \to (M_0^{m_2}(\Omega))^d$ such that

$$\left(\frac{\partial^2 \mathbf{v}_{h_2}}{\partial t^2}, \mathbf{w}_{h_2}\right) + a_0^2(\nabla \cdot \mathbf{v}_{h_2}, \nabla \cdot \mathbf{w}_{h_2}) + a_0 \left\langle \frac{\partial \mathbf{v}_{h_2}}{\partial t} \cdot \mathbf{n}, \mathbf{w}_{h_2} \cdot \mathbf{n} \right\rangle = \tag{5.12}$$

$$= \frac{1}{\rho_0}\left(\frac{\partial}{\partial t}\mathbf{F}_{h_1}, \mathbf{w}_{h_2}\right) - \frac{a_0}{\rho_0} < g, \mathbf{w}_{h_2} \cdot \mathbf{n} >$$

$$\forall \mathbf{w}_{h_2} \in (M_0^{m_2}(\Omega))^d, 0 < t < T,$$

$$\mathbf{v}_{h_2}(0,x) \text{ approximates } \mathbf{v}_1(x) \text{ well,}$$

$$\frac{\partial \mathbf{v}_{h_2}}{\partial t}(0,x) \text{ approximates } \mathbf{v}_2(x) \text{ well.}$$

Definition 2. *Let a vector function* $\mathbf{u} \in H_{div}(\Omega)$. *Then its* H_{div}*-projection* $\hat{\mathbf{u}}$ *is defined by the formula*

$$a_0^2(\nabla \cdot \hat{\mathbf{u}}, \nabla \cdot \mathbf{w}_{h_2}) + (\hat{\mathbf{u}}, \mathbf{w}_{h_2}) = a_0^2(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{w}_{h_2}) + (\mathbf{u}, \mathbf{w}_{h_2}), \ \forall \ \mathbf{w}_{h_2} \in (M_0^{m_2}(\Omega))^d.$$

Assume $\mathbf{u} \in (H^l(\Omega))^d$, with $m_2 \geqslant l \geqslant 2$. Then

$$\|\mathbf{u} - \hat{\mathbf{u}}\|_1 \leqslant Ch_2^{l-1} \cdot \|\mathbf{u}\|_l. \tag{5.13}$$

Theorem 6. *Let the solution* $\mathbf{v}$ *of* (5.11) *satisfy conditions* $\mathbf{v}, \frac{\partial \mathbf{v}}{\partial t} \in L^\infty((H^l(\Omega))^d)$ *and* $\frac{\partial^2 \mathbf{v}}{\partial t^2} \in L^2((H^l(\Omega))^d)$ *for some positive integer* l, $m_2 \geqslant l \geqslant 2$. *Let the inital conditions be the* H_{div}*-projections of the corresponding initial functions :*

$$\mathbf{v}_{h_2}(0,\cdot) = \hat{\mathbf{v}}(0,\cdot), \ \frac{\partial \mathbf{v}_{h_2}}{\partial t}(0,\cdot) = \frac{\partial \hat{\mathbf{v}}}{\partial t}(0,\cdot).$$

Then the solution of (5.12) *satisfies*

$$\|\mathbf{v} - \mathbf{v}_{h_2}\|_{L^\infty(H_{div}(\Omega))} + \left\| \frac{\partial}{\partial t}(\mathbf{v} - \mathbf{v}_{h_2}) \right\|_{L^\infty((L^2(\Omega))^d)} \leqslant$$
$$\leqslant C \left(h_2^{l-1} + \left\| \frac{\partial}{\partial t}(\mathbf{F} - \mathbf{F}_{h_1}) \right\|_{L^2((L^2(\Omega_1))^d)} \right)$$

with some constant $C > 0$ *independent of* h_2.

Proof. The equation for the error has the form

$$\left(\frac{\partial^2 \mathbf{e}}{\partial t^2}, \mathbf{w}_{h_2} \right) + a_0^2 (\nabla \cdot \mathbf{e}, \nabla \cdot \mathbf{w}_{h_2}) + a_0 \left\langle \frac{\partial \mathbf{e}}{\partial t} \cdot \mathbf{n}, \mathbf{w}_{h_2} \cdot \mathbf{n} \right\rangle = \frac{1}{\rho_0} \left(\frac{\partial}{\partial t}(\mathbf{F} - \mathbf{F}_{h_1}), \mathbf{w}_{h_2} \right).$$

Decompose the error $\mathbf{e} = \mathbf{v} - \mathbf{v}_{h_2} = \mathbf{e}_1 + \mathbf{e}_2$, where $\mathbf{e}_1 = \mathbf{v} - \hat{\mathbf{v}}$ and $\mathbf{e}_2 = \hat{\mathbf{v}} - \mathbf{v}_{h_2}$. Notice that $\mathbf{e}_2 \in (M_0^{m_2}(\Omega))^d$. It's obvious that

$$\left(\frac{\partial^2 \mathbf{e}_2}{\partial t^2}, \mathbf{w}_{h_2} \right) + a_0^2 (\nabla \cdot \mathbf{e}_2, \nabla \cdot \mathbf{w}_{h_2}) + a_0 \left\langle \frac{\partial \mathbf{e}_2}{\partial t} \cdot \mathbf{n}, \mathbf{w}_{h_2} \cdot \mathbf{n} \right\rangle = -a_0^2 (\nabla \cdot \mathbf{e}_1, \nabla \cdot \mathbf{w}_{h_2}) -$$
$$- \left(\frac{\partial^2 \mathbf{e}_1}{\partial t^2}, \mathbf{w}_{h_2} \right) + \frac{1}{\rho_0} \left(\frac{\partial}{\partial t}(\mathbf{F} - \mathbf{F}_{h_1}), \mathbf{w}_{h_2} \right) - a_0 \left\langle \frac{\partial \mathbf{e}_1}{\partial t} \cdot \mathbf{n}, \mathbf{w}_{h_2} \cdot \mathbf{n} \right\rangle.$$

Using the definition of the H_{div}-projection, we obtain

$$\left(\frac{\partial^2 \mathbf{e}_2}{\partial t^2}, \mathbf{w}_{h_2} \right) + a_0^2 (\nabla \cdot \mathbf{e}_2, \nabla \cdot \mathbf{w}_{h_2}) + a_0 \left\langle \frac{\partial \mathbf{e}_2}{\partial t} \cdot \mathbf{n}, \mathbf{w}_{h_2} \cdot \mathbf{n} \right\rangle = \left(\mathbf{e}_1 - \frac{\partial^2 \mathbf{e}_1}{\partial t^2}, \mathbf{w}_{h_2} \right) +$$
$$+ \frac{1}{\rho_0} \left(\frac{\partial}{\partial t}(\mathbf{F} - \mathbf{F}_{h_1}), \mathbf{w}_{h_2} \right) - a_0 \left\langle \frac{\partial \mathbf{e}_1}{\partial t} \cdot \mathbf{n}, \mathbf{w}_{h_2} \cdot \mathbf{n} \right\rangle.$$

Next we use the energy method by setting $\mathbf{w}_{h_2} = \frac{\partial \mathbf{e}_2}{\partial t}$.

$$\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial \mathbf{e}_2}{\partial t} \right\|^2 + a_0^2 \frac{1}{2} \frac{d}{dt} \|\nabla \cdot \mathbf{e}_2\|^2 + a_0 \left| \frac{\partial \mathbf{e}_2}{\partial t} \cdot \mathbf{n} \right|^2 = \frac{1}{\rho_0} \left(\frac{\partial}{\partial t}(\mathbf{F} - \mathbf{F}_{h_1}), \frac{\partial \mathbf{e}_2}{\partial t} \right) +$$
$$+ \left(\mathbf{e}_1 - \frac{\partial^2 \mathbf{e}_1}{\partial t^2}, \frac{\partial \mathbf{e}_2}{\partial t} \right) - a_0 \left\langle \frac{\partial \mathbf{e}_1}{\partial t} \cdot \mathbf{n}, \frac{\partial \mathbf{e}_2}{\partial t} \cdot \mathbf{n} \right\rangle.$$

Using the fact that $(a, b) \leqslant \frac{1}{2\epsilon}\|a\|^2 + \frac{\epsilon}{2}\|b\|^2$ for any inner product $(\cdot,\cdot)$ and any $\epsilon > 0$, we can get

$$\frac{d}{dt} \left(\left\| \frac{\partial \mathbf{e}_2}{\partial t} \right\|^2 + a_0^2 \|\nabla \cdot \mathbf{e}_2\|^2 \right) \leqslant \frac{1}{\rho_0} \left\| \frac{\partial}{\partial t}(\mathbf{F} - \mathbf{F}_{h_1}) \right\|_{\Omega_1}^2 + \frac{1}{\rho_0} \left\| \frac{\partial \mathbf{e}_2}{\partial t} \right\|^2 +$$
$$2\|\mathbf{e}_1\|^2 + 2 \left\| \frac{\partial^2 \mathbf{e}_1}{\partial t^2} \right\|^2 + \left\| \frac{\partial \mathbf{e}_2}{\partial t} \right\|^2 + \frac{a_0}{2} \left| \frac{\partial \mathbf{e}_1}{\partial t} \cdot \mathbf{n} \right|^2.$$

Add

$$\frac{d}{dt}\|\mathbf{e}_2\|^2 \leqslant \left(\left\| \frac{\partial \mathbf{e}_2}{\partial t} \right\|^2 + \|\mathbf{e}_2\|^2 \right)$$

to the previous inequality to obtain

$$\frac{d}{dt}\left(\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2+\|\mathbf{e}_2\|^2+a_0^2\,\|\nabla\cdot\mathbf{e}_2\|^2\right)\leqslant\frac{1}{\rho_0}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|^2_{\Omega_1}+\left(2+\frac{1}{\rho_0}\right)\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2+$$

$$+\|\mathbf{e}_2\|^2+2\|\mathbf{e}_1\|^2+2\left\|\frac{\partial^2\mathbf{e}_1}{\partial t^2}\right\|^2+\frac{a_0}{2}\left|\frac{\partial \mathbf{e}_1}{\partial t}\cdot\mathbf{n}\right|^2.$$

Integrate assuming that the initial data is approximated via H_{div}-projection.

$$\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2+\|\mathbf{e}_2\|^2+a_0^2\,\|\nabla\cdot\mathbf{e}_2\|^2\leqslant\left(2+\frac{1}{\rho_0}\right)\int_0^t\left(\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|^2+\|\mathbf{e}_2\|^2\right)d\tau+$$

$$+\frac{1}{\rho_0}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|^2_{L^2((L^2(\Omega_1))^d)}+2\|\mathbf{e}_1\|^2_{L^2((L^2(\Omega))^d)}+$$

$$+2\left\|\frac{\partial^2\mathbf{e}_1}{\partial t^2}\right\|^2_{L^2((L^2(\Omega))^d)}+\frac{a_0}{2}C_{tr}^2\left\|\frac{\partial \mathbf{e}_1}{\partial t}\right\|^2_{L^2((H^1(\Omega))^d)},$$

where C_{tr} denotes the constant from the trace theorem. Applying Gronwall's lemma and extracting the square root of both sides yield

$$\left\|\frac{\partial \mathbf{e}_2}{\partial t}\right\|_{L^\infty((L^2(\Omega))^d)}+\|\mathbf{e}_2\|_{L^\infty(H_{div}(\Omega))}\leqslant$$

$$C\left(\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2((L^2(\Omega_1))^d)}+\|\mathbf{e}_1\|_{L^2((L^2(\Omega))^d)}+\right.$$

$$\left.+\left\|\frac{\partial^2\mathbf{e}_1}{\partial t^2}\right\|_{L^2((L^2(\Omega))^d)}+\left\|\frac{\partial \mathbf{e}_1}{\partial t}\right\|_{L^2((H^1(\Omega))^d)}\right)$$

with some constant $C=C(T)$ growing exponentially fast. This implies, due to the triangle inequality, that

$$\left\|\frac{\partial \mathbf{e}}{\partial t}\right\|_{L^\infty((L^2(\Omega))^d)}+\|\mathbf{e}\|_{L^\infty(H_{div}(\Omega))}\leqslant$$

$$C\left(\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2((L^2(\Omega_1))^d)}+\|\mathbf{e}_1\|_{L^\infty(H_{div}(\Omega))}+\right.$$

$$\left.+\left\|\frac{\partial^2\mathbf{e}_1}{\partial t^2}\right\|_{L^2((L^2(\Omega))^d)}+\left\|\frac{\partial \mathbf{e}_1}{\partial t}\right\|_{L^\infty((H^1(\Omega))^d)}\right).$$

Using (5.13), we obtain the statement of the theorem. □

In order to estimate $\bar{E}_2$, it is necessary to formulate a corresponding dual problem. Similarly to the case with $\bar{E}_1$, the pointwise dual problem with unknown function $\tilde{\mathbf{v}}$ has the form

$$\left\{\begin{array}{l}\tilde{\mathbf{v}}_{tt}-a_0^2\nabla(\nabla\cdot\tilde{\mathbf{v}})=0,\ \text{on } (0,T)\times\Omega\\ \tilde{\mathbf{v}}(T,\cdot)=0,\ \text{on } \Omega\\ \tilde{\mathbf{v}}_t(T,\cdot)=0,\ \text{on } \Omega\\ \nabla\cdot\tilde{\mathbf{v}}-\frac{1}{a_0}\tilde{\mathbf{v}}_t\cdot\mathbf{n}=\xi,\ \text{on } (0,T)\times\partial\Omega,\end{array}\right.\tag{5.14}$$

where

$$\xi(t,\mathbf{x}) = \begin{cases} q_h, \text{ if } \mathbf{x} \in S \\ 0, \text{ if } \mathbf{x} \in \partial\Omega/S. \end{cases}$$

The equation in the weak form will be

$$\int_0^T \left(\frac{\partial^2 \tilde{\mathbf{v}}}{\partial t^2}, \mathbf{w}\right) + a_0^2 \int_0^T (\nabla\cdot\tilde{\mathbf{v}}, \nabla\cdot\mathbf{w}) - a_0 \int_0^T \left\langle \mathbf{w}\cdot\mathbf{n}, \frac{\partial\tilde{\mathbf{v}}}{\partial t}\cdot\mathbf{n}\right\rangle = a_0^2 \int_0^T \langle \xi, \mathbf{w}\cdot\mathbf{n}\rangle.$$

Next we present a stability lemma similar to Lemma 3. We omit its proof due to its resemblence to the proof of Lemma 3.

Lemma 4. *The variational solution of the dual problem is stable and the following inequality holds :*

$$\left\|\frac{\partial\tilde{\mathbf{v}}}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))} + a_0\|\nabla\cdot\tilde{\mathbf{v}}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant a_0^{\frac{3}{2}}\|q_h\|_{L^2(0,T;L^2(S))}$$

Remark 4. *The same stability result holds for the approximate solution* $\tilde{\mathbf{v}}_{h_2}$.

We'll follow the same ideas as those used for obtaining the estimate for the error $\bar{E}_1$. Integrating the second derivative term by parts twice and then setting $\mathbf{w} = \mathbf{v} - \mathbf{v}_{h_2}$ lead to

$$a_0^2 \int_0^T \langle \xi, (\mathbf{v}-\mathbf{v}_{h_2})\cdot\mathbf{n}\rangle =$$

$$= \int_0^T \left(\frac{\partial^2(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t^2}, \tilde{\mathbf{v}}\right) + a_0^2 \int_0^T (\nabla\cdot\tilde{\mathbf{v}}, \nabla\cdot(\mathbf{v}-\mathbf{v}_{h_2})) + a_0 \int_0^T \left\langle \tilde{\mathbf{v}}\cdot\mathbf{n}, \frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}\cdot\mathbf{n}\right\rangle +$$

$$+ \left(\tilde{\mathbf{v}}(0), \frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}(0)\right) - \left((\mathbf{v}-\mathbf{v}_{h_2})(0), \frac{\partial\tilde{\mathbf{v}}}{\partial t}(0)\right) + a_0 < \tilde{\mathbf{v}}(0)\cdot\mathbf{n}, (\mathbf{v}-\mathbf{v}_{h_2})(0)\cdot\mathbf{n} > .$$

Next use Galerkin orthogonality with a test function $\mathbf{w}_{h_2}$:

$$a_0^2 \int_0^T \langle \xi, (\mathbf{v}-\mathbf{v}_{h_2})\cdot\mathbf{n}\rangle =$$

$$= \int_0^T \left(\frac{\partial^2(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t^2}, \tilde{\mathbf{v}}-\mathbf{w}_{h_2}\right) + a_0^2 \int_0^T (\nabla\cdot(\tilde{\mathbf{v}}-\mathbf{w}_{h_2}), \nabla\cdot(\mathbf{v}-\mathbf{v}_{h_2})) +$$

$$+ a_0 \int_0^T \left\langle (\tilde{\mathbf{v}}-\mathbf{w}_{h_2})\cdot\mathbf{n}, \frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}\cdot\mathbf{n}\right\rangle + \frac{1}{\rho_0}\int_0^T \left(\frac{\partial}{\partial t}(\mathbf{F}_{h_1}-\mathbf{F}), \mathbf{w}_{h_2}\right) -$$

$$- \left((\mathbf{v}-\mathbf{v}_{h_2})(0), \frac{\partial\tilde{\mathbf{v}}}{\partial t}(0)\right) + a_0 < \tilde{\mathbf{v}}(0)\cdot\mathbf{n}, (\mathbf{v}-\mathbf{v}_{h_2})(0)\cdot\mathbf{n} > + \left(\tilde{\mathbf{v}}(0), \frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}(0)\right).$$

Let $\mathbf{w}_{h_2} = \tilde{\mathbf{v}}_{h_2}$ be the FEM solution for $\tilde{\mathbf{v}}$ in the space $(M_0^{m_2}(\Omega))^d$. We assume that at time $t = T$ the approximate solution $\tilde{\mathbf{v}}_{h_2}$ is an H_{div}-projection of the exact solution $\tilde{\mathbf{v}}$, i.e. it's zero. The same goes for $\frac{\partial\tilde{\mathbf{v}}_{h_2}}{\partial t}(T,\cdot)$ since it's an H_{div}-projection of $\frac{\partial\tilde{\mathbf{v}}}{\partial t}(T,\cdot) = 0$. Then finally

$$a_0^2 \int_0^T \langle \xi, (\mathbf{v}-\mathbf{v}_{h_2})\cdot\mathbf{n}\rangle = \left(\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}(0), \tilde{\mathbf{v}}_{h_2}(0)\right) - \left((\mathbf{v}-\mathbf{v}_{h_2})(0), \frac{\partial\tilde{\mathbf{v}}}{\partial t}(0)\right) +$$

$$+ a_0 \langle \tilde{\mathbf{v}}(0)\cdot\mathbf{n}, (\mathbf{v}-\mathbf{v}_{h_2})(0)\cdot\mathbf{n}\rangle - \int_0^T \left(\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}, \frac{\partial(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2})}{\partial t}\right) +$$

$$+ a_0^2 \int_0^T (\nabla\cdot(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2}), \nabla\cdot(\mathbf{v}-\mathbf{v}_{h_2})) + a_0 \int_0^T \left\langle (\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2})\cdot\mathbf{n}, \frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}\cdot\mathbf{n}\right\rangle +$$

$$+\frac{1}{\rho_0}\left((\mathbf{F}-\mathbf{F}_{h_1})(0),\tilde{\mathbf{v}}_{h_2}(0)\right)+\frac{1}{\rho_0}\int_0^T\left(\mathbf{F}-\mathbf{F}_{h_1},\frac{\partial\tilde{\mathbf{v}}_{h_2}}{\partial t}\right).$$

Now we have to estimate each term separately.

Theorem 7. *Assume the initial data satisfies conditions*

$$\mathbf{v}(0,\cdot)\in(H^l(\Omega))^d,\ \frac{\partial\mathbf{v}}{\partial t}(0,\cdot)\in(H^l(\Omega))^d,$$

where integer l satisfies $2\leqslant l\leqslant m_2$. Also let $\mathbf{v}_{h_2}(0,\cdot)$, $\frac{\partial\mathbf{v}_{h_2}}{\partial t}(0,\cdot)$ be H_{div}-projections of the initial data. If the exact solution $\mathbf{v}$ and the solution $\tilde{\mathbf{v}}$ of the dual problem (5.14) *satisfy regularity conditions*

$$\mathbf{v},\tilde{\mathbf{v}}\in L^\infty(0,T;H^{]\frac{l}{2}[+1}(\Omega)),$$
$$\frac{\partial\mathbf{v}}{\partial t},\frac{\partial\tilde{\mathbf{v}}}{\partial t}\in L^\infty(0,T;H^{]\frac{l}{2}[+1}(\Omega)),$$
$$\frac{\partial^2\mathbf{v}}{\partial t^2},\frac{\partial^2\tilde{\mathbf{v}}}{\partial t^2}\in L^2(0,T;H^{]\frac{l}{2}[+1}(\Omega)),$$

then

$$\bar{E}_2\leqslant C(h_2^{l-1}+h_2^{]\frac{l}{2}[-1}\left\|\frac{\partial}{\partial t}(\mathbf{F}_{h_1}-\mathbf{F})\right\|_{L^2((L^2(\Omega_1))^d)}+\ \|\mathbf{F}-\mathbf{F}_{h_1}\|_{L^1((L^2(\Omega_1))^d)}+$$
$$+\|(\mathbf{F}_{h_1}-\mathbf{F})(0,\cdot)\|)$$

with some positive constant C independent of h_2.

Proof. For each term we have estimates

$$\left|\left(\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}(0),\tilde{\mathbf{v}}_{h_2}(0)\right)\right|\leqslant\left\|\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}(0)\right\|\cdot\|\tilde{\mathbf{v}}_{h_2}(0)\|\leqslant Ch_2^l,$$

$$\left|\left((\mathbf{v}-\mathbf{v}_{h_2})(0),\frac{\partial\tilde{\mathbf{v}}}{\partial t}(0)\right)\right|\leqslant\|(\mathbf{v}-\mathbf{v}_{h_2})(0)\|\cdot\left\|\frac{\partial\tilde{\mathbf{v}}}{\partial t}(0)\right\|\leqslant Ch_2^l,$$

$$a_0\left|\langle\tilde{\mathbf{v}}(0)\cdot\mathbf{n},(\mathbf{v}-\mathbf{v}_{h_2})(0)\cdot\mathbf{n}\rangle\right|\leqslant C\|\nabla\cdot\tilde{\mathbf{v}}(0)\|\cdot\|(\mathbf{v}-\mathbf{v}_{h_2})(0)\|_1\leqslant Ch_2^{l-1},$$

$$\left|\int_0^T\left(\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t},\frac{\partial(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2})}{\partial t}\right)\right|\leqslant$$
$$\leqslant C\left\|\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}\right\|_{L^\infty((L^2(\Omega))^d)}\cdot\left\|\frac{\partial(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2})}{\partial t}\right\|_{L^\infty((L^2(\Omega))^d)}\leqslant$$
$$\leqslant C\left(h_2^{2]\frac{l}{2}[}+h_2^{]\frac{l}{2}[}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2((L^2(\Omega_1))^d)}\right),$$

$$a_0^2\left|\int_0^T(\nabla\cdot(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2}),\nabla\cdot(\mathbf{v}-\mathbf{v}_{h_2}))\right|\leqslant C\|\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2}\|_{L^\infty(H_{div}(\Omega))}\cdot\|\mathbf{v}-\mathbf{v}_{h_2}\|_{L^\infty(H_{div}(\Omega))}\leqslant$$
$$\leqslant C\left(h_2^{2]\frac{l}{2}[}+h_2^{]\frac{l}{2}[}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2((L^2(\Omega_1))^d)}\right),$$

$$a_0\left|\int_0^T\left\langle(\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2})\cdot\mathbf{n},\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}\cdot\mathbf{n}\right\rangle\right|\leqslant$$
$$\leqslant C\left\|\tilde{\mathbf{v}}-\tilde{\mathbf{v}}_{h_2}\right\|_{L^\infty(H_{div}(\Omega))}\cdot\left\|\frac{\partial(\mathbf{v}-\mathbf{v}_{h_2})}{\partial t}\right\|_{L^\infty((H_1(\Omega))^d)}\leqslant$$

$$\leqslant C\left(h_2^{2]\frac{l}{2}[-1} + h_2^{]\frac{l}{2}[-1}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2((L^2(\Omega_1))^d)}\right),$$

$$\left|\frac{1}{\rho_0}\left((\mathbf{F}-\mathbf{F}_{h_1})(0),\tilde{\mathbf{v}}_{h_2}(0)\right)\right| \leqslant C\|(\mathbf{F}-\mathbf{F}_{h_1})(0)\|\cdot\|\tilde{\mathbf{v}}_{h_2}(0)\|.$$

Finally,

$$\left|\frac{1}{\rho_0}\int_0^T\left(\mathbf{F}-\mathbf{F}_{h_1},\frac{\partial\tilde{\mathbf{v}}_{h_2}}{\partial t}\right)\right| \leqslant C\|\mathbf{F}-\mathbf{F}_{h_1}\|_{L^1((L^2(\Omega_1))^d)}\cdot\left\|\frac{\partial\tilde{\mathbf{v}}_{h_2}}{\partial t}\right\|_{L^\infty((L^2(\Omega))^d)}.$$

The estimate for $\bar{E}_2$ will be

$$\bar{E}_2 \leqslant C(h_2^{l-1} + h_2^{]\frac{l}{2}[-1}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2((L^2(\Omega_1))^d)} + \|\mathbf{F}-\mathbf{F}_{h_1}\|_{L^1((L^2(\Omega_1))^d)} + \\ +\|(\mathbf{F}-\mathbf{F}_{h_1})(0)\|).$$

□

Combining both estimates for $\overline{E}_1$ and $\overline{E}_2$, we obtain

$$|\bar{A}-\bar{A}_h| \leqslant$$

$$\leqslant C(h^k + h^{]\frac{k}{2}[-\frac{1}{2}}\|Q-Q_{h_1}\|_{L^2(L^2(\Omega_1))} + \|Q-Q_{h_1}\|_{L^1(H^{-1}(\Omega_1))} +$$

$$+h_2^{l-1}+h_2^{]\frac{l}{2}[-1}\left\|\frac{\partial}{\partial t}(\mathbf{F}-\mathbf{F}_{h_1})\right\|_{L^2((L^2(\Omega_1))^d)} + \|\mathbf{F}_{h_1}-\mathbf{F}\|_{L^1((L^2(\Omega_1))^d)} + \|(\mathbf{F}_{h_1}-\mathbf{F})(0)\|).$$

We see that in term $\overline{E}_1$ the rate of convergence is dictated by h^k whereas for the exact formula approach the convergence is of order h^{k-1}.

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