INTRODUCTION TO THE TRANSFER THEORY OF TURBULENCE

WILLIAM LAYTON*

Abstract. This expository report considers energy transfer theory of turbulence. Analysis of the Pao energy transfer model is given and it is verified that it is consistent with the important facets of the 1941 theory developed by Kolomogorov of homogeneous isotropic turbulence. A connection to shell models is noted and one implication for the Onsager conjecture about the Euler equations is pointed out.

Key words. energy transfer theory, Navier-Stokes equations, turbulence

AMS subject classification.

1. Introduction.

The basic features of the (so called) "K41" theory of homogeneous isotropic turbulence are that even with a smooth initial condition and input of energy (persistently) by a smooth body force, complex flow with many scales and irregularities develops. After either time averaging or ensemble averaging, universal features develop. These include:

- statistical equilibrium: energy input at large scales is balanced by energy dissipation which is quite concentrated at very small scales;
- universal energy spectrum through the inertial range: there is a wide range of wavenumbers, called the inertial range, through which the energy in wavenumber k satisfies

$$E(k) = 1.4(energy\_dissipation\_rate)^{2/3}k^{-5/3};$$

- beyond (acting on smaller length scales than) the inertial range the dissipation range begins. In the dissipation range $E(k)$ decays exponentially.
- The breakpoint separating the inertial range and the dissipation range is estimated in terms of time averaged flow quantities.

The first point is an assumption on the flow. The second point can be derived by several different turbulent phenomenologies such as Kraichnan's argument or even simple dimensional analysis. The third is based on a very plausible analogy with linear Oseen problems for which it can be proven. The last (the estimate of the Kolmogorov microscale) is typically derived from assuming statistical equilibrium and that the ratio of nonlinear terms to viscous term at the microscale is $O(1)$. Each of these universal properties is derived by different phenomenological simplifications. The goal of energy transfer theory is to develop a single, consistent phenomenology that explains all these points as well as giving insight into the transition between $k^{-5/3}$ in the inertial range and exponential decay in the dissipation range. This report will present energy transfer modeling of turbulence in general and the Pao energy transfer model in particular detail. Energy transfer models are strongly related to (recently popular) shell models. In fact, given a shell model an energy transfer model can readily be given for which the shell model is a method of lines discretization of it. Similarly, given an energy transfer model, discretizing the k variable yields a shell

*Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA, wjl@pitt.edu, http://www.math.pitt.edu/~wjl. Partially supported by the NSF under grant DMS-0810385.

model. Generally[1] one can organize the various competing approaches[2] to prediction of turbulence as follows.

$$\begin{array}{ccc} \Uparrow & NSE & \\ \text{Increasing} & \Updownarrow & \\ \Uparrow & LES & \\ \text{Accuracy} & \Updownarrow & \\ \Uparrow & URANS & \\ & \Updownarrow & \Downarrow \\ & RANS & \text{decreasing} \\ & \Updownarrow & \Downarrow \\ & Energy\_Transfer & \text{complexity} \\ & \Updownarrow & \Downarrow \\ & Shell & \\ \textbf{Hierarchy} & \textbf{of} & \textbf{Models} \end{array}$$

To be more precise we must first set some notation. Universal statistical or time averaged features of turbulence through the inertial range are described in part through a cascade of energy. This joint cascade is described in terms of the decomposition of time averaged (denoted by $< \cdot >$, see Section 2 for a precise definition) energy

$$E(t) := \frac{1}{L^3} \int_{(0,L)^3} \frac{1}{2} |u(x,t)|^2 dx,$$

into wavenumber shells (given precisely in Section 2). This decomposition is written

$$< E(t) >= \sum_{k \geq 1} E(k),$$

and is based on the Fourier expansion of the L periodic velocity $u(x,t)$. The other key parameter in the energy and cascade is the energy dissipation rate, given by

$$\varepsilon_E := \left\langle \frac{1}{L^3} \int_{(0,L)^3} \nu |\nabla u(x,t)|^2 dx \right\rangle .$$

Energy is conserved by the Euler equations, broken down (on average) to smaller and smaller scales by the NSE nonlinearity and dissipated primarily at small scales by the viscous term. The result is a cascade of energy through a range of wave numbers (or Fourier modes), known as the inertial range, that begins soon after the those few largest scales at which energy is input and ends when dissipation becomes the dominant effect. The energy cascade is well known to be

$$E(k) = \alpha_E \varepsilon_E^{2/3} k^{-5/3}, \text{ where } \alpha_E = \text{ the universal Kolmogorov constant.}$$

This description is observed in both nature and numerical experiments and predicted by the different phenomenologies (or simplified theories) of turbulence.

[1] Do not put too much weight on this schematic. The field is almost as complicated as the phenomena.

[2] The usual TLA's: NSE=Navier-Stokes equations, LES= large eddy simulation, URANS= Unsteady Reynolds averaged Navier-Stokes, and RANS= Reynolds averaged Navier Stokes.

(footnote to the footnote: TLA = three letter acronym)

2. Decomposition of energy and into Fourier modes. Consider the Navier-Stokes equations in a three dimensional, 2π periodic box:

$$
\begin{aligned}
&u_t + u\cdot\nabla u - \nu\triangle u + \nabla p = f \text{ and } \cdot u = 0 \text{ in } (0,2\pi)^3 \qquad (2.1)\\
&\phi(0,t) = \phi(2\pi,t) \text{ and } \int_\Omega \phi dx = 0 \text{ for } \phi = u,\ u_0,\ f,\ p.
\end{aligned}
$$

Under periodicity, the fluid velocity and its associated kinetic energy can be expanded in Fourier series (in which the sums are over all triples of integers $\mathbf{k}=(k_1,k_2,k_3)\neq(0,0,0)$[3])

$$
\begin{aligned}
u(x,t) &= \sum_{\mathbf{k}} \widehat{u}(\mathbf{k},t)e^{i\mathbf{k}\cdot x},\\
E(t) &= \sum_{\mathbf{k}} \frac{1}{2}|\widehat{u}(\mathbf{k},t)|^2.
\end{aligned}
$$

Fourier series give a natural and often used partition of the kinetic energy into wave numbers as follows. Define $|\mathbf{k}|^2 = k_1^2+k_2^2+k_3^2$ and

$$
E(k,t) := \sum_{k=|\mathbf{k}|} \frac{1}{2}|\widehat{u}(\mathbf{k},t)|^2, \text{ so that } E(t) = \sum_{1\le\mathbf{k}} E(k,t).
$$

In this definition of $E(k,t)$, the index k in the sum takes non integer values[4]. This will be no difficulty since transfer theories are further approximations in which k will be a continuous variable ranging over $1<k<\infty$.

Exact but non closed equations for $E(k,t)$ can be derived in the usual way by taking the inner product of the Navier-Stokes equations with one Fourier mode and then summing over all modes of norm k, see Davidson [D04], Frisch [F95], or Pope [P00] for details. This gives (using the Kronecker delta)

$$
\begin{aligned}
\frac{\partial}{\partial t}E(k,t) + \sum_{|\mathbf{j}|=k}\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}\left\{\widehat{u}(\mathbf{k}_1,t)\cdot\widehat{u}(\mathbf{k}_2,t)\otimes\mathbf{k}_2\cdot\overline{\widehat{u}(\mathbf{j},t)}\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{j}}\right\} +\\
+2\nu k^2 E(k,t) = \sum_{|\mathbf{j}|=k}\widehat{f}(\mathbf{j},t)\cdot\overline{\widehat{u}(\mathbf{j},t)}.
\end{aligned}
$$

Define the (discrete) transfer functions (these will be extended to continuous k next)

$$
\begin{aligned}
T(k,t) &:= \sum_{|\mathbf{j}|=k}\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}\left\{\widehat{u}(\mathbf{k}_1,t)\cdot\widehat{u}(\mathbf{k}_2,t)\otimes\mathbf{k}_2\cdot\overline{\widehat{u}(\mathbf{j},t)}\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{j}}\right\}\\
S(k,t) &:= \sum_{1\le k'\le k} T(k',t).
\end{aligned}
$$

[3] Further, since $\nabla\cdot u = 0$ and u is real, $\mathbf{k}\cdot\widehat{u}(\mathbf{k},t)=0$ and $\widehat{u}(\mathbf{k},t)=\overline{\widehat{u}(-\mathbf{k},t)}$.

[4] Many roughly equivalent variants are seen such as defining

$$
E(k,t) := \sum_{k-1/2\le|\mathbf{k}|\le k+1/2} \frac{1}{2}|\widehat{u}(\mathbf{k},t)|^2
$$

I tytransfer theory, k is extended to a continuous variable so the exact choice here will not be significant in the further development of energy transfer theory.

3. Energy transfer theory. The goal of transfer theory is to develop a closed system of differential equations for $E(k,t)$ which is (i) of much reduced complexity than the NSE in wavenumber space, (ii) predicts statistics of fully developed turbulence correctly. This case of fully developed, homogeneous, isotropic turbulence corresponds to

- smooth, persistent body forces,
- time averaged behavior of $E(k,t)$, and
- high Reynolds number with a richness of persistent scales of motion.

Accordingly, following the second point, they aim at time averaged behavior of $E(k,t)$

$$E(k) := \lim_{T\to\infty} \sup \frac{1}{T}\int_0^T E(k,t)dt.$$

and motivated by the first point, we shall suppose energy is input into the $k=1$ modes and suffices to preserve constant energy levels in them

$$E(1,t) = \frac{1}{2}U^2, \text{ for all } t>0, \text{ where } U \text{ is fixed.}$$

Exploiting the third point, energy transfer theories treat k as a continuous variable; to extend k to a continuous variable, sums are replaced by integrals in the usual way. Thus the transfer functions and the energy satisfy

$$S(k,t) = -\int_0^k T(k',t)dk' \text{ or } T(k,t) = -\frac{\partial}{\partial k}S(k,t). \tag{3.1}$$

With these small approximations we have the (following non-closed energy equation in wave number - time space:

$$\frac{\partial}{\partial t}E(k,t) + \frac{\partial}{\partial k}S(k,t) + 2\nu k^2 E(k,t) = 0 \text{ for } 1<k<\infty, t>0,$$
$$E(1,t) = \frac{1}{2}U^2 \text{ , for } t>0,$$
$$E(k,0) = E_0(k) \text{ for } 1<k<\infty \text{ where } E_0(k)\equiv 0 \text{ for large } k.$$

A transfer theory, as developed by Obukhov [O41], W. Heisenberg [H48], Kovasznay [K48], Ellison [E61] and Pao [Pao65], is simply a closure which relates $S(k,t)$ back to $E(k,t)$ either through an algebraic relation (simplest) or an extra set of integro-differential equations, [D04]. A number of transfer theories have been proposed (summarized excellently in Davidson [D04] and Monin and Yaglom [MY75]). Of these (given below) neither Obukhov's, Heisenberg's nor Kovasznay's closure correctly predict exponential decay of the energy spectrum for large enough wavenumbers, [D04] and [MY75]. All but Pao's closure are based on an assumption that energy transfer is a non-local, even global, process. This contradicts our understanding of the energy cascade's processes. Davidson [D04], page 479, summarizes the various closures by

"In fact, only Pao's hypothesis really withstands scrutiny."

The various transfer theory closures are given as follows

$$\text{Obukhov: } S(k) \doteq Const \cdot \int_k^\infty E(k')dk' \left\{ \int_0^k k'^2 E(k')dk' \right\}$$

$$\text{Ellison: } S(k) \doteq Const \cdot kE(k) \left\{ \int_0^k k'^2 E(k')dk' \right\}^{\frac{1}{2}}$$

$$\text{Heisenberg: } S(k) \doteq Const \cdot \int_k^\infty k'^{-3/2} E^{1/2}(k')dk' \int_0^k k'^2 E(k')dk'$$

$$\text{Kovasznay: } S(k) \doteq Const \cdot k^{5/2} E^{3/2}(k)$$

$$\text{Pao: } S(k) \doteq Const \cdot \varepsilon^{1/3} k^{5/3} E(k).$$

The simplest of the above and, as n noted above, so far the most successful is the transfer theory of Pao [Pao65]. To motivate his closure, Y.-H. Pao writes (page 1067 in [Pao65]):

"We visualize the transfer of turbulent energy as a cascading process[redacted] *in which the spectral elements are continuously transferred to even larger wave numbers .... Let the rate at which an energy spectral element is transferred across k be* σ *... then the energy flux across k is* $S(k) = E(k)\sigma(k)$*. We assert that the spectral element* $\sigma(k)$ *is dependent on* ε *... and on the wavenumber k .... Dimensional reasoning gives*

$$\sigma(k) = \alpha^{-1} \varepsilon^{1/3} k^{5/3}.$$

Thus ..."

To be very specific, Pao's transfer theory postulates the algebraic relation:

$$S(k,t) = \alpha^{-1} \varepsilon_0^{1/3} k^{5/3} E(k,t), \text{ where } \varepsilon_0 = 2^{-3/2} \alpha^{-1} U^3. \tag{3.2}$$

Here α is the Kolmogorov constant (with value between 1.4 and 1.6) and ε_0 is an estimate of the Pao model's prediction of its own energy dissipation rate. Because the Pao model is simple enough for exact calculation, the exact value (see Sections 2 and 3 for its calculation) given above is used.

We explore herein consequences of Pao's closure assumption above, means studying the long time averaged behavior of solutions to the following hyperbolic, initial boundary value problem:

$$\frac{\partial}{\partial t} E(k,t) + \frac{\partial}{\partial k}(\alpha^{-1} \varepsilon_0^{1/3} k^{5/3} E(k,t)) + 2\nu k^2 E(k,t) = 0 \text{ for } 1 < k < \infty, t > 0, \tag{3.3}$$

$$E(1,t) = \frac{1}{2} U^2, \text{ for } t > 0, \text{ and } \varepsilon_0 = 2^{-3/2} \alpha^{-1} U^3,$$

$$E(k,0) = E_0(k) \text{ for } 1 < k < \infty \text{ where } E_0(k) \equiv 0 \text{ for large } k.$$

The energy input defines a clear representative large scale velocity U. The natural large length scale is $L = 2\pi$. Thus, the natural Reynolds number associated with the Pao energy transfer model is

$$Re = \frac{UL}{\nu} = 2\pi \frac{U}{\nu}.$$

We shall show that the limit defining $E(t)$ exists in Section 2 and is determined by the properties of the equilibrium problem associated with (Pao Model), given by

$$\frac{\partial}{\partial k}(\alpha^{-1}\varepsilon_0^{1/3}k^{5/3}E_\infty(k)) + 2\nu k^2 E_\infty(k) = 0 \text{ for } 1 < k < \infty, \tag{3.4}$$
$$E_\infty(1) = \frac{1}{2}U^2, \text{ for } t > 0, \text{ and } \varepsilon_0 = 2^{-3/2}\alpha^{-1}U^3.$$

Concerning Pao's model we can prove the following by simply solving exactly the equilibrium problem.

PROPOSITION 3.1. *The only closure of the form*

$$S = Const.\varepsilon^a k^b E^c$$

predicting exponential decay in the dissipation range of $E_\infty(k)$ is $c = 1$, i.e., the Pao model.

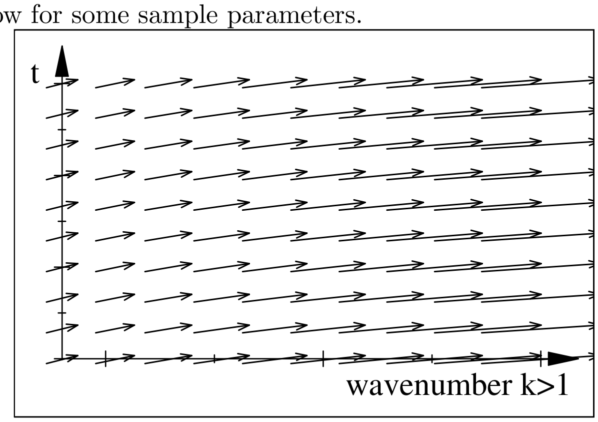
4. Analysis of Pao's Transfer Theory. The characteristic curves of the hyperbolic equation

$$\frac{\partial}{\partial t}E(k,t) + \frac{\partial}{\partial k}(\alpha^{-1}\varepsilon_0^{1/3}k^{5/3}E(k,t)) + 2\nu k^2 E(k,t) = 0, \tag{4.1}$$

are the (positive sloped) curves in the $k-t$ plane given by

$$dt - \frac{5}{3}\alpha^{-1}\varepsilon_0^{1/3}k^{2/3}dk = 0.$$

and plotted below for some sample parameters.



Slope field of characteristics

The problem (3.3) reduces to a linear ordinary differential equation along each characteristic. From this existence and uniqueness follows immediately from standard theory of hyperbolic equations, e.g., [Whi74].

PROPOSITION 4.1. *A unique solution exists to problem (3.3). For each fixed $t > 0$ the solution $E(k,t)$ has compact support in k.*

We shall soon show that $E(k,t)$ approaches the unique solution of the equilibrium problem as $t \to \infty$. That unique solution to the equilibrium problem has exponential decay as $k \to \infty$ and is easily calculated to be

$$E_\infty(k) = \frac{1}{2}U^2 k^{-\frac{5}{3}} e^{\beta} \exp(-\beta k^{\frac{4}{3}}), \quad \text{where} \quad \beta := \frac{3}{2}\frac{\nu\alpha}{\varepsilon_0^{1/3}}. \tag{4.2}$$

Proposition 4.2. *$E(k,t) \to E_\infty(k)$ exponentially fast in $L^2(1,\infty)$ as $t \to \infty$, even in the case $\nu = 0$.*

Proof. Let $w(x,t) = E(k,t) - E_\infty(k)$. Since $E(k,t)$ has compact support, $w(k,t)$ decreases exponentially in k (and thus all integrals below are convergent). subtraction gives the following equation for $w(k,t)$:

$$\frac{\partial}{\partial t} w(k,t) + \frac{\partial}{\partial k}(\alpha^{-1}\varepsilon_0^{1/3} k^{5/3} w(k,t)) + 2\nu k^2 w(k,t) = 0 \ , \tag{4.3}$$
$$w(1,t) = 0, \text{ for } t > 0, \text{ and } w(k,0) \text{ given.}$$

Multiply by $w(k,t)$ and integrate. This yields

$$\frac{d}{dt}\int_1^\infty \frac{1}{2} w(k,t)^2 dk + \int_1^\infty \left[\frac{5}{6}\alpha^{-1}\varepsilon_0^{1/3} k^{5/3} + 2\nu k^2\right] w(k,t)^2 dk = 0.$$

The term in brackets is bounded below by a positive constant, even if $\nu = 0$. Thus, we have exponential convergence to steady state. □

The fact that exponential convergence to a $k^{-5/3}$ energy spectrum occurs even for the Euler - Pao energy transfer model is perhaps relevant to the Onsager conjecture that (Onsager, 1949)

"*...in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available*".

From the last proposition and following arguments in [JLM07], it follows that time averages on $E(k,t)$ exist and correspond to the equilibrium solution.

Corollary 4.3. *The following limit exists and equals $E_\infty(k)$:*

$$E(k) := \lim_{T\to\infty} \sup \frac{1}{T}\int_0^T E(k,t)dt = E_\infty(k).$$

This corollary implies that we may check consistence of the predictions of the Pao transfer theory with the K41 theory of homogeneous, isotropic turbulent statistics through properties of the equilibrium solution, for which an explicit formula is known.

5. Consistency with the K41 Theory. We now turn to consistency of the predictions with Kolmogorov's theory of homogeneous, isotropic turbulence (often called the K41 theory), see [L08], Davidson [D04], Frisch [F95], or Pope [P00] for details. Consistency of the Pao model with K41 is known for energy input by body forces. This section verifies that consistency is not changed by keeping the range of wavenumbers to be $k > 1$ and imputing energy into (3.3) through the boundary condition at $k = 1$. to that end we consider the energy spectrum (often called the *Pao spectrum*) given by

$$E(k) = \frac{1}{2}U^2 k^{-\frac{5}{3}} e^{\beta} \exp(-\beta k^{\frac{4}{3}}), \tag{5.1}$$

$$\text{where } \ \beta \ : \ = \frac{3}{2}\frac{\nu\alpha}{\varepsilon_0^{1/3}}, \text{ and } \ \varepsilon_0 = 2^{-3/2}\alpha^{-1}U^3. \tag{5.2}$$

Since the model (3.3) has a unique solution, consistency with the predictions of the K41 theory must be evaluated through the model itself (3.3) or its explicit, time averaged solution above, without reference to the intended physical meaning of any variables or formulas. We check the models predictions of statistical equilibrium, the inertial energy range spectrum, exponential decay in the dissipation range and the prediction of the Kolmogorov microscale.

Statistical equilibrium. Statistical equilibrium in the K41 theory means that the energy input to the large scales (which is roughly $O(U^3/L)$) is balanced (after time averaging) by energy dissipation primarily at the small scales. To test of this holds for the Pao model, define (as usual) the time averaged energy dissipation rate of (3.3) to be

$$\varepsilon := \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_1^\infty 2\nu k^2 E(k,t) dk dt$$

Corollary 2.1 and the choice of ε_0 implies that $\varepsilon_0 = \varepsilon$. A form of statistical equilibrium of (3.3) follows by integrating (3.3) and time averaging. This gives

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_1^\infty \Big(E_t(k,t) + (\alpha^{-1}\varepsilon_0^{1/3} k^{5/3} E(k,t))_k + 2\nu k^2 E(k,t) \Big) dk dt = 0.$$

Using Fubini's theorem, the first term vanishes. We then have from the second term, the boundary condition at $k = 1$ and the third term

$$\alpha^{-1}\varepsilon_0^{1/3} \frac{1}{2} U^2 = \varepsilon.$$

Since $\varepsilon_0 = \varepsilon$ and $L = 2\pi$ this can be rewritten to show statistical equilibrium of the Pao transfer model as:

$$\varepsilon = \frac{\pi\sqrt{2}}{2\alpha} \frac{U^3}{L}.$$

The inertial range energy spectrum. To deduce the predictions of (3.3) about the inertial range energy spectrum we begin with the model (3.3) itself. Since $E(k,t)$ is bounded and Re is large (typically ν is small), there is a range of k ($1 < k < 1/\eta_{Pao}$, say) for which the term $2\nu k^2 E(k,t)$ is negligible. On this range, (3.3) simplifies to

$$E_t(k,t) + (\alpha^{-1}\varepsilon_0^{1/3} k^{5/3} E(k,t))_k \doteq 0.$$

Integrating over $1 < k' < k$, time averaging and using Fubini's theorem (so the first term drops out) gives

$$\int_1^k (\alpha^{-1}\varepsilon^{1/3} k'^{5/3} E(k'))_{k'} dk' \doteq 0, \text{or} \quad \alpha^{-1}\varepsilon^{1/3} k^{5/3} E(k) \doteq \alpha^{-1}\varepsilon^{1/3} \frac{1}{2} U^2.$$

From the choice of ε, $\frac{1}{2}U^2 = \alpha\varepsilon^{2/3}$. Thus, rearranging

$$E(k) \doteq \alpha\varepsilon^{2/3} k^{-5/3}, \text{ over } 1 < k < 1/\eta_{Pao}.$$

This conclusion can also be obtained directly through the exact solution, written as $E(k) = \alpha\varepsilon^{2/3} k^{-\frac{5}{3}} e^{\beta} \exp(-\beta k^{\frac{4}{3}})$.

The dissipation range. The K41 theory predicts exponential energy decay for large k and gives an estimate of the transition point at which this decay begins to be the dominant effect. Exponential decay of $E(k)$ for large k follows immediately from the closed for representation of the Pao spectrum

$$E(k) = \alpha\varepsilon^{2/3} k^{-\frac{5}{3}} e^{\beta} \exp(-\beta k^{\frac{4}{3}}).$$

Estimate for the Pao Model's Microscale. Exponential decay of $E(k)$ becomes to be significant when the exponent $\beta k^{\frac{4}{3}}$ is $O(1)$ or larger. Since

$$\beta := \frac{3}{2}\frac{\nu\alpha}{\varepsilon^{1/3}}, \text{ and } \ \varepsilon = 2^{-3/2}\alpha^{-1}U^3$$

this occurs when $\nu\varepsilon^{-1/3}k^{4/3} \geq O(1)$. Using $\varepsilon = 2^{-3/2}\alpha^{-1}U^3$ and rearranging shows that this condition is equivalent to $k \doteq (U/\nu)^{3/4} \doteq \mathrm{Re}^{3/4}/L$, as $L = 2\pi$ yielding the predicted models's microscale of

$$\eta_{Pao} \doteq \mathrm{Re}^{-3/4} L,$$

agreeing with the prediction of the Kolmogorov microscale.

6. Connection between energy transfer theories and shell models.. We conclude by noting that there is also a correspondence between energy transfer theories and shell models of turbulence. Indeed, transfer theory postulates a functional relationship between S and E, $S = \Pi(k, E)$, yielding an energy transfer model

$$\frac{\partial}{\partial t}E(k,t) + \frac{\partial}{\partial k}\Pi(k, E(k,t)) + 2\nu k^2 E(k,t) = 0.$$

Differencing the k derivative gives a shell model: $E_1(t) = \frac{1}{2}U^2$ and for $k = 2, 3, \cdots$,

$$\frac{d}{dt}E_k(t) + \Pi(k, E_k(t)) - \Pi(k-1, E_{k-1}(t)) + 2\nu k^2 E_k(t) = 0.$$

To reverse this step, given a shell model, the differences in the energy transfer function can, through its modified equation, be identified with an approximation to a k derivative and thus an associated transfer theory.

As a concrete example consider the Pao model

$$\frac{\partial}{\partial t}E(k,t) + \frac{\partial}{\partial k}(\alpha_E^{-1}\varepsilon_0^{1/3}k^{5/3}E(k,t)) + \tag{6.1}$$

$$2\nu k^2 E(k,t) = 0 \text{ for } 1 < k < \infty, t > 0, \tag{6.2}$$

$$E(1,t) = \frac{1}{2}U^2, \text{ for } t > 0, \text{ and } \varepsilon_0 = 2^{-3/2}\alpha_E^{-1}U^3,$$

$$E(k,0) = E_0(k) \text{ for } 1 < k < \infty \text{ where } E_0(k) \equiv 0 \text{ for large } k.$$

If $E(k,t)$ has a Pao spectrum we can choose in advance break wave numbers k_n that equi-distribute energy (at least after reaching statistical equilibrium). Let

$$E_n(t) \simeq E(k_n, t) \simeq E_{FIXED}, n = 2, \cdots, N.$$

Then, differencing the Pao model gives a shell model

$$\frac{d}{dt}E_n(t) + \alpha_E^{-1}(\varepsilon_0^{1/3}k_n^{5/3}E_n(t) - \varepsilon_0^{1/3}k_{n-1}^{5/3}E_{n-1}(t))/(k_n - k_{n-1}) + \tag{6.3}$$

$$2\nu k_n^2 E_n(t) = 0 \text{ for } 2 < n \leq N, t > 0, \tag{6.4}$$

$$E_1(t) = \frac{1}{2}U^2, \text{ for } t > 0, \text{ and } \varepsilon_0 = 2^{-3/2}\alpha_E^{-1}U^3,$$

$$E_n(0) = E_{n,0} \text{ for } 1 < k < \infty \text{ where } E_{n,0} \equiv 0 \text{ for large } n.$$

7. Conclusions. The Pao transfer theory is both the simplest and the most successful transfer theory in that it predicts the major statistics of isotropic turbulence successfully. Its energy spectrum $E(k,t)$ approaches its time averaged value exponentially fast and so the Pao theory cannot give useful information about essentially time dependent properties on the energy spectrum like whether backscatter occurs intermittently or whether $E(k,t)$ approaches $E(k)$ monotonically or through some sort of intermittent bursts punctuating periods of apparent equilibrium.

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