GLOBAL SOLUTIONS TO THE THREE-DIMENSIONAL FULL COMPRESSIBLE MAGNETOHYDRODYNAMIC FLOWS

XIANPENG HU AND DEHUA WANG

ABSTRACT. The equations of the three-dimensional viscous, compressible, and heat conducting magnetohydrodynamic flows are considered in a bounded domain. The viscosity coefficients and heat conductivity can depend on the temperature. A solution to the initial-boundary value problem is constructed through an approximation scheme and a weak convergence method. The existence of a global variational weak solution to the three-dimensional full magnetohydrodynamic equations with large data is established.

1. INTRODUCTION

Magnetohydrodynamics, or MHD, studies the dynamics of electrically conducting fluids and the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field. The applications of magnetohydrodynamics cover a very wide range of physical areas from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. Astrophysical problems include solar structure, especially in the outer layers, the solar wind bathing the earth and other planets, and interstellar magnetic fields. The primary geophysical problem is planetary magnetism, produced by currents deep in the planet, a problem that has not been solved to any degree of satisfaction. Magnetohydrodynamics is of importance in connection with many engineering problems as well, such as sustained plasma confinement for controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, magnetohydrodynamic power generation, electro-magnetic casting of metals, and plasma accelerators for ion thrusters for spacecraft propulsion. Due to their practical relevance, magnetohydrodynamic problems have long been the subject of intense cross-disciplinary research, but except for relatively simplified special cases, the rigorous mathematical analysis of such problems remains open.

In magnetohydrodynamic flows, magnetic fields can induce currents in a moving conductive fluid, which create forces on the fluid, and also change the magnetic field itself. There is a complex interaction between the magnetic and fluid dynamic phenomena, and both hydrodynamic and electrodynamic effects have to be considered. The set of equations which describe compressible viscous magnetohydrodynamics are a combination of the compressible Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. In this paper, we consider the full system of partial differential equations for the three-dimensional viscous compressible magnetohydrodynamic flows in the Eulerian

1991 *Mathematics Subject Classification.* 35Q36, 35D05, 76W05.

Key words and phrases. Magnetohydrodynamics, magnetic field, heat conductivity, global variational weak solution, large data.

coordinates ([19, 20]):

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \tag{1.1a}$$

$$(\rho\mathbf{u})_t + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = (\nabla\times\mathbf{H})\times\mathbf{H} + \mathrm{div}\Psi, \tag{1.1b}$$

$$\mathcal{E}_t + \mathrm{div}\big(\mathbf{u}(\mathcal{E}' + p)\big) = \mathrm{div}\big((\mathbf{u}\times\mathbf{H})\times\mathbf{H} + \nu\mathbf{H}\times(\nabla\times\mathbf{H}) + \mathbf{u}\Psi + \kappa\nabla\theta\big), \tag{1.1c}$$

$$\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \qquad \mathrm{div}\mathbf{H} = 0, \tag{1.1d}$$

where ρ denotes the density, $\mathbf{u}\in\mathbb{R}^3$ the velocity, $\mathbf{H}\in\mathbb{R}^3$ the magnetic field, and θ the temperature; Ψ is the viscous stress tensor given by

$$\Psi = \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \lambda\,\mathrm{div}\mathbf{u}\,\mathbf{I},$$

and $\mathcal{E}$ is the total energy given by

$$\mathcal{E} = \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) + \frac{1}{2}|\mathbf{H}|^2 \text{ and } \mathcal{E}' = \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right),$$

with e the internal energy, $\frac{1}{2}\rho|\mathbf{u}|^2$ the kinetic energy, and $\frac{1}{2}|\mathbf{H}|^2$ the magnetic energy. The equations of state $p = p(\rho,\theta)$, $e = e(\rho,\theta)$ relate the pressure p and the internal energy e to the density and the temperature of the flow; $\mathbf{I}$ is the 3×3 identity matrix, and $\nabla\mathbf{u}^T$ is the transpose of the matrix $\nabla\mathbf{u}$. The viscosity coefficients λ,μ of the flow satisfy $2\mu + 3\lambda > 0$ and $\mu > 0$; $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, $\kappa > 0$ is the heat conductivity. Equations (1.1a), (1.1b), (1.1c) describe the conservation of mass, momentum, and energy, respectively. It is well-known that the electromagnetic fields are governed by the Maxwell's equations. In magnetohydrodynamics, the displacement current can be neglected ([19, 20]). As a consequence, the equation (1.1d) is called the induction equation, and the electric field can be written in terms of the magnetic field $\mathbf{H}$ and the velocity $\mathbf{u}$,

$$\mathbf{E} = \nu\nabla\times\mathbf{H} - \mathbf{u}\times\mathbf{H}.$$

Although the electric field $\mathbf{E}$ does not appear in the MHD system (1.1), it is indeed induced according to the above relation by the moving conductive flow in the magnetic field.

There have been a lot of studies on magnetohydrodynamics by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges; see [3, 4, 6, 7, 10, 16, 15, 20, 25] and the references cited therein. In particular, the one-dimensional problem has been studied in many papers, for examples, [3, 4, 7, 15, 18, 23, 25] and so on. However, many fundamental problems for MHD are still open. For example, even for the one-dimensional case, the global existence of classical solutions to the full perfect MHD equations with large data remains unsolved when all the viscosity, heat conductivity, and magnetic diffusivity coefficients are constant, although the corresponding problem for the Navier-Stokes equations was solved in [17] long time ago. The reason is that the presence of the magnetic field and its interaction with the hydrodynamic motion in the MHD flow of large oscillation cause serious difficulties. In this paper we consider the global weak solution to the three-dimensional MHD problem with large data, and investigate the fundamental problem of global existence.

More precisely, we study the initial-boundary value problem of (1.1) in a bounded spatial domain $\Omega\subset\mathbb{R}^3$ with the initial data:

$$(\rho,\rho\mathbf{u},\mathbf{H},\theta)|_{t=0} = (\rho_0, m_0, \mathbf{H}_0, \theta_0)(x), \quad x\in\Omega, \tag{1.2}$$

and the no-slip boundary conditions on the velocity and the magnetic field, and the thermally insulated boundary condition on the heat flux $q = -\kappa\nabla\theta$:

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{H}|_{\partial\Omega} = 0, \quad q|_{\partial\Omega} = 0. \tag{1.3}$$

The aim of this paper is to construct the solution of the initial-boundary value problem of (1.1)-(1.3) and establish the global existence theory of variational weak solutions. In Hu-Wang [16], we studied global weak solutions to the initial-boundary value problem of the isentropic case for the three-dimensional MHD flow, while in this paper we study the full nonisentropic case. We are interested in the case that the viscosity and heat conductivity coefficients $\mu = \mu(\theta), \lambda = \lambda(\theta), \kappa = \kappa(\theta)$ are positive functions of the temperature θ; and the magnetic diffusivity coefficient $\nu > 0$ is assumed to be a constant in order to avoid unnecessary technical details. As for the pressure $p = p(\rho, \theta)$, it will be determined through a general constitutive equation:

$$p = p(\rho, \theta) = p_e(\rho) + \theta p_\theta(\rho) \tag{1.4}$$

for certain functions $p_e,\, p_\theta \in C[0, \infty) \cap C^1(0, \infty)$. The basic principles of classical thermodynamics imply that the internal energy e and pressure p are interrelated through Maxwell's relationship:

$$\frac{\partial e}{\partial \rho} = \frac{1}{\rho^2}\left(p - \theta\frac{\partial p}{\partial \theta}\right), \quad \frac{\partial e}{\partial \theta} = \frac{\partial Q}{\partial \theta} = c_v(\theta),$$

where $c_v(\theta)$ denotes the specific heat and $Q = Q(\theta)$ is a function of θ. Thus, the constitutive relation (1.4) implies that the internal energy e can be decomposed as a sum:

$$e(\rho, \theta) = P_e(\rho) + Q(\theta), \tag{1.5}$$

where

$$P_e(\rho) = \int_1^\rho \frac{p_e(\xi)}{\xi^2} d\xi, \quad Q(\theta) = \int_0^\theta c_v(\xi) d\xi.$$

If the flow is smooth, multiplying equation (1.1b) by $\mathbf{u}$ and (1.1d) by $\mathbf{H}$, and summing them together, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{H}|^2\right) + \operatorname{div}\left(\frac{1}{2}\rho|\mathbf{u}|^2\mathbf{u}\right) + \nabla p \cdot \mathbf{u} \\ &= \operatorname{div}\Psi \cdot \mathbf{u} + (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u} + \nabla \times (\mathbf{u} \times \mathbf{H}) \cdot \mathbf{H} - \nabla \times (\nu \nabla \times \mathbf{H}) \cdot \mathbf{H}. \end{aligned} \tag{1.6}$$

Subtracting (1.6) from (1.1c), we obtain the internal energy equation:

$$\partial_t(\rho e) + \operatorname{div}(\rho \mathbf{u} e) + (\operatorname{div}\mathbf{u})p = \nu|\nabla \times \mathbf{H}|^2 + \Psi : \nabla\mathbf{u} + \operatorname{div}(\kappa \nabla\theta), \tag{1.7}$$

using

$$\operatorname{div}(\nu \mathbf{H} \times (\nabla \times \mathbf{H})) = \nu|\nabla \times \mathbf{H}|^2 - \nabla \times (\nu\nabla \times \mathbf{H}) \cdot \mathbf{H},$$

and

$$\operatorname{div}((\mathbf{u} \times \mathbf{H}) \times \mathbf{H}) = (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u} + \nabla \times (\mathbf{u} \times \mathbf{H}) \cdot \mathbf{H}, \tag{1.8}$$

where $\Psi : \nabla\mathbf{u}$ denotes the scalar product of two matrices (see (4.10)). Multiplying equation (1.1a) by $(\rho P_e(\rho))'$ yields

$$\partial_t(\rho P_e(\rho)) + \operatorname{div}(\rho P_e(\rho)\mathbf{u}) + p_e(\rho)\operatorname{div}\mathbf{u} = 0, \tag{1.9}$$

and subtracting this equality from (1.7), we get the following thermal energy equation:

$$\partial_t(\rho Q(\theta)) + \operatorname{div}(\rho Q(\theta)\mathbf{u}) - \operatorname{div}(\kappa(\theta)\nabla\theta) = \nu|\nabla \times \mathbf{H}|^2 + \Psi : \nabla\mathbf{u} - \theta p_\theta(\rho)\operatorname{div}\mathbf{u}. \tag{1.10}$$

We note that in [6], Ducomet and Feireisl studied, using the entropy method, the full compressible MHD equations with an additional Poisson's equation under the assumption that the viscosity coefficients depend on the temperature and the magnetic field, and the pressure behaves like the power law ρ^γ with $\gamma = \frac{5}{3}$ for large density. We also remark that, for the mathematical analysis of incompressible MHD equations, we refer the reader to the work [11] and the references cited therein; and for the related studies on the multi-dimensional compressible Navier-Stokes equations, we refer to [8, 9, 14, 22] and particularly [8, 9] for the nonisentropic case. In this paper, we consider compressible MHD flow with more general pressure, and use the thermal equation (1.10) as in [8] instead of the entropy equation used in [6], thus the methods of this paper differ significantly from those in [6]. There are several major difficulties in studying the global solutions of the initial-boundary value problem of (1.1)-(1.3) with large data, due to the interaction from the magnetic field, large oscillations and concentrations of solutions, and poor *a priori* estimates available for MHD. To deal with the possible density oscillation, we use the weak continuity property of the effective viscous flux , first established by Lions [22] for the barotropic compressible Navier-Stokes system with constant viscosities (see also Feireisl [9] and Hoff [13]). More precisely, for fixed $T > 0$, assuming

$$\begin{cases}(\rho_n, b(\rho_n), p_n) \to (\rho, \overline{b(\rho)}, \overline{p}) \text{ weakly in } L^1(\Omega\times(0,T)),\\ (\mathbf{u}_n, \mathbf{H}_n) \to (\mathbf{u}, \mathbf{H}) \text{ weakly in } L^2([0,T]; W_0^{1,2}(\Omega)),\end{cases}$$

we will prove that, for some function b,

$$\big(p_n - (\lambda(\theta_n) + 2\mu(\theta_n))\mathrm{div}\mathbf{u}_n\big)b(\rho_n) \to \big(\overline{p} - (\overline{\lambda(\theta)} + 2\overline{\mu(\theta)})\mathrm{div}\mathbf{u}\big)b(\rho)$$

weakly in $L^1(\Omega\times(0,T))$, where $\overline{f}$ denote a weak limit of a sequence $\{f_n\}_{n=1}^\infty$ in $L^1(\Omega\times(0,T))$. To overcome the difficulty from the concentration in the temperature in order to pass to limit in approximation solutions, we use the renormalization of the thermal energy equation (1.10). More precisely, multiplying (1.10) by $h(\theta)$ for some function h, we obtain,

$$\begin{aligned}&\partial_t(\rho Q_h(\theta)) + \mathrm{div}(\rho Q_h(\theta)\mathbf{u}) - \Delta K_h(\theta)\\ &= \nu|\nabla\times\mathbf{H}|^2 h(\theta) + h(\theta)\Psi : \nabla\mathbf{u} - h(\theta)\theta p_\theta(\rho)\mathrm{div}\mathbf{u} - h'(\theta)\kappa(\theta)|\nabla\theta|^2,\end{aligned} \tag{1.11}$$

where

$$Q_h(\theta) = \int_0^\theta c_\upsilon(\xi)h(\xi)d\xi, \quad K_h(\theta) = \int_0^\theta \kappa(\xi)h(\xi)d\xi.$$

The idea of renormalization was used in Feireisl [8, 9], and is similar to that in DiPerna and Lions [5]. In addition, we also need to overcome the difficulty arising from the presence of the magnetic field and its coupling and interaction with the fluid variables.

We organize the rest of this paper as follows. In Section 2, we introduce a variational formulation of the full compressible MHD equations, and also state the main existence result (Theorem 2.1). In Section 3, we will formally derive a series of *a priori* estimates on the solution. In order to construct a sequence of approximation solutions, a three-level approximation scheme from [16] for isentropic MHD flow will be adopted in Section 4. Finally, in Section 5, our main result will be proved through a vanishing viscosity and vanishing artificial pressure limit passage using the weak convergence method.

2. Variational Formulation and Main Result

In this section, we give the definition of the variational solution to the initial-boundary value problem (1.1)-(1.3) and state the main result.

First we remark that, as shown later, the optimal estimates we can expect on the magnetic field $\mathbf{H}$ and the velocity $\mathbf{u}$ are in H^1-norms, which can not ensure the convergence of the terms $|\nabla\times\mathbf{H}|^2$ and $\Psi : \nabla\mathbf{u}$ in L^1 of equation (1.10), or even worse, in the sense of distributions. In other words, the compactness on the temperature does not seem to be sufficient to pass to the limit in the thermal energy equation. Thus, we will replace the thermal energy equality (1.10) by two inequalities in the sense of distributions to be in accordance with the second law of thermodynamics. More precisely, instead of (1.10), we only require that the following two inequalities hold:

$$\partial_t(\rho Q(\theta)) + \mathrm{div}(\rho Q(\theta)\mathbf{u}) - \Delta K(\theta) \geq \nu|\nabla\times\mathbf{H}|^2 + \Psi : \nabla\mathbf{u} - \theta p_\theta(\rho)\mathrm{div}\mathbf{u}, \tag{2.1}$$

in the sense of distributions, and

$$E[\rho, \mathbf{u}, , \theta, \mathbf{H}](t) \leq E[\rho, \mathbf{u}, \theta, \mathbf{H}](0) \text{ for } t \geq 0, \tag{2.2}$$

with the total energy

$$E[\rho, \mathbf{u}, \theta, \mathbf{H}] = \int_\Omega \left(\rho \left(P_e(\rho) + Q(\theta) + \frac{1}{2}|\mathbf{u}|^2 \right) + \frac{1}{2}|\mathbf{H}|^2 \right) dx,$$

and

$$K(\theta) = \int_0^\theta \kappa(\xi) d\xi.$$

Now we give the definition of variational solutions to the full MHD equations as follows.

Definition 2.1. A vector $(\rho, \mathbf{u}, \theta, \mathbf{H})$ is said to be a variational solution to the initial-boundary value problem (1.1)-(1.3) of the full compressible MHD equations on the time interval $(0, T)$ for any fixed $T > 0$ if the following conditions hold:

- The density $\rho \geq 0$, the velocity $\mathbf{u} \in L^2([0,T]; W_0^{1,2}(\Omega))$, and the magnetic field $\mathbf{H} \in L^2([0,T]; W_0^{1,2}(\Omega)) \cap C([0,T]; L^2_{weak}(\Omega))$ satisfy the equations (1.1a), (1.1b), and (1.1d) in the sense of distributions, and
$$\int_0^T \int_\Omega (\rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla \varphi) \, dxdt = 0,$$
for any $\varphi \in C^\infty(\Omega \times [0,T])$ with $\varphi(x, 0) = \varphi(x, T) = 0$ for $x \in \Omega$;
- The temperature θ is a non-negative function satisfying
$$\begin{aligned} &\int_0^T \int_\Omega (\rho Q(\theta) \partial_t \varphi + \rho Q(\theta) \mathbf{u} \cdot \nabla \varphi + K(\theta) \Delta \varphi) \, dxdt \\ &\leq \int_0^T \int_\Omega (\theta p_\theta(\rho) \mathrm{div} \mathbf{u} - \nu |\nabla \times \mathbf{H}|^2 - \Psi : \nabla \mathbf{u}) \varphi \, dxdt \end{aligned}$$
for any $\varphi \in C_0^\infty(\Omega \times (0,T))$ with $\varphi \geq 0$;
- The energy inequality (2.2) holds for a.e $t \in (0, T)$, with
$$E[\rho, \mathbf{u}, \theta, \mathbf{H}](0) = \int_\Omega \left(\rho_0 P_e(\rho_0) + \rho_0 Q(\theta_0) + \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\rho_0} + \frac{1}{2}|\mathbf{H}_0|^2 \right) dx;$$
- The functions ρ, $\rho\mathbf{u}$, and $\mathbf{H}$ satisfy the initial conditions in the following weak sense:
$$\operatorname{ess} \lim_{t \to 0^+} \int_\Omega (\rho, \rho \mathbf{u}, \mathbf{H})(x,t) \eta(x) \, dx = \int_\Omega (\rho_0, m_0, \mathbf{H}_0) \eta \, dx,$$
for any $\eta \in \mathcal{D}(\Omega) := C_0^\infty(\Omega)$.

Now we are ready to state the main result of this paper.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\tau}$ for some $\tau > 0$. Suppose that the following conditions hold: the pressure p is given by the equation* (1.4) *where p_e, p_θ are C^1 functions on $[0, \infty)$ and*

$$\begin{cases} p_e(0) = 0, \quad p_\theta(0) = 0, \\ p_e'(\rho) \geq a_1 \rho^{\gamma - 1}, \quad p_\theta'(\rho) \geq 0 \quad \text{for all } \rho > 0, \\ p_e(\rho) \leq a_2 \rho^\gamma, \quad p_\theta(\rho) \leq a_3(1 + \rho^{\frac{\gamma}{3}}) \quad \text{for all } \rho \geq 0, \end{cases} \tag{2.3}$$

with some constants $\gamma > \frac{3}{2}$, $a_1 > 0$, $a_2 > 0$, and $a_3 > 0$; $\kappa = \kappa(\theta)$ is a C^1 function on $[0, \infty)$ such that

$$\underline{\kappa}(1 + \theta^\alpha) \leq \kappa(\theta) \leq \overline{\kappa}(1 + \theta^\alpha), \tag{2.4}$$

for some constants $\alpha > 2$, $\underline{\kappa} > 0$, and $\overline{\kappa} > 0$; the viscosity coefficients μ and λ are C^1 functions of θ and globally Lipschitz on $[0, \infty)$ satisfying

$$0 < \underline{\mu} \leq \mu(\theta) \leq \overline{\mu}, \quad 0 \leq \lambda(\theta) \leq \overline{\lambda}, \tag{2.5}$$

for some positive constants $\underline{\mu}$, $\overline{\mu}$, $\overline{\lambda}$; $\nu > 0$ *is a constant; there exist two positive constants* $\underline{c_\upsilon}$, $\overline{c_\upsilon}$ *such that*

$$0 < \underline{c_\upsilon} \leq c_\upsilon(\theta) \leq \overline{c_\upsilon}; \tag{2.6}$$

and finally, the initial data satisfy

$$\begin{cases} \rho_0 \in L^\gamma(\Omega), \quad \rho_0 \geq 0 \text{ on } \Omega, \\ \theta_0 \in L^\infty(\Omega), \quad \theta_0 \geq \underline{\theta} > 0 \text{ on } \Omega, \\ \frac{|m_0|^2}{\rho_0} \in L^1(\Omega), \\ \mathbf{H}_0 \in L^2(\Omega), \quad \mathrm{div}\mathbf{H}_0 = 0 \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{2.7}$$

Then, the initial-boundary value problem (1.1)-(1.3) *of the full compressible MHD equations has a variational solution* $(\rho, \mathbf{u}, \theta, \mathbf{H})$ *on* $\Omega \times (0, T)$ *for any given* $T > 0$, *and*

$$\begin{cases} \rho \in L^\infty([0,T]; L^\gamma(\Omega)) \cap C([0,T]; L^1(\Omega)), \\ \mathbf{u} \in L^2([0,T]; W_0^{1,2}(\Omega)), \quad \rho\mathbf{u} \in C([0,T]; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega)), \\ \theta \in L^{\alpha+1}(\Omega \times (0,T)), \quad \rho Q(\theta) \in L^\infty([0,T]; L^1(\Omega)), \\ \theta p_\theta \in L^2(\Omega \times (0,T)), \quad \rho Q(\theta)\mathbf{u} \in L^1(\Omega \times (0,T)), \\ \ln(1+\theta) \in L^2([0,T]; W^{1,2}(\Omega)), \quad \theta^{\frac{\alpha}{2}} \in L^2([0,T]; W^{1,2}(\Omega)), \\ \mathbf{H} \in L^2([0,T]; W_0^{1,2}(\Omega)) \cap C([0,T]; L_{weak}^2(\Omega)). \end{cases} \tag{2.8}$$

Remark 2.1. In addition, the solution constructed in Theorem 2.1 will satisfy the continuity equation in the sense of renormalized solutions, that is, the integral identity

$$\int_0^T \int_\Omega (b(\rho)\partial_t\varphi + b(\rho)\mathbf{u} \cdot \nabla\varphi + (b(\rho) - b'(\rho)\rho)\mathrm{div}\mathbf{u}\,\varphi)\, dxdt = 0$$

holds for any

$$b \in C^1[0,\infty), \quad |b'(z)z| \leq cz^{\frac{\gamma}{2}} \text{ for z larger than some positive } z_0.$$

and any test function $\varphi \in C^\infty([0,T] \times \overline{\Omega})$ with $\varphi(x,0) = \varphi(x,T) = 0$ for $x \in \Omega$.

Remark 2.2. The growth restrictions imposed on κ, μ, λ, and c_υ may not be optimal, and $\gamma > \frac{3}{2}$ is a necessary condition to ensure the convergence of nonlinear term $\rho\mathbf{u} \otimes \mathbf{u}$ in the sense of distributions. In particular, our result includes the case of constant viscosity coefficients, with the assumption that the coefficient $\lambda \geq 0$.

Remark 2.3. Our method also works for the case with nonzero external force f in the momentum equation. As it is obvious that in our analysis the presence of the external force does not add any additional difficulty, and usually can be dealt with by using classical Young's inequality under suitable assumptions on the integrability of the external force f.

3. A Priori Estimates

To prove Theorem 2.1, we first need to obtain sufficient *a priori* estimates on the solution. The total energy conservation (2.2) implies

$$\begin{aligned} &\int_\Omega \left(\rho \left(P_e(\rho) + Q(\theta) + \frac{1}{2}|\mathbf{u}|^2 \right) + \frac{1}{2}|\mathbf{H}|^2 \right) dx \\ &\leq \int_\Omega \left(\rho_0 P_e(\rho_0) + \rho_0 Q(\theta_0) + \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + \frac{1}{2}|\mathbf{H}_0|^2 \right) dx. \end{aligned} \tag{3.1}$$

But the assumption (2.3) implies that there is a positive constant c such that

$$\rho P_e(\rho) \geq c\rho^\gamma, \text{ for any } \rho \geq 0.$$

Thus, (3.1) implies that $\rho^\gamma, \rho Q(\theta), \frac{1}{2}\rho|\mathbf{u}|^2$ and $\frac{1}{2}|\mathbf{H}|^2$ are bounded in $L^\infty([0,T];L^1(\Omega))$. Hence,

$$\rho \in L^\infty([0,T];L^\gamma(\Omega)), \quad \rho\mathbf{u} \in L^\infty([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega)).$$

Next, in order to obtain estimates on the temperature, we introduce the entropy

$$s(\rho,\theta) = \int_1^\theta \frac{c_\upsilon(\xi)}{\xi}d\xi - P_\theta(\rho), \quad \text{with } P_\theta(\rho) = \int_1^\rho \frac{p_\theta(\xi)}{\xi^2}d\xi.$$

If the flow is smooth and the temperature is strictly positive, then by direct calculation, using (1.1a) and (1.10), we obtain

$$\partial_t(\rho s) + \mathrm{div}(\rho s\mathbf{u}) + \mathrm{div}\left(\frac{q}{\theta}\right) = \frac{1}{\theta}\left(\nu|\nabla\times\mathbf{H}|^2 + \Psi : \nabla\mathbf{u}\right) - \frac{q\cdot\nabla\theta}{\theta^2}. \tag{3.2}$$

Integrating (3.2), we get

$$\begin{aligned}&\int_0^T\int_\Omega \left(\frac{1}{\theta}\left(\nu|\nabla\times\mathbf{H}|^2 + \Psi : \nabla\mathbf{u}\right) + \frac{\kappa(\theta)|\nabla\theta|^2}{\theta^2}\right)dxdt \\ &= \int_\Omega \rho s(x,t)\,dx - \int_\Omega \rho s(x,0)\,dx.\end{aligned} \tag{3.3}$$

On the other hand, assumptions (2.3) imply, using Young's inequality,

$$|\rho P_\theta(\rho)| \le c + \rho P_e(\rho) \quad \text{for some } c > 0. \tag{3.4}$$

Moreover, we have

$$\rho\int_1^\theta \frac{c_\upsilon(\xi)}{\xi}d\xi \le \rho Q(\theta) \quad \text{for all } \theta > 0,\ \rho \ge 0, \tag{3.5}$$

since

$$\int_1^\theta \frac{c_\upsilon(\xi)}{\xi}d\xi \le 0, \quad \text{if } 0 < \theta \le 1,$$

and

$$\int_1^\theta \frac{c_\upsilon(\xi)}{\xi}d\xi \le \int_1^\theta c_\upsilon(\xi)d\xi = Q(\theta) - Q(1) \le Q(\theta), \quad \text{if } \theta > 1.$$

Assuming that $\rho s(\cdot,0) \in L^1(\Omega)$, then from (3.3)-(3.5) , using the assumption (2.4) and the estimates from (3.1), we get

$$\int_0^T\int_\Omega |\nabla\theta^{\frac{\alpha}{2}}|^2 + |\nabla\ln\theta|^2\,dxdt \le C,$$

which, combining the Sobolev's imbedding theorem, implies

$$\ln\theta \text{ and } \theta^{\frac{\alpha}{2}} \text{ are bounded in } L^2([0,T];W^{1,2}(\Omega)). \tag{3.6}$$

Finally, we turn to the estimates on the velocity and the magnetic field. Indeed, integrating (1.10) over $\Omega\times(0,T)$, we get

$$\begin{aligned}&\int_0^T\int_\Omega \left(\Psi : \nabla\mathbf{u} + \nu|\nabla\times\mathbf{H}|^2\right)dxdt \\ &= \int_0^T\int_\Omega \theta p_\theta(\rho)\mathrm{div}\mathbf{u}\,dxdt + \int_\Omega \rho Q(\theta)(x,T)dx - \int_\Omega \rho Q(\theta)(x,0)\,dx.\end{aligned} \tag{3.7}$$

Noticing that, using Hölder inequality, one has

$$\|\theta p_\theta(\rho)\|_{L^2(\Omega)} \le \|\theta\|_{L^6(\Omega)}\|p_\theta(\rho)\|_{L^3(\Omega)}. \tag{3.8}$$

Thus, from the assumption (2.3) and estimate (3.6), we have

$$\theta p_\theta(\rho) \in L^2(\Omega\times(0,T)).$$

The relation (3.7) together with (3.1), (3.8), gives rise to the estimate

$$\int_0^T\int_\Omega \left(\Psi : \nabla\mathbf{u} + \nu|\nabla\times\mathbf{H}|^2\right) dxdt \le C(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{H}_0).$$

The assumption (2.5), the fact $\|\nabla\times\mathbf{H}\|_{L^2} = \|\nabla\mathbf{H}\|_{L^2}$ when $\mathrm{div}\mathbf{H} = 0$, and Sobolev's imbedding theorem give that

$$\mathbf{u}, \mathbf{H} \text{ are bounded in } L^2([0,T]; W_0^{1,2}(\Omega)).$$

In summary, if $\rho s(\cdot, 0) \in L^1(\Omega)$, the system (1.1a), (1.1b), (1.1d), (1.10) with the initial-boundary conditions (1.3) and our assumptions (2.3)-(2.7) yield the following estimates:

$$\begin{cases} \rho P_e(\rho), \rho Q(\theta) \text{ are bounded in } L^\infty([0,T]; L^1(\Omega)); \\ \rho \text{ is bounded in } L^\infty([0,T]; L^\gamma(\Omega)), \\ \rho\mathbf{u} \text{ is bounded in } L^\infty([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega)); \\ \ln\theta \text{ and } \theta^{\frac{\alpha}{2}} \text{ are bounded in } L^2([0,T]; W^{1,2}(\Omega)); \\ \mathbf{u}, \mathbf{H} \text{ are bounded in } L^2([0,T]; W_0^{1,2}(\Omega)). \end{cases} \tag{3.9}$$

4. The Approximation Scheme and Approximation Solutions

Similarly as Section 4 in [16] and Section 3 in [8], we introduce an approximate problem which consists of a system of regularized equations:

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = \varepsilon\Delta\rho, \\ (\rho\mathbf{u})_t + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho,\theta) + \delta\nabla\rho^\beta + \varepsilon\nabla\mathbf{u}\cdot\nabla\rho = (\nabla\times\mathbf{H})\times\mathbf{H} + \mathrm{div}\Psi, \\ \partial_t((\rho+\delta)Q(\theta)) + \mathrm{div}(\rho Q(\theta)\mathbf{u}) - \Delta K(\theta) + \delta\theta^{\alpha+1} \\ \qquad = (1-\delta)(\nu|\nabla\times\mathbf{H}|^2 + \Psi:\nabla\mathbf{u}) - \theta p_\theta(\rho)\mathrm{div}\mathbf{u}, \\ \mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \quad \mathrm{div}\mathbf{H} = 0, \end{cases} \tag{4.1}$$

with the initial-boundary conditions

$$\begin{cases} \nabla\rho\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \rho|_{t=0} = \rho_{0,\delta}, \\ \mathbf{u}|_{\partial\Omega} = 0, \quad \rho\mathbf{u}|_{t=0} = m_{0,\delta}, \\ \nabla\theta|_{\partial\Omega} = 0, \quad \theta|_{t=0} = \theta_{0,\delta}, \\ \mathbf{H}|_{\partial\Omega} = 0, \quad \mathbf{H}|_{t=0} = \mathbf{H}_0. \end{cases} \tag{4.2}$$

where ε and δ are two positive parameters, $\beta > 0$ is a fixed constant, and $\mathbf{n}$ is the unit outer normal of $\partial\Omega$. The initial data are chosen in such a way that

$$\begin{cases} \rho_{0,\delta} \in C^3(\overline{\Omega}), \quad 0 < \delta \le \rho_{0,\delta} \le \delta^{-\frac{1}{2\beta}}; \\ \rho_{0,\delta} \to \rho_0 \text{ in } L^\gamma(\Omega), \quad |\{\rho_{0,\delta} < \rho_0\}| \to 0, \quad \text{as } \delta \to 0; \\ \delta\int_\Omega \rho_{0,\delta}^\beta\, dx \to 0, \quad \text{as } \delta\to 0; \\ m_{0,\delta} = \begin{cases} m_0, & \text{if } \rho_{0,\delta} \ge \rho_0, \\ 0, & \text{if } \rho_{0,\delta} < \rho_0; \end{cases} \\ \theta_{0,\delta} \in C^3(\overline{\Omega}), \quad 0 < \underline{\theta} \le \theta_{0,\delta} \le \overline{\theta}; \\ \theta_{0,\delta} \to \theta_0 \text{ in } L^1(\Omega) \quad \text{as } \delta \to 0. \end{cases} \tag{4.3}$$

Noticing that the terms $\nu|\nabla\times\mathbf{H}|^2$ and $\Psi : \nabla\mathbf{u}$ are nonnegative, and $\theta = 0$ is a subsolution of the third equation in (4.1), we can conclude that, using the maximum principle, $\theta(t, x) \ge 0$ for all $t \in (0, T)$ and $x \in \Omega$.

From Lemma 3.2 in [16] and Proposition 7.2 in [9], we see that the approximate problem (4.1)-(4.2) with fixed positive parameters ε and δ can be solved by means of a modified

Faedo-Galerkin method (cf. Chapter 7 in [9]). Thus, we state without proof the following result (cf. Proposition 3.1 in [8]):

Proposition 4.1. Under the hypotheses of Theorem 2.1, and let β be large enough, then the approximate problem (4.1)-(4.2) has a solution $(\rho, \mathbf{u}, \theta, \mathbf{H})$ on $\Omega \times (0,T)$ for any fixed $T > 0$ satisfying the following properties:

- $\rho \geq 0$, $\mathbf{u} \in L^2([0,T]; W_0^{1,2}(\Omega))$, $\mathbf{H} \in L^2([0,T]; W_0^{1,2}(\Omega))$, the first equation in (4.1) is satisfied a.e on $\Omega \times (0,T)$, the second and fourth equations in (4.1) are satisfied in the sense of distributions on $\Omega \times (0,T)$ (denoted by $\mathcal{D}'(\Omega \times (0,T))$), $\mathbf{u}$ and $\mathbf{H}$ are bounded in $L^2([0,T]; W_0^{1,2}(\Omega))$, and, for some $r > 1$,
$$\begin{aligned} &\rho_t,\ \Delta\rho \in L^r(\Omega \times (0,T)), \quad \rho\mathbf{u} \in C([0,T]; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega)), \\ &\delta \int_0^T\!\!\int_\Omega \rho^{\beta+1}\,dxdt \leq c_1(\varepsilon,\delta), \quad \varepsilon \int_0^T\!\!\int_\Omega |\nabla\rho|^2\,dxdt \leq c_2, \end{aligned} \tag{4.4}$$
where c_2 is a constant independent of ε.
- The energy inequality
$$\begin{aligned} &\int_0^T\!\!\int_\Omega (-\psi_t)\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{H}|^2 + \frac{a_2}{\gamma-1}\rho^\gamma + \frac{\delta}{\beta-1}\rho^\beta + (\rho+\delta)Q(\theta)\right)dxdt \\ &\quad + \delta \int_0^T\!\!\int_\Omega \psi\left(\Psi : \nabla\mathbf{u} + \nu|\nabla\times\mathbf{u}|^2 + \theta^{\alpha+1}\right)dxdt \\ &\leq \int_\Omega \left(\frac{1}{2}\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}} + \frac{1}{2}|\mathbf{H}_0|^2 + \frac{a_2}{\gamma-1}\rho_{0,\delta}^\gamma + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta + (\rho_{0,\delta}+\delta)Q(\theta_{0,\delta})\right)dx \\ &\quad + \int_0^T\!\!\int_\Omega \psi p_b(\rho)\mathrm{div}\mathbf{u}\,dxdt \end{aligned}$$
holds for any $\psi \in C^\infty([0,T])$ satisfying
$$\psi(\cdot,0) = 1, \quad \psi(\cdot,T) = 0, \quad \psi_t \leq 0 \text{ on } \Omega,$$
where, $p_e(\rho)$ has been decomposed as
$$p_e(\rho) = a_2\rho^\gamma - p_b(\rho),$$
with $p_b \in C^1[0,\infty)$, $p_b \geq 0$;
- The temperature $\theta \geq 0$ satisfies that
$$\theta \in L^{\alpha+1}(\Omega \times (0,T)), \quad \theta^{\frac{\alpha}{2}} \in L^2([0,T]; W^{1,2}(\Omega)),$$
and the thermal energy inequality holds in the following renormalized sense:
$$\begin{aligned} &\int_0^T\!\!\int_\Omega \left((\rho+\delta)Q_h(\theta)\partial_t\varphi + \rho Q_h(\theta)\mathbf{u}\cdot\nabla\varphi + K_h(\theta)\Delta\varphi - \delta h(\theta)\theta^{\alpha+1}\varphi\right)dxdt \\ &\leq \int_0^T\!\!\int_\Omega \left((\delta-1)h(\theta)(\Psi : \nabla\mathbf{u} + \nu|\nabla\times\mathbf{H}|^2) + h'(\theta)\kappa(\theta)|\nabla\theta|^2\right)\varphi\,dxdt \\ &\quad + \int_0^T\!\!\int_\Omega h(\theta)\theta p_\theta(\rho)\mathrm{div}\mathbf{u}\varphi\,dxdt - \int_\Omega (\rho_{0,\delta}+\delta)Q_h(\theta_{0,\delta})\varphi(x,0)\,dx \\ &\quad + \varepsilon \int_0^T\!\!\int_\Omega \nabla\rho\cdot\nabla\big((Q_h(\theta) - Q(\theta)h(\theta))\varphi\big)dxdt, \end{aligned} \tag{4.5}$$
for any function $h \in C^\infty(\mathbb{R}^+)$ satisfying
$$\begin{aligned} &h(0) > 0, \quad h \text{ non-increasing on } [0,\infty), \quad \lim_{\xi\to\infty} h(\xi) = 0, \\ &h''(\xi)h(\xi) \geq 2(h'(\xi))^2, \text{ for all } \xi \geq 0, \end{aligned} \tag{4.6}$$

and any test function $\varphi \in C^2(\Omega \times [0,T])$ satisfying

$$\varphi \geq 0, \quad \varphi(\cdot, T) = 0, \quad \nabla\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Remark 4.1. In fact, we have

$$\|\nabla\theta^{\frac{\alpha}{2}}\|_{L^2(\Omega\times(0,T))} \leq c(\delta).$$

Comparing with the term $\nu|\nabla \times \mathbf{H}|^2$ in (4.5), we need to pay more attention to the term $\Psi : \nabla\mathbf{u}$, because the later involves temperature-dependent coefficients and thus can not be dealt with by the standard weak lower semi-continuity. Indeed, the hypothesis (4.6) was imposed in [9] in order to make the function

$$(\theta, \nabla\mathbf{u}) \mapsto h(\theta)\Psi : \nabla\mathbf{u}$$

convex, and, consequently, weakly lower semi-continuous (the stress tensor Ψ in [9] depends on $\nabla\mathbf{u}$ only). In accordance with our *new* context, the following lemma is useful:

Lemma 4.1. *Let $g(\theta)$ be a bounded, continuous and non-negative function from $[0,\infty)$ to $\mathbb{R}$. Suppose that θ_n and $\mathbf{u}_n$ are two sequences of functions defined on Ω and*

$$\theta_n \to \theta \text{ a.e in } \Omega,$$

and

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } W^{1,2}(\Omega).$$

Then,

$$\int_\Omega g(\theta)h(\theta)|\nabla\mathbf{u}|^2\,dx \leq \liminf_{n\to\infty} \int_\Omega g(\theta_n)h(\theta_n)|\nabla\mathbf{u}_n|^2\,dx. \tag{4.7}$$

In particular,

$$\int_\Omega h(\theta)\Psi : \nabla\mathbf{u}\,dx \leq \liminf_{n\to\infty} \int_\Omega h(\theta_n)\Psi(\mathbf{u}_n) : \nabla\mathbf{u}_n\,dx. \tag{4.8}$$

Proof. First we show that $\sqrt{g(\theta_n)}\nabla\mathbf{u}_n$ converges weakly to $\sqrt{g(\theta)}\nabla\mathbf{u}$ in L^2. Indeed, since $\sqrt{g(\theta_n)}\nabla\mathbf{u}_n$ is uniformly bounded in L^2, it is enough to show

$$\int_\Omega \sqrt{g(\theta_n)}\nabla\mathbf{u}_n\phi\,dx \to \int_\Omega \sqrt{g(\theta)}\nabla\mathbf{u}\,\phi\,dx, \text{ for all } \phi \in C^\infty(\Omega). \tag{4.9}$$

Since $\theta_n \to \theta$ a.e in Ω, then $\sqrt{g(\theta_n)}\phi \to \sqrt{g(\theta)}\phi$ a.e in Ω for all $\phi \in C^\infty(\Omega)$. Thus by Lebesgue's dominated convergence theorem, we know that

$$\sqrt{g(\theta_n)}\phi \to \sqrt{g(\theta)}\phi \text{ in } L^2(\Omega),$$

and (4.9) follows.

Next, by virtue of Corollary 2.2 in [9], it is enough to observe that the function $\Phi : (\theta, \xi) \mapsto h(\theta)\xi^2$ is convex and continuous on $\mathbb{R}^+ \times \mathbb{R}$. Computing the Hessian matrix of Φ, we get

$$\det\{\partial^2_{\theta,\xi}\Phi\} = 2\xi^2(h''(\theta)h(\theta) - 2(h'(\theta))^2) \geq 0,$$

and

$$\text{trace}\{\partial^2_{\theta,\xi}\Phi\} = \xi^2 h''(\theta) + 2h(\theta) \geq 0,$$

provided $\theta > 0$ and h satisfies (4.6). Thus the Hessian matrix is positively definite; therefore Φ is convex and continuous. Then, (4.7) is a direct application of Corollary 2.2 in [9].

Finally, from (4.7) and the calculation:

$$\Psi : \nabla\mathbf{u} = \sum_{i,j=1}^{3} \frac{\mu(\theta)}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right)^2 + \lambda(\theta)|\text{div}\mathbf{u}|^2, \tag{4.10}$$

(4.8) follows. ∎

In the next two steps, in order to obtain the variational solution of the initial-boundary value problem (1.1)-(1.3), we need to take the vanishing limits of the artificial viscosity $\varepsilon \to 0$ and artificial pressure coefficient $\delta \to 0$ in the approximate solutions of (4.1)-(4.2). As seen in [8, 9, 16], the techniques used in those two procedures are rather similar. Moreover, in some sense, the step of taking $\varepsilon \to 0$ is much easier than the step of taking $\delta \to 0$ due to the higher integrability of ρ. Hence we will omit the step of taking $\varepsilon \to 0$ (readers can refer to Section 5 in [16] or Section 4 in [8]), and focus on the step of taking $\delta \to 0$. Thus, we state without proof the result as $\varepsilon \to 0$ as follows.

Proposition 4.2. Let $\beta > 0$ be large enough and $\delta > 0$ be fixed, then the initial-boundary value problem (1.1)-(1.3) for full compressible MHD equations admits an approximate solution $(\rho, \mathbf{u}, \theta, \mathbf{H})$ with parameter δ (as the limit of the solutions to (4.1)-(4.2) when $\varepsilon \to 0$) in the following sense:

- The density ρ is a non-negative function, and
$$\rho \in C([0,T]; L^{\beta}_{weak}(\Omega)),$$
satisfying the initial condition in (4.3). The velocity $\mathbf{u}$ and the magnetic field $\mathbf{H}$ belong to $L^2([0,T]; W^{1,2}_0(\Omega))$. The equation (1.1a) and (1.1d) are satisfied in $\mathcal{D}'(\Omega \times (0,T))$ and
$$\delta \int_0^T\int_\Omega \rho^{\beta+1}\, dxdt \le c(\delta).$$
Moreover, $\rho, \mathbf{u}$ also solve equation (1.1a) in the sense of renormalized solutions;
- The functions $\rho, \mathbf{u}, \theta, \mathbf{H}$ solve a modified momentum equation
$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla(p(\rho,\theta) + \delta\rho^\beta) = (\nabla\times\mathbf{H})\times\mathbf{H} + \operatorname{div}\Psi, \tag{4.11}$$
in $\mathcal{D}'(\Omega \times (0,T))$. Furthermore, the momentum
$$\rho\mathbf{u} \in C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega))$$
satisfies the initial condition in (4.3);
- The energy inequality
$$\begin{aligned}
&\int_0^T\int_\Omega (-\psi_t)\left(\frac12\rho|\mathbf{u}|^2 + \frac12|\mathbf{H}|^2 + \rho P_e(\rho) + \frac{\delta}{\beta-1}\rho^\beta + (\rho+\delta)Q(\theta)\right) dxdt\\
&\quad + \delta\int_0^T\int_\Omega \psi\left(\Psi : \nabla\mathbf{u} + \nu|\nabla\times\mathbf{u}|^2 + \theta^{\alpha+1}\right) dxdt \\
&\le \int_\Omega \left(\frac12\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}} + \frac12|\mathbf{H}_0|^2 + \rho_{0,\delta}P_e(\rho_{0,\delta}) + \frac{\delta}{\beta-1}\rho^\beta_{0,\delta} + (\rho_{0,\delta}+\delta)Q(\theta_{0,\delta})\right) dx
\end{aligned}\tag{4.12}$$
holds for any $\psi \in C^\infty([0,T])$ satisfying
$$\psi(\cdot,0) = 1, \quad \psi(\cdot,T) = 0, \quad \psi_t \le 0;$$
- The temperature θ is a non-negative function, and
$$\theta \in L^{\alpha+1}(\Omega\times(0,T)), \quad \theta^{\frac{\alpha+1-\omega}{2}} \in L^2([0,T]; W^{1,2}(\Omega)), \quad \omega \in (0,1], \tag{4.13}$$
satisfying the thermal energy inequality in the following renormalized sense:
$$\begin{aligned}
&\int_0^T\int_\Omega \left((\rho+\delta)Q_h(\theta)\partial_t\varphi + \rho Q_h(\theta)\mathbf{u}\cdot\nabla\varphi + K_h(\theta)\Delta\varphi - \delta h(\theta)\theta^{\alpha+1}\varphi\right) dxdt\\
&\le \int_0^T\int_\Omega \left((\delta-1)h(\theta)(\Psi : \nabla\mathbf{u} + \nu|\nabla\times\mathbf{H}|^2) + h'(\theta)\kappa(\theta)|\nabla\theta|^2\right)\varphi dxdt\\
&\quad + \int_0^T\int_\Omega h(\theta)\theta p_\theta(\rho)\operatorname{div}\mathbf{u}\varphi\, dxdt - \int_\Omega (\rho_{0,\delta}+\delta)Q_h(\theta_{0,\delta})\varphi(0)\, dx,
\end{aligned}\tag{4.14}$$

for any admissible function $h \in C^\infty(\mathbb{R}^+)$ satisfying (4.6) and any test function $\varphi \in C^2(\Omega \times [0,T])$ satisfying

$$\varphi \geq 0, \quad \varphi(\cdot, T) = 0, \quad \nabla\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Remark 4.2. In Proposition 4.2, the second estimate in (4.13) can be explained as follows. Taking

$$h(\theta) = \frac{1}{(1+\theta)^\omega}, \quad \omega \in (0,1], \quad \varphi(t,x) = \psi(t), \quad 0 \leq \psi \leq 1, \quad \psi \in \mathcal{D}(0,T),$$

in (4.5), we obtain

$$\begin{aligned}
&\omega \int_0^T \int_\Omega \frac{\kappa(\theta_\varepsilon)}{(1+\theta_\varepsilon)^{\omega+1}} |\nabla\theta_\varepsilon|^2 \psi \, dxdt \\
&\leq - \int_0^T\int_\Omega (\rho_\varepsilon + \delta) Q_h(\theta_\varepsilon)\psi_t \, dxdt + \delta \int_0^T\int_\Omega h(\theta_\varepsilon)\theta_\varepsilon^{\alpha+1}\psi \, dxdt \\
&\quad + \int_0^T\int_\Omega \theta_\varepsilon p_\theta(\rho_\varepsilon)|\mathrm{div}\mathbf{u}_\varepsilon|\psi \, dxdt \\
&\quad + \varepsilon \int_0^T\int_\Omega |\nabla\rho_\varepsilon \cdot \nabla\left((Q_h(\theta_\varepsilon) - Q(\theta_\varepsilon)h(\theta_\varepsilon))\varphi\right)| \, dxdt.
\end{aligned}$$

Observing that

$$\int_0^T\int_\Omega \left|\nabla(1+\theta_\varepsilon)^{\frac{\alpha+1-\omega}{2}}\right|^2 \psi \, dxdt \leq c \int_0^T\int_\Omega \frac{\kappa(\theta_\varepsilon)}{(1+\theta_\varepsilon)^{\omega+1}} |\nabla\theta_\varepsilon|^2 \psi \, dxdt,$$

and

$$\begin{aligned}
&\int_0^T\int_\Omega \theta_\varepsilon p_\theta(\rho_\varepsilon)|\mathrm{div}\mathbf{u}_\varepsilon|\psi \, dxdt \\
&\leq c\|\theta_\varepsilon\|_{L^2([0,T];L^6(\Omega))}\|\mathbf{u}_\varepsilon\|_{L^2([0,T];W^{1,2}(\Omega))}\|p_\theta(\rho_\varepsilon)\|_{L^\infty([0,T];L^3(\Omega))},
\end{aligned}$$

and, by the hypothesis (2.6), we have

$$\begin{aligned}
&\varepsilon \int_0^T\int_\Omega |\nabla\rho_\varepsilon \cdot \nabla[(Q_h(\theta_\varepsilon) - Q(\theta_\varepsilon)h(\theta_\varepsilon))\varphi]| \, dxdt \\
&\leq c\varepsilon\|\psi\|_{L^\infty}\|\nabla\rho_\varepsilon\|_{L^2} \left\| \frac{Q(\theta_\varepsilon)}{(1+\theta_\varepsilon)^{\omega+1}}\nabla\theta_\varepsilon \right\|_{L^2} \\
&\leq c\varepsilon\|\nabla\rho_\varepsilon\|_{L^2} \left\|(1+\theta_\varepsilon)^{\frac{\alpha-1-\omega}{2}}\nabla\theta_\varepsilon\right\|_{L^2} \\
&= c\varepsilon\|\nabla\rho_\varepsilon\|_{L^2} \left\|\nabla(1+\theta_\varepsilon)^{\frac{\alpha+1-\omega}{2}}\right\|_{L^2},
\end{aligned}$$

where, the following property is used

$$\frac{Q(\theta)^2}{(1+\theta)^{\omega+1}}|\nabla\theta|^2 \leq c(1+\theta)^\alpha|\nabla\theta|^2.$$

By Young's inequality, Remark 4.1, and the energy inequality in Proposition 4.1, one has

$$(1+\theta_\varepsilon)^{\frac{\alpha+1-\omega}{2}} \in L^2([0,T]; W^{1,2}(\Omega)).$$

Thus,

$$\theta_\varepsilon^{\frac{\alpha+1-\omega}{2}} \in L^2([0,T]; W^{1,2}(\Omega)),$$

since

$$\theta_\varepsilon^{\frac{\alpha-1-\omega}{2}}|\nabla\theta_\varepsilon| \leq (1+\theta_\varepsilon)^{\frac{\alpha-1-\omega}{2}}|\nabla\theta_\varepsilon|.$$

In the next Section, we shall take the limit of the other artificial term: the artificial pressure, as $\delta \to 0$.

5. The Limit of Vanishing Artificial Pressure

In this section, we take the limit as $\delta \to 0$ to eliminate the δ-dependent terms appearing in (4.1), while in the previous Section passing to the limit as $\varepsilon \to 0$ has been done. Denote by $\{\rho_\delta, \mathbf{u}_\delta, \theta_\delta, \mathbf{H}_\delta\}_{\delta>0}$ the sequence of approximate solutions obtained in Proposition 4.2. In addition to the possible oscillation effects on density, the concentration effects on temperature is also a major issue of this section. To deal with these difficulties, we employ a variant of well-known Feireisl-Lions method [8, 9, 22] in our *new* context.

5.1. **Energy estimates.** The main object in this subsection is to find sufficient *a priori* estimates. First, our choice of the initial data (4.3) implies that, as $\delta \to 0$,

$$\int_\Omega \left(\frac{1}{2}\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}} + \frac{1}{2}|\mathbf{H}_0|^2 + \rho_{0,\delta}P_e(\rho_{0,\delta}) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta + (\rho_{0,\delta}+\delta)Q(\theta_{0,\delta}) \right) dx$$
$$\to E[\rho, \mathbf{u}, \theta, \mathbf{H}](0) = \int_\Omega \left(\rho_0 P_e(\rho_0) + \rho_0 Q(\theta_0) + \frac{1}{2}\frac{|m_0|^2}{\rho_0} + \frac{1}{2}|\mathbf{H}_0|^2 \right) dx.$$

Hence, from the energy inequality (4.12), we can conclude that

$$\rho_\delta \quad \text{is bounded in } L^\infty([0,T]; L^\gamma(\Omega)), \tag{5.1}$$

$$\sqrt{\rho_\delta}\mathbf{u}_\delta,\ \mathbf{H}_\delta \quad \text{are bounded in } L^\infty([0,T]; L^2(\Omega)), \tag{5.2}$$

$$(\rho_\delta + \delta)Q(\theta_\delta) \quad \text{is bounded in } L^\infty([0,T]; L^1(\Omega)), \tag{5.3}$$

$$\delta \int_0^T\!\!\int_\Omega \left(\rho_\delta^\beta + \theta_\delta^{\alpha+1} \right) dxdt \le c, \tag{5.4}$$

for some constant c, which is independent of δ.

Now, we take

$$\varphi(x,t) = \frac{T - \frac{1}{2}t}{T}, \quad h(\theta) = \frac{1}{1+\theta}$$

in (4.14) to obtain

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \left(\frac{1-\delta}{1+\theta_\delta}(\Psi_\delta : \nabla\mathbf{u}_\delta + \nu|\nabla\times\mathbf{H}_\delta|^2) + \frac{\kappa(\theta_\delta)}{(1+\theta_\delta)^2}|\nabla\theta_\delta|^2 \right) dxdt \\ &\le 2\int_0^T\!\!\int_\Omega \frac{\theta_\delta}{1+\theta_\delta} p_\theta(\rho_\delta)\mathrm{div}\mathbf{u}_\delta\, dxdt + 2\delta \int_0^T\!\!\int_\Omega \theta_\delta^\alpha\, dxdt \\ &\quad - 2\int_\Omega (\rho_{0,\delta}+\delta)Q_1(\theta_{0,\delta})\, dx + \int_\Omega (\rho_\delta+\delta)Q_1(\theta_\delta)(T)\, dx, \end{aligned} \tag{5.5}$$

where

$$Q_1(\theta) = \int_0^\theta \frac{c_v(\xi)}{1+\xi} d\xi \le Q(\theta).$$

Using the estimates (5.3) and (5.4), we deduce from (5.5) that

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \left(\frac{1-\delta}{1+\theta_\delta}(\Psi_\delta : \nabla\mathbf{u}_\delta + \nu|\nabla\times\mathbf{H}_\delta|^2) + \frac{\kappa(\theta_\delta)}{(1+\theta_\delta)^2}|\nabla\theta_\delta|^2 \right) dxdt \\ &\le c\left(1 + \int_0^T\!\!\int_\Omega p_\theta(\rho_\delta)\mathrm{div}\mathbf{u}_\delta\, dxdt \right), \end{aligned}$$

for some constant c which is independent of δ, and here the second term on right-hand side can be rewritten with the help of the renormalized continuity equation as

$$\int_0^T\int_\Omega p_\theta(\rho_\delta)\mathrm{div}\mathbf{u}_\delta\,dxdt=\int_0^T\int_\Omega \partial_t(\rho_\delta P_\theta(\rho_\delta))\,dxdt.$$

By the hypothesis (2.3) and the estimate (5.1), one has

$$\int_0^T\int_\Omega \partial_t(\rho_\delta P_\theta(\rho_\delta))dxdt\le c(\Omega)\left(1+\int_\Omega \rho_\delta^{\frac{\gamma}{3}}\,dx\right)\le c(\Omega)\left(1+\int_\Omega \rho_\delta^{\gamma}\,dx\right)\le c(\Omega,T).$$

Consequently, we can conclude that

$$\frac{\kappa(\theta_\delta)}{(1+\theta_\delta)^2}|\nabla\theta_\delta|^2\in L^1(\Omega\times(0,T)),$$

which, combining with the hypothesis (2.4), gives us that

$$\nabla\ln(1+\theta_\delta),\ \nabla\theta_\delta^{\frac{\alpha}{2}}\ \text{are bounded in } L^2(\Omega\times(0,T)).$$

Thus, combining with Sobolev's imbedding theorem, we obtain that

$$\ln(1+\theta_\delta),\ \theta_\delta^{\frac{\alpha}{2}}\ \text{are bounded in } L^2([0,T];W^{1,2}(\Omega)). \tag{5.6}$$

Moreover, in view of the hypothesis (2.3), we get

$$\theta_\delta p_\theta(\rho_\delta)\ \text{is bounded in } L^2(\Omega\times(0,T)). \tag{5.7}$$

With (5.7) in hand, we can repeat the same procedure as above, taking now

$$h(\theta)=\frac{1}{(1+\theta)^\omega},\quad \omega\in(0,1),\quad \varphi(x,t)=\frac{T-\frac12 t}{T},$$

and finally we can get

$$\int_0^T\int_\Omega\left(\frac{1-\delta}{(1+\theta_\delta)^\omega}(\Psi_\delta:\nabla\mathbf{u}_\delta+\nu|\nabla\times\mathbf{H}_\delta|^2)+\omega\frac{\kappa(\theta_\delta)}{(1+\theta_\delta)^{1+\omega}}|\nabla\theta_\delta|^2\right)dxdt\le c, \tag{5.8}$$

for some constant c which is independent of δ.

Letting $\omega\to0$ and using the monotone convergence theorem, we deduce that

$$\mathbf{u}_\delta,\ \mathbf{H}_\delta\ \text{are bounded in } L^2([0,T];W_0^{1,2}(\Omega)). \tag{5.9}$$

Moreover, from (5.8), we have

$$(1+\theta_\delta)^{\frac{\alpha+1-\omega}{2}}\ \text{is bounded in } L^2([0,T];W^{1,2}(\Omega)),\ \text{for any } \omega\in(0,1]. \tag{5.10}$$

In particular, this implies that

$$\theta_\delta\ \text{is bounded in } L^2([0,T];L^{3\alpha+2}(\Omega)). \tag{5.11}$$

Using Hölder inequality, we have

$$\|\theta_\delta^{\alpha+\frac43}\|_{L^1(D)}\le\|\theta_\delta^{\alpha+\frac23}\|_{L^3(D)}\|\theta_\delta^{\frac23}\|_{L^{\frac32}(D)}\le\|\theta_\delta^{\alpha+\frac23}\|_{L^3(D)}\|\theta_\delta\|_{L^1(D)}^{\frac23}$$

for any $D\subset\Omega$, which, together with (5.10), (5.3) and the hypothesis (2.6), yields

$$\int_{\{\rho_\delta\ge d\}}\theta_\delta^{\alpha+\frac43}\,dxdt\le c(d),$$

for any $d>0$. In particular,

$$\int_{\{\rho_\delta\ge d\}}\theta_\delta^{\alpha+1}\,dxdt\le c(d) \tag{5.12}$$

for any $d>0$.

5.2. **Temperature estimates.** In order to pass to the limit in the term $K(\theta)$, our aim in this subsection is to derive uniform estimates on θ_δ in $L^{\alpha+1}(\Omega\times(0,T))$. To this end, we will follow the argument in [9].

To begin with, we have

$$\int_{\{\rho_\delta\geq d\}}\rho_\delta\,dx \geq M_\delta - d|\Omega| \geq \frac{M}{2}-d|\Omega|,$$

where M_δ denotes the total mass,

$$M_\delta=\int_\Omega \rho_\delta\,dx,$$

independent of $t\in[0,T]$, and

$$M=\int_\Omega \rho_0\,dx>0.$$

On the other hand, Hölder inequality yields

$$\int_{\{\rho_\delta\geq d\}}\rho_\delta\,dxdt \leq \|\rho_\delta\|_{L^\gamma(\Omega)}|\{\rho_\delta\geq d\}|^{\frac{\gamma-1}{\gamma}}.$$

Consequently, there exists a function $\Lambda=\Lambda(d)$ independent of $\delta>0$ such that

$$|\{\rho_\delta\geq d\}|\geq \Lambda(d)>0 \text{ for all } t\in[0,T], \text{ if } 0\leq d<\frac{M}{2|\Omega|}.$$

Fix $0<d<\frac{M}{4|\Omega|}$ and choose a function $b\in C^\infty(R)$ such that

$$b \text{ is non-increasing;}\quad b(z)=0 \text{ for } z\leq d,\quad b(z)=-1 \text{ if } z\geq 2d.$$

For each $t\in[0,T]$, let $\eta=\eta(t)$ be the unique solution of the Neumann problem:

$$\begin{cases}\Delta\eta=b(\rho_\delta(t))-\frac{1}{|\Omega|}\int_\Omega b(\rho_\delta(t))\,dx \text{ in } \Omega,\\ \nabla\eta\cdot\mathbf{n}|_{\partial\Omega}=0,\quad \int_\Omega \eta\,dx=0,\end{cases}\tag{5.13}$$

where $\eta=\eta(t)$ is a function of spatial variable $x\in\Omega$ with t as a parameter. Since the right-hand side of (5.13) has a bound which is independent of δ, there is a constant $\underline{\eta}$ such that

$$\eta=\eta(t)\geq\underline{\eta} \text{ for all } \delta>0 \text{ and } t\in[0,T].$$

Now, we take

$$\varphi(x,t)=\psi(t)(\eta-\underline{\eta}),\quad 0\leq\psi\leq1,\quad \psi\in\mathcal{D}(0,T),$$

as a test function in (4.14) to obtain

$$\begin{aligned}&\int_0^T\!\!\int_\Omega K_h(\theta_\delta)\left(b(\rho_\delta(t))-\frac{1}{|\Omega|}\int_\Omega b(\rho_\delta(t))\right)\psi\,dxdt\\ &\leq 2\|\eta\|_{L^\infty(\Omega\times(0,T))}\int_0^T\!\!\int_\Omega\left(\delta h(\theta_\delta)\theta_\delta^{1+\alpha}+h(\theta_\delta)\theta_\delta p_\theta(\rho_\delta)|\mathrm{div}\mathbf{u}_\delta|\right)dxdt\\ &\quad+\|\nabla\eta\|_{L^\infty(\Omega\times(0,T))}\int_0^T\!\!\int_\Omega \rho_\delta Q_h(\theta_\delta)|\mathbf{u}_\delta|\,dxdt\\ &\quad+\int_0^T\!\!\int_\Omega\left((\rho_\delta+\delta)Q_h(\theta_\delta)(\underline{\eta}-\eta)\psi_t-(\rho_\delta+\delta)Q_h(\theta_\delta)\psi\partial_t\eta\right)dxdt.\end{aligned}\tag{5.14}$$

Next, taking

$$h(\theta)=\frac{1}{(1+\theta)^\omega}$$

for $0 < \omega < 1$, letting $\omega \to 0$, using Lebesgue's dominated convergence theorem and the estimates (5.1), (5.3), (5.4), and (5.7), we obtain

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega K(\theta_\delta)\left(b(\rho_\delta(t)) - \frac{1}{|\Omega|}\int_\Omega b(\rho_\delta(t))\right)dxdt \\
&\le c\left(1 + \int_0^T\!\!\int_\Omega (\rho_\delta+\delta)\theta_\delta|\partial_t\eta|\,dxdt\right).
\end{aligned} \tag{5.15}$$

On the other hand,

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega K(\theta_\delta)\left(b(\rho_\delta(t)) - \frac{1}{|\Omega|}\int_\Omega b(\rho_\delta(t))\right)dxdt \\
&= \int_{\{\rho_\delta<d\}} K(\theta_\delta)\left(b(\rho_\delta(t)) - \frac{1}{|\Omega|}\int_\Omega b(\rho_\delta(t))\right)dxdt \\
&\quad + \int_{\{\rho_\delta\ge d\}} K(\theta_\delta)\left(b(\rho_\delta(t)) - \frac{1}{|\Omega|}\int_\Omega b(\rho_\delta(t))\right)dxdt.
\end{aligned}$$

where, by virtue of (5.12), the second integral on the right-hand side is bounded by a constant independent of δ. Furthermore,

$$-\frac{1}{|\Omega|}\int_\Omega b(\rho_\delta)\,dx \ge -\frac{1}{|\Omega|}\int_{\{\rho_\delta\ge 2d\}} b(\rho_\delta)\,dx = \frac{|\{\rho_\delta \ge 2d\}|}{|\Omega|} \ge \frac{\Lambda(2d)}{|\Omega|} > 0.$$

Thus, we obtain

$$\begin{aligned}
&\int_{\{\rho_\delta<d\}} K(\theta_\delta)\left(b(\rho_\delta(t)) - \frac{1}{|\Omega|}\int_\Omega b(\rho_\delta(t))\right)dxdt \\
&\ge \frac{\Lambda(2d)}{|\Omega|}\int_{\{\rho_\delta<d\}} K(\theta_\delta)\,dxdt.
\end{aligned} \tag{5.16}$$

Combining (5.15), (5.16) together, we get

$$\int_{\{\rho_\delta<d\}} K(\theta_\delta)\,dxdt \le c\left(1 + \int_0^T\!\!\int_\Omega (\rho_\delta+\delta)\theta_\delta|\partial_t\eta|\,dxdt\right) \tag{5.17}$$

with c independent of δ.

Finally, since ρ_δ is a renormalized solution of the equation (1.1a), we have,

$$\begin{aligned}
\Delta(\partial_t\eta) &= \partial_t b(\rho_\delta) - \frac{1}{|\Omega|}\int_\Omega \partial_t b(\rho_\delta)\,dx \\
&= (b(\rho_\delta) - b'(\rho_\delta)\rho_\delta)\mathrm{div}\mathbf{u}_\delta - \mathrm{div}(b(\rho_\delta)\mathbf{u}_\delta) + \frac{1}{|\Omega|}\int_\Omega (b'(\rho_\delta)\rho_\delta - b(\rho_\delta))\mathrm{div}\mathbf{u}_\delta\,dx.
\end{aligned}$$

Hence,

$$\partial_t\eta \text{ is bounded in } L^2([0,T];W^{1,2}(\Omega)).$$

Consequently, using (5.17), we conclude that

$$\int_{\{\rho_\delta<d\}} K(\theta_\delta)\,dxdt \le c, \text{ c independent of } \delta$$

which, together with (5.12) and the hypothesis (2.4), yields

$$\theta_\delta \text{ is bounded in } L^{1+\alpha}(\Omega\times(0,T)). \tag{5.18}$$

5.3. **Refined pressure estimates.** Our goal now is to improve estimates on pressure. We follow step by step the argument of Section 5.1 in [16], that is, using the Bogovskii operator "$\mathrm{div}^{-1}[\ln(1+\rho_\delta)]$" as a test function for the modified momentum equation (4.11). Similarly to Lemma 5.1 in [16], we have the estimate

$$\int_\Omega (p(\rho_\delta,\theta_\delta)+\delta\rho_\delta^\beta)\ln(1+\rho_\delta)\,dx \le c, \tag{5.19}$$

and hence

$$p(\rho_\delta,\theta_\delta)\ln(1+\rho_\delta) \text{ is bounded in } L^1(\Omega\times(0,T)). \tag{5.20}$$

Let us define the set

$$J_k^\delta = \{(x,t)\in(0,T)\times\Omega : \rho_\delta(x,t)\le k\} \text{ for } k>0 \text{ and } \delta\in(0,1).$$

In view of (5.1) and the hypothesis (2.3), there exists a constant $s\in(0,\infty)$ such that for all $\delta\in(0,1)$ and $k>0$,

$$|\{(0,T)\times\Omega - J_k^\delta\}| \le \frac{s}{k}.$$

We have the following estimate:

$$\begin{aligned}\int_0^T\!\!\int_\Omega \delta\rho_\delta^\beta\,dxdt &= \int_{J_k^\delta}\delta\rho_\delta^\beta\,dxdt + \int_{\Omega\times(0,T)-J_k^\delta}\delta\rho_\delta^\beta\,dxdt\\ &\le T\delta k^\beta|\Omega| + \delta\int_0^T\!\!\int_\Omega \chi_{\Omega\times(0,T)-J_k^\delta}\rho_\delta^\beta\,dxdt.\end{aligned}\tag{5.21}$$

Then, by the Hölder inequality in Orlicz spaces (cf. [1]) and the estimate (5.19), we obtain

$$\begin{aligned}&\delta\int_0^T\!\!\int_\Omega \chi_{\Omega\times(0,T)-J_k^\delta}\rho_\delta^\beta\,dxdt \le \delta\|\chi_{\Omega\times(0,T)-J_k^\delta}\|_{L_N}\max\{1,\int_0^T\!\!\int_\Omega M(\rho_\delta^\beta)\,dxdt\}\\ &\le \delta\left(N^{-1}\left(\frac{k}{s}\right)\right)^{-1}\max\{1,\int_0^T\!\!\int_\Omega 2(1+\rho_\delta^\beta)\ln(1+\rho_\delta^\beta)\,dxdt\}\\ &\le \delta\left(N^{-1}\left(\frac{k}{s}\right)\right)^{-1}\max\{1,(4\ln 2)T|\Omega| + 4\beta\int_0^T\!\!\int_{\Omega\cap\{\rho_\delta\ge1\}}\rho_\delta^\beta\ln(1+\rho_\delta)\,dxdt\}\\ &\le \left(N^{-1}\left(\frac{k}{s}\right)\right)^{-1}\max\{\delta,(4\ln 2)\delta T|\Omega| + 4\delta\beta\int_0^T\!\!\int_\Omega\rho_\delta^\beta\ln(1+\rho_\delta)\,dxdt\},\end{aligned}\tag{5.22}$$

where $L_M(\Omega)$, and $L_N(\Omega)$ are two Orlicz Spaces generated by two complementary N-functions

$$M(s) = (1+s)\ln(1+s) - s,\quad N(s) = e^s - s - 1,$$

respectively. Due to (5.19), we know, if $\delta<1$,

$$\max\{\delta,(4\ln 2)\delta T|\Omega| + 4\delta\beta\int_0^T\!\!\int_\Omega\rho_\delta^\beta\ln(1+\rho_\delta)\,dxdt\}\le c,$$

for some $c>0$ which is independent of δ. Combining (5.21) with (5.22), we obtain the following estimate

$$\left|\int_0^T\!\!\int_\Omega\delta\rho_\delta^\beta\,dxdt\right| \le T\delta k^\beta|\Omega| + c\left(N^{-1}\left(\frac{k}{s}\right)\right)^{-1},$$

where c does not depend on δ and k. Consequently

$$\limsup_{\delta\to0}\left|\int_0^T\!\!\int_\Omega\delta\rho_\delta^\beta\,dxdt\right| \le c\left(N^{-1}\left(\frac{k}{s}\right)\right)^{-1}. \tag{5.23}$$

The right-hand side of (5.23) tends to zero as $k \to \infty$. Thus, we have

$$\lim_{\delta \to 0} \int_0^T \int_\Omega \delta \rho_\delta^\beta \, dxdt = 0,$$

which yields

$$\delta \rho_\delta^\beta \to 0 \text{ in } \mathcal{D}'(\Omega \times (0,T)). \tag{5.24}$$

5.4. **Strong convergence of the temperature.** Since ρ_δ, $\mathbf{u}_\delta$ satisfy the continuity equation (1.1a), then

$$\rho_\delta \to \rho \text{ in } C([0,T]; L^\gamma_{weak}(\Omega)), \tag{5.25}$$

$$\mathbf{u}_\delta \to \mathbf{u} \text{ weakly in } L^2([0,T]; W_0^{1,2}(\Omega)), \tag{5.26}$$

and thus

$$\rho_\delta \mathbf{u}_\delta \to \rho \mathbf{u} \text{ weakly-* in } L^\infty([0,T]; L^{\frac{2\gamma}{1+\gamma}}(\Omega)), \tag{5.27}$$

where the limit functions $\rho \geq 0$, $\mathbf{u}$ satisfy the continuity equation (1.1a) in $\mathcal{D}'(\Omega \times (0,T))$. Similarly, since ρ_δ, $\mathbf{u}_\delta$, θ_δ, $\mathbf{H}_\delta$ satisfy the momentum equation (4.11), we have

$$\rho_\delta \mathbf{u}_\delta \to \rho \mathbf{u} \text{ in } C([0,T]; L^{\frac{2\gamma}{1+\gamma}}_{weak}(\Omega))$$

and

$$\rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \to \rho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^2([0,T]; L^{\frac{6\gamma}{3+4\gamma}}(\Omega)).$$

From the hypothesis (2.3), equation (1.1d), and the estimates (5.1), (5.9), we can assume

$$p_\theta(\rho_\delta) \to \overline{p_\theta(\rho)} \text{ weakly in } L^\infty([0,T]; L^3(\Omega)),$$

$$\mathbf{H}_\delta \to \mathbf{H} \text{ weakly in } L^2([0,T]; W_0^{1,2}(\Omega)) \cap C([0,T]; L^2_{weak}(\Omega)),$$

with $\text{div}\mathbf{H} = 0$ in $\mathcal{D}'(\Omega \times (0,T))$. Hence, ρ, $\rho\mathbf{u}$, $\mathbf{H}$ satisfy the initial data (1.2). Due to the estimate (5.10), we can also assume

$$\theta_\delta \to \theta \text{ weakly in } L^2([0,T]; W^{1,2}(\Omega)),$$

with $\theta \geq 0$ in $\mathcal{D}'(\Omega \times (0,T))$, since

$$|\nabla \theta_\delta| \leq (1+\theta_\delta)^{\frac{\alpha-1-\omega}{2}} |\nabla \theta_\delta|, \text{ for } \omega \in (0,1).$$

Thus

$$\theta_\delta p_\theta(\rho_\delta) \to \theta \overline{p_\theta(\rho)} \text{ weakly in } L^2([0,T]; L^2(\Omega)), \tag{5.28}$$

$$(\nabla \times \mathbf{H}_\delta) \times \mathbf{H}_\delta \to (\nabla \times \mathbf{H}) \times \mathbf{H} \text{ in } \mathcal{D}'(\Omega \times (0,T)). \tag{5.29}$$

Similarly,

$$\nabla \times (\mathbf{u}_\delta \times \mathbf{H}_\delta) \to \nabla \times (\mathbf{u} \times \mathbf{H}) \text{ in } \mathcal{D}'(\Omega \times (0,T)), \tag{5.30}$$

$$\nabla \times (\nu \nabla \times \mathbf{H}_\delta) \to \nabla \times (\nu \nabla \times \mathbf{H}) \text{ in } \mathcal{D}'(\Omega \times (0,T)). \tag{5.31}$$

In view of (5.6) and the hypothesis (2.6), we can assume

$$Q_h(\theta_\delta) \to \overline{Q_h(\theta)} \text{ weakly in } L^2([0,T]; W^{1,2}(\Omega)), \tag{5.32}$$

$$M(\theta_\delta) \to \overline{M(\theta)} \text{ weakly in } L^2([0,T]; W^{1,2}(\Omega)), \tag{5.33}$$

for any $M \in C^1[0,\infty)$ satisfying the growth restriction

$$|M'(\xi)| \leq c(1 + \xi^{\frac{\alpha}{2}-1}),$$

and, consequently,

$$(\rho_\delta + \delta) Q_h(\theta_\delta) \to \rho \overline{Q_h(\theta)} \text{ weakly in } L^2([0,T]; L^{\frac{6\gamma}{\gamma+6}}(\Omega)), \tag{5.34}$$

since $L^\gamma \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$, if $\gamma > \frac{3}{2}$.

At this stage we need a variant of the celebrated Aubin-Lions Lemma (cf. Lemma 6.3 in [9]):

Lemma 5.1. *Let* $\{\theta_n\}_{n=1}^\infty$ *be a sequence of functions such that*

$$\{\theta_n\}_{n=1}^\infty \text{ is bounded in } L^2([0,T];L^q(\Omega)) \cap L^\infty([0,T];L^1(\Omega)), \text{ with } q > \frac{6}{5},$$

and assume that

$$\partial_t \theta_n \geq \chi_n \text{ in } \mathcal{D}'(\Omega \times (0,T)),$$

where

$$\chi_n \text{ are bounded in } L^1([0,T];W^{-m,r}(\Omega))$$

for certain $m \geq 1$, $r > 1$. *Then* $\{\theta_n\}_{n=1}^\infty$ *contains a subsequence such that*

$$\theta_n \to \theta \text{ in } L^2([0,T];W^{-1,2}(\Omega)).$$

With this lemma in hand, we can show the following property:

Lemma 5.2. *Let* $h = \frac{1}{1+\theta}$, *then*

$$(\rho_\delta + \delta)Q_h(\theta_\delta) \to \rho\overline{Q_h(\theta)} \text{ in } L^2([0,T];W^{-1,2}(\Omega)).$$

Proof. Substituting $h = \frac{1}{1+\theta}$ into (4.14), we get

$$\partial_t \left((\rho_\delta + \delta)Q_h(\theta_\delta)\right) \geq -\operatorname{div}(\rho_\delta Q_h(\theta_\delta)\mathbf{u}_\delta) + \operatorname{div}\left(\frac{\kappa(\theta_\delta)}{1+\theta_\delta}\nabla\theta_\delta\right) - \delta\frac{\theta_\delta^{\alpha+1}}{1+\theta_\delta} - \frac{\theta_\delta}{1+\theta_\delta}p_\theta(\rho_\delta)\operatorname{div}\mathbf{u}_\delta,$$

in $\mathcal{D}'(\Omega \times (0,T))$. Since

$$\frac{\theta_\delta}{1+\theta_\delta}p_\theta(\rho_\delta)|\operatorname{div}\mathbf{u}_\delta| \leq p_\theta(\rho_\delta)|\operatorname{div}\mathbf{u}_\delta|,$$

we know that, in view of (2.3) and (5.1),

$$\frac{\theta_\delta}{1+\theta_\delta}p_\theta(\rho_\delta)\operatorname{div}\mathbf{u}_\delta \text{ is bounded in } L^2([0,T];L^r(\Omega)), \text{ for some } r > 1,$$

and, consequently,

$$\frac{\theta_\delta}{1+\theta_\delta}p_\theta(\rho_\delta)\operatorname{div}\mathbf{u}_\delta \text{ is bounded in } L^2([0,T];W^{-k,r}(\Omega)), \text{ for all } k \geq 1.$$

Similarly, by (5.18)

$$\delta\frac{\theta_\delta^{\alpha+1}}{1+\theta_\delta} \text{ is bounded in } L^2([0,T];W^{-k,r}(\Omega)), \text{ for all } k \geq 1,$$

and

$$\operatorname{div}(\rho_\delta Q_h(\theta_\delta)\mathbf{u}_\delta) \text{ is bounded in } L^1([0,T];W^{-1,r}(\Omega)), \text{ for some } r > 1.$$

Next, by (2.4), we have

$$\frac{\kappa(\theta_\delta)}{1+\theta_\delta}|\nabla\theta_\delta| \leq c(1+\theta_\delta)^{\alpha-1}|\nabla\theta_\delta| \leq c\theta_\delta^{\frac{\alpha}{2}}|\nabla\theta_\delta^{\frac{\alpha}{2}}|, \text{ if } \theta_\delta \geq 1,$$

and

$$\frac{\kappa(\theta_\delta)}{1+\theta_\delta}|\nabla\theta_\delta| \leq c(1+\theta_\delta)^{\alpha-1}|\nabla\theta_\delta| \leq c|\nabla\theta_\delta|, \text{ if } \theta_\delta \leq 1,$$

thus, by (5.6) and (5.18)

$$\operatorname{div}\left(\frac{\kappa(\theta_\delta)}{1+\theta_\delta}\nabla\theta_\delta\right) \text{ is bounded in } L^1([0,T];W^{-1,r}(\Omega)), \text{ for some } r > 1.$$

Finally, since $Q_h(\theta) \leq Q(\theta)$, by (5.3), we deduce that

$$(\rho_\delta + \delta)Q_h(\theta_\delta) \in L^\infty([0,T];L^1(\Omega)).$$

Hence, combining Lemma 5.1 and (5.34) together, we have

$$(\rho_\delta+\delta)Q_h(\theta_\delta)\to\rho\overline{Q_h(\theta)}\ \text{in}\ L^2([0,T];W^{-1,2}(\Omega)).$$

□

Lemma 5.2 and (5.33) imply

$$(\rho_\delta+\delta)Q_h(\theta_\delta)M(\theta_\delta)\to\rho\overline{Q_h(\theta)}\quad\overline{M(\theta)}\ \text{in}\ L^1(\Omega\times(0,T)),\tag{5.35}$$

where $h(\theta)=\frac{1}{1+\theta}$. On the other hand, choosing $M(\theta)=\theta$, then $\theta Q_h(\theta)$ satisfies (5.33) since $\alpha>2$. Hence,

$$(\rho_\delta+\delta)Q_h(\theta_\delta)\theta_\delta\to\rho\overline{\theta Q_h(\theta)}\ \text{weakly in}\ L^1(\Omega\times(0,T)).\tag{5.36}$$

Properties (5.35) and (5.36) implies

$$\overline{\theta Q_h(\theta)}=\overline{Q_h(\theta)}\theta,\ \text{a.e. on}\ \{\rho>0\},$$

which yields

$$\theta_\delta\to\theta\ \text{in}\ L^1(\Omega\times(0,T)).\tag{5.37}$$

Indeed, we know that $Q_h(\theta)$ is strictly increasing and its derivative has upper bound, therefore its inverse $Q_h^{-1}(\theta)$ exists and has lower bound $1/\overline{c_v}$. Thus,

$$\begin{aligned}&\int_0^T\!\!\int_\Omega|Q_h(\theta)-Q_h(\theta_\delta)|^2\,dxdt\\ &\le\overline{c_v}\int_0^T\!\!\int_\Omega(Q_h^{-1}(Q_h(\theta))-Q_h^{-1}(Q_h(\theta_\delta)))(Q_h(\theta)-Q_h(\theta_\delta))\,dxdt\\ &=\overline{c_v}\int_0^T\!\!\int_\Omega(\theta-\theta_\delta)(Q(\theta)-Q(\theta_\delta))\,dxdt\to0,\ \text{as}\ \delta\to0.\end{aligned}$$

Therefore,

$$Q_h(\theta_\delta)\to Q_h(\theta),\ \text{in}\ L^2(\Omega\times(0,T)),\qquad\text{as}\ \delta\to0,$$

and, hence,

$$Q_h(\theta_\delta)\to Q_h(\theta),\ \text{a.e. in}\ \Omega\times(0,T),\qquad\text{as}\ \delta\to0.$$

Because $Q_h^{-1}(\theta)$ is continuous, we deduce that

$$\theta_\delta=Q_h^{-1}(Q_h(\theta_\delta))\to\theta=Q_h^{-1}(Q_h(\theta)),\ \text{a.e. in}\ \Omega\times(0,T),\qquad\text{as}\ \delta\to0,$$

which, combining Egorov's theorem, Theorem 2.10 in [9], and the weak convergence of $\{\theta_\delta\}$ to θ in $L^1(\Omega\times(0,T))$, verifies (5.37).

Finally, (5.37), together with (5.26) and Lemma 4.1, implies

$$\Psi_\delta=\mu(\theta_\delta)(\nabla\mathbf{u}_\delta+\nabla\mathbf{u}_\delta^T)+\lambda(\theta_\delta)\mathrm{div}\mathbf{u}_\delta\mathbf{I}\to\Psi=\mu(\theta)(\nabla\mathbf{u}+\nabla\mathbf{u}^T)+\lambda(\theta)\mathrm{div}\mathbf{u}\mathbf{I},\tag{5.38}$$

in $\mathcal{D}'(\Omega\times(0,T))$, and,

$$\int_0^T\!\!\int_\Omega h(\theta_\delta)\Psi_\delta:\nabla\mathbf{u}_\delta\,dxdt\ge\int_0^T\!\!\int_\Omega h(\theta)\Psi:\nabla\mathbf{u}\,dxdt.\tag{5.39}$$

5.5. **Strong convergence of the density.** In this subsection, we will adopt the technique in [9] to show the strong convergence of the density, specifically,

$$\rho_\delta \to \rho \text{ in } L^1(\Omega \times (0,T)). \tag{5.40}$$

First, due to (5.20), Proposition 2.1 in [9] and (5.28), we can assume that

$$p(\rho_\delta, \theta_\delta) \to \overline{p(\rho,\theta)} \text{ weakly in } L^1(\Omega \times (0,T)), \tag{5.41}$$

which, together with (5.24)-(5.31), implies that

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{5.42}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \overline{p(\rho,\theta)} = (\nabla \times \mathbf{H}) \times \mathbf{H} + \operatorname{div}\Psi, \tag{5.43}$$

$$\mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \quad \operatorname{div}\mathbf{H} = 0, \tag{5.44}$$

in $\mathcal{D}'(\Omega \times (0,T))$.

Similarly to Section 5.3 in [16], we use

$$\varphi_i(x,t) = \psi(t)\phi(x)\mathcal{A}_i[T_k(\rho_\delta)],\ \psi \in \mathcal{D}(0,T),\ \phi \in \mathcal{D}(\Omega),\ i = 1,2,3,$$

as test functions for the modified momentum balance equation (4.11), where $\mathcal{A}_i$ can be expressed by their Fourier symbol as

$$\mathcal{A}_i[\cdot] = \mathcal{F}^{-1}\left[\frac{-\mathrm{i}\xi_i}{|\xi|^2}\mathcal{F}[\cdot]\right],\ i = 1,2,3,$$

and $T_k(\rho)$ are cut-off functions,

$$T_k(\rho) = \min\{\rho, k\},\ k \geq 1.$$

A lengthy but straightforward computation shows that

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \psi\phi\left((p(\rho_\delta,\theta_\delta) + \delta\rho_\delta^\beta - \lambda(\theta_\delta)\operatorname{div}\mathbf{u}_\delta)T_k(\rho_\delta) - 2\mu(\theta_\delta)\frac{\partial \mathbf{u}_\delta^i}{\partial x_j}\mathcal{R}_{i,j}[T_k(\rho_\delta)]\right) dxdt \\
&= \int_0^T\!\!\int_\Omega \psi\partial_{x_i}\phi\left(\lambda(\theta_\delta)\operatorname{div}\mathbf{u}_\delta - p(\rho_\delta,\theta_\delta) - \delta\rho_\delta^\beta\right)\mathcal{A}_i[T_k(\rho_\delta)]\, dxdt \\
&\quad + \int_0^T\!\!\int_\Omega \psi\mu(\theta_\delta)\left(\frac{\partial \mathbf{u}_\delta^i}{\partial x_j} + \frac{\partial \mathbf{u}_\delta^j}{\partial x_i}\right)\partial_{x_j}\phi\mathcal{A}_i[T_k(\rho_\delta)]\, dxdt \\
&\quad - \int_0^T\!\!\int_\Omega \phi\rho_\delta u_\delta^i\Big(\psi_t\mathcal{A}_i[T_k(\rho_\delta)] + \psi\mathcal{A}_i[(T_k(\rho_\delta) - T_k'(\rho_\delta)\rho_\delta)\operatorname{div}\mathbf{u}_\delta]\Big)dxdt \\
&\quad - \int_0^T\!\!\int_\Omega \psi\rho_\delta u_\delta^i u_\delta^j\partial_{x_j}\phi\mathcal{A}_i[T_k(\rho_\delta)]\, dxdt \\
&\quad + \int_0^T\!\!\int_\Omega \psi u_\delta^i\left(T_k(\rho_\delta)\mathcal{R}_{i,j}[\rho_\delta u_\delta^j] - \phi\rho_\delta u_\delta^j\mathcal{R}_{i,j}[T_k(\rho_\delta)]\right) dxdt \\
&\quad - \int_0^T\!\!\int_\Omega \psi\phi(\nabla \times \mathbf{H}_\delta) \times \mathbf{H}_\delta \cdot \mathcal{A}[T_k(\rho_\delta)]\, dxdt,
\end{aligned} \tag{5.45}$$

where the operators $\mathcal{R}_{i,j}=\partial_{x_j}\mathcal{A}_i[v]$ and the summation convention is used to simplify notations. On the other hand, following the arguments of Section 5.3 in [16], one has

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega \psi\phi\left(\left(\overline{p(\rho,\theta)}-\overline{\lambda(\theta)\mathrm{div}\mathbf{u}}\right)\overline{T_k(\rho)}-\overline{2\mu(\theta)\frac{\partial\mathbf{u}^i}{\partial x_j}}\mathcal{R}_{i,j}[\overline{T_k(\rho)}]\right)dxdt\\
&=\int_0^T\!\!\int_\Omega \psi\partial_{x_i}\phi\left(\overline{\lambda(\theta)\mathrm{div}\mathbf{u}}-\overline{p(\rho,\theta)}\right)\mathcal{A}_i[\overline{T_k(\rho)}]\,dxdt\\
&\quad+\int_0^T\!\!\int_\Omega \psi\overline{\mu(\theta)\left(\frac{\partial\mathbf{u}^i}{\partial x_j}+\frac{\partial\mathbf{u}^j}{\partial x_i}\right)}\partial_{x_j}\phi\mathcal{A}_i[\overline{T_k(\rho)}]\,dxdt\\
&\quad-\int_0^T\!\!\int_\Omega \phi\rho u^i\left(\psi_t\mathcal{A}_i[\overline{T_k(\rho)}]+\psi\mathcal{A}_i[\overline{(T_k(\rho)-T_k'(\rho)\rho)\mathrm{div}\mathbf{u}}]\right)dxdt\\
&\quad-\int_0^T\!\!\int_\Omega \psi\rho u^iu^j\partial_{x_j}\phi\mathcal{A}_i[\overline{T_k(\rho)}]\,dxdt\\
&\quad+\int_0^T\!\!\int_\Omega \psi u^i\left(\overline{T_k(\rho)}\mathcal{R}_{i,j}[\phi\rho u^j]-\phi\rho u^j\mathcal{R}_{i,j}[\overline{T_k(\rho)}]\right)dxdt\\
&\quad-\int_0^T\!\!\int_\Omega \psi\phi(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\mathcal{A}[\overline{T_k(\rho)}]\,dxdt.
\end{aligned}\tag{5.46}
$$

Now, following the argument in Section 5.3 in [16], the Div-Curl Lemma can be used in order to show that the right-hand side of (5.45) converges to that of (5.46), that is

$$
\begin{aligned}
&\lim_{\delta\to0}\int_0^T\!\!\int_\Omega \psi\phi\left((p(\rho_\delta,\theta_\delta)-\lambda(\theta_\delta)\mathrm{div}\mathbf{u}_\delta)T_k(\rho_\delta)-2\mu(\theta_\delta)\frac{\partial\mathbf{u}_\delta^i}{\partial x_j}\mathcal{R}_{i,j}[T_k(\rho_\delta)]\right)dxdt\\
&=\int_0^T\!\!\int_\Omega \psi\phi\left((\overline{p(\rho,\theta)}-\overline{\lambda(\theta)\mathrm{div}\mathbf{u}})\overline{T_k(\rho)}-\overline{2\mu(\theta)\frac{\partial\mathbf{u}^i}{\partial x_j}}\mathcal{R}_{i,j}[\overline{T_k(\rho)}]\right)dxdt.
\end{aligned}\tag{5.47}
$$

Noting that

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega \varphi\mu(\theta_\delta)\frac{\partial\mathbf{u}_\delta^i}{\partial x_j}\mathcal{R}_{i,j}[T_k(\rho_\delta)]\,dxdt\\
&=\int_0^T\!\!\int_\Omega\left(\mathcal{R}_{i,j}\left[\varphi\mu(\theta_\delta)\frac{\partial\mathbf{u}_\delta^i}{\partial x_j}\right]-\varphi\mu(\theta_\delta)\mathcal{R}_{i,j}\left[\frac{\partial\mathbf{u}_\delta^i}{\partial x_j}\right]\right)T_k(\rho_\delta)\,dxdt\\
&\quad+\int_0^T\!\!\int_\Omega \varphi\mu(\theta_\delta)\mathrm{div}\mathbf{u}_\delta T_k(\rho_\delta)\,dxdt,
\end{aligned}
$$

for any $\varphi\in\mathcal{D}(\Omega\times(0,T))$, we have, using also (5.47),

$$
\begin{aligned}
&\lim_{\delta\to0}\int_0^T\!\!\int_\Omega \varphi\big((p(\rho_\delta,\theta_\delta)-(\lambda(\theta_\delta)+2\mu(\theta_\delta))\mathrm{div}\mathbf{u}_\delta)T_k(\rho_\delta)\big)dxdt\\
&=\int_0^T\!\!\int_\Omega \varphi\left(\left(\overline{p(\rho,\theta)}-\overline{(\lambda(\theta)+2\mu(\theta))\mathrm{div}\mathbf{u}}\right)\overline{T_k(\rho)}\right)dxdt,
\end{aligned}\tag{5.48}
$$

since Lemma 4.2 in [8] and the strong convergence of the temperature give

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega\left(\mathcal{R}_{i,j}\left[\varphi\mu(\theta_\delta)\frac{\partial\mathbf{u}_\delta^i}{\partial x_j}\right]-\varphi\mu(\theta_\delta)\mathcal{R}_{i,j}\left[\frac{\partial\mathbf{u}_\delta^i}{\partial x_j}\right]\right)T_k(\rho_\delta)\,dxdt\\
&\qquad\to\int_0^T\!\!\int_\Omega\left(\mathcal{R}_{i,j}\left[\varphi\mu(\theta)\frac{\partial\mathbf{u}^i}{\partial x_j}\right]-\varphi\mu(\theta)\mathcal{R}_{i,j}\left[\frac{\partial\mathbf{u}^i}{\partial x_j}\right]\right)T_k(\rho)\,dxdt.
\end{aligned}
$$

As in Section 5.4 of [16], we can conclude from (5.48) that there exists a constant c independent of k such that

$$\limsup_{\delta\to 0+} \|T_k(\rho_\delta) - T_k(\rho)\|_{L^{\gamma+1}((0,T)\times\Omega)} \le c.$$

This implies, in particular, that the limit functions ρ, $\mathbf{u}$ satisfy the continuity equation (1.1a) in the sense of renormalized solutions (cf. Lemma 5.4 in [16]).

Finally, following the argument as in Section 5.6 in [16], (5.40) is verified.

5.6. **Thermal energy equation.** In order to complete the proof of Theorem 2.1, we have to show that ρ, $\mathbf{u}$, θ and $\mathbf{H}$ satisfy the thermal energy equation (1.10) in the sense of Definition 2.1.

In view of (5.18) and (5.37), we have, as $\delta \to 0$,

$$\theta_\delta \to \theta \text{ in } L^p(\Omega\times(0,T)), \text{ for all } 1 \le p < 1+\alpha.$$

Hence, by the Lebesgue's dominated convergence theorem and the hypothesis (2.4), we know, as $\delta \to 0$,

$$K_h(\theta_\delta) \to K_h(\theta) \text{ in } L^1(\Omega\times(0,T)).$$

By (5.25), (5.26) and (5.40), we have, as $\delta \to 0$,

$$\begin{aligned} \rho_\delta Q_h(\theta_\delta)\mathbf{u}_\delta &\to \rho Q_h(\theta)\mathbf{u}, \\ \rho_\delta Q_h(\theta_\delta) &\to \rho Q_h(\theta), \\ h(\theta_\delta)\theta_\delta p_\theta(\rho_\delta)\mathrm{div}\mathbf{u}_\delta &\to h(\theta)\theta p_\theta(\rho)\mathrm{div}\mathbf{u}, \end{aligned}$$

in $\mathcal{D}'(\Omega\times(0,T))$.

Due to the strong convergence of the temperature (5.37) and (5.39)-(5.40), we can pass the limit as $\delta \to 0$ in (4.14) to obtain

$$\begin{aligned} &\int_0^T\!\!\int_\Omega (\rho Q_h(\theta)\partial_t\varphi + \rho Q_h(\theta)\mathbf{u}\cdot\nabla\varphi + K_h(\theta)\Delta\varphi)\,dxdt \\ &\le -\int_0^T\!\!\int_\Omega (h(\theta)(\Psi:\nabla\mathbf{u} + \nu|\nabla\times\mathbf{H}|^2)\varphi\,dxdt \\ &\quad + \int_0^T\!\!\int_\Omega h(\theta)\theta p_\theta(\rho)\mathrm{div}\mathbf{u}\varphi\,dxdt - \int_\Omega \rho_0 Q_h(\theta_0)\varphi(0)\,dx, \end{aligned} \tag{5.49}$$

since

$$\delta\left|\int_0^T\!\!\int_\Omega \theta_\delta^{1+\alpha}h(\theta_\delta)dxdt\right| \le \delta\left|\int_{\{\theta_\delta\le M\}}\theta_\delta^{1+\alpha}dxdt\right| + h(M)\delta\int_0^T\!\!\int_\Omega\theta_\delta^{1+\alpha}dxdt,$$

which tends to zero as $\delta \to 0$, because the first term on the right-hand side tends to zero for fixed M as $\delta \to 0$ while the second term can be made arbitrarily small by taking M large enough in view of (4.6) and (5.4).

Next, taking

$$h(\theta) = \frac{1}{(1+\theta)^\omega}, \quad 0<\omega<1,$$

in (5.49), letting $\omega \to 0$, and using Lebesgue's dominated convergence theorem, we get

$$\begin{aligned} &\int_0^T\!\!\int_\Omega (\rho Q(\theta)\partial_t\varphi + \rho Q(\theta)\mathbf{u}\cdot\nabla\varphi + K(\theta)\Delta\varphi)\,dxdt \\ &\le \int_0^T\!\!\int_\Omega (\theta p_\theta(\rho)\mathrm{div}\mathbf{u} - \nu|\nabla\times\mathbf{H}|^2 - \Psi:\nabla\mathbf{u})\varphi\,dxdt, \end{aligned} \tag{5.50}$$

for any $\varphi \in \mathcal{D}(\Omega\times(0,T))$ and $\varphi \ge 0$, since $\rho Q_h(\theta) \le \rho Q(\theta)$ belongs to $L^1(\Omega\times(0,T))$ by (5.1), (2.6), and (5.11).

Finally, dividing (2.1) by $1+\theta$ and using equation (1.1a), we get

$$\partial_t(\rho f(\theta)) + \mathrm{div}(\rho f(\theta)\mathbf{u}) + \mathrm{div}\left(\frac{q}{1+\theta}\right) \geq \frac{1}{1+\theta}(\nu|\nabla\times\mathbf{H}|^2 + \Psi:\nabla\mathbf{u}) - \frac{q\cdot\nabla\theta}{(1+\theta)^2} - \frac{\theta}{1+\theta}p_\theta(\rho)\mathrm{div}\mathbf{u},$$

in the sense of distributions, where

$$f(\theta) = \int_0^\theta \frac{c_\nu(\xi)}{1+\xi}\,d\xi.$$

Integrating the above inequality over $\Omega\times(0,T)$, we deduce

$$\begin{aligned}\int_0^T\int_\Omega\left(\frac{1}{1+\theta}(\nu|\nabla\times\mathbf{H}|^2 + \Psi:\nabla\mathbf{u}) + \frac{k(\theta)|\nabla\theta|^2}{(1+\theta)^2}\right)dxdt \\ \leq 2\sup_{0\leq t\leq T}\int_\Omega \rho f(\theta)\,dx + \int_0^T\int_\Omega \frac{\theta}{1+\theta}p_\theta(\rho)|\mathrm{div}\mathbf{u}|\,dxdt.\end{aligned} \tag{5.51}$$

By Hölder's inequality, the estimates (5.1), (5.9), and the hypothesis (2.3), one has

$$\int_0^T\int_\Omega \frac{\theta}{1+\theta}p_\theta(\rho)|\mathrm{div}\mathbf{u}|\,dx\,dt \leq \int_0^T\int_\Omega p_\theta(\rho)|\mathrm{div}\mathbf{u}|\,dx\,dt \leq c.$$

Similarly, by Hölder's inequality, the assumption (2.6), and the estimates (5.1), (5.18), we have

$$\int_\Omega \rho f(\theta)\,dx \leq c\int_\Omega \rho\theta\,dx \leq c.$$

Thus, (5.51) and the assumption (2.4) imply that

$$\ln(1+\theta) \in L^2([0,T];W^{1,2}(\Omega)), \quad \theta^{\frac{\alpha}{2}} \in L^2([0,T];W^{1,2}(\Omega)).$$

This completes our proof of Theorem 2.1.

Acknowledgments

Xianpeng Hu's research was supported in part by the National Science Foundation grant DMS-0604362. Dehua Wang's research was supported in part by the National Science Foundation grants DMS-0244487, DMS-0604362, and the Office of Naval Research grant N00014-01-1-0446.

References

[1] R. A. Admas, *Sobolev spaces,* Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[2] H. Cabannes, *Theoretical Magnetofluiddynamics*, Academic Press, New York, 1970.

[3] G.-Q. Chen, D. Wang, *Global solution of nonlinear magnetohydrodynamics with large initial data,* J. Differential Equations, 182 (2002), 344-376.

[4] G.-Q. Chen, D. Wang, *Existence and continuous dependence of large solutions for the magnetohydrodynamic equations,* Z. Angew. Math. Phys. 54 (2003), 608–632.

[5] R. J. DiPerna, P. L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. 98 (1989), 511-547.

[6] B. Ducomet, E. Feireisl, *The equations of Magnetohydrodynamics: On the interaction between matter and radiation in the evolution of gaseous stars,* Commun. Math. Phys. 226, 595-629(2006).

[7] J. Fan, S. Jiang, and G. Nakamura, *Vanishing shear viscosity limit in the magnetohydrodynamic equations*, Commun. Math. Phys. 270 (2007), 691-708.

[8] E. Feireisl, *On the motion of a viscous, compressible, and heat conducting fluid*, Indiana Univ. Math. J. 53 (2004) 1707-1740.

[9] E. Feireisl, *Dynamics of viscous compressible fluids,* Oxford Lecture Series in Mathematics and its Applications, 26. Oxford University Press, Oxford, 2004.

[10] H. Freistühler, P. Szmolyan, *Existence and bifurcation of viscous profiles for all intermediate magnetohydrodynamic shock waves,* SIAM J. Math. Anal., 26 (1995), 112-128.

[11] J. F. Gerebeau, C. L. Bris, T. Lelievre, *Mathematical methods for the magnetohydrodynamics of liquid metals*, Oxford University Press, Oxford, 2006.

[12] H. Goedbloed, S. Poedts, *Principles of magnetohydrodynamics with applications to laboratory and astrophysical plasmas*, Cambridge University Press,2004.

[13] D. Hoff, *Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data,* Arch. Rational Mech. Anal. 132 (1995), 1–14.

[14] D. Hoff, *Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids,* Arch. Rational Mech. Anal. 139 (1997), 303–354.

[15] D. Hoff, E. Tsyganov, *Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics,* Z. Angew. Math. Phys. 56 (2005), 791–804.

[16] X. Hu, D. Wang, *Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows*, submitted for publication.

[17] V. Kazhikhov and V. V. Shelukhin, *Unique global solution with respect to time of initial-boundary-value problems for one-dimensional equations of a viscous gas*, J. Appl. Math. Mech. 41 (1977), 273-282.

[18] S. Kawashima, M. Okada, *Smooth global solutions for the one-dimensional equations in magnetohydrodynamics,* Proc. Japan Acad. Ser. A Math. Sci., 58 (1982), 384-387.

[19] A. G. Kulikovskiy and G. A. Lyubimov, *Magnetohydrodynamics*, Addison-Wesley, Reading, Massachusetts, 1965.

[20] L. D. Laudau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed., Pergamon, New York, 1984.

[21] P. L. Lions, *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models.* Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[22] P. L. Lions, *Mathematical topics in fluid mechanics. Vol. 2. Compressible models.* Oxford Lecture Series in Mathematics and its Applications, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[23] T.-P. Liu, Y. Zeng, *Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws,* Memoirs Amer. Math. Soc. 599, 1997.

[24] A. Novotný, I. Straškraba, *Introduction to the theory of compressible flow*, Oxford University Press: Oxford, 2004.

[25] D. Wang, *Large solutions to the initial-boundary value problem for planar magnetohydrodynamics.* SIAM J. Appl. Math. 63 (2003), 1424-1441.

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA.
E-mail address: `xih15@pitt.edu`

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA.
E-mail address: `dwang@math.pitt.edu`