Theory of the NS-$\overline{\omega}$ model: A complement to the NS-α model

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Abstract

We study a new regularization of the Navier-Stokes equations, the NS-$\overline{\omega}$ model. This model has similarities to the NS-α model but its structure is more amenable to be used as a basis for numerical simulations of turbulent flows. In this report we present the model and prove existence and uniqueness of strong solutions as well as convergence (modulo a subsequence) to a weak solution of the Navier-Stokes equations as the averaging radius decreases to zero. We then apply turbulence phenomenology to the model to obtain insight into its predictions.

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1 Introduction

This report develops the theory for a regularization of the Navier-Stokes equations (NSE), the NS-$\overline{\omega}$ model, and establishes a Leray-type theory. The same techniques give a theory of a related regularization, the NS-$\alpha\&\omega$ model (1.2) below. We also apply turbulence phenomenology to obtain some insight into the three model's differences. The two related models emerge naturally from

1. the interpretation of the well-known NS-α model as a rotational Leray regularization and
2. practical consideration about the cost of a numerical simulation of the three (NS-α, NS-$\overline{\omega}$, NS-$\alpha\&\omega$).

The first regularization averages the vorticity term instead of the velocity (hence calling it the *NS-$\overline{\omega}$ model* seems natural). It is given by

$$\mathbf{u}_t + \overline{\boldsymbol{\omega}} \times \mathbf{u} - \nu \Delta \mathbf{u} + \nabla P + \chi(\mathbf{u} - \overline{\mathbf{u}}) = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{where } \overline{\boldsymbol{\omega}} := \nabla \times \overline{\mathbf{u}}. \tag{1.1}$$

Here α denotes the (user-selected) filter lengthscale and overbar denotes a filtered quantity, $\overline{\Phi} = g_\alpha \star \Phi$. For simplicity, we select a differential filter (Germano [17]), so $\overline{\Phi} := (-\alpha^2 \Delta + 1)^{-1} \Phi$. The time relaxation term $\chi(\mathbf{u} - \overline{\mathbf{u}})$, an idea of Stolz, Adams and Kleiser, e.g., [38] is added to truncate scales, [29]. For example, picking $\chi \simeq \alpha^{-2/3}$ yields the microscale of (1.1) to be $\eta_\omega \simeq \alpha$, see [29] and Section 4 (4.27).

Clearly, one can also study combinations of NS-α and NS-$\overline{\omega}$ such as NS-$\alpha\&\omega$, given by

$$\mathbf{u}_t + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{u}} - \nu \Delta \mathbf{u} + \nabla P + \chi(\mathbf{u} - \overline{\mathbf{u}}) = \mathbf{f}, \text{ and } \nabla \cdot \overline{\mathbf{u}} = 0\,, \quad \text{where} \tag{1.2}$$
$$\overline{\boldsymbol{\omega}} = \nabla \times \overline{\mathbf{u}}.$$

If $\nu = \chi = \mathbf{f} = 0$ and under appropriate boundary conditions the NS-α, NS-$\overline{\omega}$, NS-$\alpha\&\omega$ conserve a model energy and helicity given, respectively, by

$$\begin{aligned}
&\text{for NS-}\alpha, && E_\alpha(\mathbf{u}) = \frac{1}{2|\Omega|}\int_\Omega \mathbf{u}\cdot\overline{\mathbf{u}}\,d\mathbf{x}, \quad H_\alpha(\mathbf{u}) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}\cdot\nabla\times\mathbf{u}\,d\mathbf{x},\\
&\text{for NS-}\overline{\omega}, && E_\omega(\mathbf{u}) = \frac{1}{2|\Omega|}\int_\Omega \mathbf{u}\cdot\mathbf{u}\,d\mathbf{x}, \quad H_\omega(\mathbf{u}) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}\cdot\nabla\times\overline{\mathbf{u}}\,d\mathbf{x}, \quad \text{and}\\
&\text{for NS-}\alpha\&\omega, && E_{\alpha\&\omega}(\mathbf{u}) = \frac{1}{2|\Omega|}\int_\Omega \mathbf{u}\cdot\overline{\mathbf{u}}\,d\mathbf{x}, \quad H_{\alpha\&\omega}(\mathbf{u}) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}\cdot\nabla\times\overline{\mathbf{u}}\,d\mathbf{x}.
\end{aligned}$$

The differences among the forms of E and H suggest similarities and differences in the three models' phenomenologies, Section 4. There is also a difference in the treatment of the incompressibility condition in the non-periodic case, for the NS-$\overline{\omega}$, $\nabla\cdot\mathbf{u} = 0$ while for NS-α and NS-$\alpha\&\omega$, $\nabla\cdot\overline{\mathbf{u}} = 0$ is imposed. See [40] for related issues.

Concerning the precise formulation of the nonlinear term in (1.1): there is no difference between $\overline{\nabla\times\mathbf{u}}$ and $\nabla\times\overline{\mathbf{u}}$ in the periodic case. However, in the non-periodic case, computing $\nabla\times\overline{\mathbf{u}}$ requires boundary conditions for the velocity (which are known) whereas $\overline{\nabla\times\mathbf{u}}$ requires boundary conditions for the vorticity, which are typically unknown.

One main motivation for (1.1) is that regularizations are often used for numerical simulation of turbulent flow. Thus, two critical features of any regularization model are (i) its solutions must faithfully represent the qualitative properties of solutions of the NSE, and (ii) it must be amenable to efficient numerical simulation with robust methods. As an example of the motivation for the NS-$\overline{\omega}$ model, consider two robust, fully discrete, unconditionally stable algorithms, the CN (Crank-Nicolson) and CNLE (Crank-Nicolson with linear extrapolation) methods. (The numerical analysis of these two is developed in [27] for the NS-$\overline{\omega}$ model and [7], [12] and [33] for the NS-α model.) Suppressing the spatial discretization, the unconditionally stable realization of the CN method for the NS-α model reads

$$\begin{cases} \dfrac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} + \left(\nabla \times \mathbf{u}_{n+1/2}\right) \times \Phi_{n+1/2} - \nu \triangle \mathbf{u}_{n+1/2} + \nabla P_{n+1/2} = \mathbf{f}_{n+1/2}, \\ -\alpha^2 \Delta \Phi_{n+1/2} + \Phi_{n+1/2} = \mathbf{u}_{n+1/2}, \\ \nabla \cdot \Phi_{n+1} = 0. \end{cases} \tag{1.3}$$

From (1.3), each time step requires the solution of a large ($(\mathbf{u}, \Phi, p)$: $2N_{velocity} + N_{pressure}$ unknowns) coupled *nonlinear* system at every time step. We know of no other method for NS-α which is substantially more economical, $O(\Delta t^2)$ accurate and unconditionally stable. The NS-$\overline{\omega}$ model can be solved much more economically by unconditionally stable, second order accurate and linearly implicit CNLE method (adapted from [2] in [27]), given by

$$\frac{1}{\Delta t}(\mathbf{u}_{n+1} - \mathbf{u}_n) + (\frac{3}{2}\overline{\boldsymbol{\omega}}_n - \frac{1}{2}\overline{\boldsymbol{\omega}}_{n-1}) \times \frac{\mathbf{u}_{n+1}+\mathbf{u}_n}{2} + \nabla P_{n+1/2} \\ - \nu \triangle \frac{\mathbf{u}_{n+1}+\mathbf{u}_n}{2} + \chi(\mathbf{u}_{n+1} - \overline{\mathbf{u}}_n) = \mathbf{f}_{n+1/2}, \tag{1.4}$$

$$\text{where } \overline{\boldsymbol{\omega}}_n = \nabla \times \overline{\mathbf{u}}_n \tag{1.5}$$

$$\text{and } \nabla \cdot \mathbf{u}_{n+1} = 0. \tag{1.6}$$

Since $\frac{3}{2}\overline{\boldsymbol{\omega}}_n - \frac{1}{2}\overline{\boldsymbol{\omega}}_{n-1} = \overline{\boldsymbol{\omega}}(t_{n+1/2}) + O(\Delta t^2)$, (1.4)-(1.6) is a $O(\Delta t^2)$ perturbation of the Crank-Nicolson method. The method filters explicitly known velocities $\mathbf{u}_n$ and $\mathbf{u}_{n-1}$, which reduces storage and computational cost substantially. CNLE method (1.4)-(1.6) is second order accurate, unconditionally and nonlinearly stable and requires only the solution of one smaller ($(\mathbf{u}, p)$: $N_{velocity} + N_{pressure}$ unknowns) *linear* system at every time step.

The ideas we study are outgrowths of the seminal work of J. Leray [30, 31], of Geurts and Holm [18, 19] on the Leray regularization as a basis for numerical simulations, the extensive recent work on the theory of NS-α model, the early work of G. Baker [2] on CNLE= Crank-Nicolson with Linear Extrapolation methods and our previous work on the numerical analysis of approximate deconvolution models of turbulence, e.g., [23].

In Section 2 we give the notation and definitions necessary for the analysis. Section 3 develops the theory for the model. Model phenomenology is presented in Section 4, followed by conclusions.

1.1 Related models

In 1934, J. Leray [30, 31] introduced the following NSE regularization (now known as the Leray model) as a theoretical tool:

$$\mathbf{u}_t + \overline{\mathbf{u}} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ and } \nabla \cdot \mathbf{u} = 0\,, \quad \text{in } \Omega \times (0, T). \tag{1.7}$$

He chose $\overline{\mathbf{u}} = g_\alpha \star \mathbf{u}$, where g_α is a Gaussian filter associated with a length scale α. He proved existence and uniqueness of strong solutions to (1.7) and that a subsequence $\mathbf{u}_{\alpha_j}$ converges to a weak solution of the NSE as $\alpha_j \to 0$. If that weak solution is a smooth, strong solution it is not difficult to prove additionally that $\|\mathbf{u}_{NSE} - \mathbf{u}_{LerayModel}\| = O(\ \alpha^2)$ using only $\|\mathbf{u} - \overline{\mathbf{u}}\| = O(\alpha^2)$.

This model has been explored in numerical experiments. Like all models it has advantages [18] and disadvantages [20]. The higher order Leray deconvolution family of models, which includes (1.7) as $N = 0$, also seems very promising [26, 24]. The Camassa-Holm / Navier-Stokes-alpha has attracted much interest since it conserves both important integral invariants of the Euler equations (a modified energy and helicity [36]) and a complete and elegant theory of NS-α has been developed, e.g. [9]. The NS-α model is given by

$$\mathbf{u}_t + \boldsymbol{\omega} \times \overline{\mathbf{u}} - \nu \Delta \mathbf{u} + \nabla P = \mathbf{f}, \quad \nabla \cdot \overline{\mathbf{u}} = 0, \qquad \text{and } \omega = \nabla \times \mathbf{u}. \tag{1.8}$$

Recently, Ilyin, Lunasin and Titi [21] have studied a complement to the Leray regularization (in which $\overline{\mathbf{u}}\cdot\nabla\mathbf{u}$ is replaced by $\mathbf{u}\cdot\nabla\overline{\mathbf{u}}$ in (1.7)). This complementary treatment of (another form of) the NSE nonlinearity is analogous to the treatment in NS-$\overline{\omega}$ model (1.1).

2 Notation and Preliminaries

The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space $W_2^k(\Omega)$, and $\|\cdot\|_k$ denotes the norm in H^k. For functions $\mathbf{v}(\mathbf{x},t)$ defined on the entire time interval $(0,T)$, we define $(1\le m<\infty)$

$$\|\mathbf{v}\|_{\infty,k} := \operatorname{ess}\sup_{0<t<T}\|\mathbf{v}(\cdot,t)\|_k\,, \qquad \text{and} \qquad \|\mathbf{v}\|_{m,k} := \left(\int_0^T \|\mathbf{v}(\cdot,t)\|_k^m\,dt\right)^{1/m}.$$

We consider the periodic case. In the periodic case, $\Omega=(0,L)^d$, $d=2,3$ and pressure and velocity spaces are, respectively,

$$Q = L_0^2(\Omega) := \{q\in L^2(\Omega):\ \int_\Omega q\ d\mathbf{x}=0\},$$
$$X = H_\#^1(\Omega) := \{\mathbf{v}\in H^1(\Omega)\cap L_0^2(\Omega):\ \mathbf{v}\text{ is } L\text{-periodic }\},$$

while for $k>1$

$$H_\#^k(\Omega) := \{\mathbf{v}\in H_\#^1(\Omega):\ \frac{\partial^{|\alpha|}\mathbf{v}}{\partial\mathbf{x}^\alpha}\text{ is } L\text{-periodic for } 0\le|\alpha|\le k-1\}.$$

We denote the dual space of X by $X^\star$, with the norm $\|\cdot\|_\star$. The spaces of divergence free functions are denoted

$$H = H_{div} := \{\mathbf{v} \in L_0^2(\Omega) : \nabla \cdot \mathbf{v} = 0\} \quad \text{and}$$
$$V := \{\mathbf{v} \in X : (\nabla \cdot \mathbf{v}, q) = 0 \ \ \forall q \in Q\}.$$

Differential filters were introduced into turbulence modeling by Germano [17, 16] and used in the analysis of approximate deconvolution models, the rational model [15] and in NS-α [41, 10, 21, 18, 19]. They can arise, for example, as approximations to Gaussian filters of high qualitative and quantitative accuracy [15].

Definition 2.1 (Continuous differential filter)**.** *For $\boldsymbol{\phi} \in L^2(\Omega)$ and $\alpha > 0$ fixed, denote the filtering operation on $\boldsymbol{\phi}$ by $\overline{\boldsymbol{\phi}}$, where $\overline{\boldsymbol{\phi}}$ is the unique solution (in X) of*

$$-\alpha^2 \Delta \overline{\boldsymbol{\phi}} + \overline{\boldsymbol{\phi}} = \boldsymbol{\phi}. \tag{2.1}$$

We denote by $A := (-\alpha^2 \Delta + I)$, so $A^{-1}\mathbf{v} = \overline{\mathbf{v}}$.

Remark 2.2. *For non-rotational formulations of the nonlinearity, an alternative is to define the differential filter by a discrete Stokes problem so as to preserve incompressibility approximately. In this case, given $\boldsymbol{\phi} \in V$, $\overline{\boldsymbol{\phi}} \in V$ would be defined by*

$$\alpha^2(\nabla \overline{\boldsymbol{\phi}}, \nabla \boldsymbol{\chi}) + (\overline{\boldsymbol{\phi}}, \boldsymbol{\chi}) = (\boldsymbol{\phi}, \boldsymbol{\chi}) \text{ for all } \boldsymbol{\chi} \in V\,. \tag{2.2}$$

After discretization, this results in a computationally more expensive filtering operation. Herein we consider (2.1).

3 Analysis

We define the operator $\mathcal{A} \in L(V, V')$ by setting

$$\langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}), \tag{3.1}$$

for all $\mathbf{u}, \mathbf{v} \in V$. The operator $\mathcal{A}$ is an unbounded operator on $L^2(\Omega)$, with the domain $D(\mathcal{A}) = \{\mathbf{u} \in V : \Delta \mathbf{u} \in L^2(\Omega)\}$ and we denote again by $\mathcal{A}$ its restriction to $L^2(\Omega)$. We define also a continuous bilinear operator $B_\omega : X \to X'$ with

$$\langle B_\omega \mathbf{u}, \mathbf{v} \rangle = (\overline{\boldsymbol{\omega}} \times \mathbf{u}, \mathbf{v}),$$

for all $\mathbf{u}, \mathbf{v} \in X$. The following property holds

$$\langle B_\omega \mathbf{u}, \mathbf{u} \rangle = \langle B_\omega \mathbf{u}, \overline{\boldsymbol{\omega}} \rangle = 0, \tag{3.2}$$

for all $\mathbf{u}, \overline{\boldsymbol{\omega}} \in V$. In terms of $X, \mathcal{A}, B_\omega$ we can rewrite (1.1) as

$$\begin{aligned} &\frac{d\mathbf{u}}{dt}(t) + \mathcal{A}\mathbf{u}(t) + B_\omega \mathbf{u}(t) + \chi(\mathbf{u} - \overline{\mathbf{u}}) = \mathbf{f}(t), \quad t \in (0, T), \\ &\mathbf{u}(0) = \mathbf{u}_0, \end{aligned} \tag{3.3}$$

where $\mathbf{f} = \Pi \mathbf{f}$, and $\Pi : L^2(\Omega) \to H$ is the Leray-Hodge projection [11], [37].

3.1 Stability and existence for the model

The first result states that the strong solution of the model (3.3) exists globally in time, for large data and general $\nu > 0$ and that it satisfies an energy equality, while initial data and the source terms are smooth enough.

Theorem 3.1. *Consider the NS-$\overline{\omega}$ model (1.1). Let $\alpha > 0$ be fixed. For any $\mathbf{u}_0 \in V$ and $\mathbf{f} \in L^2(0, T; H)$, there exists a unique strong solution $\mathbf{u}$ to (3.3),*

$\mathbf{u} \in L^\infty(0,T;H^1_\#(\Omega)) \cap L^2(0,T;H^2_\#(\Omega))$ *and* $\mathbf{u}_t \in L^2((0,T)\times\Omega)$. *Moreover, the following energy equality holds for* $t \in [0,T]$*:*

$$E_\omega(t) + \int_0^t \epsilon_\omega(\tau)d\tau = E_\omega(0) + \int_0^t P_\omega(\tau)d\tau, \tag{3.4}$$

where

$$\begin{aligned} E_\omega(t) &= \frac{1}{2|\Omega|}\|\mathbf{u}(\cdot,t)\|^2, \\ \epsilon_\omega(t) &= \frac{\nu}{|\Omega|}\|\nabla\mathbf{u}(\cdot,t)\|^2 + \frac{\chi}{|\Omega|}(\mathbf{u}-\overline{\mathbf{u}},\mathbf{u}), \\ P_\omega(t) &= \frac{1}{|\Omega|}(\mathbf{f}(t),\mathbf{u}(t)). \end{aligned} \tag{3.5}$$

Before proving Theorem 3.1 we collect a few preliminaries. We shall use the semigroup approach proposed in [5] for the Navier-Stokes equations, based on the machinery of nonlinear differential equations of accretive type in Banach spaces.

Let us define the modified nonlinearity, $B^N_\omega : X \to X^*$, by setting

$$B^N_\omega(\mathbf{u}) = \begin{cases} B_\omega(\mathbf{u}), & \text{if } \|\nabla\mathbf{u}\| \le N, \\ \left(\frac{N}{\|\nabla\mathbf{u}\|}\right)^2 B_\omega(\mathbf{u}), & \text{if } \|\nabla\mathbf{u}\| > N. \end{cases} \tag{3.6}$$

By (2.1), we have for the case of $\|\nabla\mathbf{u}_1\|, \|\nabla\mathbf{u}_2\| \le N$

$$\begin{aligned} &\left|\langle B^N_\omega(\mathbf{u}_1) - B^N_\omega(\mathbf{u}_2), \mathbf{u}_1-\mathbf{u}_2\rangle\right| = (\overline{\boldsymbol{\omega}_1-\boldsymbol{\omega}_2}\times\mathbf{u}_1, \mathbf{u}_1-\mathbf{u}_2) + (\overline{\boldsymbol{\omega}_2}\times(\mathbf{u}_1-\mathbf{u}_2), \mathbf{u}_1-\mathbf{u}_2) \\ &\le C_0(\Omega)\|\mathbf{u}_1-\mathbf{u}_2\|_{\frac{1}{2}}\left(\|\nabla\mathbf{u}_1\|+\|\nabla\mathbf{u}_2\|\right)\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\| \le \frac{\nu}{2}\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|^2 + C_N\|\mathbf{u}_1-\mathbf{u}_2\|^2. \end{aligned}$$

In the case of $\|\nabla \mathbf{u}_1\|, \|\nabla \mathbf{u}_2\| > N$, we have by (3.2)

$$
\begin{aligned}
&\left|\langle B_\omega^N(\mathbf{u}_1) - B_\omega^N(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2\rangle\right| \\
&= \frac{N^2}{\|\nabla \mathbf{u}_1\|^2}(\overline{\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2} \times \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) \\
&\qquad\qquad + \left(\frac{N^2}{\|\nabla \mathbf{u}_1\|^2} - \frac{N^2}{\|\nabla \mathbf{u}_2\|^2}\right)(\overline{\boldsymbol{\omega}_2} \times (\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2) \\
&\le C_0(\Omega) N \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^{\frac{3}{2}} \|\mathbf{u}_1 - \mathbf{u}_2\|^{\frac{1}{2}} + C_0(\Omega) N \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2 \\
&\le \frac{\nu}{2}\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2 + C_N \|\mathbf{u}_1 - \mathbf{u}_2\|^2.
\end{aligned}
$$

For the case of $\|\nabla \mathbf{u}_2\| \le N < \|\nabla \mathbf{u}_1\|$ (similar estimates are obtained when $\|\nabla \mathbf{u}_1\| \le N < \|\nabla \mathbf{u}_2\|$) we have

$$
\begin{aligned}
&\left|\langle B_\omega^N(\mathbf{u}_1) - B_\omega^N(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2\rangle\right| \\
&= \frac{N^2}{\|\nabla \mathbf{u}_1\|^2}(\overline{\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2} \times \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - \left(1 - \frac{N^2}{\|\nabla \mathbf{u}_1\|^2}\right)(\overline{\boldsymbol{\omega}_2} \times \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \\
&\le C_0(\Omega) N \left(\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^{\frac{3}{2}} \|\mathbf{u}_1 - \mathbf{u}_2\|^{\frac{1}{2}} + \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\| \|\mathbf{u}_1 - \mathbf{u}_2\|_{\frac{1}{2}}\right) \\
&\le \frac{\nu}{2}\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2 + C_N \|\mathbf{u}_1 - \mathbf{u}_2\|^2.
\end{aligned}
$$

Combining all the cases above we conclude that

$$
\left|\langle B_\omega^N(\mathbf{u}_1) - B_\omega^N(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2\rangle\right| \le \frac{\nu}{2}\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2 + C_N \|\mathbf{u}_1 - \mathbf{u}_2\|^2. \tag{3.7}
$$

The operator B_ω^N is continuous from X to X^*. Indeed, as above we have

$$
\begin{aligned}
&\left|\langle B_\omega^N(\mathbf{u}_1) - B_\omega^N(\mathbf{u}_2), \mathbf{u}_3\rangle\right| \qquad (3.8) \\
&\le |(\overline{\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2} \times \mathbf{u}_1, \mathbf{u}_3)| + |(\overline{\boldsymbol{\omega}_2} \times (\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_3)| \le C_N \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\| \|\nabla \mathbf{u}_3\|.
\end{aligned}
$$

Now consider the operator $\Gamma_N : D(\Gamma_N) \to H$ defined by

$$\Gamma_N = \mathcal{A} + B_\omega^N, \quad D(\Gamma_N) = D(\mathcal{A}),$$

since

$$\left\|B_\omega^N(\mathbf{u})\right\|_0 \leq C_0(\Omega)\|\nabla\mathbf{u}\|^{3/2}\left\|\mathcal{A}\,\overline{\mathbf{u}}\right\|^{1/2} \leq C_N\left\|\mathcal{A}\mathbf{u}\right\|^{1/2}, \qquad \forall \mathbf{u} \in D(\mathcal{A}). \quad (3.9)$$

Lemma 3.2. *There exists $\alpha_N > 0$ such that $\Gamma_N + \alpha_N I$ is m-accretive (maximal monotone) in $H \times H$.*

Proof. By (3.7) we have that

$$((\Gamma_N + \lambda)(\mathbf{u}_1) - (\Gamma_N + \lambda)(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2) \geq \frac{\nu}{2}\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2, \text{ for all } \mathbf{u}_i \in D(\Gamma_N), \quad (3.10)$$

for $\lambda \geq C_N$. Next we consider the operator

$$F_N(\mathbf{u}) = \mathcal{A}\mathbf{u} + B_\omega^N(\mathbf{u}) + \alpha_N\mathbf{u}, \quad \text{for all } \mathbf{u} \in D(F_N),$$

with

$$D(F_N) = \{\mathbf{u} \in V : \mathcal{A}\mathbf{u} + B_\omega^N(\mathbf{u}) \in H\}.$$

By (3.8) and (3.10) we see that F_N is monotone, coercive and continuous from V to V'. We infer that F_N is maximal monotone from V to V' and the restriction to H is maximal monotone on H with the domain $D(F_N) \supseteq D(A)$ (see e.g. [6, 3]). Moreover, we have $D(F_N) = D(\mathcal{A})$. For this we use the perturbation theorem for nonlinear m-accretive operators and split F_N into a continuous and a $\boldsymbol{\omega}$-m-accretive operator on H

$$F_N^1 = (1 - \frac{\varepsilon}{2})\mathcal{A}, \quad D(F_N^1) = D(\mathcal{A}),$$
$$F_N^2 = \frac{\varepsilon}{2}\mathcal{A} + B_\omega^N + \alpha_N I, \quad D(F_N^2) = \{\mathbf{u} \in V : F_N^2\mathbf{u} \in H\}.$$

As seen above, by (3.9) we have

$$\begin{aligned} \|F_N^2\mathbf{u}\| &\leq \frac{\varepsilon}{2}\|\mathcal{A}\mathbf{u}\| + \|B_\omega^N(\mathbf{u})\| + \alpha_N\|\mathbf{u}\| \\ &\leq \varepsilon\|\mathcal{A}\mathbf{u}\| + \alpha_N\|\mathbf{u}\| + \frac{C_N^2}{2\varepsilon}, \quad \text{for all } \mathbf{u} \in D(F_N^1) = D(\mathcal{A}), \end{aligned}$$

where $0 < \varepsilon < 1$. Since $F_N^1 + F_N^2 = \Gamma_N + \alpha_N I$ we infer that $\Gamma_N + \alpha_N I$ with domain $D(\mathcal{A})$ is m-accretive in H, as claimed. □

Proof of Theorem 3.1. Since the χ term is lower order nonnegative and linear, it suffices to give the proof for $\chi = 0$. As a consequence of Lemma 3.2 (see, e.g., [3, 4]) we have that for $\mathbf{u}_0 \in D(\mathcal{A})$ and $\mathbf{f} \in W^{1,1}([0,T], H)$ the equation

$$\begin{aligned} &\frac{d\mathbf{u}}{dt}(t) + \mathcal{A}\mathbf{u}(t) + B_\omega^N(\mathbf{u}(t)) = \mathbf{f}, \;\; t \in (0,T), \\ &\mathbf{u}(0) = \mathbf{u}_0, \end{aligned} \tag{3.11}$$

has a unique strong solution $\mathbf{u}_N \in W^{1,\infty}([0,T]; H) \cap L^\infty(0,T; D(\mathcal{A}))$.

If we multiply (3.11) by $\mathbf{u}_N$, use (3.2) and integrate in time we obtain

$$\|\mathbf{u}_N(t)\|^2 + \nu\int_0^t \|\nabla\mathbf{u}_N(s)\|^2 \leq \|\mathbf{u}_0\|^2 + \frac{1}{\nu}\int_0^t \|\mathbf{f}(s)\|_{V'}^2 ds. \tag{3.12}$$

Multiplying (3.11) by $-\Delta \mathbf{u}_N$, by use of (2.1) we have

$$
\begin{aligned}
&\frac{d}{2dt}\|\nabla \mathbf{u}_N(t)\|^2 + \nu\|\Delta \mathbf{u}_N(t)\|^2 = \langle \mathbf{f}(t), \Delta \mathbf{u}_N(t)\rangle + \langle \overline{\boldsymbol{\omega}}_N \times \mathbf{u}_N, \Delta \mathbf{u}_N\rangle \\
&\leq \|\mathbf{f}(t)\|\|\Delta \mathbf{u}_N(t)\| + \|\overline{\boldsymbol{\omega}}_N\|_{1/2}\|\nabla \mathbf{u}_N\|\|\Delta \mathbf{u}_N\| \\
&\leq \frac{1}{\nu}\|\mathbf{f}\|^2 + \frac{\nu}{4}\|\Delta \mathbf{u}_N\|^2 + \|\overline{\boldsymbol{\omega}}_N\|^{1/2}\|\nabla \overline{\boldsymbol{\omega}}_N\|^{1/2}\|\nabla \mathbf{u}_N\|\|\Delta \mathbf{u}_N\| \\
&\leq \frac{1}{\nu}\|\mathbf{f}\|^2 + \frac{\nu}{4}\|\Delta \mathbf{u}_N\|^2 + \|\nabla \overline{\mathbf{u}}_N\|^{1/2}\|\nabla \overline{\mathbf{u}}_N\|_1^{1/2}\|\nabla \mathbf{u}_N\|\|\Delta \mathbf{u}_N\| \\
&\leq \frac{1}{\nu}\|\mathbf{f}\|^2 + \frac{\nu}{4}\|\Delta \mathbf{u}_N\|^2 + C(\Omega)\|\nabla \overline{\mathbf{u}}_N\|^{1/2}\|\Delta \overline{\mathbf{u}}_N\|^{1/2}\|\nabla \mathbf{u}_N\|\|\Delta \mathbf{u}_N\| \\
&\leq \frac{1}{\nu}\|\mathbf{f}\|^2 + \frac{\nu}{4}\|\Delta \mathbf{u}_N\|^2 + C(\Omega)\|\nabla \mathbf{u}_N\|^{1/2}\frac{1}{2^{1/4}\alpha^{1/2}}\|\nabla \mathbf{u}_N\|^{1/2}\|\nabla \mathbf{u}_N\|\|\Delta \mathbf{u}_N\| \\
&\leq \frac{1}{\nu}\|\mathbf{f}\|^2 + \frac{\nu}{2}\|\Delta \mathbf{u}_N\|^2 + \frac{C(\Omega)^2}{2^{1/2}\alpha\nu}\|\nabla \mathbf{u}_N\|^4,
\end{aligned}
$$

equivalently,

$$
\frac{d}{dt}\|\nabla \mathbf{u}_N(t)\|^2 + \nu\|\Delta \mathbf{u}_N(t)\|^2 \leq \frac{2}{\nu}\|\mathbf{f}\|^2 + \frac{2^{1/2}C(\Omega)^2}{\alpha\nu}\|\nabla \mathbf{u}_N\|^4.
$$

By the Gronwall inequality and (3.12), this implies

$$
\|\nabla \mathbf{u}_N(t)\| \leq C_\alpha \quad \text{for all } t \in (0,T), \tag{3.13}
$$

where C_α is independent of N. In particular, for N sufficiently large it follows from (3.6) that $B_\omega^N \equiv B_\omega$ and $\mathbf{u}_N \equiv \mathbf{u}$ is a solution to (1.1) and/or (1.2).

By a density argument (see, e.g., [4, 32]) it can be shown that if $\mathbf{u}_0 \in H$ and $\mathbf{f} \in L^2(0,T,V')$ then there exists a unique weak solution, an absolute continuous function $\mathbf{u} : [0,T] \to V'$ that satisfies $\mathbf{u} \in C([0,T];H) \cap L^2(0,T;V) \cap W^{1,2}([0,T],V')$ and (3.11) a.e. in $(0,T)$, where d/dt is considered in the strong topology of V'. □

Remark 3.3. *The pressure is recovered from the weak solution via the classical*

DeRham theorem (see [31]).

3.2 Regularity

Theorem 3.4. *Consider the NS-$\overline{\omega}$ model (1.1). Let $\chi \geq 0, m \in \mathbb{N}$, $\mathbf{u}_0 \in V \cap H^{m+1}_{\#}(\Omega)$ and $\mathbf{f} \in L^2(0,T;H^m_{\#}(\Omega))$. Then there exists a unique solution $\mathbf{u}, P$ to the equation (1.1) such that*

$$\mathbf{u} \in L^\infty(0,T;H^{m+1}_{\#}(\Omega)) \cap L^2(0,T;H^{m+2}_{\#}(\Omega)),$$
$$P \in L^2(0,T;H^{m+2}_{\#}(\Omega)).$$

Proof. We give the proof for $\chi = 0$ since the χ term is a lower order term. The result is already proved when $m = 0$ in Theorem 3.1. For any $m \in \mathbb{N}^*$, we assume that

$$\mathbf{u} \in L^\infty(0,T;H^k_{\#}(\Omega)) \cap L^2(0,T;H^{k+1}_{\#}(\Omega)), \quad \forall k = 0, 1, \ldots, m \tag{3.14}$$

so it remains to prove

$$D^{m+1}\mathbf{u} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),$$

where D^m denotes any partial derivative of total order m. We take the $(m+1)^{\text{th}}$ derivative of (1.8) and have

$$(D^{m+1}\mathbf{u})_t - \nu\Delta(D^{m+1}\mathbf{u}) + D^{m+1}(\overline{\boldsymbol{\omega}} \times \mathbf{u}) + D^{m+1}\nabla P = D^{m+1}\mathbf{f}, \quad \text{in } (0,T) \times \Omega,$$
$$\nabla \cdot (D^{m+1}\mathbf{u}) = 0, \quad \text{in } (0,T) \times \Omega,$$
$$D^{m+1}\mathbf{u}(0,\cdot) = D^{m+1}\mathbf{u}_0, \quad \text{in } \Omega,$$

with periodic boundary conditions and zero mean, and the initial conditions with zero divergence and mean. Taking $D^{m+1}\mathbf{u}$ as test functions we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|D^{m+1}\mathbf{u}\|^2 + \nu\|\nabla D^{m+1}\mathbf{u}\|^2 \\
&= \int_\Omega D^{m+1}\mathbf{f} D^{m+1}\mathbf{u} d\mathbf{x} - \int_\Omega D^{m+1}(\overline{\boldsymbol{\omega}} \times \mathbf{u}) D^{m+1}\mathbf{u} d\mathbf{x}.
\end{aligned} \tag{3.15}$$

Rewriting, we obtain

$$\begin{aligned}
&\int_\Omega D^{m+1}(\overline{\boldsymbol{\omega}} \times \mathbf{u}) D^{m+1}\mathbf{u} d\mathbf{x} = \sum_{|\alpha|\leq m+1} \begin{pmatrix} m+1 \\ \alpha \end{pmatrix} \int_\Omega D^\alpha \overline{\boldsymbol{\omega}} \times D^{m+1-\alpha}\mathbf{u} D^{m+1}\mathbf{u}\ d\mathbf{x} \\
&= \int_\Omega \big(D^{m+1}\overline{\boldsymbol{\omega}} \times \mathbf{u} D^{m+1}\mathbf{u} + (m+1) D^m \overline{\boldsymbol{\omega}} \times D\mathbf{u} D^{m+1}\mathbf{u} + \dots \\
&\quad + (m+1) D\overline{\boldsymbol{\omega}} \times D^m \mathbf{u} D^{m+1}\mathbf{u} + \overline{\boldsymbol{\omega}} \times D^{m+1}\mathbf{u} D^{m+1}\mathbf{u} \big)\ d\mathbf{x}.
\end{aligned}$$

Now we use (3.2), (2.1) and the induction assumption (3.14) to get

$$\int_\Omega D^{m+1}(\overline{\boldsymbol{\omega}} \times \mathbf{u}) D^{m+1}\mathbf{u} d\mathbf{x}\ \leq C\ \|\mathbf{u}\|_m \|\mathbf{u}\|_{m+1}^{1/2} \|\mathbf{u}\|_{m+2}^{3/2}.$$

Integrating (3.15) on $(0, T)$, using the Cauchy-Schwarz and Hölder inequalities, and the assumption (3.14) we obtain the desired result for $\mathbf{u}$. We conclude the proof mentioning that the regularity of the pressure term P is obtained via classical methods, see e.g. [39, 1]. □

Corollary 3.5.

1. *If the conditions of Theorem 3.4 hold with $m = 1$, then $\mathbf{u}$ is a unique strong solution of NS-$\overline{\omega}$ model.*

2. *If $\mathbf{u}_0, \mathbf{f} \in C^\infty_\#(\Omega)$, then $\mathbf{u}, p \in C^\infty_\#(\Omega)$.*

4 Phenomenology

In this section we apply turbulence phenomenology to study the joint energy-helicity cascade for homogeneous, isotropic turbulence generated by NS-α, NS-$\overline{\omega}$, and NS-$\alpha\&\omega$ models. This approach is due to A. Muschinsky, [35] and used for approximate deconvolution models in [25]. We consider the same problem for NS-$\overline{\omega}$, and NS-$\alpha\&\omega$ models, focusing first on the case when $\chi = 0$.

Energy cascades were proposed by Richardson in 1922 as a qualitative feature of turbulent flow. Kolmogorov's famous work (e.g. [14]) shows that energy cascades have universal, quantitative features that can be uncovered by dimensional analysis (for example). Helicity, discovered by Moreau, [34], is the second important integral invariant of the Euler equations and exhibits a cascade in turbulent flows as well. Interestingly, the various approaches to turbulent phenomenology (which all yield the same energy cascade) give different predictions of helicity cascade details. Kraichnan's dynamic argument, [22] seems to give the prediction best in accord with numerical experiments. Thus, we shall apply the dynamic argument to elucidate energy and helicity prediction of the NS-$\overline{\omega}$ model. The dynamic argument predicts that for homogeneous, isotropic turbulence through the inertial range

$$\widehat{E}(k) = C_E \epsilon^{2/3} k^{-5/3} \text{ and } \widehat{H}(k) = h C_H \gamma^{2/3} k^{-5/3}, \tag{4.1}$$

where k is the wave number, ϵ the mean energy dissipation rate, and γ the mean helicity dissipation rate, see [8] and [13]. The cascades are referred to as joint because they travel with the same speed through the wave space (i.e. the exponents of k are equal). In each case we find the energy and helicity microscale.

To begin, we present the helicity conservation of the NS-$\overline{\omega}$ model.

Theorem 4.1. *Consider the NS-$\overline{\omega}$ model (1.1). Let $\mathbf{u}_0 \in V$, $\mathbf{f} \in L^2(0,T;H)$, $\chi \geq 0$. The following helicity balance holds*

$$H_\omega(t) + \int_0^t \gamma_\omega(\tau)d\tau = H_\omega(0) + \int_0^t P_\omega(\tau)d\tau, \quad 0 \leq t \leq T, \tag{4.2}$$

where

$$\begin{aligned} H_\omega(t) &= \tfrac{1}{2|\Omega|}(\mathbf{u}(\cdot,t), \nabla \times \overline{\mathbf{u}}(\cdot,t)), \\ \gamma_\omega(t) &= \tfrac{\nu}{|\Omega|}(\nabla\times\mathbf{u}(\cdot,t), \nabla\times\nabla\times\overline{\mathbf{u}}(\cdot,t)) + \tfrac{\chi}{|\Omega|}(\mathbf{u}(\cdot,t)-\overline{\mathbf{u}}(\cdot,t), \nabla\times\overline{\mathbf{u}}(\cdot,t)), \\ P_\omega(t) &= \tfrac{1}{|\Omega|}(\mathbf{f}(\cdot,t), \nabla \times \overline{\mathbf{u}}(\cdot,t)). \end{aligned} \tag{4.3}$$

Proof. We obtain (4.3) by letting $\mathbf{v} = \frac{1}{|\Omega|}\nabla \times \overline{\mathbf{u}}$ in (1.1). Integration by parts and integration in time give

$$\begin{aligned} &\frac{1}{2|\Omega|}(\mathbf{u}\cdot\nabla\times\overline{\mathbf{u}})(T) + \frac{\nu}{|\Omega|}\int_0^T (\nabla\times\mathbf{u}, \nabla\times\nabla\times\overline{\mathbf{u}})\,d\tau \\ &\quad + \frac{\chi}{|\Omega|}\int_0^T (\mathbf{u}(t)-\overline{\mathbf{u}}(t)\nabla\times\overline{\mathbf{u}}(t))d\tau = \frac{1}{2|\Omega|}(\mathbf{u}\cdot\nabla\times\overline{\mathbf{u}})(0) + \frac{1}{|\Omega|}\int_0^T (\mathbf{f}, \nabla\times\overline{\mathbf{u}})\,d\tau. \end{aligned}$$

□

To represent the true kinetic energy and the model's kinetic energy spectrally, we expand the velocity field $\mathbf{u}(\mathbf{x},t)$ in Fourier series as follows:

$$\mathbf{u}(\mathbf{x},t) = \sum_k \sum_{|\mathbf{k}|=k} \widehat{\mathbf{u}}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ where } \widehat{\mathbf{u}}(\mathbf{k},t) = \frac{1}{L^3}\int_\Omega \mathbf{u}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} \tag{4.4}$$

and $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$ ($\mathbf{n} \in \mathbb{Z}^3$) is the wave number. Using Parceval's equality we deduce the balance equation for the kinetic energy at wave number $\mathbf{k}$

$$E(\mathbf{u})(t) = \frac{2\pi}{L}\sum_k \widehat{E}(k,t), \text{ where } \widehat{E}(k,t) = \frac{L}{2\pi}\sum_{|\mathbf{k}|=k}\frac{1}{2}|\widehat{\mathbf{u}}(\mathbf{k},t)|^2. \tag{4.5}$$

Proposition 4.2. *In Fourier modes, the model's kinetic energy and energy dissipation rate are*

$$E_\omega(\mathbf{u})(t) = \frac{2\pi}{L}\sum_k \widehat{E}_\omega(k,t), \quad \epsilon_\omega(\mathbf{u})(t) = 2\nu\frac{2\pi}{L}\sum_k k^2\widehat{E}_\omega(k,t) \tag{4.6}$$

where

$$\widehat{E}_\omega(k,t) = \widehat{E}(k,t). \tag{4.7}$$

Proof. Indeed, from (3.4) we have

$$E_\omega(\mathbf{u})(t) = \frac{1}{2|\Omega|}\|\mathbf{u}(\mathbf{x},t)\|^2\frac{2\pi}{L}\sum_k \widehat{E}(k,t). \tag{4.8}$$

Using the expansion of $\mathbf{u}(x,t)$ in Fourier series we have

$$\nabla\mathbf{u}(\mathbf{x},t) = \sum_k\sum_{|\mathbf{k}|=k} k\widehat{\mathbf{u}}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Thus

$$\epsilon_\omega(\mathbf{u})(t) = \frac{1}{2|\Omega|}\|\nabla\mathbf{u}(\mathbf{x},t)\|^2\frac{2\pi}{L}\sum_k k^2\widehat{E}(k,t). \tag{4.9}$$

□

The decomposition of helicity into wave numbers depends upon further splitting of Fourier modes into helical modes. Then, the velocity field can be expanded into helical waves $i\mathbf{k}\times\mathbf{h}_\pm = \pm k\mathbf{h}_\pm$, where $\mathbf{h}_\pm$ are orthonormal eigenvectors of the curl operator. Since $\mathbf{u}$ is incompressible, we have $\mathbf{k}\cdot\widehat{\mathbf{u}}(\mathbf{k},t) = 0$ and we can then write $\widehat{\mathbf{u}}(\mathbf{k},t) = a_+(\mathbf{k},t)\mathbf{h}_+ + a_-(\mathbf{k},t)\mathbf{h}_-$. Thus, for the spectral decomposition of helicity, we can expand $\widehat{\mathbf{u}}(\mathbf{k},t)$ in a basis of helical modes

$$\mathbf{u}(\mathbf{x},t) = \sum_{k} \sum_{|\mathbf{k}|=k} \sum_{s=\pm} a_s(\mathbf{k},t)\mathbf{h}_s(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{4.10}$$

We also have

$$(\nabla\times)^n \mathbf{u}(\mathbf{x},t) = \sum_{k} \sum_{|\mathbf{k}|=k} \sum_{s=\pm} s^n k^n a_s(\mathbf{k},t)\mathbf{h}_s(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{4.11}$$

Expanding $\mathbf{u}$ in helical modes, we get the spectral representation of helicity

$$H(\mathbf{u})(t) = \frac{2\pi}{L}\sum_{k} \widehat{H}(k,t), \quad \text{where} \quad \widehat{H}(k,t) := sk \sum_{|\mathbf{k}|=k} \sum_{s=\pm} |a_s(\mathbf{k},t)|^2. \tag{4.12}$$

Proposition 4.3. *The NS-$\overline{\omega}$ model's helicity spectrum and helicity dissipation rate are*

$$H_\omega(\mathbf{u})(t) = \frac{2\pi}{L}\sum_{k} \widehat{H}_\omega(k,t), \quad \epsilon_\omega(\mathbf{u})(t) = 2\nu\frac{2\pi}{L}\sum_{k} k^2 \widehat{H}_\omega(k,t) \tag{4.13}$$

where

$$\widehat{H}_\omega(k,t) = \frac{1}{\alpha^2 k^2 + 1}\widehat{H}(k,t). \tag{4.14}$$

Proof. Using (4.10) and (4.11) we have

$$H_\omega(\mathbf{u})(t) = \frac{2\pi}{L}\sum_{k} \frac{k}{\alpha^2 k^2 + 1} \sum_{|\mathbf{k}|=k} \sum_{\pm} s|a_s(\mathbf{k},t)|^2 = \frac{2\pi}{L}\sum_{k} \frac{1}{\alpha^2 k^2 + 1}\widehat{H}(k,t),$$

$$\gamma_\omega(\mathbf{u})(t) = \frac{2\pi}{L}\sum_{k} \frac{k^3}{\alpha^2 k^2 + 1} \sum_{|\mathbf{k}|=k} \sum_{\pm} s^3|a_s(\mathbf{k},t)|^2 = 2\nu\frac{2\pi}{L}\sum_{k} \frac{k^2}{\alpha^2 k^2 + 1}\widehat{H}(k,t).$$

This proves the claimed result for the NS-$\overline{\omega}$ model. ∎

Let $\langle\cdot\rangle$ denote time averaging

$$\langle \boldsymbol{\phi} \rangle(\mathbf{x}) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{\phi}(\mathbf{x},t)dt, \tag{4.15}$$

provided the limit exists. If the limit does not exist $\langle\cdot\rangle$ is defined replacing lim by LIM, a suitable Banach or generalized limit.

Definition 4.4. *The kinetic energy and helicity distribution functions are defined by*

$$\widehat{E}(k) = \langle \widehat{E}(k,t)\rangle,\ \widehat{E}_\omega(k) = \langle \widehat{E}_\omega(k,t)\rangle \quad \textit{and}$$
$$\widehat{H}(k) = \langle \widehat{H}(k,t)\rangle,\ \widehat{H}_\omega(k) = \langle \widehat{H}_\omega(k,t)\rangle.$$

Proposition 4.5. *In Fourier space, we have*

$$\widehat{E}_\alpha(k)\widehat{E}_{\alpha\&\omega}(k) = \frac{1}{\alpha^2k^2+1}\widehat{E}(k) \quad \textit{and} \quad \widehat{H}_\alpha(k) = \widehat{H}(k) \tag{4.16}$$
$$\widehat{E}_\omega(k) = \widehat{E}(k) \quad \textit{and} \quad \widehat{H}_\omega(k)\widehat{H}_{\alpha\&\omega}(k) = \frac{1}{\alpha^2k^2+1}\widehat{H}(k). \tag{4.17}$$

Proof. The proof follows by time averaging (4.6) and (4.13). □

We are now ready to investigate the phenomenology of a possible joint cascade of energy and helicity for NS-$\overline{\omega}$ model adapting the dynamic argument of Kraichnan, [22], following [25]. This is a phenomenological argument. For the NS-$\overline{\omega}$ model we find through the model's inertial range:

$$\widehat{E}_\omega(k) \simeq C_E\epsilon_\omega^{2/3}k^{-5/3} \text{ and } \widehat{H}_\omega(k) \simeq C_H\gamma_\omega\epsilon_\omega^{-1/3}k^{-5/3}.$$

Definition 4.6. *Let $\widehat{\Pi}_\omega$ and $\widehat{\Sigma}_\omega$ denote the total energy and helicity transfer from all wave numbers k. We say that the model exhibits a joint energy and helicity cascade if, in some inertial range, $\widehat{\Pi}_\omega$ and $\widehat{\Sigma}_\omega$ are independent of the* *wave number, i.e.* $\widehat{\Pi}_\omega(k) = \varepsilon_\omega$ *and* $\widehat{\Sigma}_\omega(k) = \gamma_\omega$.

Following Kraichnan, see [22], we make the following assumptions on $\widehat{\Pi}_\omega$ and $\widehat{\Sigma}_\omega$.

Assumption 4.7. $\widehat{\Pi}_\omega(k)$ *is proportional to the total energy* $k\widehat{E}_\omega(k)$ *and* $\widehat{\Sigma}_\omega(k)$ *is proportional to the total helicity* $k\widehat{H}_\omega(k)$ *in wave numbers of order* k *and to some effective rate of shear,* $\sigma(k)$*, which acts to distort flow structures of scale* $1/k$.

Let $\tau(k)$ be the distortion time of flow structures of scales $1/k$ due to the shearing action $\sigma(k)$ of all wave numbers $\leq k$. Then

$$\tau(k) = \frac{1}{\sigma(k)}, \text{ where } \sigma^2(k) \sim \int_0^k p^2 \widehat{E}_\omega(p)dp. \tag{4.18}$$

Assumption 4.8. *Since energy and helicity are both dissipated by the same mechanism (of viscosity), they relax over comparable time scales. Thus,* $\tau(k)$ *and* $\sigma(k)$ *are the same for energy and helicity of the model.*

Assumption 2 leads to

$$\widehat{\Pi}_\omega(k) \simeq k\widehat{E}_\omega(k)/\tau(k) \text{ and } \widehat{\Sigma}_\omega(k) \simeq k\widehat{H}_\omega(k)/\tau(k). \tag{4.19}$$

Kolmogorov's locality assumption gives that the main contribution in (4.18) is from $p \simeq k$. Thus, we have

$$\sigma(k) \sim k^{-3/2}\widehat{E}_\omega^{-1/2}(k). \tag{4.20}$$

To derive the energy spectrum, we first note that Definition 4.6, Assumption 4.8, and (4.20) imply

$$\epsilon_\omega = \widehat{\Pi}_\omega(k) \simeq k\sigma(k)\widehat{E}_\omega(k)k^{5/2}\widehat{E}_\omega^{3/2}(k) \tag{4.21}$$

and thus

$$\widehat{E}_\omega(k) \simeq \epsilon_\omega^{2/3} k^{-5/3}. \tag{4.22}$$

In the same way we derive the helicity spectrum.

$$\widehat{H}_\omega(k) \simeq \gamma_\omega \epsilon_\omega^{-1/3} k^{-5/3}. \tag{4.23}$$

Similarly, we find that for NS-α and NS-$\alpha\&\omega$ we have

$$\widehat{E}_\alpha(k) \simeq (1+\alpha^2k^2)\epsilon_\alpha^{2/3} k^{-5/3} \text{ and } \widehat{H}_\alpha(k) \simeq \gamma_\alpha \epsilon_\alpha^{-1/3} k^{-5/3}, \tag{4.24}$$

$$\widehat{E}_{\alpha\&\omega}(k) \simeq (1+\alpha^2k^2)\epsilon_{\alpha\&\omega}^{2/3} k^{-5/3} \text{ and}$$
$$\widehat{H}_{\alpha\&\omega}(k) \simeq (1+\alpha^2k^2)\gamma_{\alpha\&\omega} \epsilon_{\alpha\&\omega}^{-1/3} k^{-5/3}. \tag{4.25}$$

Thus, the NS-$\overline{\omega}$ model, like NS-α and (as can be shown) NS-$\alpha\&\omega$, gives accurate prediction of energy and helicity (in the limit of resolution of all the model's persistent scales). However, (4.22) and (4.23) have two interesting subtleties that suggest care must be taken when testing this prediction numerically. First, if $\chi = 0$ all three models likely have persistent scales smaller than $O(\alpha)$ with the accompanying possibility of energy or helicity accumulation near $h \sim 1/\alpha$ in a numerical simulation. Thus, time relaxation with the appropriate choice (see [28], formula (1.3))

$$\chi = \frac{U}{L^{1/3}} 2^{\frac{N+1}{3}} \alpha^{-2/3}, \; U = \text{global velocity scale}, \; L = \text{global length scale}, \tag{4.26}$$

is an important tool for scale truncation. Second, there is a difference between $\widehat{H}$ and $\widehat{H}_\omega$. Thus, it is vital to monitor the appropriate *model* energy and helicity statistics in any test.

4.1 Microscales when $\chi = 0$

Let η_{E_ω} and η_{H_ω} be the model's energy and helicity microscale respectively. To find η_{E_ω} we use (3.4)

$$\epsilon_\omega(t) = \nu \int_0^{k_\omega} k^2 \widehat{E}_\omega(k) dk = \nu \int_0^{k_\omega} k^2 C_E \epsilon_\omega^{2/3} k^{-5/3} dk.$$

After integration we get $k_\omega C \nu^{-3/4} \epsilon_\omega^{1/4}$. Now, set $\eta_{E_\omega} = \frac{1}{k_{E_\omega}}$. Using the same type of argument, we find that $\eta_{H_\omega} = \frac{1}{k_{H_\omega}}$, where $k_{H_\omega} = C\nu^{-3/4}\epsilon_\omega^{1/4}$. It means that, for the NS-$\overline{\omega}$ model, the end of the inertial range for energy and helicity is the same.

When $\chi > 0$ we find by adapting the argument in [29] that if χ is given by (4.26) we have

$$\eta_{E_\omega} = \eta_{H_\omega} = \alpha, \text{ if } \chi = O(\alpha^{-2/3}). \tag{4.27}$$

This further reinforces the importance of enhancing scale truncation by time relaxation in all three models.

5 Conclusions

The NSE is an exact model for the flow of a viscous, incompressible fluid. Their solution contains so much information that they become impractical for many problems within typical time and resource limitations. Thus, numerical simulations if turbulent flows are often based on various regularizations of the NSE rather than the NSE themselves. Herein, we have developed the mathematical theory of a such a regularization of the NSE, the NS-$\overline{\omega}$ model. Using the semigroup approach for nonlinear differential equations of accretive type in Banach spaces, we proved that NS-$\overline{\omega}$ admits strong solutions for large data and general $\nu > 0$. We also applied the phenomenology of homogeneous, isotropic turbulence to the NS-$\overline{\omega}$ model. By adapting the reasoning of Richardson and Kolmogorov, we establish the model's energy cascade and found the micro-scale of the model (the length-scale of the smallest persistent structure in the model's solution). We also showed that the model has a helicity cascade, linked to its energy cascade. The energy and helicity both cascade at the correct $O(k^{5/3})$ rate for inertial range wave numbers up to the cutoff wave number of $O(1/\alpha)$. This establishes consistency of the models helicity and energy cascades with the true cascades of the NSE. In parallel, we also discuss the phenomenology of the NS-$\overline{\omega}$ with the ones of the NS-α and the NS-$\alpha\&\omega$.

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