NUMERICAL ANALYSIS OF TWO PARTITIONED METHODS FOR UNCOUPLING EVOLUTIONARY MHD FLOWS

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Abstract. Magnetohydrodynamics (MHD) studies the dynamics of electrically conducting fluids, involving Navier-Stokes (NSE) equations in fluid dynamics and Maxwell equations in eletromagnetism. The physical processes of fluid flows and electricity and magnetism are quite different and numerical simulations of each sub-process can require different meshes, time steps and methods. In most terrestrial applications, MHD flows occur at low magnetic Reynold numbers. We introduce two partitioned methods to solve evolutionary MHD equations in such cases. The methods we study allow us at each time step to call NSE and Maxwell codes separately, each possibly optimized for the subproblem's respective physics. Complete error analysis and computational tests supporting the theory are given.

Key words. partitioned methods, finite element methods, magnetohydrodynamics.

1. Introduction. Broadly, MHD flows divide into plasmas and astrophysical flows at high magnetic Reynolds numbers (denoted by R_m throughout this paper) and terrestrial applications, such as liquid metals, at low R_m. We consider herein the reduced MHD or RMHD model of MHD flows at low R_m. Incompressible flow of an electrically conducting fluid in the presence of a magnetic field at low R_m is modelled by the system, see, e.g., [15, 6, 21]: Given $\mathbf{f}$, $\mathbf{B}$ and time $T > 0$, find $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, $p : \Omega \times [0, T] \to \mathbb{R}$ and $\phi : \Omega \times [0, T] \to \mathbb{R}$ such that:

$$\begin{aligned} \frac{1}{N}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \frac{1}{M^2}\Delta\mathbf{u} + \nabla p \,&=\, \mathbf{f} + (\mathbf{B}\times\nabla\phi + \mathbf{B}\times(\mathbf{B}\times\mathbf{u}))\,, \\ \Delta\phi = \nabla\cdot(\mathbf{u}\times\mathbf{B}),\ \text{and}\ \ \nabla\cdot\mathbf{u} \,&=\, 0. \end{aligned} \tag{RMHD}$$

Here Ω is a bounded, Lipschitz domain in $\mathbb{R}^d$ $(d = 3)$ and the body force $\mathbf{f}$ and the magnetic field $\mathbf{B}$ are assumed to be known with $\nabla\cdot\mathbf{B} = 0$. Further, $\mathbf{u}$ is the fluid velocity, p is pressure and ϕ is electric potential. M, N are the Hartman number and interaction parameter given by

$$M = BL\sqrt{\frac{\sigma}{\rho\nu}}, \quad N = \sigma B^2\frac{L}{\rho u}$$

where u, B, L are the characteristic velocity, magnetic field and length, respectively. The other parameters appearing above are the density ρ, the kinematic viscosity ν, and the electrical conductivity σ, all assumed constant. The system (RMHD) is supplemented by the homogeneous Dirichlet boundary conditions

$$\mathbf{u} = 0,\ \phi = 0 \text{ on } \partial\Omega\times[0, T]$$

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and the initial data

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \ \forall x \in \Omega. \tag{1.1}$$

Other boundary conditions and variable material parameters can be considered. However, constant parameters and simple boundary conditions allow us to focus on the physical coupling between u and ϕ and an algorithm allowing uncoupling of RMHD into physical subprocesses.

In this report, we propose and analyze the stability and errors of two partitioned methods for the evolutionary RMHD equations. The methods we study herein include a first order, one step scheme and a second order, two step scheme, both of which consist of implicit discretization of the subproblem terms and explicit discretization of coupling terms. These approaches solve the coupled problems by solving each subphysics problem per time step (without iteration), allowing the use of optimized NSE codes and Ohm's law codes. We prove that the new partitioned method (IMEX1) below is convergent at first order and stable over $0 \le t < \infty$. This method does *not* require a restriction on time step size Δt, even though we treat the coupling terms explicitly. The second method, (IMEX2), is second order convergent and stable over $0 \le t < \infty$, provided that the time step is small. We also present numerical tests to confirm the theoretical results.

1.1. Previous works. For a justification of using simplified MHD equations to model MHD flows in terrestrial applications, we refer to [3, 8, 16]. There are several different, almost equivalent formulations for RMHD. The one we study in this paper can be found in [15, 6, 21]. Applications of MHD in industry and engineering are abundant. They include liquid metal cooling of nuclear reactors [1, 7, 18], sea water propulsion [11] and process metallurgy [2].

The results on existence, uniqueness and finite element approximation of the steady-state MHD problems were developed through work in [20] (for two dimensional case), [15] (for small magnetic Reynolds number case) and [6] (for full MHD flows with perfectly conducting wall condition). In [12, 13, 10, 14], Meir et. al. studied variational methods and numerical approximation for solving stationary MHD equations under more physically realistic boundary conditions that account for the electromagnetic interaction of the fluid with the outside world. For further discussions on mathematical and numerical analysis of steady-state MHD flows, we refer to [5, 4].

There are much less works on time-dependent MHD. Schmidt [17] developed a formulation for evolutionary MHD and established the existence of global-in-time weak solutions via the Galerkin method. To the best of our knowledge, the first papers dealing with time discretization schemes of MHD problems were of Yuksel and Ingram [21] and Trenchea [19]. The former studied the stability and error analysis of the fully coupled, monolithic Crank-Nicolson method for reduced MHD equations while the latter introduced an unconditionally stable, first order partitioned method for full MHD based on uncoupling Elsässer variables. Our methods was presented in [9] where partial result on stability was proven (but not convergence). Herein, we review stability and give a complete and comprehensive convergence and error analysis as well as new tests.

2. Notation and preliminaries. Throughout this paper, we will use C_0 to represent a generic positive constant whose value may be different from place to place but which is independent of mesh size and time step. We denote the $L^2(\Omega)$ norms and corresponding inner products by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. $H^k(\Omega)$ is used to represent the Sobolev space $W_2^k(\Omega)$, and $\|\cdot\|_k$ denotes the norm in $H^k(\Omega)$. The space $H^{-k}(\Omega)$ denotes the dual space of $H_0^k(\Omega)$. For functions $v(x,t)$ defined on the entire time interval $(0,T)$, we define for $1 \le m < \infty$

$$\|v\|_{\infty,k} \ := \ EssSup_{[0,T]}\|v(t,\cdot)\|_k \, , \text{ and } \ \|v\|_{m,k} \ := \ \left(\int_0^T \|v(t,\cdot)\|_k^m \, dt\right)^{1/m} \ .$$

The velocity, pressure and potential spaces are $X = (H_0^1(\Omega))^d$, $Q = L_0^2(\Omega)$ and $S = H_0^1(\Omega)$, respectively. The space of divergence free functions is given by

$$V = \{\mathbf{v} \in X : \ (\nabla \cdot \mathbf{v}, q) = 0 \ \ \forall q \in Q\} \, .$$

A weak formulation of (RMHD) is: Find $\mathbf{u} : [0,T] \to X$, $p : [0,T] \to Q$ and $\phi : [0,T] \to S$ for a.e. $t \in (0,T]$ satisfying

$$\begin{aligned}
\frac{1}{N}((\mathbf{u}_t,\mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v})) &+ \frac{1}{M^2}(\nabla\mathbf{u},\nabla\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) \\
&+ (\mathbf{u}\times\mathbf{B},\mathbf{v}\times\mathbf{B}) - (\nabla\phi,\mathbf{v}\times\mathbf{B}) = (\mathbf{f},\mathbf{v}) \ \ \forall \mathbf{v} \in X, \\
(\nabla\cdot\mathbf{u},q) = 0 \quad &\forall q \in Q, \\
-(\nabla\phi,\nabla\psi) + (\mathbf{u}\times\mathbf{B},\nabla\psi) = 0 \quad &\forall \psi \in S,
\end{aligned} \tag{2.1}$$

with the initial condition (1.1) a.e. in Ω. Note that, setting $\mathbf{v} = \mathbf{u}$, $\psi = \phi$ and adding, the coupling terms exactly cancel in the monolithic sum and one verifies the stability of the continuous problem.

To make a spatial discretization of the RMHD system by the finite element method, we select finite element spaces

velocity: $X^h \subset X$, pressure: $Q^h \subset Q$, and potential: $S^h \subset S$

which are built on a conforming, edge to edge triangulation with maximum triangle parameter denoted by a subscript "h". We assume that $X^h \times Q^h$ satisfies the usual discrete inf-sup condition for the stability of the discrete pressure and X^h, Q^h, S^h satisfy approximation properties of piecewise polynomials on quasi-uniform meshes of local degrees $k, k-1, k$ respectively. The error analysis in [15, 21] indicates that the same order elements to be used for the velocity and electric potential. The discretely divergence free velocity space is denoted by

$$V^h := X^h \cap \{\mathbf{v}_h : (q_h, \nabla\cdot\mathbf{v}_h) = 0, \ \text{ for all } q_h \in Q^h\}.$$

Also define the usual, explicitly skew symmetrized trilinear form

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}((\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - (\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})).$$

The monolithic, semi-discrete approximation of (2.1) (see [21]) are maps $(\mathbf{u}_h, p_h, \phi_h) : [0,T] \to X^h \times Q^h \times S^h$ satisfying for all $\mathbf{v}_h \in X^h, q_h \in Q^h, \psi_h \in S^h$

$$\begin{aligned}
\frac{1}{N}((\mathbf{u}_{h,t},\mathbf{v}_h) + b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h)) &+ \frac{1}{M^2}(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) - (p_h,\nabla\cdot\mathbf{v}_h) \\
&+ (\mathbf{u}_h\times\mathbf{B},\mathbf{v}_h\times\mathbf{B}) - (\nabla\phi_h,\mathbf{v}_h\times\mathbf{B}) = (\mathbf{f},\mathbf{v}_h), \\
(\nabla\cdot\mathbf{u}_h,q_h) &= 0, \\
-(\nabla\phi_h,\nabla\psi_h) + (\mathbf{u}_h\times\mathbf{B},\nabla\psi_h) &= 0.
\end{aligned} \tag{2.2}$$

2.1. The implicit-explicit partitioned schemes. The methods we propose and analyze herein have the coupling terms lagged or extrapolated in a careful way that preserves stability. Thus the system at each time step uncouples into two subproblem solves. The first scheme we study is a combination of the two level implicit method with the coupling terms treated by the explicit method. We shall use the same time step in both subproblems. It reads

ALGORITHM 2.1 (First order IMEX scheme). *Given* $\mathbf{u}_h^n \in X^h, p_h^n \in Q^h, \phi_h^n \in S^h$, *find* $\mathbf{u}_h^{n+1} \in X^h, p_h^{n+1} \in Q^h, \phi_h^{n+1} \in S^h$ *satisfying*

$$\begin{aligned}\frac{1}{N}\left(\left(\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t},\mathbf{v}_h\right)+b(\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1},\mathbf{v}_h)\right)+\frac{1}{M^2}(\nabla\mathbf{u}_h^{n+1},\nabla\mathbf{v}_h)&\\ -(p_h^{n+1},\nabla\cdot\mathbf{v}_h)+(\mathbf{u}_h^{n+1}\times\mathbf{B},\mathbf{v}_h\times\mathbf{B})-(\nabla\phi_h^n,\mathbf{v}_h\times\mathbf{B})=(\mathbf{f}^{n+1},\mathbf{v}_h),&\\ (\nabla\cdot\mathbf{u}_h^{n+1},q_h)=0,&\qquad\text{(IMEX1)}\\ -(\nabla\phi_h^{n+1},\nabla\psi_h)+(\mathbf{u}_h^n\times\mathbf{B},\nabla\psi_h)=0,&\end{aligned}$$

for all $\mathbf{v}_h \in X^h, q_h \in Q^h$ *and* $\psi_h \in S^h$.

The second scheme we consider employs second order, three level BDF discretization for the subproblem terms. The coupling terms are treated by two step extrapolation in Navier-Stokes equation and by implicit method in Ohm's law. Since one needs the updated value of u_h at current time level to compute ϕ_h, this method is uncoupled but sequential: $\phi_h^n \to \mathbf{u}_h^{n+1} \to \phi_h^{n+1}$. Nevertheless, solving the subproblems sequentially may be an acceptable tradeoff for higher accuracy and preservation of stability. Computing time for the nonlinear equation of $\mathbf{u}_h$ is normally expected to dominate that for the Poisson solve for ϕ_h.

ALGORITHM 2.2 (Second order IMEX scheme). *Given* $\mathbf{u}_h^{n-1}, \mathbf{u}_h^n \in X^h, p_h^{n-1}, p_h^n \in Q^h, \phi_h^{n-1}, \phi_h^n \in S^h$, *find* $\mathbf{u}_h^{n+1} \in X^h, p_h^{n+1} \in Q^h, \phi_h^{n+1} \in S^h$ *satisfying*

$$\begin{aligned}\frac{1}{N}\left(\left(\frac{3\mathbf{u}_h^{n+1}-4\mathbf{u}_h^n+\mathbf{u}_h^{n-1}}{2\Delta t},\mathbf{v}_h\right)+b(\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1},\mathbf{v}_h)\right)+\frac{1}{M^2}(\nabla\mathbf{u}_h^{n+1},\nabla\mathbf{v}_h)&\\ -(p_h^{n+1},\nabla\cdot\mathbf{v}_h)+(\mathbf{u}_h^{n+1}\times\mathbf{B},\mathbf{v}_h\times\mathbf{B})-(\nabla(2\phi_h^n-\phi^{n-1}),\mathbf{v}_h\times\mathbf{B})=(\mathbf{f}^{n+1},\mathbf{v}_h),&\\ (\nabla\cdot\mathbf{u}_h^{n+1},q_h)=0,&\qquad\text{(IMEX2)}\\ -(\nabla\phi_h^{n+1},\nabla\psi_h)+(\mathbf{u}_h^{n+1}\times\mathbf{B},\nabla\psi_h)=0,&\end{aligned}$$

for all $\mathbf{v}_h \in X^h, q_h \in Q^h$ *and* $\psi_h \in S^h$.

3. Stability of the two partitioned methods.

In this preliminary section, we establish stability of the approximations in Algorithms IMEX1 and IMEX2. Theorem 3.1 was presented previously in [9]. We repeat its (short) proof here for completeness.

THEOREM 3.1 (Unconditional stability of Algorithm IMEX1). *Let* $(\mathbf{u}_h^n, p_h^n, \phi_h^n) \in X^h \times Q^h \times S^h$ *satisfy* (IMEX1) *for each* $n \in \{1, 2, ..., \frac{T}{\Delta t}\}$. *Then*

$$\begin{aligned}\frac{1}{N}\|\mathbf{u}_h^n\|^2+\frac{1}{N}\sum_{j=0}^{n-1}\|\mathbf{u}_h^{j+1}-\mathbf{u}_h^j\|^2+\Delta t\|\nabla\phi_h^n\|^2+\Delta t\|\mathbf{B}\times\mathbf{u}_h^n\|^2+\frac{\Delta t}{M^2}\sum_{j=0}^{n-1}\|\nabla\mathbf{u}_h^{j+1}\|^2&\\ +\Delta t\sum_{j=0}^{n-1}\left(\|-\nabla\phi_h^j+\mathbf{u}_h^{j+1}\times\mathbf{B}\|^2+\|-\nabla\phi_h^{j+1}+\mathbf{u}_h^j\times\mathbf{B}\|^2\right)&\qquad(3.1)\\ \le\frac{1}{N}\|\mathbf{u}_h^0\|^2+\Delta t\|\nabla\phi_h^0\|^2+\Delta t\|\mathbf{B}\times\mathbf{u}_h^0\|^2+M^2\Delta t\sum_{j=0}^{n-1}\|\mathbf{f}^{j+1}\|_{-1}^2.&\end{aligned}$$

Proof. In (IMEX1), we set $\mathbf{v}_h = \mathbf{u}_h^{j+1}, q_h = p_h^{j+1}, \psi_h = \phi_h^{j+1}$, add and multiply by $2\Delta t$ and sum from $j=0$ to $n-1$. This gives

$$\frac{1}{N}\|\mathbf{u}_h^n\|^2 + \frac{1}{N}\sum_{j=0}^{n-1}\|\mathbf{u}_h^{j+1} - \mathbf{u}_h^j\|^2 + \Delta t\|\nabla\phi_h^n\|^2 + \Delta t\|\mathbf{B}\times\mathbf{u}_h^n\|^2 \tag{3.2}$$
$$+ \frac{2\Delta t}{M^2}\sum_{j=0}^{n-1}\|\nabla\mathbf{u}_h^{j+1}\|^2 + \Delta t\sum_{j=0}^{n-1}\Big(\|-\nabla\phi_h^j + \mathbf{u}_h^{j+1}\times\mathbf{B}\|^2 + \|-\nabla\phi_h^{j+1} + \mathbf{u}_h^j\times\mathbf{B}\|^2\Big)$$
$$= \frac{1}{N}\|\mathbf{u}_h^0\|^2 + \Delta t\|\nabla\phi_h^0\|^2 + \Delta t\|\mathbf{B}\times\mathbf{u}_h^0\|^2 + 2\Delta t\sum_{j=0}^{n-1}(\mathbf{f}^{j+1}, \mathbf{u}_h^{j+1}).$$

Applying Young's inequality yields the result. ☐

REMARK 3.1. *Besides the electric potential ϕ, the electric current density $\mathbf{J}$ defined by $\mathbf{J} = \sigma(-\nabla\phi + \mathbf{u}\times\mathbf{B})$ is another important electromagnetic quantity to be determined in MHD flows, see [16, 3]. For IMEX1, the stability of $\mathbf{J}$ comes directly from the boundedness of $\frac{1}{N}\sum_{j=0}^{n-1}\|\mathbf{u}_h^{j+1} - \mathbf{u}_h^j\|^2$ and $\Delta t\sum_{j=0}^{n-1}\|-\nabla\phi_h^{j+1} + \mathbf{u}_h^j\times\mathbf{B}\|^2$ in (3.1).*

Next, we turn to Algorithm IMEX2. We prove that it is stable over $0 \le t < \infty$ with a condition related the time step and the problem data but independent of the spacial meshwidth, a result stated without proof in [9].

THEOREM 3.2 (Stability of Algorithm IMEX2). *Let $(\mathbf{u}_h^n, p_h^n, \phi_h^n) \in X^h \times Q^h \times S^h$ satisfy (IMEX2) for each $n \in \{1, 2, ..., \frac{T}{\Delta t}\}$. Under the time step restriction*

$$\Delta t < \frac{1}{2N\|\mathbf{B}\|_{L^\infty}^2(M^2C_P^2\|\mathbf{B}\|_{L^\infty}^2 + 1)} \tag{3.3}$$

Algorithm IMEX2 is stable

$$\frac{1}{2N}\|\mathbf{u}_h^n\|^2 + \frac{1}{2N}\|2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \frac{\Delta t}{2M^2}\sum_{j=1}^{n-1}\|\nabla\mathbf{u}_h^{j+1}\|^2 + \frac{\Delta t}{\sigma^2}\sum_{j=1}^{n-1}\|2\mathbf{J}^j - \mathbf{J}^{j-1}\|^2$$
$$\le \frac{1}{2N}\|\mathbf{u}_h^1\|^2 + \frac{1}{2N}\|2\mathbf{u}_h^1 - \mathbf{u}_h^0\|^2 + 2\Delta t M^2\sum_{j=1}^{n-1}\|\mathbf{f}^{j+1}\|_{-1}^2. \tag{3.4}$$

Proof. Set $\mathbf{v}_h = \mathbf{u}_h^{j+1}$ in the first equation of (IMEX2) and use the identity

$$\frac{1}{4}[3a^2 - 4b^2 + c^2] + \frac{1}{2}(a-b)^2 - \frac{1}{2}(b-c)^2 + \frac{1}{4}(a-2b+c)^2 = \frac{1}{2}(3a - 4b + c)a$$

with $a = \mathbf{u}_h^{j+1}, b = \mathbf{u}_h^j, c = \mathbf{u}_h^{j-1}$ to get

$$\frac{1}{4\Delta t}\cdot\frac{1}{N}\Big(3\|\mathbf{u}_h^{j+1}\|^2 - 4\|\mathbf{u}_h^j\|^2 + \|\mathbf{u}_h^{j-1}\|^2\Big)$$
$$+ \frac{1}{2\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^{j+1} - \mathbf{u}_h^j\|^2 - \frac{1}{2\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^j - \mathbf{u}_h^{j-1}\|^2$$
$$+ \frac{1}{4\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^{j+1} - 2\mathbf{u}_h^j + \mathbf{u}_h^{j-1}\|^2 + \frac{1}{M^2}\|\nabla\mathbf{u}_h^{j+1}\|^2 + \frac{1}{2}\|\mathbf{B}\times\mathbf{u}_h^{j+1}\|^2 \tag{3.5}$$
$$+ \frac{1}{2}\|-\nabla(2\phi_h^j - \phi_h^{j-1}) + \mathbf{u}_h^{j+1}\times\mathbf{B}\|^2$$
$$= \frac{1}{2}\|\nabla(2\phi_h^j - \phi_h^{j-1})\|^2 + (\mathbf{f}^{j+1}, \mathbf{u}_h^{j+1}).$$

The third equation of (IMEX2) gives

$$-(\nabla(2\phi_h^j-\phi_h^{j-1}),\nabla\psi_h)+((2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B},\nabla\psi_h)=0.$$

Setting $\psi_h=2\phi_h^j-\phi_h^{j-1}$, we have

$$\begin{aligned}\|\nabla(2\phi_h^j-\phi_h^{j-1})\|^2=&\|(2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2\\&-\|-\nabla(2\phi_h^j-\phi_h^{j-1})+(2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2.\end{aligned}\tag{3.6}$$

Plugging (3.6) into (3.5) yields

$$\begin{aligned}&\frac{1}{4\Delta t}\cdot\frac{1}{N}\left(3\|\mathbf{u}_h^{j+1}\|^2-4\|\mathbf{u}_h^j\|^2+\|\mathbf{u}_h^{j-1}\|^2\right)\\&\quad+\frac{1}{2\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^{j+1}-\mathbf{u}_h^j\|^2-\frac{1}{2\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^j-\mathbf{u}_h^{j-1}\|^2\\&\quad+\frac{1}{4\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^{j+1}-2\mathbf{u}_h^j+\mathbf{u}_h^{j-1}\|^2+\frac{1}{M^2}\|\nabla\mathbf{u}_h^{j+1}\|^2+\frac{1}{2}\|\mathbf{B}\times\mathbf{u}_h^{j+1}\|^2\\&+\frac{1}{2}\|-\nabla(2\phi_h^j-\phi_h^{j-1})+\mathbf{u}_h^{j+1}\times\mathbf{B}\|^2+\frac{1}{2}\|-\nabla(2\phi_h^j-\phi_h^{j-1})+(2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2\\&=\frac{1}{2}\|(2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2+(\mathbf{f}^{j+1},\mathbf{u}_h^{j+1}).\end{aligned}\tag{3.7}$$

Next, observe that for an arbitrary $\epsilon>0$

$$\begin{aligned}&\|(2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2=\|(-\mathbf{u}_h^{j+1}+2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2+\|\mathbf{u}_h^{j+1}\times\mathbf{B}\|^2\\&\qquad+2((-\mathbf{u}_h^{j+1}+2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B},\mathbf{u}_h^{j+1}\times\mathbf{B})\\&=\|(-\mathbf{u}_h^{j+1}+2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2+\|\mathbf{u}_h^{j+1}\times\mathbf{B}\|^2+\frac{1}{\epsilon^2}\|(-\mathbf{u}_h^{j+1}+2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2\\&\quad+\epsilon^2\|\mathbf{u}_h^{j+1}\times\mathbf{B}\|^2-\|(\frac{1}{\epsilon}(-\mathbf{u}_h^{j+1}+2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})-\epsilon\mathbf{u}_h^{j+1})\times\mathbf{B}\|^2\\&\le\left(1+\frac{1}{\epsilon^2}\right)\|\mathbf{B}\|^2_{L^\infty}\|(-\mathbf{u}_h^{j+1}+2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\|^2+\|\mathbf{u}_h^{j+1}\times\mathbf{B}\|^2\\&\quad+\epsilon^2C_P^2\|\mathbf{B}\|^2_{L^\infty}\|\nabla\mathbf{u}_h^{j+1}\|^2-\|(\frac{1}{\epsilon}(-\mathbf{u}_h^{j+1}+2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})-\epsilon\mathbf{u}_h^{j+1})\times\mathbf{B}\|^2\end{aligned}\tag{3.8}$$

where C_P is the Poincaré constant.

From (3.8), we can hide $\frac{1}{2}\|(2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2$ in the left hand side of (3.7) to obtain

$$\begin{aligned}&\frac{1}{4\Delta t}\cdot\frac{1}{N}\left(3\|\mathbf{u}_h^{j+1}\|^2-4\|\mathbf{u}_h^j\|^2+\|\mathbf{u}_h^{j-1}\|^2\right)\\&\quad+\frac{1}{2\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^{j+1}-\mathbf{u}_h^j\|^2-\frac{1}{2\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^j-\mathbf{u}_h^{j-1}\|^2\\&\quad+\left(\frac{1}{4\Delta t}\cdot\frac{1}{N}-\frac{1}{2}\left(1+\frac{1}{\epsilon^2}\right)\|\mathbf{B}\|^2_{L^\infty}\right)\|\mathbf{u}_h^{j+1}-2\mathbf{u}_h^j+\mathbf{u}_h^{j-1}\|^2\\&+\frac{1}{2}\|-\nabla(2\phi_h^j-\phi_h^{j-1})+\mathbf{u}_h^{j+1}\times\mathbf{B}\|^2+\frac{1}{2}\|-\nabla(2\phi_h^j-\phi_h^{j-1})+(2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})\times\mathbf{B}\|^2\\&+\frac{1}{2}\|(\frac{1}{\epsilon}(-\mathbf{u}_h^{j+1}+2\mathbf{u}_h^j-\mathbf{u}_h^{j-1})-\epsilon\mathbf{u}_h^{j+1})\times\mathbf{B}\|^2+\left(\frac{1}{M^2}-\frac{1}{2}\epsilon^2C_P^2\|\mathbf{B}\|^2_{L^\infty}\right)\|\nabla\mathbf{u}_h^{j+1}\|^2\\&\le(\mathbf{f}^{j+1},\mathbf{u}_h^{j+1}).\end{aligned}\tag{3.9}$$

Let $\epsilon = (C_P M \|\mathbf{B}\|_{L^\infty})^{-1}$, under the condition (3.3), we have from (3.9)

$$
\begin{aligned}
&\frac{1}{4\Delta t}\cdot\frac{1}{N}\left(3\|\mathbf{u}_h^{j+1}\|^2 - 4\|\mathbf{u}_h^j\|^2 + \|\mathbf{u}_h^{j-1}\|^2\right)\\
&\qquad + \frac{1}{2\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^{j+1}-\mathbf{u}_h^j\|^2 - \frac{1}{2\Delta t}\cdot\frac{1}{N}\|\mathbf{u}_h^j - \mathbf{u}_h^{j-1}\|^2\\
&+\frac{1}{2}\|-\nabla(2\phi_h^j - \phi_h^{j-1}) + \mathbf{u}_h^{j+1}\times\mathbf{B}\|^2 + \frac{1}{2}\|-\nabla(2\phi_h^j-\phi_h^{j-1}) + (2\mathbf{u}_h^j - \mathbf{u}_h^{j-1})\times\mathbf{B}\|^2\\
&+\frac{1}{2}\|(\frac{1}{\epsilon}(-\mathbf{u}_h^{j+1} + 2\mathbf{u}_h^j - \mathbf{u}_h^{j-1}) - \epsilon\mathbf{u}_h^{j+1})\times\mathbf{B}\|^2 + \frac{1}{2M^2}\|\nabla\mathbf{u}_h^{j+1}\|^2\\
&\le (\mathbf{f}^{j+1}, \mathbf{u}_h^{j+1}).
\end{aligned}
$$

Summing from $j = 1$ to $n-1$, multiply both sides by $2\Delta t$ and use the identity

$$
\frac{3}{2}a^2 - \frac{1}{2}b^2 + (a-b)^2 = \frac{a^2}{2} + \left(a\sqrt{2} - \frac{b}{\sqrt{2}}\right)^2
$$

we get

$$
\begin{aligned}
&\frac{1}{2N}\|\mathbf{u}_h^n\|^2 + \frac{1}{2N}\|2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \frac{\Delta t}{M^2}\sum_{j=1}^{n-1}\|\nabla\mathbf{u}_h^{j+1}\|^2\\
&+\Delta t\sum_{j=1}^{n-1}\|-\nabla(2\phi_h^j - \phi_h^{j-1}) + \mathbf{u}_h^{j+1}\times\mathbf{B}\|^2\\
&\qquad + \Delta t\sum_{j=1}^{n-1}\|-\nabla(2\phi_h^j - \phi_h^{j-1}) + (2\mathbf{u}_h^j - \mathbf{u}_h^{j-1})\times\mathbf{B}\|^2\\
&\le \frac{1}{2N}\|\mathbf{u}_h^1\|^2 + \frac{1}{2N}\|2\mathbf{u}_h^1 - \mathbf{u}_h^0\|^2 + 2\Delta t\sum_{j=1}^{n-1}(\mathbf{f}^{j+1}, \mathbf{u}_h^{j+1}).
\end{aligned}
$$

Applying Young's inequality for the term involving body force yields the energy estimate (3.4). □

4. Error analysis. We proceed to give an á priori error estimate for the partitioned methods IMEX1 and IMEX2. Due to the length and technicality of the proofs, for the compactness, we only analyze the error of the first order IMEX scheme, i.e. Algorithm IMEX1. With minor modifications (and greater length), the analogous convergence rates are obtained for Algorithm IMEX2.

Let $t^j = j\Delta t$ and $\mathbf{u}^j := \mathbf{u}(t^j)$ (and similarly for other variables). To establish the optimal error estimate for the model, we introduce the following discrete norms

$$
|||\omega|||_{\infty,k} := \max_{0\le j\le T/\Delta t}\|\omega^j\|_k, \quad |||\omega|||_{2,k} := \left(\sum_{j=0}^{T/\Delta t}\|\omega^j\|_k^2\Delta t\right)^{1/2}
$$

and assume the following regularity of the true solution

$$
\begin{aligned}
&\mathbf{u}\in L^\infty(0,T;(H^{k+1}(\Omega))^d)\cap H^1(0,T;(H^{k+1}(\Omega))^d)\cap H^2(0,T;(L^2(\Omega))^d),\\
&p\in L^2(0,T;H^{s+1}(\Omega)),\ \phi\in L^\infty(0,T;H^{k+1}(\Omega))\cap H^1(0,T;H^1(\Omega))
\end{aligned}
\tag{4.1}
$$

Denote the errors by $e_{\mathbf{u}}^j = \mathbf{u}^j - \mathbf{u}_h^j, e_\phi^j = \phi^j - \phi_h^j$ and $e_{\mathbf{J}}^j = -\nabla e_\phi^j + e_{\mathbf{u}}^j \times \mathbf{B}$. We have the following theorem

THEOREM 4.1. *For $\mathbf{u}, p, \phi$ satisfying the weak formulation (2.1) and regularity condition (4.1), and $\mathbf{u}_h^n, p_h^n, \phi_h^n$ given by Algorithm IMEX1 with $n \in \{1, 2, ..., \frac{T}{\Delta t}\}$, we have for Δt sufficiently small*

$$\begin{aligned}
&\|e_{\mathbf{u}}^n\|^2 + \sum_{j=0}^{n-1} \|e_{\mathbf{u}}^{j+1} - e_{\mathbf{u}}^j\|^2 + \frac{N\Delta t}{M^2} \sum_{j=0}^{n-1} \|\nabla e_{\mathbf{u}}^{j+1}\|^2 + N\Delta t \sum_{j=0}^{n-1} \|\nabla e_\phi^{j+1}\|^2 \\
&+ N\Delta t \sum_{j=0}^{n-1} \| - \nabla e_\phi^j + e_{\mathbf{u}}^{j+1} \times \mathbf{B}\|^2 + N\Delta t \sum_{j=0}^{n-1} \| - \nabla e_\phi^{j+1} + e_{\mathbf{u}}^j \times \mathbf{B}\|^2 \\
&\leq C_0 \Bigg(\|\mathbf{u}^0 - \mathbf{u}_h^0\|^2 + \|\nabla(\phi^0 - \phi_h^0)\|^2 + h^{2k+2} |||\mathbf{u}|||_{\infty,k+1}^2 + h^{2k} |||\phi|||_{\infty,k+1}^2 \\
&+ h^{2k+2} \|\mathbf{u}_t\|_{2,k+1}^2 + h^{2k} |||\mathbf{u}|||_{2,k+1}^2 + h^{4k} |||\mathbf{u}|||_{4,k+1}^4 + \Delta t^2 \|\phi_t\|_{2,1}^2 + h^{2s+2} |||p|||_{2,s+1}^2 \\
&\quad + \Delta t^2 \|\mathbf{u}_{tt}\|_{2,0}^2 + h^{2k} |||\phi|||_{2,k+1}^2 + h^{2k+2} |||\mathbf{u}|||_{2,k+1}^2 + \Delta t^2 \|\mathbf{u}_t\|_{2,0}^2 \Bigg).
\end{aligned} \tag{4.2}$$

Proof. At time $t^{j+1} = (j+1)\Delta t$, the true solution $(\mathbf{u}, p, \phi)$ of (2.1) satisfies

$$\begin{aligned}
&\frac{1}{N} \left(\left(\frac{\mathbf{u}^{j+1} - \mathbf{u}^j}{\Delta t}, \mathbf{v}_h \right) + b(\mathbf{u}^{j+1}, \mathbf{u}^{j+1}, \mathbf{v}_h) \right) + \frac{1}{M^2} (\nabla \mathbf{u}^{j+1}, \nabla \mathbf{v}_h) \\
&\qquad - (p^{j+1}, \nabla \cdot \mathbf{v}_h) + (\mathbf{u}^{j+1} \times \mathbf{B}, \mathbf{v}_h \times \mathbf{B}) - (\nabla \phi^j, \mathbf{v}_h \times \mathbf{B}) = (\mathbf{f}^{j+1}, \mathbf{v}_h) \\
&\qquad + (\nabla(\phi^{j+1} - \phi^j), \mathbf{v}_h \times \mathbf{B}) + \frac{1}{N} \left(\frac{\mathbf{u}^{j+1} - \mathbf{u}^j}{\Delta t} - u_t(t^{j+1}), \mathbf{v}_h \right) \quad \forall \mathbf{v}_h \in X_h, \\
&- (\nabla \phi^{j+1}, \nabla \psi_h) + (\mathbf{u}^j \times \mathbf{B}, \nabla \psi_h) = -((\mathbf{u}^{j+1} - \mathbf{u}^j) \times \mathbf{B}, \nabla \psi_h) \quad \forall \psi_h \in S_h.
\end{aligned} \tag{4.3}$$

We construct the error equations for velocity and electric potential. Decompose the velocity error

$$\mathbf{u}^{j+1} - \mathbf{u}_h^{j+1} = (\mathbf{u}^{j+1} - \tilde{\mathbf{u}}_h^{j+1}) + (\tilde{\mathbf{u}}_h^{j+1} - \mathbf{u}_h^{j+1}) =: \eta^{j+1} + \mathbf{U}_h^{j+1}$$

and the electric potential error

$$\phi^{j+1} - \phi_h^{j+1} = (\phi^{j+1} - \tilde{\phi}_h^{j+1}) + (\tilde{\phi}_h^{j+1} - \phi_h^{j+1}) =: \zeta^{j+1} + \Phi_h^{j+1}$$

where $\tilde{\mathbf{u}}_h^{j+1}$ and $\tilde{\phi}_h^{j+1}$ will be the interpolation of $\mathbf{u}^{j+1}$ and ϕ^{j+1} in V_h and S_h, respectively.

Substract (4.3) from (IMEX1) and set $\mathbf{v}_h = \mathbf{U}_h^{j+1}$ and $\psi_h = \Phi_h^{j+1}$ to obtain

$$\begin{aligned}
&\frac{1}{2\Delta t} \cdot \frac{1}{N} \left(\|\mathbf{U}_h^{j+1}\|^2 - \|\mathbf{U}_h^j\|^2 + \|\mathbf{U}_h^{j+1} - \mathbf{U}_h^j\|^2 \right) + \frac{1}{M^2} \|\nabla \mathbf{U}_h^{j+1}\|^2 + \|\mathbf{B} \times \mathbf{U}_h^{j+1}\|^2 \\
&- (\nabla \Phi_h^j, \mathbf{U}_h^{j+1} \times \mathbf{B}) = -\frac{1}{N} \left(\frac{\eta^{j+1} - \eta^j}{\Delta t}, \mathbf{U}_h^{j+1} \right) - \frac{1}{N} b(\mathbf{U}_h^{j+1}, \mathbf{u}^{j+1}, \mathbf{U}_h^{j+1}) \\
&\qquad - \frac{1}{N} b(\mathbf{u}_h^{j+1}, \eta^{j+1}, \mathbf{U}_h^{j+1}) - \frac{1}{N} b(\eta^{j+1}, \mathbf{u}^{j+1}, \mathbf{U}_h^{j+1}) + (p^{j+1} - \lambda_h^{j+1}, \nabla \cdot \mathbf{U}_h^{j+1}) \\
&\qquad - \frac{1}{M^2} (\nabla \eta^{j+1}, \nabla \mathbf{U}_h^{j+1}) - (\eta^{j+1} \times \mathbf{B}, \mathbf{U}_h^{j+1} \times \mathbf{B}) + (\nabla \zeta^j, \mathbf{U}_h^{j+1} \times \mathbf{B})
\end{aligned} \tag{4.4}$$

$$+ (\nabla(\phi^{j+1} - \phi^j), \mathbf{U}_h^{j+1} \times \mathbf{B}) + \frac{1}{N}\left(\frac{\mathbf{u}^{j+1} - \mathbf{u}^j}{\Delta t} - u_t(t^{j+1}), \mathbf{U}_h^{j+1}\right),$$

for every $\lambda_h^{j+1} \in Q^h$, and

$$\begin{aligned} -\|\nabla\Phi_h^{j+1}\|^2 + (\mathbf{U}_h^j \times \mathbf{B}, \nabla\Phi_h^{j+1}) &= (\nabla\zeta^{j+1}, \nabla\Phi_h^{j+1}) \\ &- (\eta^j \times \mathbf{B}, \nabla\Phi_h^{j+1}) - ((\mathbf{u}^{j+1} - \mathbf{u}^j) \times \mathbf{B}, \nabla\Phi_h^{j+1}). \end{aligned} \tag{4.5}$$

We have from (4.5)

$$\begin{aligned} 2\|\nabla\Phi_h^{j+1}\|^2 - (\mathbf{U}_h^j \times \mathbf{B}, \nabla\Phi_h^{j+1}) &= (\mathbf{U}_h^j \times \mathbf{B}, \nabla\Phi_h^{j+1}) - 2(\nabla\zeta^{j+1}, \nabla\Phi_h^{j+1}) \\ &+ 2(\eta^j \times \mathbf{B}, \nabla\Phi_h^{j+1}) + 2((\mathbf{u}^{j+1} - \mathbf{u}^j) \times \mathbf{B}, \nabla\Phi_h^{j+1}). \end{aligned} \tag{4.6}$$

Adding (4.4) and (4.6) and rearranging terms in the left hand side give

$$\begin{aligned} &\frac{1}{2\Delta t}\cdot\frac{1}{N}\left(\|\mathbf{U}_h^{j+1}\|^2 - \|\mathbf{U}_h^j\|^2 + \|\mathbf{U}_h^{j+1} - \mathbf{U}_h^j\|^2\right) + \frac{1}{M^2}\|\nabla\mathbf{U}_h^{j+1}\|^2 \\ &\quad + \frac{1}{2}\|\mathbf{U}_h^{j+1} \times \mathbf{B}\|^2 - \frac{1}{2}\|\mathbf{U}_h^j \times \mathbf{B}\|^2 + \frac{1}{2}\|\nabla\Phi_h^{j+1}\|^2 - \frac{1}{2}\|\nabla\Phi_h^j\|^2 \\ &+ \frac{1}{2}\| - \nabla\Phi_h^j + \mathbf{U}_h^{j+1} \times \mathbf{B}\|^2 + \frac{1}{2}\| - \nabla\Phi_h^{j+1} + \mathbf{U}_h^j \times \mathbf{B}\|^2 + \|\nabla\Phi_h^{j+1}\|^2 \\ &= -\frac{1}{N}\left(\frac{\eta^{j+1} - \eta^j}{\Delta t}, \mathbf{U}_h^{j+1}\right) - \frac{1}{N}b(\mathbf{U}_h^{j+1}, \mathbf{u}^{j+1}, \mathbf{U}_h^{j+1}) \\ &- \frac{1}{N}b(\mathbf{u}_h^{j+1}, \eta^{j+1}, \mathbf{U}_h^{j+1}) - \frac{1}{N}b(\eta^{j+1}, \mathbf{u}^{j+1}, \mathbf{U}_h^{j+1}) + (p^{j+1} - \lambda_h^{j+1}, \nabla\cdot\mathbf{U}_h^{j+1}) \\ &- \frac{1}{M^2}(\nabla\eta^{j+1}, \nabla\mathbf{U}_h^{j+1}) - (-\nabla\zeta^j + \eta^{j+1} \times \mathbf{B}, \mathbf{U}_h^{j+1} \times \mathbf{B}) + (\mathbf{U}_h^j \times \mathbf{B}, \nabla\Phi_h^{j+1}) \\ &\quad + (\nabla(\phi^{j+1} - \phi^j), \mathbf{U}_h^{j+1} \times \mathbf{B}) + \frac{1}{N}\left(\frac{\mathbf{u}^{j+1} - \mathbf{u}^j}{\Delta t} - \mathbf{u}_t(t^{j+1}), \mathbf{U}_h^{j+1}\right) \\ &\quad - 2(\nabla\zeta^{j+1}, \nabla\Phi_h^{j+1}) + 2(\eta^j \times \mathbf{B}, \nabla\Phi_h^{j+1}) + 2((\mathbf{u}^{j+1} - \mathbf{u}^j) \times \mathbf{B}, \nabla\Phi_h^{j+1}). \end{aligned} \tag{4.7}$$

We proceed to bound each term on the right hand side of (4.7), absorb like-terms into the left hand side. For an arbitrary $\varepsilon > 0$,

$$-\frac{1}{N}\left(\frac{\eta^{j+1} - \eta^j}{\Delta t}, \mathbf{U}_h^{j+1}\right) \le \frac{1}{4\varepsilon N^2}\left\|\frac{\eta^{j+1} - \eta^j}{\Delta t}\right\|_{-1}^2 + \varepsilon\|\nabla\mathbf{U}_h^{j+1}\|^2. \tag{4.8}$$

The first nonlinear term can be bounded as

$$\begin{aligned} -\frac{1}{N}b(\mathbf{U}_h^{j+1}, \mathbf{u}^{j+1}, \mathbf{U}_h^{j+1}) &\le C_0\|\mathbf{U}_h^{j+1}\|\|\mathbf{u}^{j+1}\|_2\|\nabla\mathbf{U}_h^{j+1}\| \\ &\le \frac{C_0^2}{4\varepsilon}\|\mathbf{U}_h^{j+1}\|^2\|\mathbf{u}^{j+1}\|_2^2 + \varepsilon\|\nabla\mathbf{U}_h^{j+1}\|^2. \end{aligned} \tag{4.9}$$

We now give an estimation for $-\frac{1}{N}b(\mathbf{u}_h^{j+1}, \eta^{j+1}, \mathbf{U}_h^{j+1})$:

$$\begin{aligned} -\frac{1}{N}b(\mathbf{u}_h^{j+1}, \eta^{j+1}, \mathbf{U}_h^{j+1}) &= -\frac{1}{N}b(\mathbf{u}^{j+1}, \eta^{j+1}, \mathbf{U}_h^{j+1}) \\ &+ \frac{1}{N}b(\eta^{j+1}, \eta^{j+1}, \mathbf{U}_h^{j+1}) + \frac{1}{N}b(\mathbf{U}_h^{j+1}, \eta^{j+1}, \mathbf{U}_h^{j+1}), \end{aligned}$$

where terms in the right hand side can be controlled as

$$\begin{aligned}-\frac{1}{N}b(\mathbf{u}^{j+1},\eta^{j+1},\mathbf{U}_h^{j+1}) &\le C_0\|\nabla\mathbf{u}^{j+1}\|\|\nabla\eta^{j+1}\|\|\nabla\mathbf{U}_h^{j+1}\| \\ &\le \frac{C_0^2}{4\varepsilon}\|\mathbf{u}\|_{\infty,1}^2\|\nabla\eta^{j+1}\|^2+\varepsilon\|\nabla\mathbf{U}_h^{j+1}\|^2,\end{aligned}\tag{4.10}$$

$$\frac{1}{N}b(\eta^{j+1},\eta^{j+1},\mathbf{U}_h^{j+1}) \le \frac{C_0^2}{4\varepsilon}\|\nabla\eta^{j+1}\|^4+\varepsilon\|\nabla\mathbf{U}_h^{j+1}\|^2,\tag{4.11}$$

and

$$\begin{aligned}\frac{1}{N}b(\mathbf{U}_h^{j+1},\eta^{j+1},\mathbf{U}_h^{j+1}) &\le C_0\|\mathbf{U}_h^{j+1}\|^{1/2}\|\nabla\mathbf{U}_h^{j+1}\|^{1/2}\|\nabla\eta^{j+1}\|\|\nabla\mathbf{U}_h^{j+1}\| \\ &\le C_0h^{-1/2}\|\mathbf{U}_h^{j+1}\|\|\nabla\eta^{j+1}\|\|\nabla\mathbf{U}_h^{j+1}\| \le C_0h^{1/2}\|\mathbf{U}_h^{j+1}\|\|\mathbf{u}^{j+1}\|_2\|\nabla\mathbf{U}_h^{j+1}\| \\ &\le \frac{C_0^2}{4\varepsilon}h\|\mathbf{U}_h^{j+1}\|^2\|\mathbf{u}^{j+1}\|_2^2+\varepsilon\|\nabla\mathbf{U}_h^{j+1}\|^2.\end{aligned}\tag{4.12}$$

The last nonlinear term can be bounded exactly like in (4.10). For the pressure term

$$(p^{j+1}-\lambda_h^{j+1},\nabla\cdot\mathbf{U}_h^{j+1}) \le \frac{C_0^2}{4\varepsilon}\|p^{j+1}-\lambda_h^{j+1}\|^2+\varepsilon\|\nabla\mathbf{U}_h^{j+1}\|^2.\tag{4.13}$$

We continue to deal with the remaining terms. Firstly,

$$-\frac{1}{M^2}(\nabla\eta^{j+1},\nabla\mathbf{U}_h^{j+1}) \le \frac{C_0^2}{4\varepsilon}\|\nabla\eta^{j+1}\|^2+\varepsilon\|\nabla\mathbf{U}_h^{j+1}\|^2.\tag{4.14}$$

Next, we have

$$\begin{aligned}-(-\nabla\zeta^j+\eta^{j+1}\times\mathbf{B},\mathbf{U}_h^{j+1}\times\mathbf{B}) &\le \|-\nabla\zeta^j+\eta^{j+1}\times\mathbf{B}\|\|\mathbf{U}_h^{j+1}\times\mathbf{B}\| \\ &\le C_0\|-\nabla\zeta^j+\eta^{j+1}\times\mathbf{B}\|^2+\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{U}_h^{j+1}\|^2.\end{aligned}\tag{4.15}$$

Also, observe that

$$(\mathbf{U}_h^j\times\mathbf{B},\nabla\Phi_h^{j+1}) \le \|\mathbf{U}_h^j\times\mathbf{B}\|\|\nabla\Phi_h^{j+1}\| \le \frac{1}{4\varepsilon'}\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{U}_h^j\|^2+\varepsilon'\|\nabla\Phi_h^{j+1}\|^2,\tag{4.16}$$

and

$$(\nabla(\phi^{j+1}-\phi^j),\mathbf{U}_h^{j+1}\times\mathbf{B}) \le C_0\|\nabla(\phi^{j+1}-\phi^j)\|^2+\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{U}_h^{j+1}\|^2.\tag{4.17}$$

Furthermore,

$$\begin{aligned}\frac{1}{N}\left(\frac{\mathbf{u}^{j+1}-\mathbf{u}^j}{\Delta t}-\mathbf{u}_t(t^{j+1}),\mathbf{U}_h^{j+1}\right) &\le C_0\left\|\frac{\mathbf{u}^{j+1}-\mathbf{u}^j}{\Delta t}-\mathbf{u}_t(t^{j+1})\right\|\left\|\nabla\mathbf{U}_h^{j+1}\right\| \\ &\le \frac{C_0^2}{4\varepsilon}\left\|\frac{\mathbf{u}^{j+1}-\mathbf{u}^j}{\Delta t}-\mathbf{u}_t(t^{j+1})\right\|^2+\varepsilon\left\|\nabla\mathbf{U}_h^{j+1}\right\|^2.\end{aligned}\tag{4.18}$$

We also have

$$-2(\nabla\zeta^{j+1},\nabla\Phi_h^{j+1}) \le 2\|\nabla\zeta^{j+1}\|\|\nabla\Phi_h^{j+1}\| \le \frac{1}{\varepsilon'}\|\nabla\zeta^{j+1}\|^2+\varepsilon'\|\nabla\Phi_h^{j+1}\|^2.\tag{4.19}$$

Finally, it gives

$$2(\eta^j \times \mathbf{B}, \nabla\Phi_h^{j+1}) \le 2\|\eta^j \times \mathbf{B}\|\|\nabla\Phi_h^{j+1}\| \le \frac{1}{\varepsilon'}\|\mathbf{B}\|_{L^\infty}^2\|\eta^j\|^2 + \varepsilon'\|\nabla\Phi_h^{j+1}\|^2, \quad (4.20)$$

and

$$\begin{aligned} 2((\mathbf{u}^{j+1} - \mathbf{u}^j) \times \mathbf{B}, \nabla\Phi_h^{j+1}) &\le 2\|(\mathbf{u}^{j+1} - \mathbf{u}^j) \times \mathbf{B}\|\|\nabla\Phi_h^{j+1}\| \\ &\le \frac{1}{\varepsilon'}\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{u}^{j+1} - \mathbf{u}^j\|^2 + \varepsilon'\|\nabla\Phi_h^{j+1}\|^2. \end{aligned} \quad (4.21)$$

Applying estimate (4.8)–(4.21) to (4.7) with $\varepsilon = \frac{1}{18M^2}$ and $\varepsilon' = \frac{1}{8}$ gives

$$\begin{aligned} &\frac{1}{2\Delta t}\cdot\frac{1}{N}\left(\|\mathbf{U}_h^{j+1}\|^2 - \|\mathbf{U}_h^j\|^2 + \|\mathbf{U}_h^{j+1} - \mathbf{U}_h^j\|^2\right) + \frac{1}{2M^2}\|\nabla\mathbf{U}_h^{j+1}\|^2 \\ &\quad + \frac{1}{2}\|\mathbf{U}_h^{j+1} \times \mathbf{B}\|^2 - \frac{1}{2}\|\mathbf{U}_h^j \times \mathbf{B}\|^2 + \frac{1}{2}\|\nabla\Phi_h^{j+1}\|^2 - \frac{1}{2}\|\nabla\Phi_h^j\|^2 \\ &+ \frac{1}{2}\| - \nabla\Phi_h^j + \mathbf{U}_h^{j+1} \times \mathbf{B}\|^2 + \frac{1}{2}\| - \nabla\Phi_h^{j+1} + \mathbf{U}_h^j \times \mathbf{B}\|^2 + \frac{1}{2}\|\nabla\Phi_h^{j+1}\|^2 \\ &\le \left(\frac{9}{2}C_0^2M^2\|\mathbf{u}^{j+1}\|_2^2(1+h) + 2\|\mathbf{B}\|_{L^\infty}^2\right)\|\mathbf{U}_h^{j+1}\|^2 + 2\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{U}_h^j\|^2 \\ &+ \frac{9M^2}{2N^2}\left\|\frac{\eta^{j+1} - \eta^j}{\Delta t}\right\|_{-1}^2 + 9C_0^2M^2\|\mathbf{u}\|_{\infty,1}^2\|\nabla\eta^{j+1}\|^2 + \frac{9}{2}C_0^2M^2\|\nabla\eta^{j+1}\|^4 \\ &+ \frac{9}{2}C_0^2M^2\|p^{j+1} - \lambda_h^{j+1}\|^2 + \frac{9}{2}C_0^2M^2\|\nabla\eta^{j+1}\|^2 + C_0\| - \nabla\zeta^j + \eta^{j+1} \times \mathbf{B}\|^2 \\ &\quad + C_0\|\nabla(\phi^{j+1} - \phi^j)\|^2 + \frac{9}{2}C_0^2M^2\left\|\frac{\mathbf{u}^{j+1} - \mathbf{u}^j}{\Delta t} - \mathbf{u}_t(t^{j+1})\right\|^2 \\ &\quad + 8\|\nabla\zeta^{j+1}\|^2 + 8\|\mathbf{B}\|_{L^\infty}^2\|\eta^j\|^2 + 8\|\mathbf{B}\|_{L^\infty}^2\|\mathbf{u}^{j+1} - \mathbf{u}^j\|^2. \end{aligned} \quad (4.22)$$

Let $\kappa = 9C_0^2M^2N\|\mathbf{u}\|_{\infty,2}^2(1+h) + 8\|\mathbf{B}\|_{L^\infty}^2 N$, summing from $j = 0$ to $j = n-1$ and applying the discrete Gronwall lemma yield

$$\begin{aligned} &\|\mathbf{U}_h^n\|^2 + \sum_{j=0}^{n-1}\|\mathbf{U}_h^{j+1} - \mathbf{U}_h^j\|^2 + \frac{N\Delta t}{M^2}\sum_{j=0}^{n-1}\|\nabla\mathbf{U}_h^{j+1}\|^2 + N\Delta t\sum_{j=0}^{n-1}\|\nabla\Phi_h^{j+1}\|^2 \\ &+ N\Delta t\sum_{j=0}^{n-1}\| - \nabla\Phi_h^j + \mathbf{U}_h^{j+1} \times \mathbf{B}\|^2 + N\Delta t\sum_{j=0}^{n-1}\| - \nabla\Phi_h^{j+1} + \mathbf{U}_h^j \times \mathbf{B}\|^2 \\ &\le \exp\left((n+1)\frac{\Delta t\kappa}{1 - \Delta t\kappa}\right)\Bigg(\|\mathbf{U}_h^0\|^2 + N\Delta t\|\mathbf{U}_h^0 \times \mathbf{B}\|^2 + N\Delta t\|\nabla\Phi_h^0\|^2 \\ &+ \Delta t\frac{9M^2}{N}\sum_{j=0}^{n-1}\left\|\frac{\eta^{j+1} - \eta^j}{\Delta t}\right\|_{-1}^2 + 2N\Delta t\left(9C_0^2M^2\|\mathbf{u}\|_{\infty,1}^2 + \frac{9}{2}C_0^2M^2\right)\sum_{j=0}^{n-1}\|\nabla\eta^{j+1}\|^2 \\ &+ 9\Delta tC_0^2M^2N\sum_{j=0}^{n-1}\|\nabla\eta^{j+1}\|^4 + 9\Delta tC_0^2M^2N\sum_{j=0}^{n-1}\|p^{j+1} - \lambda_h^{j+1}\|^2 \\ &+ 2N\Delta tC_0\sum_{j=0}^{n-1}\|\nabla(\phi^{j+1} - \phi^j)\|^2 + 9\Delta tC_0^2M^2N\sum_{j=0}^{n-1}\left\|\frac{\mathbf{u}^{j+1} - \mathbf{u}^j}{\Delta t} - \mathbf{u}_t(t^{j+1})\right\|^2 \end{aligned} \quad (4.23)$$

$$
+ 2N\Delta t(2C_0+8)\sum_{j=0}^{n} \|\nabla\zeta^j\|^2 + 2N\Delta t(2C_0+8)\|\mathbf{B}\|^2_{L^\infty} \sum_{j=0}^{n} \|\eta^j\|^2
$$

$$
+ 16N\Delta t\|\mathbf{B}\|^2_{L^\infty} \sum_{j=0}^{n-1} \|\mathbf{u}^{j+1} - \mathbf{u}^j\|^2 \Bigg)
$$

provided that $\Delta t < 1/\kappa$.

We next bound the right hand side of (4.23). First,

$$
\begin{aligned}
&\|\mathbf{U}_h^0\|^2 + N\Delta t\|\mathbf{U}_h^0 \times \mathbf{B}\|^2 + N\Delta t\|\nabla\Phi_h^0\|^2 \\
\le\, & 2\|\mathbf{u}^0 - \mathbf{u}_h^0\|^2 + 2\|\eta^0\|^2 + 2N\Delta t\|(\mathbf{u}^0 - \mathbf{u}_h^0) \times \mathbf{B}\|^2 + 2N\Delta t\|\eta^0 \times \mathbf{B}\|^2 \\
& \qquad + 2N\Delta t\|\nabla(\phi^0 - \phi_h^0)\|^2 + 2N\Delta t\|\nabla\zeta^0\|^2 \\
\le\, & (2 + 2N\Delta t\|\mathbf{B}\|^2_{L^\infty})\|\mathbf{u}^0 - \mathbf{u}_h^0\|^2 + 2N\Delta t\|\nabla(\phi^0 - \phi_h^0)\|^2 \\
& \quad + C_0(2 + 2N\Delta t\|\mathbf{B}\|^2_{L^\infty})h^{2k+2}|||\mathbf{u}|||^2_{\infty,k+1} + 2N\Delta t C_0 h^{2k}|||\phi|||^2_{\infty,k+1} \\
\le\, & C_0\|\mathbf{u}^0 - \mathbf{u}_h^0\|^2 + C_0\|\nabla(\phi^0 - \phi_h^0)\|^2 + C_0 h^{2k+2}|||\mathbf{u}|||^2_{\infty,k+1} + C_0 h^{2k}|||\phi|||^2_{\infty,k+1}.
\end{aligned}
\tag{4.24}
$$

The next term can be controlled as follows

$$
\Delta t\frac{9M^2}{N}\sum_{j=0}^{n-1}\left\|\frac{\eta^{j+1}-\eta^j}{\Delta t}\right\|_{-1}^2 \le C_0\sum_{j=0}^{n-1}\int_{t^j}^{t^{j+1}}\|\eta_t\|^2 dt \le C_0 h^{2k+2}\|\mathbf{u}_t\|^2_{2,k+1}. \tag{4.25}
$$

We also have

$$
\begin{aligned}
&9N\Delta t C_0^2 M^2\big(2\|\mathbf{u}\|^2_{\infty,1}+1\big)\sum_{j=0}^{n-1}\|\nabla\eta^{j+1}\|^2 \\
&\qquad \le C_0\Delta t\sum_{j=0}^{n-1} h^{2k}\|\mathbf{u}^{j+1}\|^2_{k+1} = C_0 h^{2k}|||\mathbf{u}|||^2_{2,k+1}.
\end{aligned}
\tag{4.26}
$$

Observe that

$$
9\Delta t C_0^2 M^2 N\sum_{j=0}^{n-1}\|\nabla\eta^{j+1}\|^4 \le C_0\Delta t\sum_{j=0}^{n-1} h^{4k}\|\mathbf{u}^{j+1}\|^4_{k+1} = C_0 h^{4k}|||\mathbf{u}|||^4_{4,k+1}, \tag{4.27}
$$

and

$$
2N\Delta t C_0\sum_{j=0}^{n-1}\|\nabla(\phi^{j+1}-\phi^j)\|^2 \le C_0\Delta t^2\sum_{j=0}^{n-1}\int_{t^j}^{t^{j+1}}\|\nabla\phi_t\|^2 dt \le C_0\Delta t^2\|\phi_t\|^2_{2,1}. \tag{4.28}
$$

Let λ_h^{j+1} be the interpolation of p^{j+1} in Q_h, we have

$$
9\Delta t C_0^2 M^2 N\sum_{j=0}^{n-1}\|p^{j+1}-\lambda_h^{j+1}\|^2 \le C_0 h^{2s+2}|||p|||^2_{2,s+1}. \tag{4.29}
$$

Moreover, it gives

$$
9\Delta t C_0^2 M^2 N\sum_{j=0}^{n-1}\left\|\frac{\mathbf{u}^{j+1}-\mathbf{u}^j}{\Delta t}-\mathbf{u}_t(t^{j+1})\right\|^2 \le C_0\Delta t^2\sum_{j=0}^{n-1}\int_{t^j}^{t^{j+1}}\|\mathbf{u}_{tt}\|^2 dt = C_0\Delta t^2\|\mathbf{u}_{tt}\|^2_{2,0}. \tag{4.30}
$$

On the other hand, we can see that

$$2N\Delta t(2C_0+8)\sum_{j=0}^{n}\|\nabla\zeta^j\|^2 \le C_0\Delta t\sum_{j=0}^{n}h^{2k}\|\phi^j\|_{k+1}^2 = C_0 h^{2k}|||\phi|||_{2,k+1}^2. \tag{4.31}$$

Finally, we have

$$2N\Delta t(2C_0+8)\|\mathbf{B}\|_{L^\infty}^2\sum_{j=0}^{n}\|\eta^j\|^2 \le C_0\Delta t\sum_{j=0}^{n}h^{2k+2}\|\mathbf{u}^j\|_{k+1}^2 = C_0 h^{2k+2}|||\mathbf{u}|||_{2,k+1}^2, \tag{4.32}$$

and

$$16N\Delta t\|\mathbf{B}\|_{L^\infty}^2\sum_{j=0}^{n-1}\|\mathbf{u}^{j+1}-\mathbf{u}^j\|^2 \le C_0\Delta t^2\sum_{j=0}^{n-1}\int_{t^j}^{t^{j+1}}\|\mathbf{u}_t\|^2dt \le C_0\Delta t^2\|\mathbf{u}_t\|_{2,0}^2. \tag{4.33}$$

Combining (4.23)–(4.33) gives

$$\begin{aligned}
&\|\mathbf{U}_h^n\|^2 + \sum_{j=0}^{n-1}\|\mathbf{U}_h^{j+1}-\mathbf{U}_h^j\|^2 + \frac{N\Delta t}{M^2}\sum_{j=0}^{n-1}\|\nabla\mathbf{U}_h^{j+1}\|^2 + N\Delta t\sum_{j=0}^{n-1}\|\nabla\Phi_h^{j+1}\|^2 \\
&+ N\Delta t\sum_{j=0}^{n-1}\|-\nabla\Phi_h^j + \mathbf{U}_h^{j+1}\times\mathbf{B}\|^2 + N\Delta t\sum_{j=0}^{n-1}\|-\nabla\Phi_h^{j+1} + \mathbf{U}_h^j\times\mathbf{B}\|^2 \\
&\le C_0\Bigg(\|\mathbf{u}^0-\mathbf{u}_h^0\|^2 + \|\nabla(\phi^0-\phi_h^0)\|^2 + h^{2k+2}|||\mathbf{u}|||_{\infty,k+1}^2 + h^{2k}|||\phi|||_{\infty,k+1}^2 \\
&+ h^{2k+2}\|\mathbf{u}_t\|_{2,k+1}^2 + h^{2k}|||\mathbf{u}|||_{2,k+1}^2 + h^{4k}|||\mathbf{u}|||_{4,k+1}^4 + \Delta t^2\|\phi_t\|_{2,1}^2 + h^{2s+2}|||p|||_{2,s+1}^2 \\
&\quad + \Delta t^2\|\mathbf{u}_{tt}\|_{2,0}^2 + h^{2k}|||\phi|||_{2,k+1}^2 + h^{2k+2}|||\mathbf{u}|||_{2,k+1}^2 + \Delta t^2\|\mathbf{u}_t\|_{2,0}^2\Bigg).
\end{aligned} \tag{4.34}$$

To obtain the error estimate given in (4.2), we add both sides of (4.34) with

$$\begin{aligned}
Extra_terms = &\|\eta^n\|^2 + \sum_{j=0}^{n-1}\|\eta^{j+1}-\eta^j\|^2 + \frac{N\Delta t}{M^2}\sum_{j=0}^{n-1}\|\nabla\eta^{j+1}\|^2 + N\Delta t\sum_{j=0}^{n-1}\|\nabla\zeta^{j+1}\|^2 \\
&+ N\Delta t\sum_{j=0}^{n-1}\|-\nabla\zeta^j + \eta^{j+1}\times\mathbf{B}\|^2 + N\Delta t\sum_{j=0}^{n-1}\|-\nabla\zeta^{j+1} + \eta^j\times\mathbf{B}\|^2
\end{aligned}$$

and apply the triangle inequality for the left hand side, noticing that the upcoming new terms are already contained in the right hand side of the model. ◻

Consequently, for Taylor-Hood elements, i.e. $k=2$, $s=1$, we have the following result.

Corollary 4.2. *Consider Algorithm IMEX1. Under the assumptions of Theorem 4.1, suppose that (X^h, Q^h) is given by P2-P1 Taylor-Hood approximation elements and S^h is P2 finite element. Then, there is a positive constant C_0 such that*

$$|||e_{\mathbf{u}}|||_{\infty,0}^2 + |||\nabla e_{\mathbf{u}}|||_{2,0}^2 + |||\nabla e_\phi|||_{2,0}^2 + |||e_{\mathbf{J}}|||_{2,0}^2 \le C_0(\Delta t^2 + h^4).$$

5. Numerical experiments. We present two numerical experiments to test the algorithms proposed herein. First, given exact solutions, we verify the convergence rates of our methods. Second, we will test the stability in case M and N are large. The code was implemented using the software package *FreeFEM++*.

5.1. Test 1. A verification of convergence rates for a smooth exact solution was presented in [9]. We test herein a solution that is more oscillatory than the test in [21]. Taking the time interval $0 \le t \le 1$, $M = 20, N = 16$ and setting the imposed magnetic field $\mathbf{B} = (0, 0, 1)$, we consider true solution $(\mathbf{u}, p, \phi)$ given by

$$\mathbf{u}(x, y, t) = (5\cos(5x)\sin(5y), -5\sin(5x)\cos(5y), 0)e^{-5t},$$
$$p(x, y, t) = 0,$$
$$\phi(x, y, t) = (\cos(5x)\cos(5y) + x^2 - y^2)e^{-5t}.$$

defined on the domain $\Omega = [0, \pi]^2$, satisfying $\Delta\phi = \nabla \cdot (\mathbf{u} \times \mathbf{B})$. We utilize piecewise quadratic for velocity and piecewise linear for pressure for the Navier-Stokes equation and continuous piecewise quadratic finite elements for the Ohm's law. Convergence rates are calculated from the errors at two successive values of h in the usual manner by postulating $e(h) = Ch^\beta$ and solving for β via $\beta = \ln(e(h_1)/e(h_2))/\ln(h_1/h_2)$. The boundary condition on the problem is inhomogeneous Dirichlet: $\mathbf{u}_h = \mathbf{u}$ on $\partial\Omega$. The initial data and source terms are chosen to correspond the exact solution.

For this section, we denote $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|\cdot\|_2 = \|\cdot\|_{L^2(0,T;L^2(\Omega))}$. From Tables 5.1 and 5.2, IMEX1 is first order and IMEX2 is second order.

The performance of numerical methods we studied herein is also compared with the monolithic, fully implicit methods (same discretization of subdomain terms but *implicit* discretization of coupling terms). Specifically, using the same test problem, Table 5.3 compares the errors $\|\mathbf{u} - \mathbf{u}_h\|_\infty + \|\phi - \phi_h\|_\infty$ produced by IMEX1 and Backward Euler method (BE), and compares those errors of IMEX2 and second order, implicit BDF method (BDF).

h	Δt	$\|\mathbf{u}-\mathbf{u}_h\|_\infty$	Rate	$\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|_2$	Rate	$\|\nabla\phi-\nabla\phi_h\|_2$	Rate
1/20	1/160	9.196e-1	–	5.361e+0	–	8.046e-1	–
1/40	1/320	5.307e-1	0.793	2.856e+0	0.908	4.455e-1	0.853
1/60	1/480	3.644e-1	0.927	1.935e+0	0.960	3.031e-1	0.950
1/80	1/640	2.769e-1	0.955	1.462e+0	0.974	2.293e-1	0.970
1/120	1/960	1.870e-1	0.968	9.826e-1	0.980	1.542e-1	0.979

TABLE 5.1

The convergence performance for Algorithm IMEX1.

h	Δt	$\|\mathbf{u}-\mathbf{u}_h\|_\infty$	Rate	$\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|_2$	Rate	$\|\nabla\phi-\nabla\phi_h\|_2$	Rate
1/20	1/160	1.209e-1	–	1.725e+0	–	2.137e-1	–
1/40	1/320	1.187e-2	3.348	4.147e-1	2.056	5.227e-2	2.032
1/60	1/480	3.417e-3	3.071	1.769e-1	2.101	2.338e-2	1.984
1/80	1/640	1.516e-3	2.825	9.738e-2	2.075	1.321e-2	1.985
1/120	1/960	5.782e-4	2.377	4.253e-2	2.043	5.897e-3	1.989

TABLE 5.2

The convergence performance for Algorithm IMEX2.

h	Δt	IMEX1	BE	IMEX2	BDF
1/20	1/160	1.073e+0	9.573e-2	1.407e-1	1.238e-1
1/40	1/320	6.195e-1	4.295e-2	1.372e-2	1.339e-2
1/60	1/480	4.254e-1	3.104e-2	3.942e-3	3.873e-3
1/80	1/640	3.233e-1	2.376e-2	1.758e-3	1.597e-3
1/120	1/960	2.183e-1	1.602e-2	6.710e-4	4.592e-4

TABLE 5.3
Errors $\|u-u_h\|_\infty + \|\phi-\phi_h\|_\infty$ of IMEX1, IMEX2 and corresponding monolithic methods.

5.2. Test 2. Many important applications of MHD in laboratory and industry involve large Hartmann number and interaction parameter, see, e.g., [16, 3]. The theory shows that Algorithm IMEX1 is unconditionally stable. However, the time step condition for stability of Algorithm IMEX2 looks pessimistic in these cases. In the following experiment, we test and compare the performance of our methods for such flows. We confirm the unconditional stability of Algorithm IMEX1 and show that Algorithm IMEX2 to be stable for much larger time steps than predicted by Theorem 3.2.

Let $\Omega = [0, 10^{-1}]^2$ and $\mathbf{B} = (0, 0, 1)$. A test for liquid aluminium was performed in [9]. Herein, we consider the flow of liquid sodium at 100°C, which involves larger M and N:

$$\sigma = 1.03 \cdot 10^7 \text{ mho/m}, \qquad \rho = 928 \text{ kg/m}^3,$$
$$\nu = 7.39 \cdot 10^{-7} \text{ m}^2/\text{s}, \qquad \eta = 7.72 \cdot 10^{-2} \text{ m}^2/\text{s}.$$

We take the characteristic values of length, velocity and magnetic field to be $L = 0.1$m, $u = 0.05$m/s, $B = 1$T, practical for laboratory and industrial flows. The Reynolds number, magnetic Reynolds number, Hartmann number and interaction parameter are then $Re = 6766$, $R_m = 0.064736$, $M = 12255$, $N = 22198$ correspondingly.

We take the source term $\mathbf{f}$ and the boundary condition to be 0 and the initial condition is given by

$$\mathbf{u}_0(x, y) = (10\pi\cos(10\pi x)\sin(10\pi y), -10\pi\sin(10\pi x)\cos(10\pi y), 0),$$
$$\phi_0(x, y) = (\cos(10\pi x)\cos(10\pi y) + x^2 - y^2).$$

For a system lacking of external energy exchange and body forces, the system energy decays over time. The energy $E^j = \frac{1}{2}\left(\|\mathbf{u}_h^j\|^2 + \|\phi_h^j\|^2\right)$ is computed using two different methods studied herein, on $h = 1/10$. For each algorithm, the time step is chosen purposely to give us an estimate of practical time step restriction for the stability of the method. The results are showed in Figure 5.1.

Figure 5.1 confirms the unconditional stability of IMEX1 established in Theorem 3.1. It also indicates that the experimental stability condition for IMEX2 is $\Delta t \lesssim 1/5000$, which, while still restrictive, is significantly better than the condition in Theorem 3.2.

6. Conclusion. In this paper, we give a complete analysis on stability and errors of a promising approach to solving the MHD problems at low magnetic Reynolds numbers. Our algorithms lag or extrapolate the coupling terms to previous time levels at which their values are known; therefore, at each time step, the multi-physics problem is uncoupled and solved non-iteratively. Compared to monolithic methods, our methods allow the use of legacy and optimized codes for subproblems.

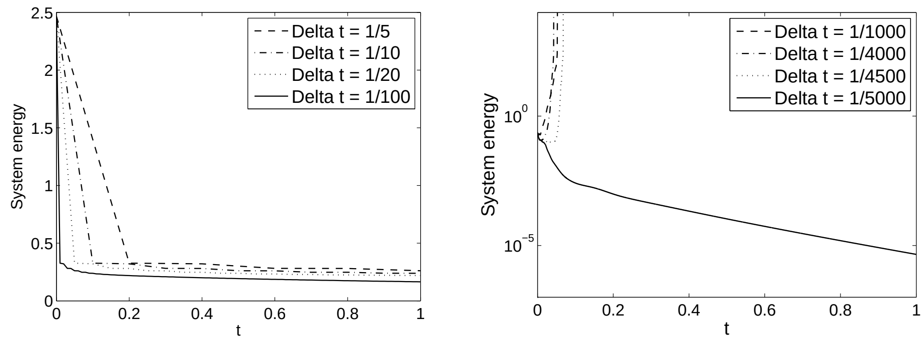

FIG. 5.1. *The decay of system energy computed by IMEX1 (left) and IMEX2 (right) with several different time steps chosen.*

Normally, for uncoupling a coupled problem, the price to be paid is stability. The first order scheme is, surprisingly, unconditionally stable. However, the time step condition of second order scheme, while independent of h, may be too restrictive in some applications involving small or large physical parameters. Open problems in elaborating this approach to MHD flows include higher order partitioned methods that are long time stable with *improved time step restrictions* with respect to the physical parameters. Another important question which naturally arises is developing partitioned methods for general MHD flows, which occur in both astrophysics and terrestrial applications and whose coupling terms are nonlinear.

REFERENCES

[1] L. Barleon, V. Casal, and L. Lenhart, *MHD flow in liquid-metal-cooled blankets*, Fusion Engineering and Design (1991), no. 14, 401–412.

[2] P. A. Davidson, *Magnetohydrodynamics in material processing*, Annu. Rev. Fluid Mech. **31** (1999), 273–300.

[3] ———, *An introduction to magnetohydrodynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.

[4] M. Discacciati, *Numerical approximation of a steady MHD problem*, Domain decomposition methods in science and engineering XVII, no. 60, Springer, Berlin, 2008, Lect. Notes Comput. Sci. Eng., pp. 313–320.

[5] J. L. Guermond and P. D. Minev, *Mixed finite element approximation of an MHD problem involving conducting and insulating regions: The 3D case*, Numer. Methods Partial Differential Equations **19** (2003), no. 6, 709–731.

[6] M.D. Gunzburger, A.J. Meir, and J. Peterson, *On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics*, Math. Comp. **56** (1991), no. 194, 523–563.

[7] H. Hashizume, *Numerical and experimental research to solve MHD problem in liquid-blanket system*, Fusion Engineering and Design **81** (2006), no. 8–14, 1431–1438.

[8] L. D. Landau and E. M. Lifshitz, *Electrodynamics of continuous media*, Course of Theoretical Physics, Vol. 8. Translated from the Russian by J. B. Sykes and J. S. Bell, Pergamon Press, Oxford, 1960.

[9] W. Layton, H. Tran, and C. Trenchea, *Stability of partitioned methods for magnetohydrodynamics flows at small magnetic Reynolds numbers*, Contemp. Math. (2012), to appear.

[10] W. J. Layton, A. J. Meir, and P. G. Schmidt, *A two-level discretization method for the stationary MHD equations*, Electron. Trans. Numer. Anal. **6** (1997), 198–210 (electronic).

[11] T. F. Lin, J. B. Gilbert, and R. Kossowsky, *Sea water magnetohydrodynamic propulsion for next-generation undersea vehicles*, ONR annual report for period from Feb. 1, 1989 to Jan. 31, 1990, under contract no. N00014-89-J-1693, NTIS AD Number: A218318 (1990).

[12] A. J. Meir, *The equations of stationary, incompressible magnetohydrodynamics with mixed*

boundary conditions, Comput. Math. Appl. **25** (1993), no. 12, 13–29.

[13] A. J. Meir and P. G. Schmidt, *Variational methods for stationary MHD flow under natural interface conditions*, Nonlinear Anal. **26** (1996), no. 4, 659–689.

[14] ———, *Analysis and numerical approximation of a stationary MHD flow problem with non-ideal boundary*, SIAM J. Numer. Anal. **36** (1999), no. 4, 1304–1332.

[15] J. Peterson, *On the finite element approximation of incompressible flows of an electrically conducting fluid*, Numer. Methods Partial Differential Equations **4** (1988), no. 1, 57–68.

[16] P. H. Roberts, *An introduction to magnetohydrodynamics*, Elsevier, USA, 1967.

[17] P. G. Schmidt, *A Galerkin method for time-dependent MHD flow with nonideal boundaries*, Commun. Appl. Anal. **3** (1999), no. 3, 383–398.

[18] S. Smolentsev, R. Moreau, L. Bühler, and C. Mistrangelo, *MHD thermofluid issues of liquid-metal blankets: Phenomena and advances*, Fusion Engineering and Design **85** (2010), no. 7–9, 1196–1205.

[19] C. Trenchea, *Unconditional stability of a partitioned IMEX method for magnetohydrodynamics flows*, (2012), submitted.

[20] N. Winowich and W. Hughes, *A finite element analysis of two dimensional MHD flow*, Liquid-Metal Flows and Magnetohydrodynamics, AIAA, New York, 1983.

[21] G. Yuksel and R. Ingram, *Numerical analysis of a finite element, Crank-Nicolson discretization for MHD flow at small magnetic Reynolds number*, Tech. report, University of Pittsburgh, 2011.