PARTITIONED TIME STEPPING METHOD FOR FULLY EVOLUTIONARY STOKES-DARCY FLOW WITH BEAVERS-JOSEPH INTERFACE CONDITIONS

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Abstract. In this report, a partitioned time stepping algorithm for transient flow in a porous medium coupled to a free flow in embedded conduits is analyzed. The coupled flow is modeled by the *fully evolutionary* Stokes-Darcy problem. This method requires only solve one, uncoupled Stokes and Darcy sub-physics and sub-domain per time step. On the interface between the matrix and conduit, Beavers-Joseph interface conditions, instead of the simplified Beavers-Joseph-Saffman condition, are imposed. Under a modest time step restriction of the form $\triangle t \leq C$ where $C = C$*(physical parameters)* we prove stability of the method. We also derive error estimates. Numerical tests illustrate the validity of the theoretical results.

Key words. Fully evolutionary Stokes-Darcy problem; partitioned time stepping method; Beavers-Joseph interface conditions; error estimate

AMS subject classifications. 65M55, 65M70

1. Introduction.

The transport of substances coupling between surface water and groundwater is an important problem of great current interest. In many countries, groundwater is a major source of drinkable and industrial water. Groundwater systems are so tightly bonded with the lives of human beings that they are also very susceptible to contamination.

Generally speaking, in conduit domain, the Stokes equation, are commonly used. In the matrix domain, one popular choice is to use Darcy law. For the coupled Stokes-Darcy model, two boundary conditions are well-accepted: the continuity of the normal velocity across the interface which is a consequence of the the conservation of mass, and the balance of force normal to the interface (2.6). Actually, there have been a few studies of the numerical solutions of the coupled Stokes-Darcy equations, see [5, 6, 10, 11, 13]. All the work, however, consider only the steady state case and utilize the simplified interface conditions such as the Beavers-Joseph-Saffman-Jone condition. Among the fewer papers (so far) on the numerical analysis of the *fully evolutionary* Stokes-Darcy problem (consider herein), Mu and Zhu [12] study a partitioned method which we build upon herein. Cao, Gunzburger, Hu, Hua, Wang and Zhao [3, 4] study a fully, monolithically coupled implicit method for the much harder and physically more accurate case of Beavers-Joseph coupling condition (without Saffman's simplication), which we considered herein. The main mathematical difficulty in adopting the Beavers-Joseph interface conditions is that the bilinear form in the weak formulation is not coercive. The remedy is a novel rescaling (which can be interpreted as pre-conditioning) of the Darcy equation [4], which turns out that the bilinear form for the new system satisfies a Gårding-type inequality for a sufficiently large scaling factor η. This essentially leads to the well-posedness of the system.

In this report, we propose a partitioned time stepping method for fully evolutionary Stokes-Darcy problem with the classical empirical Beavers-Joseph interface condition which was proposed in the seminal work [2]. This method requires only solve one, uncoupled Stokes and Darcy sub-physics and sub-domain solve per time step. Most importantly, both subdomain solvers are used as a black box, each time step involves passing information across the interface followed by solving the individual subproblems independently. We still rescale Darcy equation with the scaling factor η as the coupled scheme [3] does, moreover, we prove that sufficient large μ enables us to complete convergence and error analysis of the partitioned method.

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The organization of the paper is as follows: in Section 2, we provide the formulation and the coupled method for the fully evolutionary Stokes-Darcy system. We present the partitioned sheme and analyze its stability in Section 3. In section 4, we analyze the error estimations for velocity and pressure. Numerical tests are reported in Section 5, followed by conclusion in Seciton 6.

2. Stokes-Darcy system with Beavers-Joseph interface condition.

2.1. Formulation of the problem. Specifically, let us consider a conduit-matrix system consist of two domain, the conduit domain Ω_c and the matrix domain Ω_m, see the figure 2.1, where $\Omega_c, \Omega_m \subset \mathbb{R}^d (d = 2, 3)$ are bounded domains, $\Omega_c \cap \Omega_m = \emptyset$ and $\Gamma_{cm} = \overline{\Omega}_m \cap \overline{\Omega}_c, \Gamma_c = \partial\Omega_c \setminus \Gamma_{cm}$ and $\Gamma_m = \partial\Omega_m \setminus \Gamma_{cm}$.

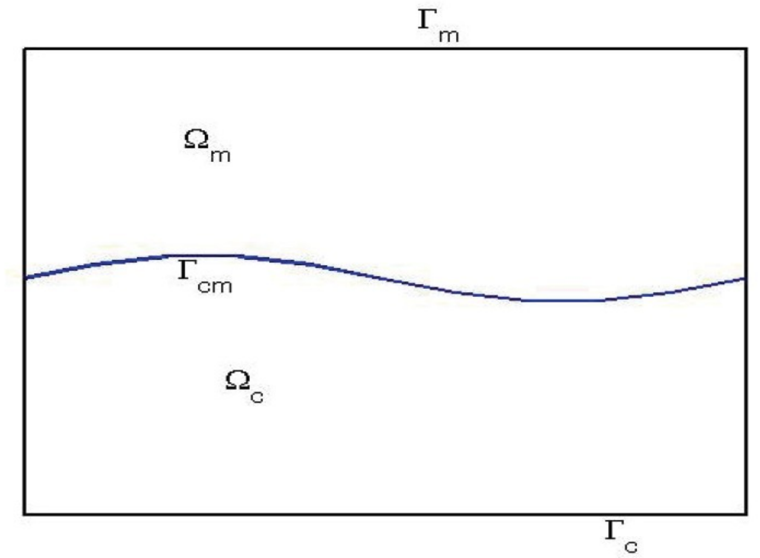

FIG. 2.1. *The global domain Ω consisting of the matrix region Ω_m and the conduit region Ω_c, separated by the interface Γ_{cm}.*

In the matrix domain Ω_m, the flow is governed by

$$\begin{aligned} S\partial_t\phi_m + \nabla \cdot \mathbf{v}_m &= f_2 && \texttt{in } \Omega_m, \\ \mathbf{v}_m &= -\mathbb{K}\nabla\phi_m && \texttt{in } \Omega_m, \\ \phi_m(0) &= \phi_0, && \texttt{in } \Omega_m, \end{aligned} \tag{2.1}$$

which includes, in the first equation, the saturated flow model and, in the second equation, Darcy's law [1]. In (2.1), $\partial_t := \frac{\partial}{\partial t}, \mathbf{v}_m$ denotes the specific discharge, ϕ_m the hydraulic (piezometric) head, S the mass storativity coefficient, $\mathbb{K}(\mathbf{x})$ denotes the hydraulic conductivity tensor of the porous media, which is assumed to be symmetric and positive definite but could be location dependent (heterogeneous), and f_2 a sink/source term. The unknown ϕ_m denotes the hydraulic (piezometric) head, which is linearly related to the dynamic pressure of the fluid p_m, defined as $\phi_m = z + \frac{p_m}{\rho g}$, where ρ denotes the density, g the gravitational acceleration, and z the relative depth from an arbitrary fixed reference height. By substituting the second equation in (2.1) into the first one, we obtain the parabolic equation that governs the hydraulic head:

$$S\partial_t\phi_m + \nabla \cdot (-\mathbb{K}\nabla\phi_m) = f_2 \;\; \texttt{in } \Omega_m. \tag{2.2}$$

In the following, we will refer to (2.2) simply as the Darcy equation. We impose the homogeneous Dirichlet condition along the boundary of the matrix:

$$\phi_m = 0 \quad \texttt{on } \Gamma_m. \tag{2.3}$$

In the conduit domain Ω_c, the flow is governed by the Stokes equations:

$$\begin{aligned} \partial_t \mathbf{v}_c &= \nabla \cdot (-p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{v}_c)) + \mathbf{f}_1 && \texttt{in } \Omega_c, \\ \nabla \cdot \mathbf{v}_c &= 0 && \texttt{in } \Omega_c, \\ \mathbf{v}_c(0) &= \mathbf{v}_0 && \texttt{in } \Omega_c, \end{aligned} \tag{2.4}$$

where $\mathbf{v}_c$ denotes the fluid velocity, p the kinematic pressure, $\mathbb{D}(\mathbf{v}_c)) := \frac{1}{2}(\nabla\mathbf{v}_c + (\nabla\mathbf{v}_c)^T)$ the deformation tensor, ν the kinematic viscosity of the fluid, and $\mathbf{f}_1$ a general body forcing term that includes gravitational acceleration. For the sake of simplicity, the homogeneous Dirichlet condition is imposed on the boundary of the conduit:

$$\mathbf{v}_c = 0 \quad \texttt{on } \Gamma_c. \tag{2.5}$$

We use the subscripts m and c to indicate where the variables belong. We omit these subscripts in what follows whenever there is no possibility for confusion.

In addition to the boundary conditions (2.3) and (2.5) imposed along the boundary of the matrix or conduit, respectively, we apply the Beavers-Joseph interface boundary conditions on Γ_{cm} that coupled the solutions in the two domains:

$$\begin{aligned} \mathbf{v}_c \cdot \mathbf{n}_{cm} &= \mathbf{v}_m \cdot \mathbf{n}_{cm} && \texttt{on } \Gamma_{cm}, \\ -\mathbf{n}_{cm}^T \mathbb{T}(\mathbf{v}_c, p)\mathbf{n}_{cm} &= g(\phi_m - z) && \texttt{on } \Gamma_{cm}, \\ -P_\tau(\mathbb{T}(\mathbf{v}_c, p)\mathbf{n}_{cm}) &= \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\Pi)}} P_\tau(\mathbf{v}_c - \mathbf{v}_m) && \texttt{on } \Gamma_{cm}, \end{aligned} \tag{2.6}$$

where $\mathbf{n}_{cm}$ denotes the unit normal vector on Γ_{cm} pointing from Ω_c to Ω_m, the stress tensor $\mathbb{T}(\mathbf{v}_c, p) := -p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{v}_c)$. $P_\tau(\cdot)$ the projection onto the local tangent plane on Γ_{cm}, g the gravitational acceleration, α denotes a constant parameter which depends on the properties of the porous material as well as the geometrical setting of the coupled problem, Π represents the intrinsic permeability that satisfies the relation $\mathbb{K} = \frac{\Pi g}{\nu}$. It should be notice that Π and $\mathbb{K}$ differ by a factor of a constant. Thus, all the assumptions on $\mathbb{K}$ such as symmetric positive definiteness also carry over to Π.

The first two interface boundary conditions in (2.6) are quite natural, as discussed in [4]. The first condition guarantees the conservation of the mass, i.e., the exchange of fluid between the two domains is conservative. The second condition represents the balance of two driving forces, the kinematic pressure in the matrix and the normal component of the normal stress in the free flow, in the normal direction along the interface. The last equation in (2.6) is the well-known Beavers-Joseph condition [2]. However, whether the Beavers-Joseph interface condition leads to a well-posed problem is still unclear. If the term $\mathbf{v}_c - \mathbf{v}_m$ is replaced by $\mathbf{v}_c$, the the Beavers-Joseph condition reduces to the Beavers-Joseph-Saffman-Jones condition [8, 14] which is prevalently used.

2.2. Weak formulation of the fully evolutionary Stokes-Darcy model. For $s > \frac{1}{2}$, define the Hilbert spaces

$$\begin{aligned} \mathbf{H}^s_{c,0} &:= \{\mathbf{w} \in (H^s(\Omega_c))^d \mid \mathbf{w} = 0 \texttt{ on } \Gamma_c\}, \\ H^s_{m,0} &:= \{\varphi \in H^s(\Omega_m) \mid \varphi = 0 \texttt{ on } \Gamma_m\}, \\ Q &:= L^2(\Omega_c), \end{aligned}$$

and the product Hilbert spaces

$$\underline{\mathbf{L}}^2 := (L^2(\Omega_c))^d \times L^2(\Omega_m),$$
$$\underline{\mathbf{H}}^s := \mathbf{H}^s_{c,0} \times H^s_{m,0}.$$

A norm on Q is given by

$$||q||_0 := ||q||_{L^2(\Omega_c)}$$

for $q \in Q$ and a norm in $\underline{\mathbf{H}}^s$ is given by

$$||\underline{\mathbf{w}}||_s := (||\mathbf{w}||^2_{(H^s(\Omega_c))^d} + ||\varphi||^2_{H^s(\Omega_m)})^{1/2}$$

for $\underline{\mathbf{w}} = (\mathbf{w}, \varphi) \in \underline{\mathbf{H}}^s$. In what follows, we use $\underline{\mathbf{W}}$ to denote $\underline{\mathbf{H}}^1$, and $\underline{\mathbf{V}}$ the divergence free subspace of $\underline{\mathbf{W}}$, i.e.,

$$\underline{\mathbf{V}} := \mathbf{H}^1_{c,div} \times H^1_{m,0},$$

where $\mathbf{H}^1_{c,div} = \{\mathbf{w} \in \mathbf{H}^1_{c,0} \mid div\mathbf{w} = 0\}$. In particularly, we use the notations of the norm hereafter,

$$||\mathbf{u}||_0 := ||\mathbf{u}||_{L^2(\Omega_c)}, ||\nabla\mathbf{u}||_0 = ||\mathbf{u}||_1 := ||\mathbf{u}||_{\mathbf{H}^1_{c,0}(\Omega_c)},$$
$$||\phi||_0 := ||\phi||_{L^2(\Omega_m)}, ||\nabla\phi||_0 = ||\phi||_1 := ||\phi||_{H^1_{m,0}(\Omega_m)}.$$

If we define the bilinear forms $a : \underline{\mathbf{W}} \times \underline{\mathbf{W}} \to \mathbb{R}$ and $b : \underline{\mathbf{W}} \times Q \to \mathbb{R}$ in the following way, for $\underline{\mathbf{u}} = (\mathbf{u}, \phi)$ and $\underline{\mathbf{v}} = (\mathbf{v}, \psi)$ in $\underline{\mathbf{W}}$ and q in Q,

$$\begin{aligned} a(\underline{\mathbf{u}}, \underline{\mathbf{v}}) := {}& 2\nu \int_{\Omega_c} \mathbf{u} : \mathbf{v} d\Omega_c + \frac{1}{S}\int_{\Omega_m} (\mathbb{K}\nabla\phi)\cdot\nabla\psi d\Omega_m \\ &+ g\int_{\Gamma_{cm}} \phi\mathbf{v}\cdot\mathbf{n}_{cm} d\Gamma_{cm} - \frac{1}{S}\int_{\Gamma_{cm}} \mathbf{u}\cdot\mathbf{n}_{cm}\psi d\Gamma_{cm} \\ &+ \int_{\Gamma_{cm}} \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\Pi)}} P_\tau(\mathbf{u} + \mathbb{K}\nabla\phi)\cdot\mathbf{v} d\Gamma_{cm}, \end{aligned} \tag{2.7}$$

and

$$b(\underline{\mathbf{u}}, q) := -\int_{\Omega_c} q\nabla\cdot\mathbf{u} d\Omega_c. \tag{2.8}$$

then the weak formulation for the Stokes-Darcy problem is: seek $\underline{\mathbf{u}} = (\mathbf{u}, \phi) \in \underline{\mathbf{W}}$ and $p \in Q$ such that

$$\begin{aligned} <\partial_t\underline{\mathbf{u}}, \underline{\mathbf{v}}> + a(\underline{\mathbf{u}}, \underline{\mathbf{v}}) + b(\underline{\mathbf{v}}, p) = <\underline{\mathbf{F}}, \underline{\mathbf{v}}> \quad & \forall\ \underline{\mathbf{v}} \in \underline{\mathbf{W}}, \\ b(\underline{\mathbf{u}}, q) = 0 \quad & \forall\ q \in Q, \\ \underline{\mathbf{u}}(0) = \underline{\mathbf{u}}_0, & \end{aligned} \tag{2.9}$$

The difficulty with the (2.9) is that the bilinear form a is not coercive, to overcome this difficulty, Cao and his co-workers [4] multiply (2.2) with a scaling factor η to drive a new bilinear form for the weak formulation. Obviously, the scaling factor does not change the Darcy equation itself. However, the interface conditions can be modified accordingly in order to preserve the solution of the Stokes-Darcy problem. To this end, they modified the variational formulation as follows: seek $\underline{\mathbf{u}} = (\mathbf{u}, \phi) \in \underline{\mathbf{W}}$ and $p \in Q$ such that

$$
\begin{aligned}
< \partial_t \underline{\mathbf{u}}, \underline{\mathbf{v}} >_\eta + a_\eta(\underline{\mathbf{u}}, \underline{\mathbf{v}}) + b(\underline{\mathbf{v}}, p) = < \underline{\mathbf{F}}, \underline{\mathbf{v}} >_\eta \quad & \forall\ \underline{\mathbf{v}} \in \underline{\mathbf{W}}, \\
b(\underline{\mathbf{u}}, q) = 0 \quad & \forall\ q \in Q, \\
\underline{\mathbf{u}}(0) = \underline{\mathbf{u}}_0, &
\end{aligned} \tag{2.10}
$$

where

$$
< \partial_t \underline{\mathbf{u}}, \underline{\mathbf{v}} >_\eta := < \partial_t \mathbf{u}, \mathbf{v} > + < \eta \partial_t \phi, \psi >,
$$

and new bilinear form

$$
\begin{aligned}
a_\eta(\underline{\mathbf{u}}, \underline{\mathbf{v}}) := {} & 2\nu \int_{\Omega_c} \mathbf{u} : \mathbf{v} d\Omega_c + \frac{\eta}{S} \int_{\Omega_m} (\mathbb{K}\nabla\phi) \cdot \nabla\psi d\Omega_m \\
& + g \int_{\Gamma_{cm}} \phi \mathbf{v} \cdot \mathbf{n}_{cm} d\Gamma_{cm} - \frac{\eta}{S} \int_{\Gamma_{cm}} \mathbf{u} \cdot \mathbf{n}_{cm} \psi d\Gamma_{cm} \\
& + \int_{\Gamma_{cm}} \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\Pi)}} P_\tau(\mathbf{u} + \mathbb{K}\nabla\phi) \cdot \mathbf{v} d\Gamma_{cm},
\end{aligned} \tag{2.11}
$$

as well as the linear functional $\underline{\mathbf{F}} : \underline{\mathbf{W}} \to \mathbb{R}$ by

$$
< \underline{\mathbf{F}}, \underline{\mathbf{w}} >_\eta := < \mathbf{f}_1, \mathbf{w} >_c + \frac{\eta}{S} < f_2, \varphi >_m + g \int_{\Gamma_{cm}} z \mathbf{w} \cdot \mathbf{n}_{cm} d\Gamma_{cm},
$$

where $\mathbf{f}_1$ and f_2 are functionals on $\mathbf{H}^1_{c,0}$ and $H^1_{m,0}$, respectively, and $< \cdot, \cdot >_c$ and $< \cdot, \cdot >_m$ are the dualities induced by the L^2 inner product on Ω_c and Ω_m, respectively. The last integral results from the second equation in (2.6). The effect of the integral is to add the hydrostatic pressure profile of the problem. For convenience of discussion, it is omitted hereafter, although it is taken into account in the numerical tests.

Note that without further assumptions on the regularity of the domain spaces of $a_\eta(\cdot, \cdot)$, we have that $\nabla\phi \in L^2(\Omega_m)$ and thus does not have a well-defined trace on $\partial\Omega_m$ for a general hydraulic conductivity tensor $\mathbb{K}$. Nevertheless, if the hydraulic conductivity is isotropic everywhere, i.e., when the permeability tensor $\Pi(\mathbf{x}) = k(\mathbf{x})\mathbb{I}$, where k is a scalar function and $\mathbb{I}$ is the identity matrix, then the last term of $a_\eta(\cdot, \cdot)$ is wall defined in the sense that(see [4, 9] for more details)

$$
\begin{aligned}
& \int_{\Gamma_{cm}} \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\Pi)}} P_\tau(\mathbf{u} + \mathbb{K}\nabla\phi) \cdot \mathbf{v} d\Gamma_{cm} \\
& = \nu\alpha\sqrt{d} \int_{\Gamma_{cm}} \frac{1}{\sqrt{trace(\Pi)}} (P_\tau(\mathbf{u}) + \frac{g}{\nu} k P_\tau(\nabla\phi)) \cdot P_\tau \mathbf{v} d\Gamma_{cm} \\
& = \nu\alpha \int_{\Gamma_{cm}} \{ \frac{1}{\sqrt{k}} P_\tau(\mathbf{u}) \cdot P_\tau(\mathbf{v}) + \frac{g}{\nu} \sqrt{k} \nabla_\tau \phi \cdot P_\tau(\mathbf{v}) \} d\Gamma_{cm}
\end{aligned} \tag{2.12}
$$

Here $\nabla_\tau \phi = \frac{\partial\phi}{\partial\tau_1}\tau_1 + \frac{\partial\phi}{\partial\tau_2}\tau_2$ which is exactly the tangential derivative, and the integral of $\nabla_\tau\phi \cdot P_\tau(\mathbf{v})$ on Γ_{cm} is understood to be the value of the functional $\nabla_\tau\phi|_{\Gamma_{cm}} \in (H^{1/2}_{00}(\Gamma_{cm}))'$ applied to $P_\tau(\mathbf{v})|_{\Gamma_{cm}} \in H^{1/2}_{00}(\Gamma_{cm})$.

A slightly simpler approach is to take the Leray-Hopf projection, and we work on the divergence free subspace only, i.e., see $\underline{\mathbf{u}} = (\mathbf{u}, \phi) \in \underline{\mathbf{V}}$ and $p \in Q$ such that

$$
< \partial_t \underline{\mathbf{u}}, \underline{\mathbf{v}} >_\eta + a_\eta(\underline{\mathbf{u}}, \underline{\mathbf{v}}) = < \underline{\mathbf{F}}, \underline{\mathbf{v}} >_\eta \quad \forall\ \underline{\mathbf{v}} \in \underline{\mathbf{V}} \tag{2.13}
$$

for almost all $t, 0 < t \leq T$. From [4, 3], we know that when η is sufficiently large, the weak solution for the Stokes-Darcy problem uniquely exists.

In order to derive the coupled Backward-Euler discretization for Stokes-Darcy problem, we partition Ω_c and Ω_m into mesh $\{\mathcal{T}_j^h\}(j = c, m)$ with $\overline{\Omega_j} = \cup_{K \in \{\mathcal{T}_j^h\}} \bar{K}$. We assume that the cells $K \in \{\mathcal{T}_j^h\}$ are affine equivalent and the grids of $\{\mathcal{T}_c^h\}$ and $\{\mathcal{T}_m^h\}$ match along Γ_{cm}. On the other hand, we divide the time interval $[0, T]$ into N subintervals $[t^n, t^{n+1}](n = 0, 1, \cdots, N-1)$, satisfying

$$0 = t^0 < t^1 < \cdots < t^{N-1} < t^N = T.$$

Let $\triangle t_n = t^n - t^{n-1}$ be the time step with the biggest one $\triangle t = max_{1 \leq n \leq N} \triangle t_n$.

We introduce the finite element spaces $\underline{\mathbf{W}}^h$ and Q^h which are div-stable: there exists a constant $\beta > 0$, independent of h, such that

$$\underline{\mathbf{W}}^h = \mathbf{H}_c^h \times H_m^h \subset \underline{\mathbf{W}}, Q^h \subset Q, \tag{2.14}$$

$$\inf_{0 \neq q^h \in Q^h} \sup_{0 \neq \mathbf{v}_h \in \underline{\mathbf{W}}^h} \frac{b(\mathbf{v}^h, q^h)}{||\mathbf{v}^h||_1 ||q^h||_0} > \beta. \tag{2.15}$$

and

$$\underline{\mathbf{V}}^h = \{\underline{\mathbf{v}}^h \in \underline{\mathbf{W}}^h \mid b(\underline{\mathbf{v}}^h, q^h) = 0, \ \forall\, q^h \in Q^h\}. \tag{2.16}$$

We also assume Korn's inequality (see [4])

$$(\mathbb{D}(\mathbf{v}^h), \mathbb{D}(\mathbf{v}^h)) \geq C_1 ||\mathbf{v}^h||_1^2 \qquad \forall\, \mathbf{v}^h \in \underline{\mathbf{W}}^h, \tag{2.17}$$

and the trace inequality

$$||\underline{\mathbf{v}}^h||_{L^2(\Gamma_{cm})} \leq C_2 ||\underline{\mathbf{v}}^h||_0^{\frac{1}{2}} ||\underline{\mathbf{v}}^h||_1^{\frac{1}{2}} \qquad \forall\, \mathbf{v}^h \in \underline{\mathbf{W}}^h, \tag{2.18}$$

If using Poincar$e$ inequality to the right-hand side of the trace inequality, we have

$$||\underline{\mathbf{v}}^h||_{L^2(\Gamma_{cm})} \leq C_3 ||\underline{\mathbf{v}}^h||_1 \qquad \forall\, \mathbf{v}^h \in \underline{\mathbf{W}}^h, \tag{2.19}$$

here C_1, C_2, C_3 are the strictly positive constants independent of $\mathbb{K}, \nu$ *and* α but depend on the domain Ω.

Based on the weak form (2.10), [3] proposed a fully, monolithically coupled implicit Euler scheme as follows:
Algorithm 2.1(Coupled scheme): given $(\underline{\mathbf{u}}_h^0, p_h^0) \in \underline{\mathbf{W}}^h \times Q^h$, find $(\underline{\mathbf{u}}_h^{n+1}, p_h^{n+1}) \in \underline{\mathbf{W}}^h \times Q^h$ such that

$$< \frac{\underline{\mathbf{u}}_h^{n+1} - \underline{\mathbf{u}}_h^n}{\triangle t}, \underline{\mathbf{v}}_h >_\eta + a_\eta(\underline{\mathbf{u}}_h^{n+1}, \underline{\mathbf{v}}_h) + b(\underline{\mathbf{v}}_h, p_h^{n+1}) = < \underline{\mathbf{F}}^{n+1} \underline{\mathbf{v}}_h >_\eta \quad \forall\, \underline{\mathbf{v}}_h \in \underline{\mathbf{W}}^h, \tag{2.20}$$

$$b(\underline{\mathbf{u}}_h^{n+1}, q_h) = 0 \qquad\qquad \forall\, q_h \in Q^h, \tag{2.21}$$

for $n = 0, 1, \cdots, N-1$, where $\underline{\mathbf{F}}^{n+1} := \underline{\mathbf{F}}(t^{n+1})$.

Under the certain assumptions, Cao and his co-worker have derived the error estimation as follows [3]:

$$||\underline{\mathbf{u}}(t^{n+1}) - \underline{\mathbf{u}}_h^{n+1}||_{0,\eta} \leq C(h^2 + \triangle t), \tag{2.22}$$

where $||\underline{\mathbf{v}}||_{0,\eta} := (||\mathbf{v}||^2_{(L^2(\Omega_c))^d} + ||\eta^{\frac{1}{2}} \psi||^2_{L^2(\Omega_m)})^{1/2}$.

The main purpose of this report is to present a partitioned time stepping method for the Stokes-Darcy problem, which requires only to solve one, uncoupled Stokes and Darcy subproblem in each sub-domain per time step. We will analyze its error estimations below and compare it with result of the above coupled method.

3. Partitioned time stepping method. In this section, we present the partitioned time stepping method for the Stokes-Darcy problem and analyze its stability in sequence.

Algorithm 3.1(Partitioned scheme):

Step 1: In Ω_c, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{H}_c^h \times Q^h$ satisfies

$$(\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\triangle t}, \mathbf{v}_h) + 2\nu(\mathbb{D}(\mathbf{u}_h^{n+1}), \mathbb{D}(\mathbf{v}_h)) + b(\mathbf{v}_h, p_h^{n+1}) = (f_1^{n+1}, \mathbf{v}_h)$$

$$-g\int_{\Gamma_{cm}} \phi_h^n \mathbf{v}_h \cdot \mathbf{n}_{cm} d\Gamma_{cm} - \int_{\Gamma_{cm}} \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\Pi)}} P_\tau(\mathbf{u}_h^{n+1} + \mathbb{K}\nabla\phi_h^n)\cdot \mathbf{v}_h d\Gamma_{cm} \quad \forall\ \mathbf{v}_h \in \mathbf{H}_c^h, \tag{3.1}$$

$$b(\mathbf{u}_h^{n+1}, q_h) = 0 \qquad \forall\ q_h \in Q^h, \tag{3.2}$$

Step 2: In Ω_m, find $\phi_h^{n+1} \in H_m^h$ satisfies

$$\eta(\frac{\phi_h^{n+1}-\phi_h^n}{\triangle t}, \psi_h) + \frac{\eta}{S}(\mathbb{K}\nabla\phi_h^{n+1}, \nabla\psi_h) = \frac{\eta}{S}(f_2^{n+1}, \psi_h) + \frac{\eta}{S}\int_{\Gamma_{cm}} \mathbf{u}_h^n \cdot \mathbf{n}_{cm}\psi_h d\Gamma_{cm} \tag{3.3}$$

for all $\psi_h \in H_m^h$.

3.1. Stability of the method. Under a modest time step restriction, we prove stability of the partitioned time stepping method for the Stokes-Darcy problem.

THEOREM 3.1. *Suppose that the scaling parameter η satisfies that $\eta \geq \frac{4S\alpha^2 g}{C_1}$, and the time step size $\triangle t$ satisfies the following condition:*

$$(\frac{\eta C_2^2}{2\sqrt{S^3 C_1 g k}} + \frac{C_2^2\sqrt{Sg^3}}{\eta\sqrt{2C_1 k}})\triangle t \leq 1,$$

then we have

$$\begin{aligned}
&||\mathbf{u}_h^N||_0^2 + \eta||\phi_h^N||_0^2 + \sum_{n=0}^{N-1}(||\mathbf{u}_h^{n+1}-\mathbf{u}_h^n||_0^2 + \eta||\phi_h^{n+1}-\phi_h^n||_0^2) + \frac{C_1\nu\triangle t}{2}||\mathbf{u}_h^N||_1^2 + \frac{\eta g k\triangle t}{2S\nu}||\phi_h^N||_1^2 \\
&\quad \leq C(T)(\frac{\triangle t}{2C_1\nu}\sum_{n=0}^{N-1}||\mathbf{f}_1^{n+1}||_{H^{-1}(\Omega_c)}^2 + \frac{\eta\nu\triangle t}{Sgk}\sum_{n=0}^{N-1}||f_2^{n+1}||_{H^{-1}(\Omega_m)}^2 \\
&\quad\quad + ||\mathbf{u}_0||_0^2 + \eta||\phi_0||_0^2 + \frac{C_1\nu\triangle t}{2}||\mathbf{u}_0||_1^2 + \frac{\eta g k\triangle t}{2S\nu}||\phi_0||_1^2),
\end{aligned} \tag{3.4}$$

where $C(T)$ denote a constant which depends on the the final time T, C_1 and C_2 are the constants which are related to Korn's inequality and trace inequality, respectively.

Proof. Setting $\mathbf{v}_h = 2\triangle t\mathbf{u}_h^{n+1}$ in (3.1) and using the identity $2(a-b,a) = |a|^2-|b|^2+|a-b|^2$, as well as $b(\mathbf{u}_h^{n+1}, p_h^{n+1}) = 0$, we have

$$\begin{aligned}
&||\mathbf{u}_h^{n+1}||_0^2 - ||\mathbf{u}_h^n||_0^2 + ||\mathbf{u}_h^{n+1}-\mathbf{u}_h^n||_0^2 + 4\nu\triangle t||\mathbb{D}(\mathbf{u}_h^{n+1})||_0^2 + \frac{2\nu\alpha\triangle t}{\sqrt{k}}||P_\tau(\mathbf{u}_h^{n+1})||_{L^2(\Gamma_{cm})}^2 \\
&\quad = 2\triangle t(\mathbf{f}_1^{n+1}, \mathbf{u}_h^{n+1}) - 2\triangle t g\int_{\Gamma_{cm}} \phi_h^n \mathbf{u}_h^{n+1}\cdot\mathbf{n}_{cm} d\Gamma_{cm} \\
&\quad - 2\alpha g\sqrt{k}\triangle t < \nabla_\tau\phi_h^n, P_\tau(\mathbf{u}_h^{n+1}) >_{(H_{00}^{1/2}(\Gamma_c m))', H_{00}^{1/2}(\Gamma_c m)}
\end{aligned} \tag{3.5}$$

Choosing $\psi = 2\triangle t\phi_h^{n+1}$ in (3.3) and using the identity $2(a-b,a) = |a|^2-|b|^2+|a-b|^2$ gives

$$\begin{aligned}
&\eta(||\phi_h^{n+1}||_0^2 - ||\phi_h^n||_0^2 + ||\phi_h^{n+1}-\phi_h^n||_0^2) + \frac{2\eta g k\triangle t}{S\nu}||\phi_h^{n+1}||_1^2 \\
&\quad = \frac{2\eta\triangle t}{S}(f_2^{n+1}, \phi_h^{n+1}) + \frac{2\eta\triangle t}{S}\int_{\Gamma_{cm}} \mathbf{u}_h^n\cdot\mathbf{n}_{cm}\phi_h^{n+1} d\Gamma_{cm}.
\end{aligned} \tag{3.6}$$

Adding these two equations and using Korn's inequality (2.17), we have

$$
\begin{aligned}
&||\mathbf{u}_h^{n+1}||_0^2 - ||\mathbf{u}_h^n||_0^2 + ||\mathbf{u}_h^{n+1} - \mathbf{u}_h^n||_0^2 + \eta(||\phi_h^{n+1}||_0^2 - ||\phi_h^n||_0^2 + ||\phi_h^{n+1} - \phi_h^n||_0^2) \\
&\quad +4C_1\nu\triangle t||\mathbf{u}_h^{n+1}||_1^2 + \frac{2\eta gk\triangle t}{S\nu}||\phi_h^{n+1}||_1^2 + \frac{2\nu\alpha\triangle t}{\sqrt{k}}||P_\tau(\mathbf{u}_h^{n+1})||_{L^2(\Gamma_{cm})}^2 \\
&\leq 2\triangle t(\mathbf{f}_1^{n+1}, \mathbf{u}_h^{n+1}) + \frac{2\eta\triangle t}{S}(f_2^{n+1}, \phi_h^{n+1}) \\
&\quad +2\triangle t(\frac{\eta}{S}\int_{\Gamma_{cm}} \mathbf{u}_h^n \cdot \mathbf{n}_{cm}\phi_h^{n+1} d\Gamma_{cm} - g\int_{\Gamma_{cm}} \phi_h^n \mathbf{u}_h^{n+1} \cdot \mathbf{n}_{cm} d\Gamma_{cm}) \\
&\quad -2\alpha g\sqrt{k}\triangle t < \nabla_\tau \phi_h^n, P_\tau(\mathbf{u}_h^{n+1}) >_{(H_{00}^{1/2}(\Gamma_c m))', H_{00}^{1/2}(\Gamma_{cm})} . \qquad (3.7)
\end{aligned}
$$

For the first two terms on the right-hand side, using Young and Hölder inequalities, we have

$$
\begin{aligned}
2\triangle t(\mathbf{f}_1^{n+1}, \mathbf{u}_h^{n+1}) + \frac{2\eta\triangle t}{S}(f_2^{n+1}, \phi_h^{n+1}) &\leq 2C_1\nu\triangle t||\mathbf{u}_h^{n+1}||_1^2 + \frac{\triangle t}{2C_1\nu}||\mathbf{f}_1^{n+1}||_{H^{-1}(\Omega_c)}^2 \\
&\quad + \frac{\eta gk\triangle t}{S\nu}||\phi_h^{n+1}||_1^2 + \frac{\eta\nu\triangle t}{Sgk}||f_2^{n+1}||_{H^{-1}(\Omega_m)}^2. \qquad (3.8)
\end{aligned}
$$

For the first two interface boundary term, by using Young, trace (2.18) and Hölder inequalities, it follows that

$$
\begin{aligned}
&2\triangle t(\frac{\eta}{S}\int_{\Gamma_{cm}} \mathbf{u}_h^n \cdot \mathbf{n}_{cm}\phi_h^{n+1} d\Gamma_{cm} - g\int_{\Gamma_{cm}} \phi_h^n \mathbf{u}_h^{n+1} \cdot \mathbf{n}_{cm} d\Gamma_{cm}) \\
&\leq \frac{2\eta\triangle t}{S}||\mathbf{u}_h^n \cdot \mathbf{n}_{cm}||_{L^2(\Gamma_{cm})}||\phi_h^{n+1}||_{L^2(\Gamma_{cm})} + 2\triangle tg||\mathbf{u}_h^{n+1} \cdot \mathbf{n}_{cm}||_{L^2(\Gamma_{cm})}||\phi_h^n||_{L^2(\Gamma_{cm})} \\
&\leq \frac{2\eta C_2\triangle t}{S}||\mathbf{u}_h^n||_0^{1/2}||\mathbf{u}_h^n||_1^{1/2}||\phi_h^{n+1}||_0^{1/2}||\phi_h^{n+1}||_1^{1/2} + 2C_2\triangle tg||\mathbf{u}_h^{n+1}||_0^{1/2}||\mathbf{u}_h^{n+1}||_1^{1/2}||\phi_h^n||_0^{1/2}||\phi_h^n||_1^{1/2} \\
&\leq \sqrt{\frac{C_1\eta gk}{S}}\triangle t||\mathbf{u}_h^n||_1||\phi_h^{n+1}||_1 + \sqrt{\frac{\eta^3 C_2^4}{S^3 C_1 gk}}\triangle t||\mathbf{u}_h^n||_0||\phi_h^{n+1}||_0 \\
&\quad +\sqrt{\frac{C_1\eta gk}{2S}}\triangle t||\mathbf{u}_h^{n+1}||_1||\phi_h^n||_1 + \sqrt{\frac{2Sg^3C_2^4}{C_1\eta k}}\triangle t||\mathbf{u}_h^{n+1}||_0||\phi_h^n||_0 \\
&\leq \frac{C_1\nu\triangle t}{2}||\mathbf{u}_h^n||_1^2 + \frac{\eta gk\triangle t}{2S\nu}||\phi_h^{n+1}||_1^2 + \frac{C_1\nu\triangle t}{2}||\mathbf{u}_h^{n+1}||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||\phi_h^n||_1^2 \\
&\quad +\frac{\eta C_2^2}{2\sqrt{S^3C_1gk}}\triangle t(||\mathbf{u}_h^n||_0^2 + \eta||\phi_h^{n+1}||_0^2) + \frac{C_2^2\sqrt{Sg^3}}{\eta\sqrt{2C_1k}}\triangle t(||\mathbf{u}_h^{n+1}||_0^2 + \eta||\phi_h^n||_0^2).
\end{aligned}
$$

For the last term on the right-hand side, by using the imbedding inequality and Hölder inequality, it leads to

$$
\begin{aligned}
&2\ \alpha\ g\sqrt{k}\triangle t < \nabla_\tau\phi_h^n, P_\tau(\mathbf{u}_h^{n+1}) >_{(H_{00}^{1/2}(\Gamma_c m))', H_{00}^{1/2}(\Gamma_c m)} \\
&\quad \leq 2\alpha g\sqrt{k}\triangle t||\nabla\phi_h^n||_{(H_{00}^{1/2}(\Gamma_c m))'}||P_\tau(\mathbf{u}_h^{n+1})||_{H_{00}^{1/2}(\Gamma_c m)} \\
&\quad \leq 2\alpha g\sqrt{k}\triangle t||\phi_h^n||_{H^1(\Omega_m)}||\mathbf{u}_h^{n+1}||_{H^1(\Omega_c)} \\
&\quad \leq C_1\nu\triangle t||\mathbf{u}_h^{n+1}||_1^2 + \frac{\alpha^2 g^2 k\triangle t}{C_1\nu}||\phi_h^n||_1^2. \qquad (3.9)
\end{aligned}
$$

Combining these estimates with (3.7), we obtain

$$
||\mathbf{u}_h^{n+1}||_0^2 - ||\mathbf{u}_h^n||_0^2 + ||\mathbf{u}_h^{n+1} - \mathbf{u}_h^n||_0^2 + \eta(||\phi_h^{n+1}||_0^2 - ||\phi_h^n||_0^2 + ||\phi_h^{n+1} - \phi_h^n||_0^2)
$$

$$+\frac{C_1\nu\triangle t}{2}(||\mathbf{u}_h^{n+1}||_1^2-||\mathbf{u}_h^n||_1^2)+\frac{\eta gk}{2S\nu}\triangle t||\phi_h^{n+1}||_1^2-(\frac{\alpha^2g^2k}{C_1\nu}+\frac{\eta gk}{4S\nu})\triangle t||\phi_h^n||_1^2$$
$$\leq\frac{\triangle t}{2C_1\nu}||\mathbf{f}_1^{n+1}||_{H^{-1}(\Omega_c)}^2+\frac{\eta\nu\triangle t}{Sgk}||f_2^{n+1}||_{H^{-1}(\Omega_m)}^2$$
$$+\frac{\eta C_2^2}{2\sqrt{S^3C_1gk}}\triangle t(||\mathbf{u}_h^n||_0^2+\eta||\phi_h^{n+1}||_0^2)+\frac{C_2^2\sqrt{Sg^3}}{\eta\sqrt{2C_1k}}\triangle t(||\mathbf{u}_h^{n+1}||_0^2+\eta||\phi_h^n||_0^2).$$

Assuming that $\frac{\alpha^2g^2k}{C_1\nu}+\frac{\eta gk}{4S\nu}\leq\frac{\eta gk}{2S\nu}$, i.e., $\eta\geq\frac{4S\alpha^2g}{C_1}$, we also denote $\tilde{C}=\frac{\eta C_2^2}{2\sqrt{S^3C_1gk}}+\frac{C_2^2\sqrt{Sg^3}}{\eta\sqrt{2C_1k}}$, then summing over n from $n=0$ to $N-1$, we arrive at

$$||\mathbf{u}_h^N||_0^2+\eta||\phi_h^N||_0^2+\sum_{n=0}^{N-1}(||\mathbf{u}_h^{n+1}-\mathbf{u}_h^n||_0^2+\eta||\phi_h^{n+1}-\phi_h^n||_0^2)+\frac{C_1\nu\triangle t}{2}||\mathbf{u}_h^N||_1^2+\frac{\eta gk\triangle t}{2S\nu}||\phi_h^N||_1^2$$
$$\leq\tilde{C}\triangle t\sum_{n=0}^{N-1}(||\mathbf{u}_h^{n+1}||_0^2+\eta||\phi_h^{n+1}||_0^2)+\frac{\triangle t}{2C_1\nu}\sum_{n=0}^{N-1}||\mathbf{f}_1^{n+1}||_{H^{-1}(\Omega_c)}^2+\frac{\eta\nu\triangle t}{Sgk}\sum_{n=0}^{N-1}||f_2^{n+1}||_{H^{-1}(\Omega_m)}^2$$
$$+||\mathbf{u}_0||_0^2+\eta||\phi_0||_0^2+\frac{C_1\nu\triangle t}{2}||\mathbf{u}_0||_1^2+\frac{\eta gk\triangle t}{2S\nu}||\phi_0||_1^2. \tag{3.10}$$

It follows from Gronwall inequality that when $\tilde{C}\triangle t\leq 1$,

$$||\mathbf{u}_h^N||_0^2+\eta||\phi_h^N||_0^2+\sum_{n=0}^{N-1}(||\mathbf{u}_h^{n+1}-\mathbf{u}_h^n||_0^2+\eta||\phi_h^{n+1}-\phi_h^n||_0^2)+\frac{C_1\nu\triangle t}{2}||\mathbf{u}_h^N||_1^2+\frac{\eta gk\triangle t}{2S\nu}||\phi_h^N||_1^2$$
$$\leq C(T)(\frac{\triangle t}{2C_1\nu}\sum_{n=0}^{N-1}||\mathbf{f}_1^{n+1}||_{H^{-1}(\Omega_c)}^2+\frac{\eta\nu\triangle t}{Sgk}\sum_{n=0}^{N-1}||f_2^{n+1}||_{H^{-1}(\Omega_m)}^2$$
$$+||\mathbf{u}_0||_0^2+\eta||\phi_0||_0^2+\frac{C_1\nu\triangle t}{2}||\mathbf{u}_0||_1^2+\frac{\eta gk\triangle t}{2S\nu}||\phi_0||_1^2). \tag{3.11}$$

□

4. Error Estimate. In this section, we analyze the convergence rate of the method. First of all, we assume that the true solution satisfies the following assumptions: for $\underline{\mathbf{w}}(t)=(\mathbf{u}(t),\phi(t))$, we have

$$\underline{\mathbf{w}}_t(t)\in L^2(0,T;\underline{\mathbf{W}}),\quad \underline{\mathbf{w}}_{tt}(t)\in L^2(0,T;\underline{\mathbf{L}}^2)\cap L^\infty(0,T;\underline{\mathbf{W}}),\quad \underline{\mathbf{w}}_{ttt}(t)\in L^\infty(0,T;\underline{\mathbf{L}}^2).$$

Let us define a projection operator $P_h:(\underline{\mathbf{w}}(t),p(t))\in\underline{\mathbf{W}}\times Q\rightarrow(P_h\underline{\mathbf{w}}(t),P_hp(t))\in\underline{\mathbf{W}}^h\times Q^h,\forall\ t\in[0,T]$ by

$$\begin{aligned} a_\eta(P_h\underline{\mathbf{w}}(t),\underline{\mathbf{v}}_h)+b(\underline{\mathbf{v}}_h,P_hp(t))&=a_\eta(\underline{\mathbf{w}}(t),\underline{\mathbf{v}}_h)+b(\underline{\mathbf{v}}_h,p(t)) && \forall\ \underline{\mathbf{v}}_h\in\underline{\mathbf{W}}^h,\\ b(P_h\underline{\mathbf{w}}(t),q_h)&=0 && \forall\ q_h\in Q^h.\end{aligned}$$

Note that P_h is a linear operator, and under a certain smoothness assumption on $(\mathbf{w}(t),p(t))$, the following approximation properties hold:

$$||P_h\underline{\mathbf{w}}(t)-\underline{\mathbf{w}}(t)||_0\leq Ch^2, \tag{4.1}$$
$$||P_h\underline{\mathbf{w}}(t)-\underline{\mathbf{w}}(t)||_1\leq Ch, \tag{4.2}$$
$$||P_hp(t)-p(t)||_0\leq Ch. \tag{4.3}$$

Furthermore, we will use the following notations, we denote $(P_h\underline{\mathbf{w}}(t), P_h p(t))$ by $(\bar{\underline{\mathbf{w}}}(t), \bar{p}(t))$ for simplification. Then

$$\begin{aligned}
e_1^{n+1} &= \mathbf{u}(t^{n+1}) - \bar{\mathbf{u}}^{n+1} + \bar{\mathbf{u}}^{n+1} - \mathbf{u}_h^{n+1} \triangleq \varepsilon_1^{n+1} + \delta_1^{n+1}, \\
e_2^{n+1} &= \phi(t^{n+1}) - \bar{\phi}^{n+1} + \bar{\phi}^{n+1} - \phi_h^{n+1} \triangleq \varepsilon_2^{n+1} + \delta_2^{n+1}, \\
\mu^{n+1} &= p(t^{n+1}) - \bar{p}^{n+1} + \bar{p}^{n+1} - p_h^{n+1} \triangleq \varepsilon_\mu^{n+1} + \delta_\mu^{n+1}.
\end{aligned}$$

In particular, $(\delta_1^0, \delta_2^0) = (0,0)$. Obviously, from the definition of P_h, we have

$$||\varepsilon_1^{n+1}||_0 + ||\varepsilon_2^{n+1}||_0 \le Ch^2, ||\varepsilon_1^{n+1}||_1 + ||\varepsilon_2^{n+1}||_1 \le Ch, ||\varepsilon_\mu^{n+1}||_0 \le Ch.$$

For convenience, let us introduce the following nations. We denote the backward divided difference operator d_t by

$$d_t \underline{\mathbf{w}}_h^{n+1} = \frac{\underline{\mathbf{w}}_h^{n+1} - \underline{\mathbf{w}}_h^n}{\triangle t}, \quad for\ n = 0, \cdots, N-1. \tag{4.4}$$

We also denote

$$d_{tt} \underline{\mathbf{w}}_h^{n+1} = \frac{d_t\underline{\mathbf{w}}_h^{n+1} - d_t\underline{\mathbf{w}}_h^n}{\triangle t}, \quad for\ n = 1, \cdots, N-1. \tag{4.5}$$

In the following, we need to estimate the $\delta_1^{n+1}, \delta_2^{n+1}$ and δ_μ^{n+1} in the according norms. Note that $(\bar{\underline{\mathbf{w}}}^{n+1}, \bar{p}^{n+1}) \in \underline{\mathbf{V}}^h \times Q^h$ satisfies the following equations: for all $\underline{\mathbf{v}}_h = (\mathbf{v}_h, \psi_h) \in \underline{\mathbf{V}}^h, q \in Q^h$,

$$\begin{aligned}
&(\frac{\bar{\mathbf{u}}^{n+1} - \bar{\mathbf{u}}^n}{\triangle t}, \mathbf{v}_h) + 2\nu(\mathbb{D}(\bar{\mathbf{u}}^{n+1}), \mathbb{D}(\mathbf{v}_h)) + b(\mathbf{v}_h, \bar{p}_h^{n+1}) = (\theta_u^{n+1}, \mathbf{v}_h) + (\mathbf{f}_1^{n+1}, \mathbf{v}_h) \\
&\qquad - g\int_{\Gamma_{cm}} \bar{\phi}^{n+1}\mathbf{v}_h \cdot \mathbf{n}_{cm} d\Gamma_{cm} - \int_{\Gamma_{cm}} \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\Pi)}} P_\tau(\bar{\mathbf{u}}^{n+1} + \mathbb{K}\nabla\bar{\phi}^{n+1}) \cdot \mathbf{v}_h d\Gamma_{cm},
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&\eta(\frac{\bar{\phi}^{n+1} - \bar{\phi}^n}{\triangle t}, \psi_h) + \frac{\eta}{S}(\mathbb{K}\nabla\bar{\phi}^{n+1}, \nabla\psi_h) = \eta(\theta_\phi^{n+1}, \psi_h) + \frac{\eta}{S}(f_2^{n+1}, \psi_h) \\
&\qquad + \frac{\eta}{S}\int_{\Gamma_{cm}} \bar{\mathbf{u}}^{n+1} \cdot \mathbf{n}_{cm}\psi_h d\Gamma_{cm},
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
\theta_u^{n+1} &= \frac{\bar{\mathbf{u}}^{n+1} - \bar{\mathbf{u}}^n}{\triangle t} - \mathbf{u}_t(t^{n+1}) \\
&= [\frac{\bar{\mathbf{u}}^{n+1} - \bar{\mathbf{u}}^n}{\triangle t} - \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\triangle t}] - [\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\triangle t} - \mathbf{u}_t(t^{n+1})] \\
&\triangleq \theta_{u,1}^{n+1} + \theta_{u,2}^{n+1},
\end{aligned}$$

and

$$\begin{aligned}
\theta_\phi^{n+1} &= \frac{\bar{\phi}^{n+1} - \bar{\phi}^n}{\triangle t} - \phi_t(t^{n+1}) \\
&= [\frac{\bar{\phi}^{n+1} - \bar{\phi}^n}{\triangle t} - \frac{\phi(t^{n+1}) - \phi(t^n)}{\triangle t}] - [\frac{\phi(t^{n+1}) - \phi(t^n)}{\triangle t} - \phi_t(t^{n+1})] \\
&\triangleq \theta_{\phi,1}^{n+1} + \theta_{\phi,2}^{n+1}.
\end{aligned}$$

It is easy to verify that the following properties of $\theta_{u,1}^{n+1}, \theta_{u,2}^{n+1}, \theta_{\phi,1}^{n+1}, \theta_{\phi,2}^{n+1}$ hold:

$$\theta_{u,1}^{n+1} = (P_h - I)\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\triangle t} = \frac{1}{\triangle t}\int_{t^n}^{t^{n+1}} (P_h - I)\mathbf{u}_t(t)dt,$$

which implies

$$||\theta_{u,1}^{n+1}||_0^2 = \frac{1}{\triangle t^2}\int_\Omega (\int_{t^n}^{t^{n+1}} (P_h - I)\mathbf{u}_t(t)dt)^2 dx \leq \frac{1}{\triangle t}\int_{t^n}^{t^{n+1}} ||(P_h - I)\mathbf{u}_t||_0^2 dt. \tag{4.8}$$

and

$$\triangle t\theta_{u,2}^{n+1} = \mathbf{u}(t^{n+1}) - \mathbf{u}(t^n) - \triangle t\mathbf{u}_t(t^{n+1}) = \int_{t^n}^{t^{n+1}} (t^{n+1} - t)\mathbf{u}_{tt}(t)dt,$$

which means

$$||\theta_{u,2}^{n+1}||_0^2 = \frac{1}{\triangle t^2}\int_\Omega (\int_{t^n}^{t^{n+1}} (t^{n+1} - t)\mathbf{u}_{tt}(t)dt)^2 dx \leq \triangle t\int_{t^n}^{t^{n+1}} ||\mathbf{u}_{tt}||_0^2 dt. \tag{4.9}$$

In the same way, we can obtain the similar results for $\theta_{\phi,1}^{n+1}$ and $\theta_{\phi,2}^{n+1}$,

$$||\theta_{\phi,1}^{n+1}||_0^2 \leq \frac{1}{\triangle t}\int_{t^n}^{t^{n+1}} ||(P_h - I)\phi_t||_0^2 dt, \tag{4.10}$$

$$||\theta_{\phi,2}^{n+1}||_0^2 \leq \triangle t\int_{t^n}^{t^{n+1}} ||\phi_{tt}||_0^2 dt. \tag{4.11}$$

We also need to estimates the following two terms in error analysis.

$$\begin{aligned} ||\bar{\mathbf{u}}^{n+1} - \bar{\mathbf{u}}^n||_1^2 &\leq C||\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)||_1^2 = C\int_\Omega (\int_{t^n}^{t^{n+1}} \nabla\mathbf{u}_t(t))^2 d\mathbf{x} \\ &\leq C\int_\Omega (\int_{t^n}^{t^{n+1}} |\nabla\mathbf{u}_t(t)|^2 dt \int_{t^n}^{t^{n+1}} 1dt)d\mathbf{x} \leq C\triangle t\int_{t^n}^{t^{n+1}} ||\mathbf{u}_t||_1^2 dt, \end{aligned} \tag{4.12}$$

$$||\bar{\phi}^{n+1} - \bar{\phi}^n||_1^2 \leq C\triangle t\int_{t^n}^{t^{n+1}} ||\phi_t||_1^2 dt. \tag{4.13}$$

Under the preparation above, we can obtain the following error estimate for velocity.

THEOREM 4.1. *(Error for the velocity) Under the assumption of Theorem 3.1 for the scaling parameter and the time step size, then we have*

$$\begin{aligned} ||\mathbf{u}(t^N) - \mathbf{u}_h^N||_0^2 + \eta||\phi(t^N) - \phi_h^N||_0^2 + \frac{C_1\nu\triangle t}{2}(||\mathbf{u}(t^N) - \mathbf{u}_h^N||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||\phi(t^N) - \phi_h^N||_1^2 \\ \leq C(T)(h^4 + \triangle t^2). \end{aligned}$$

Proof. Subtracting (3.1)-(3.3) from (4.6)-(4.7), we have

$$\begin{aligned} (\frac{\delta_1^{n+1} - \delta_1^n}{\triangle t}, \mathbf{v}_h) + 2\nu(\mathbb{D}(\delta_1^{n+1}), \mathbb{D}(\mathbf{v}_h)) + b(\mathbf{v}_h, \delta_\mu^{n+1}) = -g\int_{\Gamma_{cm}} (\bar{\phi}^{n+1} - \phi_h^n)\mathbf{v}_h \cdot \mathbf{n}_{cm} d\Gamma_{cm} \\ + (\theta_u^{n+1}, \mathbf{v}_h) - \int_{\Gamma_{cm}} \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\Pi)}} P_\tau(\delta_1^{n+1} + \mathbb{K}(\nabla\bar{\phi}^{n+1} - \nabla\phi_h^n)) \cdot \mathbf{v}_h d\Gamma_{cm}, \end{aligned} \tag{4.14}$$

$$\eta(\frac{\delta_2^{n+1}-\delta_2^n}{\triangle t},\psi_h)+\frac{\eta}{S}(\mathbb{K}\nabla\delta_2^{n+1},\nabla\psi_h)$$

$$=\eta(\theta_\phi^{n+1},\psi_h)+\frac{\eta}{S}\int_{\Gamma_{cm}}(\bar{\mathbf{u}}^{n+1}-\mathbf{u}_h^n)\cdot\mathbf{n}_{cm}\psi_h d\Gamma_{cm}. \tag{4.15}$$

Setting $\mathbf{v}_h=2\triangle t\delta_1^{n+1}$ in (4.14) and using the identity $2(a-b,a)=|a|^2-|b|^2+|a-b|^2$, as well as $b(\delta_1^{n+1},\delta_\mu^{n+1})=0$, we have

$$\begin{aligned}
&||\delta_1^{n+1}||_0^2-||\delta_1^n||_0^2+||\delta_1^{n+1}-\delta_1^n||_0^2+4\nu\triangle t||\mathbb{D}(\delta_1^{n+1})||_0^2+\frac{2\nu\alpha\triangle t}{\sqrt{k}}||P_\tau(\delta_1^{n+1})||^2_{L^2(\Gamma_{cm})}\\
&\quad=2\triangle t(\theta_u^{n+1},\delta_1^{n+1})-2g\triangle t\int_{\Gamma_{cm}}(\bar{\phi}^{n+1}-\bar{\phi}^n+\delta_2^n)\delta_1^{n+1}\cdot\mathbf{n}_{cm}d\Gamma_{cm}\\
&\quad-2\alpha g\sqrt{k}\triangle t<\nabla_\tau(\bar{\phi}^{n+1}-\bar{\phi}^n+\delta_2^n),P_\tau(\delta_1^{n+1})>_{(H_{00}^{1/2}(\Gamma_cm))',H_{00}^{1/2}(\Gamma_cm)}.
\end{aligned}\tag{4.16}$$

Choosing $\psi_h=2\triangle t\delta_2^{n+1}$ in (4.15) and using the identity $2(a-b,a)=|a|^2-|b|^2+|a-b|^2$, we have

$$\begin{aligned}
&\eta(||\delta_2^{n+1}||_0^2-||\delta_2^n||_0^2+||\delta_2^{n+1}-\delta_2^n||_0^2)+\frac{2\eta gk\triangle t}{S\nu}||\delta_2^{n+1}||_1^2\\
&\quad=2\triangle t\eta(\theta_\phi^{n+1},\delta_2^{n+1})+\frac{2\eta\triangle t}{S}\int_{\Gamma_{cm}}(\bar{\mathbf{u}}^{n+1}-\bar{\mathbf{u}}^n+\delta_1^n)\cdot\mathbf{n}_{cm}\delta_2^{n+1}d\Gamma_{cm}.
\end{aligned}\tag{4.17}$$

Adding these two equations together and using Korn's inequality (2.17), we have

$$\begin{aligned}
&||\delta_1^{n+1}||_0^2-||\delta_1^n||_0^2+||\delta_1^{n+1}-\delta_1^n||_0^2+\eta(||\delta_2^{n+1}||_0^2-||\delta_2^n||_0^2+||\delta_2^{n+1}-\delta_2^n||_0^2)\\
&\qquad+4C_1\nu\triangle t||\delta_1^{n+1}||_1^2+\frac{2\eta gk\triangle t}{S\nu}||\delta_2^{n+1}||_1^2\\
&\quad\le 2\triangle t(\theta_u^{n+1},\delta_1^{n+1})+2\triangle t\eta(\theta_\phi^{n+1},\delta_2^{n+1})\\
&\quad-2g\triangle t\int_{\Gamma_{cm}}(\bar{\phi}^{n+1}-\bar{\phi}^n)\delta_1^{n+1}\cdot\mathbf{n}_{cm}d\Gamma_{cm}+\frac{2\eta\triangle t}{S}\int_{\Gamma_{cm}}(\bar{\mathbf{u}}^{n+1}-\bar{\mathbf{u}}^n)\cdot\mathbf{n}_{cm}\delta_2^{n+1}d\Gamma_{cm}\\
&\quad-2\alpha g\sqrt{k}\triangle t<\nabla_\tau(\bar{\phi}^{n+1}-\bar{\phi}^n),P_\tau(\delta_1^{n+1})>_{(H_{00}^{1/2}(\Gamma_cm))',H_{00}^{1/2}(\Gamma_cm)}\\
&\quad-2g\triangle t\int_{\Gamma_{cm}}\delta_2^n\delta_1^{n+1}\cdot\mathbf{n}_{cm}d\Gamma_{cm}+\frac{2\eta\triangle t}{S}\int_{\Gamma_{cm}}\delta_1^n\cdot\mathbf{n}_{cm}\delta_2^{n+1}d\Gamma_{cm}\\
&\quad-2\alpha g\sqrt{k}\triangle t<\nabla_\tau(\delta_2^n),P_\tau(\delta_1^{n+1})>_{(H_{00}^{1/2}(\Gamma_cm))',H_{00}^{1/2}(\Gamma_cm)}.
\end{aligned}\tag{4.18}$$

For the first four terms on the right-hand side, by using (4.8)-(4.11) with Young's, Poincaré and Hölder inequalities, it gives that

$$\begin{aligned}
&2\triangle t(\theta_u^{n+1},\delta_1^{n+1})+2\triangle t\eta(\theta_\phi^{n+1},\delta_2^{n+1})\\
&\quad\le\frac{3\nu C_1\triangle t}{2}||\delta_1^{n+1}||_1^2+\frac{\eta gk\triangle t}{2S\nu}||\delta_2^{n+1}||_1^2+\frac{2C(\Omega)\triangle t}{3C_1\nu}||\theta_u^{n+1}||_0^2+\frac{2S\nu\eta C(\Omega)\triangle t}{gk}||\theta_\phi^{n+1}||_0^2\\
&\quad\le\frac{3\nu C_1\triangle t}{2}||\delta_1^{n+1}||_1^2+\frac{\eta gk\triangle t}{2S\nu}||\delta_2^{n+1}||_1^2\\
&\qquad+\frac{2C(\Omega)}{3C_1\nu}\int_{t^n}^{t^{n+1}}||(P_h-I)\mathbf{u}_t||_0^2dt+\frac{2C(\Omega)\triangle t^2}{3C_1\nu}\int_{t^n}^{t^{n+1}}||\mathbf{u}_{tt}||_0^2dt\\
&\qquad+\frac{2S\nu\eta C(\Omega)}{gk}\int_{t^n}^{t^{n+1}}||(P_h-I)\phi_t||_0^2dt+\frac{2S\nu\eta C(\Omega)\triangle t^2}{gk}\int_{t^n}^{t^{n+1}}||\phi_{tt}||_0^2dt.
\end{aligned}\tag{4.19}$$

For the second two terms on the right-hand side of (4.18, by using trace inequality (2.19) and (4.12)-(4.13), we have

$$-2g\triangle t\int_{\Gamma_{cm}}(\bar\phi^{n+1}-\bar\phi^{n})\delta_1^{n+1}\cdot\mathbf{n}_{cm}d\Gamma_{cm}+\frac{2\eta\triangle t}{S}\int_{\Gamma_{cm}}(\bar{\mathbf{u}}^{n+1}-\bar{\mathbf{u}}^{n})\cdot\mathbf{n}_{cm}\delta_2^{n+1}d\Gamma_{cm}$$
$$\le\frac{C_1\nu\triangle t}{4}||\delta_1^{n+1}||_1^2+\frac{\eta gk\triangle t}{2S\nu}||\delta_2^{n+1}||_1^2+\frac{4g^2C_3^2\triangle t}{C_1\nu}||\bar\phi^{n+1}-\bar\phi^{n}||_1^2+\frac{2\eta\nu C_3^2\triangle t}{gkS}||\bar{\mathbf{u}}^{n+1}-\bar{\mathbf{u}}^{n}||_1^2$$
$$\le\frac{C_1\nu\triangle t}{4}||\delta_1^{n+1}||_1^2+\frac{\eta gk\triangle t}{2S\nu}||\delta_2^{n+1}||_1^2+\frac{4Cg^2C_3^2\triangle t^2}{C_1\nu}\int_{t^n}^{t^{n+1}}||\phi_t||_1^2dt+\frac{2C\eta\nu C_3^2\triangle t^2}{gkS}\int_{t^n}^{t^{n+1}}||\mathbf{u}_t||_1^2dt.$$

Moreover, by using imbedding inequality, the fifth term on the right-hand side of (4.18) can be estimated as follows,

$$\begin{aligned}&-2\alpha g\sqrt{k}\triangle t<\nabla_\tau(\bar\phi^{n+1}-\bar\phi^{n}),P_\tau(\delta_1^{n+1})>_{(H_{00}^{1/2}(\Gamma_cm))',H_{00}^{1/2}(\Gamma_cm)}\\&\le\frac{C_1\nu\triangle t}{4}||\delta_1^{n+1}||_1^2+\frac{4\alpha^2g^2k\triangle t}{\nu C_1}||\bar\phi^{n+1}-\bar\phi^{n}||_1^2\\&\le\frac{C_1\nu\triangle t}{4}||\nabla\delta_1^{n+1}||_0^2+\frac{4C\alpha^2g^2k\triangle t^2}{\nu C_1}\int_{t^n}^{t^{n+1}}||\phi_t||_1^2dt.\end{aligned}\tag{4.20}$$

For the last three terms on the right-hand side of (4.18), from the trace inequality (2.18), we have

$$\begin{aligned}&\frac{2\eta\triangle t}{S}\int_{\Gamma_{cm}}\delta_1^n\cdot\mathbf{n}_{cm}\delta_2^{n+1}d\Gamma_{cm}-2g\triangle t\int_{\Gamma_{cm}}\delta_2^n\delta_1^{n+1}\cdot\mathbf{n}_{cm}d\Gamma_{cm}\\&\quad-2\alpha g\sqrt{k}\triangle t<\nabla_\tau(\delta_2^n),P_\tau(\delta_1^{n+1})>_{(H_{00}^{1/2}(\Gamma_cm))',H_{00}^{1/2}(\Gamma_cm)}\\&\le\frac{C_1\nu\triangle t}{2}||\delta_1^n||_1^2+\frac{\eta gk\triangle t}{2S\nu}||\delta_2^{n+1}||_1^2+\frac{C_1\nu\triangle t}{2}||\delta_1^{n+1}||_1^2\\&\quad+\frac{\eta gk\triangle t}{4S\nu}||\delta_2^n||_1^2+C_1\nu\triangle t||\delta_1^{n+1}||_1^2+\frac{\alpha^2g^2k\triangle t}{C_1\nu}||\delta_2^n||_1^2\\&\quad+\frac{\eta C_2^2\triangle t}{2\sqrt{S^3C_1gk}}(||\delta_1^n||_0^2+\eta||\delta_2^{n+1}||_0^2)+\frac{C_2^2\sqrt{Sg^3}\triangle t}{\eta\sqrt{2C_1k}}(||\delta_1^{n+1}||_0^2+\eta||\delta_2^n||_0^2).\end{aligned}\tag{4.21}$$

Combining (4.19)-(4.21) with (4.18), we arrive at

$$\begin{aligned}&||\delta_1^{n+1}||_0^2-||\delta_1^n||_0^2+||\delta_1^{n+1}-\delta_1^n||_0^2+\eta(||\delta_2^{n+1}||_0^2-||\delta_2^n||_0^2+||\delta_2^{n+1}-\delta_2^n||_0^2)\\&\quad+\frac{C_1\nu\triangle t}{2}(||\delta_1^{n+1}||_1^2-||\delta_1^n||_1^2)+\frac{\eta gk\triangle t}{2S\nu}||\delta_2^{n+1}||_1^2-(\frac{\eta gk}{4S\nu}+\frac{\alpha^2g^2k}{C_1\nu})\triangle t||\delta_2^n||_1^2\\&\le\frac{\eta C_2^2\triangle t}{2\sqrt{S^3C_1gk}}(||\delta_1^n||_0^2+\eta||\delta_2^{n+1}||_0^2)+\frac{C_2^2\sqrt{Sg^3}\triangle t}{\eta\sqrt{2C_1k}}(||\delta_1^{n+1}||_0^2+\eta||\delta_2^n||_0^2)\\&\quad+\frac{3C(\Omega)}{2C_1\nu}\int_{t^n}^{t^{n+1}}||(P_h-I)\mathbf{u}_t||_0^2dt+\frac{3C(\Omega)\triangle t^2}{2C_1\nu}\int_{t^n}^{t^{n+1}}||\mathbf{u}_{tt}||_0^2dt\\&\quad+\frac{2S\nu C(\Omega)\eta}{gk}\int_{t^n}^{t^{n+1}}||(P_h-I)\phi_t||_0^2dt+\frac{2S\nu C(\Omega)\eta\triangle t^2}{gk}\int_{t^n}^{t^{n+1}}||\phi_{tt}||_0^2dt\\&\quad+\frac{4Cg^2C_3^2\triangle t^2}{C_1\nu}\int_{t^n}^{t^{n+1}}||\phi_t||_1^2dt+\frac{2C\eta\nu C_3^2\triangle t^2}{gkS}\int_{t^n}^{t^{n+1}}||\mathbf{u}_t||_1^2dt\\&\quad+\frac{4C\alpha^2g^2k\triangle t^2}{C_1\nu}\int_{t^n}^{t^{n+1}}||\phi_t||_1^2dt.\end{aligned}\tag{4.22}$$

Thus, as the stability condition, we assume that $\eta \geq \frac{4S\alpha^2 g}{C_1}$ and using the above definition of $\tilde{C}$, summing over n from $n=0$ to $N-1$, it follows that

$$
\begin{aligned}
&||\delta_1^N||_0^2 + \eta||\delta_2^N||_0^2 + \sum_{n=0}^{N-1}(||\delta_1^{n+1}-\delta_1^n||_0^2 + \eta||\delta_2^{n+1}-\delta_2^n||_0^2) + \frac{C_1\nu\triangle t}{2}||\delta_1^N||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||\delta_2^N||_1^2 \\
&\quad \leq \tilde{C}\triangle t\sum_{n=0}^{N-1}(||\delta_1^{n+1}||_0^2 + \eta||\delta_2^{n+1}||_0^2) + ||\delta_1^0||_0^2 + \eta||\delta_2^0||_0^2 + \frac{C_1\nu\triangle t}{2}||\delta_1^0||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||\delta_2^0||_1^2 \\
&\qquad + \frac{3C(\Omega)}{2C_1\nu}||(P_h - I)\mathbf{u}_t||^2_{L^2(0,T;L^2)} + \frac{3C(\Omega)\triangle t^2}{2C_1\nu}||\mathbf{u}_{tt}||^2_{L^2(0,T;L^2)} \\
&\qquad + \frac{2S\nu C(\Omega)\eta}{gk}||(P_h - I)\phi_t||^2_{L^2(0,T;L^2)} + \frac{2S\nu C(\Omega)\eta\triangle t^2}{gk}||\phi_{tt}||^2_{L^2(0,T;L^2)} \\
&\qquad + \frac{4Cg^2C_3^2\triangle t^2}{C_1\nu}||\phi_t||^2_{L^2(0,T'H^1)} + \frac{C\eta\nu C_3^2\triangle t^2}{gkS}||\mathbf{u}_t||^2_{L^2(0,T;H^1)} + \frac{4C\alpha^2g^2k\triangle t^2}{C_1\nu}||\phi_t||^2_{L^2(0,T;H^1)}.
\end{aligned}
$$

It follows from Gronwall inequality that when $\tilde{C}\triangle t \leq 1$,

$$
\begin{aligned}
&||\delta_1^N||_0^2 + \eta||\delta_2^N||_0^2 + \sum_{n=0}^{N-1}(||\delta_1^{n+1}-\delta_1^n||_0^2 + \eta||\delta_2^{n+1}-\delta_2^n||_0^2) + \frac{C_1\nu\triangle t}{2}||\delta_1^N||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||\delta_2^N||_1^2 \\
&\quad \leq C(T)(||\delta_1^0||_0^2 + \eta||\delta_2^0||_0^2 + \frac{C_1\nu\triangle t}{2}||\delta_1^0||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||\delta_2^0||_1^2 \\
&\qquad + \frac{3C(\Omega)}{2C_1\nu}||(P_h - I)\mathbf{u}_t||^2_{L^2(0,T;L^2)} + \frac{3C(\Omega)\triangle t^2}{2C_1\nu}||\mathbf{u}_{tt}||^2_{L^2(0,T;L^2)} \\
&\qquad + \frac{2S\nu C(\Omega)\eta}{gk}||(P_h - I)\phi_t||^2_{L^2(0,T;L^2)} + \frac{2S\nu C(\Omega)\eta\triangle t^2}{gk}||\phi_{tt}||^2_{L^2(0,T;L^2)} \\
&\qquad + \frac{4Cg^2C_3^2\triangle t^2}{C_1\nu}||\phi_t||^2_{L^2(0,T;H^1)} + \frac{C\eta\nu C_3^2\triangle t^2}{gkS}||\mathbf{u}_t||^2_{L^2(0,T;H^1)} + \frac{4C\alpha^2g^2k\triangle t^2}{C_1\nu}||\phi_t||^2_{L^2(0,T;H^1)}).
\end{aligned}
$$

Finally, from triangle inequality and the approximation properties (4.1)-(4.2), as well as the assumptions for the error of initial data, we obtain the final result (4.14). ☐

Next, we analyze the convergence of pressure for the decoupled scheme. Note that

$$
||p(t^{n+1}) - p_h^{n+1}||_0 \leq ||\varepsilon_\mu^{n+1}||_0 + ||\delta_\mu^{n+1}||_0, \tag{4.23}
$$

so we only need to estimate $||\delta_\mu^{n+1}||_0$, to this end, let us start with the following lemma which estimates the error for $||d_t\delta_1^{n+1}||_0$.

Lemma 4.2. *Under the assumption of Theorem 4.1 for the scaling parameter η and time step size $\triangle t$, we have*

$$
\begin{aligned}
&||d_t\delta_1^N||_0^2 + \eta\sum_{n=1}^{N-1}||d_t\delta_1^{n+1} - d_t\delta_1^n||_0^2 + \eta||d_t\delta_2^N||_0^2 + \eta\sum_{n=1}^{N-1}||d_t\delta_2^{n+1} - d_t\delta_2^n||_0^2 \\
&\quad + \frac{C_1\nu\triangle t}{2}\sum_{n=1}^{N-1}||d_t\delta_1^{n+1}||_1^2 + \frac{\eta kg\triangle t}{4S\nu}\sum_{n=1}^{N-1}||d_t\delta_2^{n+1}||_1^2 \leq C(\triangle t + \triangle t^{-1}h^4).
\end{aligned} \tag{4.24}
$$

Proof. Subtracting equation (4.14)-(4.15) on the two adjacent levels, we have

$$
(\frac{d_t\delta_1^{n+1} - d_t\delta_1^n}{\triangle t}, \mathbf{v}_h) + 2\nu(\mathbb{D}(d_t\delta_1^{n+1}), \mathbb{D}(\mathbf{v}_h)) + b(\mathbf{v}_h, d_t\delta_\mu^{n+1}) = (d_t\theta_u^{n+1}, \mathbf{v}_h)
$$

$$- \frac{g}{\triangle t}\int_{\Gamma_{cm}}(\bar{\phi}^{n+1}-\phi_h^n-\bar{\phi}^n+\phi_h^{n-1})\mathbf{v}_h\cdot\mathbf{n}_{cm}d\Gamma_{cm}$$

$$- \int_{\Gamma_{cm}}\frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\Pi)}}P_\tau(d_t\delta_1^{n+1}+\frac{1}{\triangle t}\mathbb{K}\nabla(\bar{\phi}^{n+1}-\phi_h^n-\bar{\phi}^n+\phi_h^{n-1}))\mathbf{v}_h d\Gamma_{cm}, \tag{4.25}$$

$$\eta(\frac{d_t\delta_2^{n+1}-d_t\delta_2^n}{\triangle t},\psi_h)+\frac{\eta}{S}(\mathbb{K}\nabla d_t\delta_2^{n+1},\nabla\psi_h)=\eta(d_t\theta_\phi^{n+1},\psi_h)$$

$$+ \frac{\eta}{S\triangle t}\int_{\Gamma_{cm}}(\bar{\mathbf{u}}^{n+1}-\mathbf{u}_h^n-\bar{\mathbf{u}}^n+\mathbf{u}_h^{n-1})\cdot\mathbf{n}_{cm}\psi_h d\Gamma_{cm}. \tag{4.26}$$

Taking $\mathbf{v}_h=2\triangle t d_t\delta_1^{n+1}$ and $\psi_h=2\triangle t d_t\delta_2^{n+1}$ and observing that $b(d_t\delta_1^{n+1},\delta_\mu^{n+1}-\delta_\mu^n)=0$, after adding the resulting equations together, using the Korn's inequality (2.17), we arrive at

$$\begin{aligned}
&||d_t\delta_1^{n+1}||_0^2-||d_t\delta_1^n||_0^2+||d_t\delta_1^{n+1}-d_t\delta_1^n||_0^2+4C_1\nu\triangle t||d_t\delta_1^{n+1}||_1^2+\frac{2\nu\alpha\triangle t}{\sqrt{k}}||P_\tau(d_t\delta_1^{n+1})||^2_{L^2(\Gamma_{cm})}\\
&\quad+\eta(||d_t\delta_2^{n+1}||_0^2-||d_t\delta_2^n||_0^2+||d_t\delta_2^{n+1}-d_t\delta_2^n||_0^2)+\frac{2\eta kg\triangle t}{S\nu}||d_t\delta_2^{n+1}||_1^2\\
&\le 2\ \triangle t(d_t\theta_u^{n+1},d_t\delta_1^{n+1})+2\triangle t\eta(d_t\theta_\phi^{n+1},d_t\delta_2^{n+1})\\
&\quad-2g\int_{\Gamma_{cm}}(\bar{\phi}^{n+1}-\phi_h^n+\bar{\phi}^n-\phi_h^{n-1})d_t\delta_1^{n+1}\cdot\mathbf{n}_{cm}d\Gamma_{cm}\\
&\quad+\frac{2\eta}{S}\int_{\Gamma_{cm}}(\bar{\mathbf{u}}^{n+1}-\mathbf{u}_h^n-\bar{\mathbf{u}}^n+\mathbf{u}_h^{n-1})\cdot\mathbf{n}_{cm}d_t\delta_2^{n+1}d\Gamma_{cm}\\
&\quad-2\int_{\Gamma_{cm}}\frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\Pi)}}P_\tau(\mathbb{K}\nabla(\bar{\phi}^{n+1}-\phi_h^n-\bar{\phi}^n+\phi_h^{n-1}))d_t\delta_1^{n+1}d\Gamma_{cm}\\
&\le 2\ \triangle t(d_t\theta_u^{n+1},d_t\delta_1^{n+1})+2\triangle t\eta(d_t\theta_\phi^{n+1},d_t\delta_2^{n+1})\\
&\quad-2g\int_{\Gamma_{cm}}(\bar{\phi}^{n+1}-2\bar{\phi}^n+\bar{\phi}^{n-1})d_t\delta_1^{n+1}\cdot\mathbf{n}_{cm}d\Gamma_{cm}-g\triangle t\int_{\Gamma_{cm}}d_t\delta_2^n d_t\delta_1^{n+1}\cdot\mathbf{n}_{cm}d\Gamma_{cm}\\
&\quad+\frac{2\eta}{S}\int_{\Gamma_{cm}}(\bar{\mathbf{u}}^{n+1}-2\bar{\mathbf{u}}^n+\bar{\mathbf{u}}^{n-1})\cdot\mathbf{n}_{cm}d_t\delta_2^{n+1}d\Gamma_{cm}+\frac{2\eta\triangle t}{S}\int_{\Gamma_{cm}}d_t\delta_1^n\cdot\mathbf{n}_{cm}d_t\delta_2^{n+1}d\Gamma_{cm}\\
&\quad-2\alpha g\sqrt{k}<\nabla_\tau(\bar{\phi}^{n+1}-2\bar{\phi}^n+\bar{\phi}^{n-1}),P_\tau(d_t\delta_1^{n+1})>_{(H_{00}^{1/2}(\Gamma_cm))',H_{00}^{1/2}(\Gamma_cm)}\\
&\quad-2\alpha g\sqrt{k}\triangle t<\nabla_\tau(d_t\delta_2^n),P_\tau(d_t\delta_1^{n+1})>_{(H_{00}^{1/2}(\Gamma_cm))',H_{00}^{1/2}(\Gamma_cm)}
\end{aligned} \tag{4.27}$$

Note that

$$d_t\theta_u^{n+1}=d_t\theta_{u,1}^{n+1}+d_t\theta_{u,2}^{n+1},\quad d_t\theta_\phi^{n+1}=d_t\theta_{\phi,1}^{n+1}+d_t\theta_{\phi,2}^{n+1},$$

where

$$d_t\theta_{u,1}^{n+1}=(P_h-I)\frac{\mathbf{u}(t^{n+1})-2\mathbf{u}(t^n)+\mathbf{u}(t^{n-1})}{\triangle t^2}$$

$$d_t\theta_{\phi,1}^{n+1}=(P_h-I)\frac{\phi(t^{n+1})-2\phi(t^n)+\phi(t^{n-1})}{\triangle t^2}$$

and

$$d_t\theta_{u,2}^{n+1}=\frac{1}{\triangle t^2}\{[\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)-\triangle t\mathbf{u}_t(t^{n+1})]-[\mathbf{u}(t^n)-\mathbf{u}(t^{n-1})-\triangle t\mathbf{u}_t(t^n)]\}$$

$$= -\frac{1}{2}(\mathbf{u}_{tt}(s_{n0}) - \mathbf{u}_{tt}(s_{n1})),$$

$$d_t\theta_{\phi,2}^{n+1} = \frac{1}{\triangle t^2}\{[\phi(t^{n+1}) - \phi(t^n) - \triangle t\phi_t(t^{n+1})] - [\phi(t^n) - \phi(t^{n-1}) - \triangle t\phi_t(t^n)]\}$$

$$= -\frac{1}{2}(\phi_{tt}(s_{n2}) - \phi_{tt}(s_{n3})),$$

for some $s_{n0}, s_{n2} \in (t^n, t^{n+1}), s_{n1}, s_{n3} \in (t^{n-1}, t^n)$, thus

$$\triangle t||d_t\theta_{u,1}^{n+1}||_0^2 = \triangle t \max_{t^{n-1}\le t\le t^{n+1}} ||(P_h - I)\mathbf{u}_{tt}(t)||_0^2. \tag{4.28}$$

Furthermore,

$$\triangle t||d_t\theta_{u,2}^{n+1}||_0^2 \le \frac{\triangle t^3}{4} \max_{t^{n-1}\le t\le t^{n+1}} ||\mathbf{u}_{ttt}(t)||_0^2. \tag{4.29}$$

Similarly,

$$\triangle t||d_t\theta_{\phi,1}^{n+1}||_0^2 = \triangle t \max_{t^{n-1}\le t\le t^{n+1}} ||(P_h - I)\phi_{tt}(t)||_0^2, \tag{4.30}$$

$$\triangle t||d_t\theta_{\phi,2}^{n+1}||_0^2 \le \frac{\triangle t^3}{4} \max_{t^{n-1}\le t\le t^{n+1}} ||\phi_{ttt}(t)||_0^2. \tag{4.31}$$

Furthermore, we will use the following estimate results:

$$||\bar{\phi}^{n+1} - 2\bar{\phi}^n + \bar{\phi}^n||_1 \le ||\phi(t^{n+1}) - 2\phi(t^n) + \phi(t^{n-1})||_1 \le \triangle t^2 \max_{t^{n-1}\le t\le t^{n+1}} ||\phi_{tt}(t)||_1,$$

$$||\bar{\mathbf{u}}^{n+1} - 2\bar{\mathbf{u}}^n + \bar{\mathbf{u}}^{n-1}||_1^2 \le ||\mathbf{u}(t^{n+1}) - 2\mathbf{u}(t^n) + \mathbf{u}(t^{n-1})||_1 \le \triangle t^2 \max_{t^{n-1}\le t\le t^{n+1}} ||\mathbf{u}_{tt}(t)||_1.$$

Now we start to estimate the terms on the right-hand side of (4.27). For the first two terms, using Young and Poincar*é* inequality, together with the above estimate (4.28)-(4.31), we obtain

$$\begin{aligned}
&2\triangle t(d_t\theta_u^{n+1}, d_t\delta_1^{n+1}) + 2\triangle t\eta(d_t\theta_\phi^{n+1}, d_t\delta_2^{n+1})\\
&\le \frac{3C_1\nu\triangle t}{2}||d_t\delta_1^{n+1}||_1^2 + \frac{\eta kg\triangle t}{2S\nu}||d_t\delta_2^{n+1}||_1^2 + \frac{2C(\Omega)\triangle t}{3C_1\nu}||d_t\theta_u^{n+1}||_0^2 + \frac{2\eta S\nu\triangle t}{kg}||d_t\theta_\phi^{n+1}||_0^2\\
&\le \frac{3C_1\nu\triangle t}{2}||d_t\delta_1^{n+1}||_1^2 + \frac{\eta kg\triangle t}{2S\nu}||d_t\delta_2^{n+1}||_1^2\\
&\quad + \frac{2C(\Omega)\triangle t}{3C_1\nu}(\max_{t^{n-1}\le t\le t^{n+1}} ||(P_h - I)\mathbf{u}_{tt}(t)||_0^2 + \frac{\triangle t^2}{4}\max_{t^{n-1}\le t\le t^{n+1}} ||\mathbf{u}_{ttt}(t)||_0^2)\\
&\quad + \frac{2\eta S\nu\triangle t}{kg}(\max_{t^{n-1}\le t\le t^{n+1}} ||(P_h - I)\phi_{tt}(t)||_0^2 + \frac{\triangle t^2}{4}\max_{t^{n-1}\le t\le t^{n+1}} ||\phi_{ttt}(t)||_0^2),
\end{aligned} \tag{4.32}$$

where $C(\Omega)$ is the parameter depending on the domain Ω. For the third term and fifth term on the right-hand side of (4.27), by using trace inequality (2.19), we have

$$\begin{aligned}
&\frac{2\eta}{S}\int_{\Gamma_{cm}} (\bar{\mathbf{u}}^{n+1} - 2\bar{\mathbf{u}}^n + \bar{\mathbf{u}}^{n-1})\cdot \mathbf{n}_{cm} d_t\delta_2^{n+1} d\Gamma_{cm} - 2g\int_{\Gamma_{cm}} (\bar{\phi}^{n+1} - 2\bar{\phi}^n + \bar{\phi}^{n-1}) d_t\delta_1^{n+1}\cdot \mathbf{n}_{cm} d\Gamma_{cm}\\
&\le \frac{2\eta C_3}{S}||\bar{\mathbf{u}}^{n+1} - 2\bar{\mathbf{u}}^n + \bar{\mathbf{u}}^{n-1}||_1||d_t\delta_2^{n+1}||_1 + 2gC_3||\bar{\phi}^{n+1} - 2\bar{\phi}^n + \bar{\phi}^n||_1||d_t\delta_1^{n+1}||_1\\
&\le \frac{\eta kg\triangle t}{2S\nu}||d_t\delta_2^{n+1}||_1^2 + \frac{C_1\nu\triangle t}{4}||d_t\delta_1^{n+1}||_1^2\\
&\quad + \frac{2\eta\nu C_3^2}{gkS\triangle t}||\bar{\mathbf{u}}^{n+1} - 2\bar{\mathbf{u}}^n + \bar{\mathbf{u}}^{n-1}||_1^2 + \frac{4g^2C_3^2}{C_1\nu\triangle t}||\bar{\phi}^{n+1} - 2\bar{\phi}^n + \bar{\phi}^n||_0^2
\end{aligned}$$

$$
\leq \frac{\eta kg\triangle t}{2S\nu}||d_t\delta_2^{n+1}||_1^2 + \frac{C_1\nu\triangle t}{4}||d_t\delta_1^{n+1}||_1^2
$$

$$
+\frac{2\eta\nu C_3^2\triangle t^3}{gkS}\max_{t^{n-1}\leq t\leq t^{n+1}}||\mathbf{u}_{tt}(t)||_1^2 + \frac{4g^2C_3^2\triangle t^3}{C_1\nu}\max_{t^{n-1}\leq t\leq t^{n+1}}||\phi_{tt}(t)||_1^2. \tag{4.33}
$$

From trace inequality (2.18), the forth term and sixth term on the right-hand side of (4.27) can be estimated as follows,

$$
\frac{2\eta\triangle t}{S}\int_{\Gamma_{cm}} d_t\delta_1^n\cdot\mathbf{n}_{cm}d_t\delta_2^{n+1}d\Gamma_{cm} - g\triangle t\int_{\Gamma_{cm}} d_t\delta_2^n d_t\delta_1^{n+1}\cdot\mathbf{n}_{cm}d\Gamma_{cm}
$$

$$
\leq \frac{C_1\nu\triangle t}{2}||d_t\delta_1^n||_1^2 + \frac{\eta gk\triangle t}{2S\nu}||d_t\delta_2^{n+1}||_1^2 + \frac{C_1\nu\triangle t}{2}||d_t\delta_1^{n+1}||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||d_t\delta_2^n||_1^2
$$

$$
+\frac{\eta C_2^2\triangle t}{2\sqrt{S^3C_1gk}}(||d_t\delta_1^n||_0^2 + \eta||d_t\delta_2^{n+1}||_0^2) + \frac{C_2^2\sqrt{Sg^3}\triangle t}{\eta\sqrt{2C_1k}}(||d_t\delta_1^{n+1}||_0^2 + \eta||d_t\delta_2^n||_0^2). \tag{4.34}
$$

For the last two terms on the right-hand side of (4.27), by using imbedding inequality, we obtain

$$
-2\alpha g\sqrt{k} < \nabla_\tau(\bar\phi^{n+1} - 2\bar\phi^n + \bar\phi^{n-1}), P_\tau(d_t\delta_1^{n+1}) >_{(H_{00}^{1/2}(\Gamma_cm))', H_{00}^{1/2}(\Gamma_cm)}
$$

$$
\leq 2\alpha g\sqrt{k}||\bar\phi^{n+1} - 2\bar\phi^n + \bar\phi^{n-1}||_1||d_t\delta_1^{n+1}||_1
$$

$$
\leq \frac{C_1\nu\triangle t}{4}||d_t\delta_1^{n+1}||_1^2 + \frac{4\alpha^2g^2k}{C_1\nu\triangle t}||\bar\phi^{n+1} - 2\bar\phi^n + \bar\phi^{n-1}||_1^2
$$

$$
\leq \frac{C_1\nu\triangle t}{4}||d_t\delta_1^{n+1}||_1^2 + \frac{4\alpha^2g^3k\triangle t^3}{C_1\nu}\max_{t^{n-1}\leq t\leq t^{n+1}}||\phi_{tt}(t)||_1^2. \tag{4.35}
$$

and

$$
-2\alpha g\sqrt{k}\triangle t < \nabla_\tau(d_t\delta_2^n), P_\tau(d_t\delta_1^{n+1}) >_{(H_{00}^{1/2}(\Gamma_cm))', H_{00}^{1/2}(\Gamma_cm)}
$$

$$
\leq 2\alpha g\sqrt{k}\triangle t||d_t\delta_2^n||_1||d_t\delta_1^{n+1}||_1
$$

$$
\leq C_1\nu\triangle t||d_t\delta_1^{n+1}||_1^2 + \frac{\alpha^2g^2k\triangle t}{C_1\nu}||d_t\delta_2^n||_1^2. \tag{4.36}
$$

Combining (4.32)-(4.36) with (4.27), we have

$$
||d_t\delta_1^{n+1}||_0^2 - ||d_t\delta_1^n||_0^2 + ||d_t\delta_1^{n+1} - d_t\delta_1^n||_0^2 + \eta(||d_t\delta_2^{n+1}||_0^2 - ||d_t\delta_2^n||_0^2 + ||d_t\delta_2^{n+1} - d_t\delta_2^n||_0^2)
$$

$$
+\frac{C_1\nu\triangle t}{2}(||d_t\delta_1^{n+1}||_1^2 - ||d_t\delta_1^n||_1^2) + \frac{\eta kg\triangle t}{2S\nu}||d_t\delta_2^{n+1}||_1^2 - (\frac{\eta kg\triangle t}{4S\nu} + \frac{\alpha^2g^2k\triangle t}{C_1\nu})||d_t\delta_2^n||_1^2
$$

$$
\leq \frac{\eta C_2^2\triangle t}{2\sqrt{S^3C_1gk}}(||d_t\delta_1^n||_0^2 + \eta||d_t\delta_2^{n+1}||_0^2) + \frac{C_2^2\sqrt{Sg^3}\triangle t}{\eta\sqrt{2C_1k}}(||d_t\delta_1^{n+1}||_0^2 + \eta||d_t\delta_2^n||_0^2)
$$

$$
+\frac{2C(\Omega)\triangle t}{3C_1\nu}(\max_{t^{n-1}\leq t\leq t^{n+1}}||(P_h - I)\mathbf{u}_{tt}(t)||_0^2 + \frac{\triangle t^2}{4}\max_{t^{n-1}\leq t\leq t^{n+1}}||\mathbf{u}_{ttt}(t)||_0^2)
$$

$$
+\frac{2\eta S\nu\triangle t}{kg}(\max_{t^{n-1}\leq t\leq t^{n+1}}||(P_h - I)\phi_{tt}(t)||_0^2 + \frac{\triangle t^2}{4}\max_{t^{n-1}\leq t\leq t^{n+1}}||\phi_{ttt}(t)||_0^2)
$$

$$
+\frac{2\eta\nu C_3^2\triangle t^3}{gkS}\max_{t^{n-1}\leq t\leq t^{n+1}}||\mathbf{u}_{tt}(t)||_1^2 + \frac{4g^2C_3^2\triangle t^3}{C_1\nu}\max_{t^{n-1}\leq t\leq t^{n+1}}||\phi_{tt}(t)||_1^2
$$

$$
+\frac{4\alpha^2g^2k\triangle t^3}{C_1\nu}\max_{t^{n-1}\leq t\leq t^{n+1}}||\phi_{tt}(t)||_1^2. \tag{4.37}
$$

As the same as Theorem 4.1, assuming that $\eta \geq \frac{4S\alpha^2 g}{C_1}$ and using the above definition of $\tilde{C}$, summing over n from $n = 1$ to $N - 1$, we have

$$
\begin{aligned}
&||d_t\delta_1^N||_0^2 + \sum_{n=1}^{N-1} ||d_t\delta_1^{n+1} - d_t\delta_1^n||_0^2 + \eta||d_t\delta_2^N||_0^2 + \eta \sum_{n=1}^{N-1} ||d_t\delta_2^{n+1} - d_t\delta_2^n||_0^2 \\
&\qquad + \frac{C_1\nu\triangle t}{2}||d_t\delta_1^N||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||d_t\delta_2^N||_1^2 \\
&\leq \tilde{C}\triangle t \sum_{n=1}^{N-1} (||d_t\delta_1^{n+1}||_0^2 + \eta||d_t\delta_2^{n+1}||_0^2) \\
&\quad + ||d_t\delta_1^1||_0^2 + \eta||d_t\delta_2^1||_0^2 + \frac{C_1\nu\triangle t}{2}||d_t\delta_1^1||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||d_t\delta_2^1||_1^2 \\
&\quad + \frac{2C(\Omega)}{3C_1\nu}(\max_{0\leq t\leq T} ||(P_h - I)\mathbf{u}_{tt}(t)||_0^2 + \frac{\triangle t^2}{4} \max_{0\leq t\leq T} ||\mathbf{u}_{ttt}(t)||_0^2) \\
&\quad + \frac{2\eta S\nu}{kg}(\max_{0\leq t\leq T} ||(P_h - I)\phi_{tt}(t)||_0^2 + \frac{\triangle t^2}{4} \max_{0\leq t\leq T} ||\phi_{ttt}(t)||_0^2) \\
&\quad + \frac{2\eta\nu C_3^2\triangle t^2}{gkS} \max_{0\leq t\leq T} ||\mathbf{u}_{tt}(t)||_1^2 + \frac{4g^2C_3^2\triangle t^2}{C_1\nu} \max_{0\leq t\leq T} ||\phi_{tt}(t)||_1^2 \\
&\quad + \frac{4\alpha^2g^2k\triangle t^2}{C_1\nu} \max_{0\leq t\leq T} ||\phi_{tt}(t)||_1^2,
\end{aligned} \tag{4.38}
$$

then it follows from the Gronwall inequality that when $\tilde{C}\triangle t \leq 1$,

$$
\begin{aligned}
&||d_t\delta_1^N||_0^2 + \sum_{n=1}^{N-1} ||d_t\delta_1^{n+1} - d_t\delta_1^n||_0^2 + \eta||d_t\delta_2^N||_0^2 + \eta \sum_{n=1}^{N-1} ||d_t\delta_2^{n+1} - d_t\delta_2^n||_0^2 \\
&\qquad + \frac{C_1\nu\triangle t}{2}||d_t\delta_1^N||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||d_t\delta_2^N||_1^2 \\
&\leq C(T)\{||d_t\delta_1^1||_0^2 + \eta||d_t\delta_2^1||_0^2 + \frac{C_1\nu\triangle t}{2}||d_t\delta_1^1||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||d_t\delta_2^1||_1^2 \\
&\quad + \frac{2C(\Omega)}{3C_1\nu}(\max_{0\leq t\leq T} ||(P_h - I)\mathbf{u}_{tt}(t)||_0^2 + \frac{\triangle t^2}{4} \max_{0\leq t\leq T} ||\mathbf{u}_{ttt}(t)||_0^2) \\
&\quad + \frac{2\eta S\nu}{kg}(\max_{0\leq t\leq T} ||(P_h - I)\phi_{tt}(t)||_0^2 + \frac{\triangle t^2}{4} \max_{0\leq t\leq T} ||\phi_{ttt}(t)||_0^2) \\
&\quad + \frac{2\eta\nu C_3^2\triangle t^2}{gkS} \max_{0\leq t\leq T} ||\mathbf{u}_{tt}(t)||_1^2 + \frac{4g^2C_3^2\triangle t^2}{C_1\nu} \max_{0\leq t\leq T} ||\phi_{tt}(t)||_1^2 \\
&\quad + \frac{4\alpha^2g^2k\triangle t^2}{C_1\nu} \max_{0\leq t\leq T} ||\phi_{tt}(t)||_1^2\}.
\end{aligned} \tag{4.39}
$$

For the four terms on the right-hand side, by using (4.22) with $n = 0$, we have

$$
\begin{aligned}
&(2 - \tilde{C}\triangle t)(||\delta_1^1||_0^2 + \eta||\delta_2^1||_0^2) + \frac{C_1\nu\triangle t}{2}||\delta_1^1||_1^2 + \frac{\eta gk\triangle t}{2S\nu}||\delta_2^1||_1^2 \\
&\qquad \leq \frac{3C(\Omega)\triangle t}{2C_1\nu}||(P_h - I)\mathbf{u}_t||_0^2 + \frac{3C(\Omega)\triangle t^3}{2C_1\nu}||\mathbf{u}_{tt}||_0^2 \\
&\qquad + \frac{2S\nu C(\Omega)\eta\triangle t}{gk}||(P_h - I)\phi_t||_0^2 + \frac{2S\nu C(\Omega)\eta\triangle t^3}{gk}||\phi_{tt}||_0^2 \\
&\qquad + \frac{4Cg^2C_3^2\triangle t^3}{C_1\nu}||\phi_t||_1^2 + \frac{2C\eta\nu C_3^2\triangle t^3}{gkS}||\mathbf{u}_t||_1^2 + \frac{4C\alpha^2g^2k\triangle t^3}{C_1\nu}||\phi_t||_1^2.
\end{aligned} \tag{4.40}
$$

Thus, when $\tilde{C}\triangle t \leq 1$, which means $2 - \tilde{C}\triangle t \geq 1$, by applying the approximate properties of P_h, the above inequality reduces to

$$
\begin{aligned}
&||\delta_1^1||_0^2 + \eta||\delta_2^1||_0^2 + \frac{C_1\nu\triangle t}{2}||\delta_1^1||_1^2 + \frac{\eta gk\triangle t}{2S\nu}||\delta_2^1||_1^2 \\
&\quad \leq \frac{3C(\Omega)\triangle th^4}{2C_1\nu} + \frac{3C(\Omega)\triangle t^3}{2C_1\nu}||\mathbf{u}_{tt}||_0^2 + \frac{2S\nu C(\Omega)\eta\triangle th^4}{gk} + \frac{2S\nu C(\Omega)\eta\triangle t^3}{gk}||\phi_{tt}||_0^2 \\
&\quad + \frac{4Cg^2C_3^2\triangle t^3}{C_1\nu}||\phi_t||_1^2 + \frac{2C\eta\nu C_3^2\triangle t^3}{gkS}||\mathbf{u}_t||_1^2 + \frac{4C\alpha^2g^2k\triangle t^3}{C_1\nu}||\phi_t||_1^2. \qquad (4.41)
\end{aligned}
$$

Thus

$$
\begin{aligned}
&||d_t\delta_1^1||_0^2 + \eta||d_t\delta_2^1||_0^2 + \frac{C_1\nu\triangle t}{2}||d_t\delta_1^1||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||d_t\delta_2^1||_1^2 \\
&\quad = ||\frac{\delta_1^1 - \delta_1^0}{\triangle t}||_0^2 + \eta||\frac{\delta_2^1 - \delta_2^0}{\triangle t}||_0^2 + \frac{C_1\nu\triangle t}{2}||\frac{\delta_1^1 - \delta_1^0}{\triangle t}||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||\frac{\delta_2^1 - \delta_2^0}{\triangle t}||_1^2 \\
&\quad = \frac{1}{\triangle t^2}[||\delta_1^1||_0^2 + \eta||\delta_2^1||_0^2 + \frac{C_1\nu\triangle t}{2}||\delta_1^1||_1^2 + \frac{\eta gk\triangle t}{4S\nu}||\delta_2^1||_1^2] \\
&\quad \leq \hat{C}(\triangle t^{-1}h^4 + \triangle t), \qquad (4.42)
\end{aligned}
$$

where $\hat{C}$ is a constant depends on $S, \nu, g, k, \eta, \alpha, C_1, C_3, \Omega$. Finally, combining (4.42) and the approximate properties of P_h with (4.39), we claim the theorem. □

Theorem 4.3. *(Error for the pressure) Under the assumptions of Theorem 4.1, we have*

$$
||p(t^{n+1}) - p_h^{n+1}||_0 \leq C(\triangle t^{\frac{1}{2}} + \triangle t^{-\frac{1}{2}}h^2). \qquad (4.43)
$$

Proof. From (4.14), we have

$$
\begin{aligned}
b(\mathbf{v}_h, \delta_\mu^{n+1}) &= -(d_t\delta_1^{n+1}, \mathbf{v}_h) - 2\nu(\mathbb{D}(\delta_1^{n+1}), \mathbb{D}(\mathbf{v}_h)) - g\int_{\Gamma_{cm}}(\bar{\phi}^{n+1} - \phi_h^n)\mathbf{v}_h \cdot \mathbf{n}_{cm}d\Gamma_{cm} \\
&\quad + (\theta_u^{n+1}, \mathbf{v}_h) - \int_{\Gamma_{cm}}\frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\Pi)}}P_\tau(\delta_1^{n+1} + \mathbb{K}(\nabla\bar{\phi}^{n+1} - \nabla\phi_h^n)) \cdot \mathbf{v}_hd\Gamma_{cm} \\
&\quad \leq ||\mathbf{v}_h||_1(||d_t\delta_1^{n+1}||_0 + (2\nu + \frac{\nu\alpha}{\sqrt{k}})||\delta_1^{n+1}||_1 + (g + g\alpha\sqrt{k})||\bar{\phi}^{n+1} - \phi_h^n||_1 + ||\theta_u^{n+1}||_0). \qquad (4.44)
\end{aligned}
$$

Therefore, from the discrete inf-sup condition (2.14), it follows that

$$
\begin{aligned}
||\delta_\mu^{n+1}||_0 &\leq C\beta^{-1}(||d_t\delta_1^{n+1}||_0 + (2\nu + \frac{\nu\alpha}{\sqrt{k}})||\delta_1^{n+1}||_1 + (g + g\alpha\sqrt{k})||\bar{\phi}^{n+1} - \phi_h^n||_1 + ||\theta_u^{n+1}||_0) \\
&\leq C\beta^{-1}[||d_t\delta_1^{n+1}||_0 + (2\nu + \frac{\nu\alpha}{\sqrt{k}})||\delta_1^{n+1}||_1 + ||\theta_u^{n+1}||_0 \\
&\quad + (g + g\alpha\sqrt{k})(\triangle t\max_{t^n \leq t \leq t^{n+1}}||\phi_t(t)||_1 + ||\delta_2^n||_1)].
\end{aligned}
$$

By using (4.8)-(4.9) and Theorem 4.1 and Lemma 4.2, we have

$$
||\delta_\mu^{n+1}||_0 \leq C(\triangle t^{1/2} + \triangle t^{-\frac{1}{2}}h^2). \qquad (4.45)
$$

Thus, by using triangle inequality, (4.43) follows from (4.45) and (4.3). □

5. Numerical tests. In this section, we present some results of numerical tests which confirm the theoretical analysis.

Assume $\Omega_m = [0,1] \times [1,2]$ and $\Omega_c = [0,1] \times [0,1]$ with interface $\Gamma_{cm} = (0,1) \times \{1\}$. The exact solution is given by

$$\begin{aligned}
&(u1,u2) = ([x^2(y-1)^2 + y]cos(t), [-\frac{2}{3}x(y-1)^3 + 2 - \pi sin(\pi x)]cos(t)),\\
&p = [2 - \pi sin(\pi x)]sin(0.5\pi y)cos(t),\\
&\phi = [2 - \pi sin(\pi x)][1 - y - cos(\pi y)]cos(t).
\end{aligned}$$

Here the initial conditions, boundary conditions, and the forcing terms follows the solution.

The finite element spaces are constructed by using the well-known MINI elements ($P1b{-}P1$) for the Stokes problem and the linear Lagrangian elements ($P1$) for the Darcy flow. The code was implemented by using the software package FreeFEM++[7]. For the monolithically coupled scheme, the GMRES routine is used to solve the (non-symmetric) coupled system. For the uncoupled scheme, a multi-frontal Gauss LU factorization is implemented to solve the SPD sub-systems. For simplicity, we set $\alpha = 0.1$ and $\eta = 10$.

For the simplicity of notations, we denote $(\mathbf{u}^{h,m}, p^{h,m}, \phi^{h,m})$ the solutions for the monolithically coupled scheme Algorithm 2.1, and accordingly, we denote

$$e_u^{h,m} = \mathbf{u}^{h,m} - \mathbf{u}(t^m), e_p^{h,m} = p^{h,m} - p(t^m), e_\phi^{h,m} = \phi^{h,m} - \phi(t^m).$$

On the other hand, $(\mathbf{u}_h^m, p_h^m, \phi_h^m)$ denotes the solutions for the partitioned scheme Algorithm 3.1, accordingly, we denote

$$e_{h,u}^m = \mathbf{u}_h^m - \mathbf{u}(t^m), e_{h,p}^m = p_h^m - p(t^m), e_{h,\phi}^m = \phi_h^m - \phi(t^m).$$

First, we compare the convergence performance and CPU time for both the coupled scheme and the partitioned scheme. In Table 5.1-5.2, we consider both schemes at time $t^m = 1.0$, with varying mesh h but fixed time step $\triangle t$. Two schemes achieve similar precision, although the monolithically coupled scheme is slightly more accurate than the partitioned scheme. However, the monolithically coupled scheme required much more CPU time than the partitioned scheme. The relative advantage of the partitioned scheme increases as the mesh size decreases. On the other hand, in Table 5.3-5.4, we consider both schemes at the same time $t^m = 1.0$, with varying time step $\triangle t$ but fixed mesh $h = \frac{1}{8}$. Two schemes almost get the same accuracy, but the coupled scheme still needs much more CPU time than the partitioned scheme. In all, we can conclude that the partitioned scheme is comparable with the coupled scheme, but much cheaper and more efficient than the coupled one.

Next, we focus on the partitioned scheme, and demonstrate its orders of convergence with respect to the spacing h and the time step $\triangle t$. Following [12], we introduce a more accurate approach to examine the orders of convergence with respect to the time step $\triangle t$ or the mesh size h due to the approximation errors $O(\triangle t^\gamma) + O(h^\mu)$. For example, assuming

$$\mathbf{v}_h^{\triangle t}(\mathbf{x}, t^m) \approx \mathbf{v}(\mathbf{x}, t^m) + C_1(\mathbf{x}, t^m)\triangle t^\gamma + C_2(\mathbf{x}, t^m)h^\mu,$$

it follows that

$$\begin{aligned}
&\mathbf{v}_h^{\triangle t}(\mathbf{x}, t^m) - \mathbf{v}_h^{\frac{\triangle t}{2}}(\mathbf{x}, t^m) \approx C_1(\mathbf{x}, t^m)(1 - \frac{1}{2^\gamma})\triangle t^\gamma,\\
&\mathbf{v}_h^{\triangle t}(\mathbf{x}, t^m) - \mathbf{v}_{\frac{h}{2}}^{\triangle t}(\mathbf{x}, t^m) \approx C_2(\mathbf{x}, t^m)(1 - \frac{1}{2^\mu})h^\mu.
\end{aligned} \tag{5.1}$$

Thus,

$$\rho_{\mathbf{v},h,i} = \frac{||\mathbf{v}_h^{\triangle t}(\mathbf{x}, t^m) - \mathbf{v}_{\frac{h}{2}}^{\triangle t}(\mathbf{x}, t^m)||_i}{||\mathbf{v}_{\frac{h}{2}}^{\triangle t}(\mathbf{x}, t^m) - \mathbf{v}_{\frac{h}{4}}^{\triangle t}(\mathbf{x}, t^m)||_i} \approx \frac{4^\mu - 2^\mu}{2^\mu - 1} = 2^\mu.$$

$$\rho_{\mathbf{v},\triangle t,i} = \frac{||\mathbf{v}_h^{\triangle t}(\mathbf{x},t^m) - \mathbf{v}_h^{\frac{\triangle t}{2}}(\mathbf{x},t^m)||_i}{||\mathbf{v}_h^{\frac{\triangle t}{2}}(\mathbf{x},t^m) - v_h^{\frac{\triangle t}{4}}(\mathbf{x},t^m)||_i} \approx \frac{4^\gamma - 2^\gamma}{2^\gamma - 1} = 2^\gamma.$$

Here, $\mathbf{v}$ can be $\mathbf{u}$, p or ϕ and i can be 0 or 1. Thus, while $\rho_{\mathbf{v},h,i}$, $\rho_{\mathbf{v},\triangle t,i}$ approach to 4.0 and 2.0, it means that the convergence orders will approach to 2.0 and 1.0, respectively.

Using these definitions, in Table 5.5, we study the convergence orders for the partitioned scheme with a fixed time step $\triangle t = 0.01$ and varying spacing $h = 1/2, 1/4, 1/8, 1/16, 1/32$. Observe from Table 5.5 that $\rho_{u,h,0}$, $\rho_{\phi,h,0}$ is a little larger than 4.0, and $\rho_{u,h,1}$, $\rho_{p,h,0}$, $\rho_{\phi,h,1}$ approach to 2.0, which suggest that the concerned orders of convergence in space for $\mathbf{u}$ and ϕ in L^2-norm are all $O(h^2)$ and in H^1-norm are all $O(h)$, the pressure p in L^2-norm is $O(h)$. However, in Table 5.6, we study the convergence order with a fixed spacing $h = 1/8$ and varying time step $\triangle t = 0.2,\ 0.1,\ 0.05,\ 0.025,\ 0.0125$. The numerical experiments strongly suggest that the orders of convergence in time for all variables should be $O(\triangle t)$, which implies that the error estimates for $\mathbf{u}$ and ϕ in L^2-norm is optimal, however, the error estimates for the H^1-norm of $\mathbf{u}$ and ϕ might not be optimal for the partitioned scheme, and may be further improved from $O(\triangle t^{1/2})$ to $O(\triangle t)$ by a finer analysis, this is an open problem for further work.

Table 5.1

The convergence performance and CPU time of the coupled scheme at time $t^m = 1.0$, with varying mesh h but fixed time step $\triangle t = 0.01$.

h	$\|\|e_u^{h,m}\|\|_0$	$\|\|e_u^{h,m}\|\|_1$	$\|\|e_p^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	CPU
$\frac{1}{2}$	0.267314	1.55801	1.23503	0.153039	1.37635	3.369
$\frac{1}{4}$	0.076279	1.06164	0.91369	0.058034	0.86910	7.569
$\frac{1}{8}$	0.026057	0.44886	0.36556	0.011133	0.38736	30.810
$\frac{1}{16}$	0.022761	0.28795	0.25357	0.003486	0.19702	138.184
$\frac{1}{32}$	0.023625	0.24447	0.21791	0.002030	0.10135	621.395

Table 5.2

The convergence performance and CPU time of the partitioned scheme at time $t^m = 1.0$, with varying mesh h but fixed time step $\triangle t = 0.01$.

h	$\|\|e_u^{h,m}\|\|_0$	$\|\|e_u^{h,m}\|\|_1$	$\|\|e_p^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	CPU
$\frac{1}{2}$	0.267313	1.55801	1.23624	0.153051	1.37634	1.758
$\frac{1}{4}$	0.073254	0.83724	0.57534	0.046875	0.79330	2.625
$\frac{1}{8}$	0.026164	0.46064	0.37295	0.013239	0.40969	12.726
$\frac{1}{16}$	0.022889	0.28703	0.25593	0.003836	0.19585	40.404
$\frac{1}{32}$	0.023669	0.24438	0.21924	0.002398	0.10168	170.097

Table 5.3

The convergence performance and CPU time of the coupled scheme at time $t^m = 1.0$, with varying time step $\triangle t$ but fixed mesh $h = \frac{1}{8}$.

$\triangle t$	$\|\|e_u^{h,m}\|\|_0$	$\|\|e_u^{h,m}\|\|_1$	$\|\|e_p^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	CPU
0.2	0.025843	0.448095	0.358632	0.011945	0.387217	3.057
0.1	0.025958	0.448506	0.362193	0.011494	0.387284	5.242
0.05	0.026019	0.448707	0.364065	0.011297	0.387323	8.986
0.025	0.026045	0.448806	0.364996	0.011190	0.387343	13.728
0.0125	0.026053	0.448848	0.365446	0.011142	0.387353	23.727

Table 5.4

The convergence performance and CPU of the partitioned scheme at time $t^m = 1.0$, with varying time step $\triangle t$ but fixed mesh $h = \frac{1}{8}$.

$\triangle t$	$\|\|e^m_{h,u}\|\|_0$	$\|\|e^m_{h,u})\|\|_1$	$\|\|e^m_{h,p}\|\|_0$	$\|\|e^m_{h,\phi}\|\|_0$	$\|\|e^m_{h,\phi}\|\|_0$	CPU
0.2	0.026371	0.470571	0.402374	0.014093	0.409713	0.873
0.1	0.026215	0.465084	0.386249	0.013490	0.409691	1.248
0.05	0.026176	0.462564	0.378732	0.013205	0.409692	2.168
0.025	0.026168	0.461379	0.375187	0.013068	0.409696	4.352
0.0125	0.026167	0.460808	0.373477	0.012999	0.409699	8.642

Table 5.5

Convergence orders of $O(h^\mu)$ of the partitioned scheme at time $t^m = 1.0$, with varying mesh h but fixed time step $\triangle t = 0.01$.

h	$\|\|u^m_h - u^m_{\frac{h}{2}}\|\|_0$	$\rho_{u,h,0}$	$\|\|u^m_h - u^m_{\frac{h}{2}}\|\|_1$	$\rho_{u,h,1}$	$\|\|p^m_h - p^m_{\frac{h}{2}}\|\|_0$	$\rho_{p,h,0}$
$\frac{1}{2}$	0.215206	3.69422	1.65540	1.91733	1.00526	1.85609
$\frac{1}{4}$	0.058255	3.67166	0.86339	1.89901	0.54160	2.03378
$\frac{1}{8}$	0.015866	3.99297	0.45465	2.06079	0.26630	2.34875
$\frac{1}{16}$	0.003974		0.22062		0.11338	
h	$\|\|\phi^m_h - \phi^m_{\frac{h}{2}}\|\|_0$	$\rho_{\phi,h,0}$	$\|\|\phi^m_h - \phi^m_{\frac{h}{2}}\|\|_0$	$\rho_{\phi,h,1}$		
$\frac{1}{2}$	0.133931	3.40479	1.30793	1.67328		
$\frac{1}{4}$	0.039336	3.61042	0.78165	1.87898		
$\frac{1}{8}$	0.010895	4.85791	0.41599	2.05800		
$\frac{1}{16}$	0.002243		0.20213			

Table 5.6

Convergence orders of $O(\triangle t^\gamma)$ of the partitioned at time $t^m = 1.0$, with varying time step $\triangle t$ but fixed mesh $h = \frac{1}{8}$.

$\triangle t$	$\|\|u^m_{\triangle t} - u^m_{\frac{\triangle t}{2}}\|\|_0$	$\rho_{u,\triangle t,0}$	$\|\|u^m_{\triangle t} - u^m_{\frac{\triangle t}{2}}\|\|_1$	$\rho_{u,\triangle t,1}$	$\|\|p^m_{\triangle t} - p^m_{\frac{\triangle t}{2}}\|\|_0$	$\rho_{p,\triangle t,0}$
0.2	1.82652e-3	1.88614	2.23601e-2	1.87735	3.85061e-2	1.87848
0.1	9.68391e-4	1.94752	1.19104e-2	1.94395	2.04985e-2	1.94289
0.05	4.97242e-4	1.97488	6.12692e-3	1.97327	1.05506e-2	1.97240
0.025	2.51783e-4		3.10496e-3		5.34910e-3	
$\triangle s$	$\|\|\phi^m_{\triangle s} - \phi^m_{\frac{\triangle s}{2}}\|\|_0$	$\rho_{\phi,\triangle s,0}$	$\|\|\phi^m_{\triangle s} - \phi^m_{\frac{\triangle s}{2}}\|\|_0$	$\rho_{\phi,\triangle s,1}$		
0.2	8.55442e-4	2.04615	3.71811e-3	2.02567		
0.1	4.18074e-4	2.02602	1.83549e-3	2.01520		
0.05	2.06352e-4	2.01349	9.10823e-4	2.00804		
0.025	1.02485e-4		4.53588e-4			

At last, It is also of practical interest to compare the effects of the Beavers-Joseph interface conditions with the simplified Beavers-Joseph-Saffman conditions. [12] have studied the decoupled scheme with the simplified Beavers-Joseph-Saffman conditions. Here, for simplicity, we set $\alpha = 0.1$, and solve the Stokes-Darcy problem with simplified Beavers-Joseph-Saffman conditions by using the method provided in [12] and list the experiment results in Table 5.7. Comparing Table 5.7 with Table 5.1, it is easy to see that, while α is small enough, both decoupled scheme in [12] and the partitioned scheme Algorithm 3.1 obtain the good approximation solutions, and the convergence performance are almost similar, which means the Beaver-Joseph interface conditions are also reasonable.

Table 5.7

The convergence performance of the partitioned scheme in [12] with simplified Beavers-Joseph-Saffman condition at time $t^m = 1.0$, with varying mesh size but fixed time step $\triangle t = 0.01$.

h	$\|\|e_u^{h,m}\|\|_0$	$\|\|e_u^{h,m}\|\|_1$	$\|\|e_p^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	CPU
$\frac{1}{2}$	0.267313	1.55801	1.23624	0.153051	1.37634	2.087
$\frac{1}{4}$	0.073890	0.83457	0.58647	0.046805	0.79332	4.407
$\frac{1}{8}$	0.025638	0.44917	0.32231	0.012886	0.40971	11.969
$\frac{1}{16}$	0.021387	0.25669	0.20009	0.003190	0.19572	43.932
$\frac{1}{32}$	0.021871	0.20074	0.16927	0.001629	0.10131	177.778

6. conclusions. In this report, we propose a partitioned time stepping method for the fully evolutionary Stokes-Darcy problem with Beavers-Joseph interface condition. we conclude that if we choose the scaling parameter η large enough and the time step $\triangle t$ small enough, then the partitioned method is stable and convergent.

REFERENCES

[1] J. Bear, *Dynamics of Fluids in Porous Media*, Dover, 1972.

[2] G. Beavers and D. Joseph, *Boundary conditions at a naturally permeable wall*, J. Fluid Mech., 30(1967), pp. 197-207.

[3] Y. Cao, M. Gunzburger, X. Hu, F. Hua, X. Wang and W. Zhao, *Finite element approxiations for Stokes-Darcy model with Beavers-Joseph interface boundary condition*, SIAM J. Numer. Anal. 6(2010), pp. 4239-4256.

[4] Y. Cao, M. Gunzburger, F. Hua and X. Wang, *Coupled Stokes-Darcy model with Beavers-Joseph interface boundary condition*, Commun. Math. Sci., 8(2010), pp. 1-25.

[5] M. Discacciati, E. Miglio and A. Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows*, Appl. Numer. Math., 43(2002), pp. 57-74.

[6] M. Discacciati, A. Quarteroni and A. Valli, *Robin-Robin domain decomposition methods for the Stokes-Darcy coupling*, SIAM J. Numer. Anal., 45(2007), pp. 1246-1268.

[7] F. Hecht, O. Pironneau, and K. Ohtsuka, FreeFEM++, http://www.freefem.org/ff++/ftp/ (2010).

[8] I. P. Jones, Reynolds number flow past a porous spherical shell, Proc. Camb. Phil. Soc., 73, 231-238, 1973.

[9] J.-L.Lions and E. Magenes, *Problémes aux limites nonhomogènes at applications*, Vol. 1, Dunod, Paris, 1968.

[10] W. J. Layton, F. Schieweck and I. Yotov, *Coupling fluid flow with porous media flow*, SIAM J. Numer. Anal., 40(2003), pp. 2195-2218.

[11] M. Mu and J. Xu, *A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow*, SIAM J. Numer. Anal., 45(2007), pp. 1801-1813.

[12] M. Mu and X. H. Zhu, *Decoupled schemes for a non-stationary mixed Stokes-Darcy model*, Math. Comput., 79(2010), pp. 707-731.

[13] B. Rivière and I. Yotov, *Locally conservative coupling of Stokes and Darcy flows*, SIAM J. Numer. Anal., 42(2005), pp. 1959-1977.

[14] P. Saffman, *On the boundary condition at the interface of a porous medium*, Stud. in Appl. Math., 1, 77-84, 1971.