ON TAYLOR / EDDY SOLUTIONS OF APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE

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Abstract. This article shows that, so called, general Green-Taylor solutions, also called Taylor solutions or eddy solutions of the $2d$ Navier Stokes equations are also exact solutions to approximate deconvolution models of turbulence. Thus these special structures in flows exist as exact features in the models studied and their persistence / transient behavior is exactly determined by their stability not by the effects of modelling or truncation errors.

Key words. Green-Taylor vortex, eddy solution, approximate deconvolution model, large eddy simulation

AMS subject classifications. primary: 76F65, Secondary: 35Q40

1. Introduction. Consider the $2d$ Navier-Stokes equations[1] (NSE) under 2π periodic boundary conditions: for $\Omega = (0, 2\pi)^2,\ t > 0$

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = 0 \text{ and } \nabla \cdot u = 0 \tag{1.1}$$

One fundamental exact solution of (1.1) is the Green-Taylor vortex. In the simplest case, it is an array of signed vortices which decay in place as t increases. These have been used as a test problem in CFD, e.g., Chorin [Chorin68], Kim and Moin [KM68], Brachet [Brachet91], Orszag [Ors74] (and many others since) and have been used to explore the analytic structure of some turbulence models by Berselli [B05] (a remarkable and interesting paper) and Barbatto, Berselli and Grisanti [BBG07]. In this note we follow this latter approach and use general *Green-Taylor vortices* of the NSE, also called *eddy solutions* by Walsh [Walsh92] and *Taylor solutions* by Berselli [B05], and Barbatto, Berselli and Grisanti [BBG07] to explore similarly when a relatively new approach to turbulence models can replicate these fundamental vortex structures.

First we recall from Walsh [Walsh92] (see also Berker's classic article [Berker63]) the definition of Taylor / eddy solutions of the NSE.

DEFINITION 1.1. *Let $\boldsymbol{\phi}(x)$ be 2π periodic and satisfy*

$$-\triangle \boldsymbol{\phi} = \lambda \boldsymbol{\phi}, \nabla \cdot \boldsymbol{\phi} = 0. \tag{1.2}$$

Then

$$u(x,t) = e^{-\nu\lambda t}\boldsymbol{\phi}(x)$$

with pressure p satisfying

$$\nabla p = -u \cdot \nabla u$$

is a Taylor solution / eddy solution of the NSE.

Walsh [Walsh92] showed that for such a u, $\nabla \times (u \cdot \nabla u) = 0$ and thus such a p exists with (u, p) satisfying (1.1). We note that $3d$ generalizations, corresponding to

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[1] The usual abbreviations: NSE = Navier-Stokes equations, CFD = Computational Fluid Dynamics, LES = Large Eddy Simulation, ADM = Approximate Deconvolution Model.

viscous extensions of abc (Arnold-Beltrami-Childress) flows, have been given by Ross Ethier and Steinman [ES94].

REMARK 1.2. *The construction of Taylor / eddy solutions can begin with scalar eigenfunctions of the Laplacian. As noted by Walsh [Walsh92] if ψ is 2π periodic and satisfies*

$$-\triangle\psi = \lambda\psi$$

then $\boldsymbol{\phi} = (\psi_{x_2}, -\psi_{x_1})$ satisfies (1.2). All the eigenvalues of this problem are given by

$$\lambda = n^2 + m^2, \text{ where } n, m \in \mathbb{N}$$

The most commonly seen example is

$$\lambda = n^2 \text{ and } \psi = \cos(nx_1)\cos(nx_2).$$

For this choice, $u(x,t)$ consists of a checkerboard / rectangular array of $+/-$ vortices which decay in place as $t \to \infty$. Walsh [Walsh92] has shown that much more intricate spacial patterns occur when

$$\lambda = n^2 + m^2 = k^2 + l^2, \text{ where } n, m \neq k, l.$$

Walsh [Walsh92] gives the example

$$\lambda = 3^2 + 4^2 = 5^2, \text{ and } \psi = \frac{1}{4}\cos(3x_1)\sin(4x_2) - \frac{1}{5}\cos(5x_2) - 15\sin(5x_1). \quad (1.3)$$

We prove herein in Section 2 that Taylor / eddy solutions of the NSE are also exact solutions of the general Approximate Deconvolution Model (ADM) of turbulence (2.1) below, proving that ADMs ' structure allows persistence of a large class of special solutions of the NSE. The spacial structure of the ADM solutions is exactly the same as for Taylor / eddy solutions of the NSE and the decay exponents are slightly larger for the ADM Taylor / eddy solution than for the corresponding NSE Taylor / eddy solution.

2. Approximation deconvolution models of turbulence. Simulation of the pointwise velocity in turbulent flows by solving the NSE (1.1) down to the last persistent scale of motion is not feasible within time and resource constraints in many important applications. The normal approach is instead to derive (approximate) equations for local spacial averages (denoted by $\overline{u}(x,t)$) and to solve these for the (approximate) averaged velocities (denoted herein by $w(x,t)$). The class of models we study herein, Approximate Deconvolution Models (ADMs) were pioneered by Geurts [Geu97] and Stolz, Adams, Kleiser and their co-workers [AS01], [AS02], [SA99], [SAK01a], [SAK01b]. A sound theoretical foundation exists supporting their practical effectiveness, e.g., [BIL04], [LMNR06], [LN07], [S07]. In large eddy simulation (LES) the averages are defined by a local spacial filter (with filter radius denoted by δ); see [BIL04], Sagaut [S01] for details and examples of models and filters. Herein we select a differential filter (an idea of Germano [Ger86]) denoted by overbar or the action of the filter operator G given by

$$\overline{u} = Gu, G := (-\delta^2\triangle + 1)^{-1}.$$

Averaging (1.1) leads to the non-closed Space Filtered NSE

$$\overline{u}_t + \nabla \cdot \overline{uu} - \nu \triangle \overline{u} + \nabla \overline{p} = 0, \nabla \cdot \overline{u} = 0.$$

In ADMs an approximate filter inverse / deconvolution operator D is constructed and used to close the SFNSE by $\overline{uu} =$ (formally) $\overline{G^{-1}(\overline{u})G^{-1}(\overline{u})} \simeq \overline{D(\overline{u})D(\overline{u})}$. There are several used. We assume the chosen deconvolution operator D can be expressed as

$$D = f(G), \text{ where } f : \mathbb{R} \to \mathbb{R}.$$

Significant examples of deconvolution operators where $D = f(G)$ include

- Modified Tikhonov, (Stanculescu [S07]) : $D = ((1-\mu)G + \mu I)^{-1}$, for $\mu > 0$,
- van Cittert (e.g., [BB98], [LMNR06]): $D = \sum_{n=0}^{N}(I-G)^n$, N fixed,
- Optimized or unoptimized Tikhonov and iterated versions thereof.

This closure approximation with an added time relaxation / secondary regularization term $\chi(w - D(\overline{w}))$ (see [LN07], [R89], [ST92] for background on this term), the general ADM which is

$$w_t + \nabla \cdot \overline{D(w)D(w)} - \nu \triangle w + \nabla q + \chi(w - D(\overline{w})) = 0, \nabla \cdot w = 0. \tag{2.1}$$

Here w denotes the resulting *approximation* of $\overline{u}$. By analogy to spectral methods for periodic problems (which are currently the "gold standard" for problems to which they apply), the secondary regularization term $\chi(w - D(\overline{w}))$ is comparable to spectral vanishing viscosity in damping scales near the cutoff length. The closure model of the nonlinear term $\nabla \cdot \overline{D(w)D(w)}$ is comparable to the 3/2 rule in providing a more accurate representation of the restriction of the nonlinear term to the resolved scales.

An abstract theory of the general ADM is now in place, Stanculescu [S07]. We turn to particular features of ADMs; we prove next that Taylor / eddy solutions of the NSE are also exact solutions of the above general ADM. The spacial structure is exactly the same as for Taylor / eddy solutions of the NSE and the decay exponents are slightly larger for the ADM Taylor / eddy solution than for the corresponding NSE Taylor / eddy solution.

Theorem 2.1. *If $u(x,t) = e^{-\nu\lambda t}\boldsymbol{\phi}(x)$ is a Taylor / eddy solution of the NSE then (w,q) where*

$$\begin{aligned} w(x,t) &= e^{-\alpha t}\boldsymbol{\phi}(x), \alpha = -[\nu\lambda + \chi(1 - f(\frac{1}{\delta^2\lambda+1})\frac{1}{\delta^2\lambda+1})], \\ \nabla q &= -\overline{D(w)\cdot\nabla D(w)} \end{aligned}$$

is a Taylor / eddy solution of the ADM. Further $\nabla \times \left(\overline{D(w)\cdot\nabla D(w)}\right) = 0$ so q exists. The energy and enstrophy of Taylor / eddy solutions decay exponentially at the same rates:

$$\frac{Energy(t)}{Energy(0)} = e^{-2\lambda\nu t}, \ \ and \ \ \frac{Enstrophy(t)}{Enstrophy(0)} = \lambda e^{-2\lambda\nu t}$$

Proof. The proof is an extension of the construction in the NSE case. Indeed, we show that $\nabla \times \overline{D(\boldsymbol{\phi})\cdot\nabla D(\boldsymbol{\phi})} = 0$ so the ADM pressure q exists and then it is simple to verify by direct substitution that $w(x,t)$ satisfies

$$w_t - \nu\triangle w + \nabla q + \chi(w - D(\overline{w})) = 0, \nabla \cdot w = 0.$$

First we recall from Walsh [Walsh92] that $\nabla \times (\boldsymbol{\phi} \cdot \nabla \boldsymbol{\phi}) = 0$. Indeed, let $\boldsymbol{\phi} = (u, v)$. It is easy to verify by a direct calculate that

$$\frac{\partial}{\partial x_2}(uu_x + vu_{x_2}) - \frac{\partial}{\partial x_1}(uv_{x_1} + vv_{x_2}) = 0$$

using $-\triangle(u, v) = \lambda(u, v)$ and $u_{x_1} + v_{x_2} = 0$.

Since differential and convolution operators commute in the absence of boundaries (including periodic boundary conditions), it follows that $\nabla \times (\boldsymbol{\phi} \cdot \nabla \boldsymbol{\phi}) = 0$ and $\nabla \times \left(\overline{\boldsymbol{\phi} \cdot \nabla \boldsymbol{\phi}}\right) = 0$ are equivalent. Consider $\nabla \times \left(\overline{D(w) \cdot \nabla D(w)}\right)$. This is zero provided $\nabla \times D(w) \cdot \nabla D(w) = 0$. Let $\widetilde{w} = D(w)$. we claim that $-\triangle \widetilde{w} = \lambda \widetilde{w}, \nabla \cdot \widetilde{w} = 0$ so by the (above) same argument as the NSE case $\nabla \times (\boldsymbol{\phi} \cdot \nabla \boldsymbol{\phi})$ and thus (going backwards) $\nabla \times \left(\overline{D(w) \cdot \nabla D(w)}\right) = 0$. Indeed, in the absence of boundaries all the operators involved commute. Thus

$$\begin{aligned} &\nabla \cdot \widetilde{w} = \nabla \cdot D(w) = e^{-\alpha t} \nabla \cdot D(\boldsymbol{\phi}) = e^{-\alpha t} \nabla \cdot f(G) \boldsymbol{\phi} = e^{-\alpha t} f(G) \nabla \cdot \boldsymbol{\phi} = 0, \\ &-\triangle \widetilde{w} = e^{-\alpha t} D(-\triangle \boldsymbol{\phi}) = e^{-\alpha t} \lambda D(\boldsymbol{\phi}) = \lambda e^{-\alpha t} D(\boldsymbol{\phi}) = \lambda \widetilde{w}. \end{aligned}$$

The claimed decay of energy and enstrophy is directly calculated from the exact solution. ∎

3. Comments. For most families of deconvolution operators the zeroth order member is $D = I$ (i.e., approximate u by $\overline{u}$). For this simplest case we have the rate constant

$$\alpha = -\lambda \left(\nu + \frac{\chi \delta^2}{\lambda \delta^2 + 1} \right).$$

Thus the effect of time relaxation / secondary regularization here is (i) to increase the viscosity coefficient slightly from ν to roughly $\nu + \chi \delta^2$, and (ii) due to the denominator in the second term, the additional damping acts slightly stronger on larger spacial scales (smaller λ). This seems paradoxical since the intent of time relaxation is to damp smaller scales exponentially in time. We note however that in Taylor / eddy solutions all scales are damped exponentially so a clear conclusion cannot be drawn from this point.

A close look at the above proof shows that the following also holds.

THEOREM 3.1. *If $u(x, t)$ be a 2π periodic solution of the 2d or 3d NSE (1.1) with $\nabla \times (u \cdot \nabla u) = 0$. then this same $u(x, t)$ is an exact solution of the ADM (2.1) with $\chi = 0$.*

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