FLUID-POROUS INTERFACE CONDITIONS WITH THE "INERTIA TERM" $\frac{1}{2}|U_{FLUID}|^2$ ARE NOT GALILEAN INVARIANT

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Abstract. Consider a body of fluid in motion sitting above a porous medium saturated by the same fluid. This is modelled by the Navier-Stokes equations in the fluid region, the Darcy model of flow averages in the porous media, various boundary conditions away from the interface I and the interface conditions of mass balance across I, force balance across I and the Beavers-Joseph condition on tangential velocities. A simple á priori bound does not hold for this coupled problem and so a theory exactly paralleling the NSE cannot be developed. Some have explored adding a term $\frac{1}{2}|u_{fluid}|^2$ to one interface condition which exactly cancels the term preventing a basic á priori bound. In this note we show that this additive term cannot be correct because the resulting interface condition violates Galilean invariance (suitable adapted to the coupled problem).

Key words. Navier-Stokes-Darcy, Beavers-Joseph-Saffman condition, inertia term, Galilean invariance

1. An Additive Inertia Term Violates Galilean Invariance. Consider a body of fluid in motion sitting across an interface I above a porous medium saturated by the same fluid. The fluid velocity satisfies the Navier-Stokes equations in the fluid domain Ω_{fluid}

$$\begin{aligned}\frac{\partial u_{fluid}}{\partial t} + u_{fluid}\cdot\nabla u_{fluid} - \nabla\cdot T(u_{fluid}, p_{fluid}) = f_1, \text{ where} \\ T(u_{fluid}, p_{fluid}) = -p_{fluid}I + 2\nu\nabla^s u_{fluid}, \text{ in } \Omega_{fluid},\end{aligned} \tag{1.1}$$

and suitable averages of the fluid velocity are assumed to satisfy the Darcy model in the porous medium

$$S\frac{\partial p_{porous}}{\partial t} - \nabla\cdot u_{porous} = f_2, \text{ and } u_{porous} = -k\nabla p_{porous}, \text{ in } \Omega_{porous}. \tag{1.2}$$

S, k and ν are the positive mass storage (per unit density) coefficient, hydraulic conductivity and dynamic viscosity. Let the unit normal to I from Ω_i be denoted by n_i and the unit tangents vectors on I by τ. On the interface I we apply the accepted (but not the only accepted) interface conditions consisting of *mass in = mass out* and *balance of forces*

$$u_{fluid}\cdot n_{fluid} = u_{porous}\cdot n_{porous}, \text{ on } I, \tag{1.3}$$

$$n_{fluid}\cdot T\cdot n_{fluid} = p_{porous}, \text{ on } I. \tag{1.4}$$

The specification of the interface conditions is completed by either the Beavers-Joseph or Beavers-Joseph-Saffman condition on the tangential velocities. The former is

$$-\alpha k^{-\frac{1}{2}}(u_{fluid} - u_{porous})\cdot\tau = 2\nu n_{fluid}\cdot\nabla^s u_{fluid}\cdot\tau. \tag{1.5}$$

Saffman [Saff71] proved that in certain cases[1] in the Beavers-Joseph condition the velocity u_{porous} is negligible (see also e.g., [BJ67], [JM00], [PS98]). Further, traces of

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[1] Saffman considered the restrictions: one-dimensional flow, uniform pressure gradient, no mass exchange between fluid region and the porous medium, uniform media and the zero permeability limit.

weak solutions in the porous region are not strong enough for its tangential component to be well defined. Thus it is often dropped giving the Beavers-Joseph-Saffman condition

$$-\alpha k^{-\frac{1}{2}} u_{fluid} \cdot \tau = 2\nu n_{fluid} \cdot \nabla^s u_{fluid} \cdot \tau. \tag{1.6}$$

Of the two variants, the Beavers-Joseph condition is more generally correct and has recently been proven to lead to a strongly well posed coupled problem in the remarkable paper [Cao08].

For simplicity we let

$$I = \{(x, y, z) | z = 0\}, \Omega_{fluid} = \{(x, y, z) | z > 0\}, \Omega_{porous} = \{(x, y, z) | z < 0\}.$$

For reasons of mathematical convenience and elegance, adding an additional "inertia term" $\frac{1}{2}|u_{fluid}|^2$ to either (1.6) or (1.4) has been explored to cancel the term that arises when one attempts a standard á priori estimate for possible solutions of the coupled problem. This addition is mechanically questionable because it does not arise from forces fluids exert on surfaces which are represented in the Cauchy traction vector, [Serrin]. Beyond this critique, we observe herein that adding the term $\frac{1}{2}|u_{fluid}|^2$ violates the basic principle of mechanics known since 1632, e.g., [G32], [T91], that the laws of mechanics be the same in all inertial frames of reference to the extent allowed by the geometry. This implies that the fluids model and the Beavers-Joseph interface condition should be invariant under (1.7) below. The Darcy model is relative to the laboratory frame of reference (since the porous matrix is fixed) and thus the Darcy equations change when considered under a moving frame of reference.

Indeed, the physical problem, the interface I, (1.1) and the interface conditions (1.5), (1.4) and (1.3) are all invariant under any Galilean transformation with z component zero of the form[2]

$$\begin{aligned} \widetilde{x} &= x + \overrightarrow{U} t + \overrightarrow{b}, \widetilde{t} = t \\ \overrightarrow{U} &= (U, V, 0), \overrightarrow{b} = (b_1, b_2, 0). \end{aligned} \tag{1.7}$$

Thus any augmentation of these equations for mathematical reasons must also be similarly invariant, e.g., [G32], [T91]. Consider then the Beavers-Joseph condition with the additive "inertia term"

$$-\alpha k^{-\frac{1}{2}} (u_{fluid} - u_{porous}) \cdot \tau = 2\nu n_{fluid} \cdot \nabla^s u_{fluid} \cdot \tau - \frac{1}{2}|u_{fluid}|^2. \tag{1.8}$$

Let the velocities in the base coordinate system and transformed one be denoted (with subscripts 1 or 2) by u and $\widetilde{u}$. Under (1.7) we relate them via

$$\widetilde{u} = u + U, \frac{\partial}{\partial \widetilde{x}} = \frac{\partial}{\partial x}, \text{ and } \frac{\partial}{\partial \widetilde{t}} = \frac{\partial}{\partial t} - U\frac{\partial}{\partial x}.$$

CONCLUSION 1.1. *Consider the Beavers-Joseph condition replaced by (1.8). The resulting interface condition is not Galilean invariant under transformations (1.7).*

[2] For example in the Beavers-Joseph condition, replace $u_j \Leftarrow u_j + U$:

$$\begin{aligned} -\alpha k^{-\frac{1}{2}} (u_1 + U - (u_2 + U)) \cdot \tau - 2\nu n_1 \cdot \nabla^s (u_1 + U) \cdot \tau = \\ -\alpha k^{-\frac{1}{2}} (u_1 - u_2) \cdot \tau - 2\nu n_1 \cdot \nabla^s u_1 \cdot \tau = 0 \end{aligned}$$

Proof. Since the condition without the extra term $\frac{1}{2}|u_{fluid}|^2$ is invariant, the model with $\frac{1}{2}|u_{fluid}|^2$ can only be Galilean invariant provided the extra term satisfies

$$\frac{1}{2}|u_{fluid} + \overrightarrow{U}|^2 = \frac{1}{2}|u_{fluid}|^2 \text{ for all } \overrightarrow{U} = (U, V, 0).$$

which is never true. $\square$

2. Comments. The problem posed by the Navier-Stokes' nonlinear term with $u_{fluid} \cdot n \neq 0$ on some portion of the boundary occurs also when coupled to porous media flow. Adding to the interface conditions the term $\frac{1}{2}|u_{fluid}|^2$ is not a good path for the future development of coupled models of fluid flow with porous media flow. When this term is small, the coupled problem is well posed without it and when it is significant, the coupled problem is mechanically incorrect.

It is possible that the solution to the problem of constructing a comprehensive mathematical theory of the coupled problem lies in a development not paralleling that of the Navier-Stokes equations, [Cao08]. It is also possible that the solution involves incorporating into the models a better understanding of the physical problem. Possibly this means a nonlinear coupling condition and possibly a more detailed model of flow in the porous media region. In particular, it is possible that if the flow across the interface is large enough for this extra term to be significant, the Darcy model itself is no longer a valid approximation of the flow averages near the interface. If so, either a separate model of the interface layer or a more complex porous media model might be necessary. Either possibility could remove the mathematical difficulty of existence of traces of weak solutions of u_{porous} in the Beavers-Joseph interface condition.

The other fix, adding the same inertia term into the force balance (1.4) fails invariance for the same reason. As a last note, the following modification of the Beavers-Joseph condition, although still not arising from any forces in fluids exert on surfaces, is Galilean invariant since $U \cdot n_{fluid} = 0$

$$-\alpha k^{-\frac{1}{2}}\left(u_{fluid} - u_{porous}\right) \cdot \tau = 2\nu n_{fluid} \cdot \nabla^s u_{fluid} \cdot \tau - \frac{1}{2}|u_{fluid} \cdot n_{fluid}|^2. \qquad (2.1)$$

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