ENERGY AND HELICITY DISSIPATION RATES OF THE NS-ALPHA AND NS-ALPHA-DECONVOLUTION MODELS

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Abstract. We consider the NS-alpha and the family of high accuracy, NS-alpha-deconvolution models of turbulence on $\Omega = [0, L_\Omega]^3$ subject to periodic boundary conditions. For body force driven turbulence, we prove directly from the model equations of motion the following bounds on the time averaged modified energy dissipation rate, $\langle \varepsilon_{\alpha,\mathrm{N}}(w_{\alpha,\mathrm{N}})\rangle$, and unmodified helicity dissipation rate, $\langle \gamma(w_{\alpha,\mathrm{N}})\rangle$, for the N^{th} model ($N = 0, 1, 2, \dots$)

$$\langle \varepsilon_{\alpha,\mathrm{N}}(w_{\alpha,\mathrm{N}})\rangle \leq C_1(1+N)\frac{U_N^3}{L_N}, \quad \text{and} \quad |\,\langle \gamma(w_{\alpha,\mathrm{N}})\rangle\,| \leq C_2(1+N)\frac{U_N^3}{L_N^2}.$$

Here, N is the degree of the approximate deconvolution operator, U_N and L_N are global velocity and length scales, and C_1 and C_2 are constants that don't depend on U_N.

Key words. energy dissipation rate, helicity, helicity dissipation rate, large eddy simulation, turbulence, approximate deconvolution model, NS-alpha

1. Introduction. The three-dimensional (3D) Navier-Stokes equations (NSE) have remained intractable to both rigorous mathematical analysis as well as direct numerical simulation for many important applications. For this and other reasons, there is a continuing interest in both turbulence models and regularizations of the NSE. One such is the NS-alpha model, given by

$$\begin{gathered} w_t - \overline{w} \times \nabla \times w - \nu \triangle w + \nabla q = f \\ \overline{w} = A^{-1} w \text{ and } \nabla \cdot \overline{w} = 0, \end{gathered} \tag{1.1}$$

where $A = I - \alpha^2 \triangle$. We consider (1.1) and related models (1.2) under periodic boundary and zero mean conditions

$$\begin{gathered} w(x + L_\Omega e_j, t) = w(x,t) \quad j = 1,2,3 \quad \text{and}, \\ \textstyle\int_\Omega \phi \, dx = 0 \ \text{ for } \ \phi = w,\, w_0,\, f,\, q. \end{gathered}$$

With periodic boundary conditions, the condition $\nabla \cdot \overline{w} = 0$ in (1.1 is equivalent to $\nabla \cdot w = 0$. We suppose throughout that the data $w_0(x)$ and $f(x)$ are periodic with zero mean, smooth and satisfy

$$\nabla \cdot w_0 = 0 \text{ and } \nabla \cdot f = 0.$$

Among its many attractive properties, the NS-alpha model conserves both a modified kinetic energy [FHT02] and unmodified helicity [R08]. Its main limitations as a basis for numerical simulation of parctical flows are that (i) its accuracy is limited to $O(\alpha^2)$, even in smooth flow regions, and (ii) its microscale is significantly smaller than the averaging radius α, [FHT02].

The NS-alpha-deconvolution models [R08] are, for $N = 0, 1, 2, \dots$, a family of models $O(\alpha^{2N+2})$ consistent with NSE that includes NS-alpha as the $N = 0$ case. These models are given by

$$w_t - D_N(\overline{w}) \times \nabla \times w - \nu \triangle w + \nabla q = f, \text{ and } \nabla \cdot D_N \overline{w} = 0. \tag{1.2}$$

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In (1.2), D_N is an approximate deconvolution operator given by $D_N = \sum_{n=0}^{N}(I - A^{-1})^n$ and discussed in Section 2.1. Since $D_0 = I$, (1.2) includes (1.1) as the zeroth order case. The operator $D_N : L^2 \to L^2$ is bounded, symmetric, and positive definite and approximates the filter inverse with high accuracy on smooth functions ϕ [DE06]:

$$\phi = D_N\overline{\phi} + O(\alpha^{2N+2}).$$

Energy and helicity are two fundamental integral invariants of the 3D Euler equations (Moreau [M61]). Helicity is given by

$$H(w(t)) := \frac{1}{|\Omega|}\int_\Omega w(t)\cdot(\nabla\times w(t))dx =: (w, \nabla\times w).$$

(Inner products used herein are divided by volume.) The helicity of a flow vanishes if and only if the flow has a reflectional symmetry, and the helicity magnitude can be interpreted as the degree to which vortex lines are knotted and intertwined (defined precisely in terms of the total circulation and the Gauss linking number of interlocking vortex filaments), Moffatt [M84], Moffatt and Tsoniber [MT92]. Much less is known about helicity than energy and its mathematical study is more difficult than that of energy because more derivatives are involved and neither H nor its dissipation rate γ have one sign.

A similarity theory of coupled helicity and energy cascades with universal statistics has been developed for the NSE, Brissaud, Frisch, Leorat, Lesieur and Mazure [BFL73], Andre and Lesieur [AL77], Chen, Chen and Eyink [CCE03], Ditlevsen and Giuliani [DG01a], [DG01b] and it has been observed in turbulent flows, Bourne and Orszag [BO97]. An analogous theory for the related family of approximate deconvolution turbulence models appears in [LMNR08]. In these theories, in the inertial range (after suitable averaging), the only quantities that distinguish one flow from another in these two cascades are their energy and helicity dissipation rates. Thus, a model's prediction of average energy and helicity dissipation rates is critical for evaluating the model's physical fidelity.

Bounds for time averaged energy and helicity dissipation rates for the NS-alpha and NS-alpha-deconvolution models are derived herein and mirror both the analogous rates for the underlying solution of the NSE and estimates derived by dimensional analysis. This work builds on the fundamental advances of Constantin and Doering [CD92], Doering and Gibbon [DG95], Foias [F97], Wang [W97], and Doering and Foias [DF02] providing analytic estimates of turbulent flow statistics for the NSE.

For the NSE, an energy equality can be derived by multiplying the NSE through by velocity and integrating. The result is an equality in which energy can be identified as $(w, w)/2$. For NS-alpha, instead of multiplying through by velocity, an energy equality arises from multiplying by $\overline{w}$ and integrating [FHT02]. This equality gives rise to a modified energy given by $(w, \overline{w})/2$. The models (1.2) share a similar modified energy equality arising from multiplication by $D_N(\overline{w})$. This energy balance gives rise to expressions for modified energy and energy dissipation that are most naturally expressed in terms of weighted inner products and norms, developed in Section 3.

Definition 1.1. *For D_N a bounded, self-adjoint, positive definite operator, define the deconvolution weighted inner product and norm:*

$$(u, v)_{D_N} := \frac{1}{|\Omega|}\int_\Omega u\cdot D_N(v)dx, \text{ and } \|u\|_{D_N}^2 := (u, u)_{D_N}. \tag{1.3}$$

It is known that $\|\cdot\|_{D_N}$ is equivalent to the usual L^2 norm, Lemma 2.1 below.

The energy and energy dissipation rate of the model are identified from the balance equation (3.1) as

$$E_{\alpha,\mathrm{N}}(w(t)) := \frac{1}{2}(w(t), \overline{w}(t))_{D_N}, \quad \text{and} \quad \varepsilon_{\alpha,\mathrm{N}}(w(t)) := \nu(\nabla w(t), \nabla \overline{w}(t))_{D_N}.$$

Both these quantities are shown to be non-negative in Section 3.

An helicity balance equation can be derived formally for the NSE by multiplying by $\nabla \times w$ and integrating. Similarly, for NS-alpha-deconvolution models, (see Section 5) the quantity

$$\gamma(w(t)) := \nu(\nabla \times w(t), \nabla \times (\nabla \times w(t)))$$

plays the role of dissipation in the balance equation.

The present work focuses on long time averages of energy and helicity dissipation rates. Let $\langle \cdot \rangle$ denote long time averaging (defined in Section 2). K41 phenomenology, *e.g.*, Frisch [F95], Pope [P00] suggests the scaling of the energy and helicity dissipation rates for the NSE, $\langle \varepsilon \rangle$ and $\langle \gamma \rangle$, satisfy

$$\langle \varepsilon \rangle \approx C_\varepsilon \frac{U^3}{L}, \text{ and } |\langle \gamma \rangle| \approx C_\gamma \frac{U^3}{L^2}. \tag{1.4}$$

In Section 4, we prove that the NS-alpha-deconvolution family's energy dissipation rate satisfies

$$\langle \varepsilon_{\alpha,\mathrm{N}}(w) \rangle \le \frac{U_N^3}{L_N}\left(42 + 36N + \frac{1}{\mathrm{Re}_N}(1 + \frac{\alpha^2}{L_N^2})\right). \tag{1.5}$$

Here N denotes the degree of deconvolution and U_N, L_N denote natural velocity and length scales associated with the largest scales of the model (1.2), defined precisely in Section 2. Re_N denotes the Reynolds number based on U_N, L_N, and ν. Estimate (1.5) is consistent, as $\alpha \to 0$, with the dimensional estimate (1.4) for the NSE.

In Section 5, we prove that the NS-alpha-deconvolution family's helicity dissipation rate, $|\langle \gamma(w) \rangle|$, satisfies

$$|\langle \gamma(w) \rangle| \le 2(1 + \frac{\alpha^2}{L_N^2})\left((18N + 21) + \frac{1}{\mathrm{Re}_N}(1 + \frac{\alpha^2}{L_N^2})\right)\frac{U_N^3}{L_N^2}. \tag{1.6}$$

Estimate (1.6) is also consistent, as $\alpha \to 0$, with the dimensional estimate (1.4) for the NSE.

REMARK 1.1. *A time relaxation term of the form $\chi(w - D_N(\overline{w}))$, with χ a constant, can be included in (1.2) to improve numerical behavior. Such a term could easily be included in the analysis below.*

2. Notation and preliminaries. The deconvolution operator we consider was studied by van Cittert in 1934, e.g., Bertero and Boccacci [BB98], and its use in LES pioneered by Adams, Kleiser and Stolz [AS01], [SA99], [AS02], [SAK01a], [SAK01b], [SAK02].

2.1. Approximate deconvolution operators. A filtering or convolution operator $A(w) = \overline{w}$ is a bounded map: $A : L^2(\Omega) \to L^2(\Omega)$. The deconvolution problem is to approximate w given $(\overline{w} + \text{noise})$. If (as in the case we study) A is smoothing, its inverse cannot be bounded. An unbounded inverse, even an exact inverse, would magnify the noise catastrophically. An approximate deconvolution operator D is an approximate inverse $\overline{w} \mapsto D(\overline{w}) \approx w$ which:

- Is a bounded operator on $L^2(\Omega)$;
- Approximates w in some useful (typically asymptotic) sense; and,
- Satisfies other conditions necessary for the application at hand.

In particular, the most desirable conditions for an approximate deconvolution operator in flow modeling include:

- On the large scales the operator has very high accuracy;
- Its calculational expense is modest;
- The deconvolution operator D and the operator $I-DA^{-1}$ are both self-adjoint and positive semidefinite, Stanculescu [S07]; and,
- It commutes with filtering and differentiation.

Let A be the differential operator $A = (-\alpha^2\triangle + I)$ for a small constant α. The N^{th} van Cittert approximate deconvolution operator D_N is defined by N steps (typically, $1 \leq N \leq 7$) of Picard iteration, [BB98], for the fixed point problem:

$$\text{given } \overline{w}, \text{ solve } w = w + \{\overline{w} - A^{-1}w\} \text{ for } w.$$

Eliminating intermediate steps gives a formula for D_N.

$$D_N\phi = \sum_{n=0}^{N}(I - A^{-1})^n\phi = (A - A(I - A^{-1})^{N+1})\phi. \tag{2.1}$$

Lemma 2.1. *Consider the approximate deconvolution operator*

$$D_N : L^2(\Omega) \to L^2(\Omega)$$

Both D_N and $I - D_N A^{-1}$ are bounded, self-adjoint, positive definite operators. D_N satisfies

$$\|\phi\|^2 \leq \|\phi\|_{D_N}^2 \leq (N+1)\|\phi\|^2, \quad \forall \phi \in L^2(\Omega) .$$

Proof. The first half is in [BIL06], for example, and the second half is a consequence of (2.1) and the facts that A and $I - A^{-1}$ are both positive definite. ∎

2.2. Scaling and time averages. For each fixed N, define the following quantities. When $N = 0$, these quantities will sometimes be written without subscripts. The scale of the body force is defined by

$$F_N := \|f\|_{D_N}, \quad \overline{F}_N := \|\overline{f}\|_{D_N}$$

The global length scale associated with the power input at the large scales, *i.e.,* with $f(x)$, is

$$\begin{aligned} L_N := \min\{L_\Omega, &\frac{F_N}{||\nabla f||_{L^\infty(\Omega)}}, \frac{F_N}{\|\nabla f\|}, \frac{F_N}{\|\nabla \times f\|}, \frac{F_N^{\frac{1}{2}}}{\|\triangle f\|^{\frac{1}{2}}}, \frac{F_N^{\frac{1}{3}}}{\|\nabla \times \triangle f\|^{\frac{1}{3}}}, \\ &\frac{\overline{F}_N}{||\nabla \overline{f}||_{L^\infty(\Omega)}}, \frac{\overline{F}_N}{\|\nabla \overline{f}\|}, \frac{\overline{F}_N}{\|\nabla \times \overline{f}\|}, \frac{\overline{F}_N^{\frac{1}{2}}}{\|\triangle \overline{f}\|^{\frac{1}{2}}}, \frac{\overline{F}_N^{\frac{1}{3}}}{\|\nabla \times \triangle \overline{f}\|^{\frac{1}{3}}}\} \end{aligned}$$

The long time average of a function $\phi(t)$ can be defined in several natural ways. For this work, the long time average is taken as the limit of the finite window time average;

$$\langle \phi \rangle_{[0,T]} := \frac{1}{T}\int_0^T \phi(t)dt, \text{ and}$$
$$\langle \phi \rangle := \lim_{T\to\infty} \langle \phi \rangle_{[0,T]}.$$

With this definition,

$$\langle \phi + \psi \rangle \leq \langle \phi \rangle + \langle \psi \rangle,$$

but equality does not hold unless, *e.g.*, $\lim_{T\to\infty} \langle \phi \rangle_{[0,T]}$ exists.

Foias, Jolly, and Manley [FJM05] have recently proved estimates of finite time averages of two-dimensional turbulent statistics—an important extension since data is taken from experiments over long but finite time windows.

Assuming w is a solution of (1.2) and that the long time averages of w exist, denote the quantities

$$U_N = \langle w \rangle, \, \overline{U}_N = \langle \overline{w} \rangle, \text{ and } \mathrm{Re_N} := \frac{L_N U_N}{\nu},$$

where $\mathrm{Re_N}$ is the global Reynolds number and ν is the kinematic viscosity.

2.3. Basic estimates and identites. Two useful estimates are presented in the next lemma. The first of these appears in [DE06]. The second involves the curl operator, $\mathbf{curl} w = \nabla \times w$, repeated application of which will be denoted with exponents, as in $\mathbf{curl}^3 w = \nabla \times (\nabla \times (\nabla \times w))$.

LEMMA 2.2. *With A and D_N defined as above, the following estimates hold.*

$$\left\| \alpha^2 \triangle A^{-1} w \right\|_{D_N} \leq \|w\|_{D_N} \tag{2.2}$$

and, for $0 \leq m \leq 2N$,

$$\left\| \alpha^m \mathbf{curl}^m (A^{-1})^{N+1} w \right\|_{D_N} \leq \|w\|_{D_N} \begin{cases} \left(\frac{1}{2}\right)^{m/2} & \text{for } 1 \leq m \leq N+1 \\ \left(\frac{1}{2}\right)^{N+1-m/2} & \text{for } N+1 \leq m \leq 2N+1 \end{cases} \tag{2.3}$$

Proof. The estimate (2.2) follows from Fourier series expansion using Parseval's equality. Note that D_N is positive definite by Lemma 2.1, so it has a square root whose Fourier series coefficients, denoted $\widehat{(D_N^{1/2})}_k$, are the square roots of the coefficients of D_N.

$$\left\| \alpha^2 \triangle A^{-1} w \right\|_{D_N} = \left\| \frac{\alpha^2 |k|^2}{(1 + \alpha^2 |k|^2)} \widehat{(D_N^{1/2})}_k \widehat{w}_k \right\| \leq \left\| \widehat{(D_N^{1/2})}_k \widehat{w}_k \right\| = \|w\|_{D_N} \tag{2.4}$$

For (2.3), the case $m = 0$ reduces to $\|\overline{w}\|_{D_N} \leq \|w\|_{D_N}$, whose proof follows from (2.4) with $\alpha^2 |k|^2$ replaced by one.

For (2.3) with $1 \leq m \leq 2N$, taking a Fourier series expansion yields the expression

$$\left\| \alpha^m \mathbf{curl}^m (A^{-1})^{N+1} w \right\|_{D_N} \leq \left\| \frac{\alpha^m |k|^m}{(1 + \alpha^2 |k|^2)^{N+1}} \widehat{(D_N^{1/2})}_k \widehat{w}_k \right\|.$$

For $x \geq 0$ and $m < 2N + 2$, the function $\phi(x) = x^m / (1 + x^2)^{N+1}$ has a maximum value when $x = \sqrt{m/2N + 2 - m}$. For $1 \leq m \leq N + 1$, x is smaller than 1, and for $N + 1 \leq m < 2N + 1$, x is larger than 1. ϕ can be rewritten in the form

$$\phi(x) = \left(\frac{x^2}{1 + x^2} \right)^{m/2} \left(\frac{1}{1 + x^2} \right)^{N+1-m/2}.$$

For $x \le 1$, the second factor is smaller than 1 and the first is monotone and takes its maximum when $x = 1$, so that $|\phi(x)| \le (1/2)^{m/2}$. For $x \ge 1$, the first factor is smaller than 1 and the second is monotone and takes its maximum when $x = 1$, so that $|\phi(x)| \le (1/2)^{N+1-m/2}$. ◻

The following lemma expresses the Poincaré inequality in deconvolution weighted form. Its proof is a standard application of Parseval's equality.

LEMMA 2.3. *Each differentiable, periodic function w with zero mean satisfies the estimate*

$$\|w\|_{D_N} \le (L_\Omega/\pi) \|\nabla w\|_{D_N} . \tag{2.5}$$

Proof. Again denoting the Fourier coefficients of w for multi-index k by $\widehat{w}_k$, and the Fourier coefficients of the square root of D_N by $\widehat{(D_N^{1/2})}_k$, Parseval's equality yields the following calculations.

$$\|w\|_{D_N}^2 = \sum_{|k|\ge 1} |\widehat{(D_N^{1/2})}_k \widehat{w}_k|^2,$$

where the $|k| = 0$ term vanishes because w has zero mean. Clearly,

$$\sum_{|k|\ge 1} |\widehat{(D_N^{1/2})}_k \widehat{w}_k|^2 \le \frac{L_\Omega^2}{\pi^2} \sum_{|k|\ge 1} |\frac{k\pi}{L_\Omega} \widehat{(D_N^{1/2})}_k \widehat{w}_k|^2 = \frac{L_\Omega^2}{\pi^2} \|\nabla w\|_{D_N}^2 . \tag{2.6}$$

◻

The following useful vector identities can be found in, for example, Gibbs [Gib09, pp. 161, 169].

$$w \times (\nabla \times w) = -\nabla \cdot (ww) + \frac{1}{2}\nabla|w|^2, \quad \text{and} \tag{2.7}$$

$$\begin{aligned} \nabla \times (\nabla \times w) &= -\triangle w + \nabla(\nabla \cdot w) \\ &= -\triangle w, \quad \text{if } \nabla \cdot w = 0. \end{aligned} \tag{2.8}$$

A final vector identity will be used in the sequel. It is a straightforward consequence of (2.7). Its importance lies in the fact that $\triangle = -\mathbf{curl}^2$ for divergence-free functions, so that $\triangle^N \phi \times (\nabla \times \phi)$ can be written as the divergence of a (dyadic) tensor plus the gradient of a scalar.

LEMMA 2.4. *For sufficiently differentiable vector functions ϕ, and $N \ge 0$, the following identity holds.*

$$\begin{aligned} \mathbf{curl}^{2N}\phi \times \mathbf{curl}\phi = &\sum_{k=1}^{N} \nabla \cdot \big((\mathbf{curl}^{2N-k}\phi)(\mathbf{curl}^k\phi) + (\mathbf{curl}^k\phi)(\mathbf{curl}^{2N-k}\phi)\big) \\ &- \sum_{k=1}^{N} \nabla \big((\mathbf{curl}^{2N-k}\phi)\cdot(\mathbf{curl}^k\phi)\big) \\ &- \nabla \cdot \big((\mathbf{curl}^N\phi)(\mathbf{curl}^N\phi)\big) + \frac{1}{2}\nabla(|\mathbf{curl}^N\phi|^2) \end{aligned} \tag{2.9}$$

Proof. An induction proof could be started with the $N = 0$ case, which is merely (2.7). It is instructive, however, to consider the case $N = 1$.

$$\begin{aligned}\mathbf{curl}^2\phi \times \mathbf{curl}\phi &= -\mathbf{curl}\phi \times \mathbf{curl}(\mathbf{curl}\phi)\\ &= \nabla\cdot((\mathbf{curl}\phi)(\mathbf{curl}\phi)) - \frac{1}{2}\nabla(|\mathbf{curl}\phi|^2)\end{aligned}$$

because of (2.7) with w replaced by $\mathbf{curl}\phi$.

From the induction hypothesis, we have

$$\begin{aligned}\mathbf{curl}^{2N+2}\phi \times \mathbf{curl}\phi = &(\mathbf{curl}^{2N+1}\phi - \mathbf{curl}\phi) \times (\mathbf{curl}(\mathbf{curl}^{2N+1}\phi - \mathbf{curl}\phi))\\ &- \mathbf{curl}\phi \times (\mathbf{curl}(\mathbf{curl}\phi)) - \mathbf{curl}^{2N+1}\phi \times (\mathbf{curl}(\mathbf{curl}^{2N+1}\phi))\\ &+ \mathbf{curl}^{2N}(\mathbf{curl}\phi) \times (\mathbf{curl}(\mathbf{curl}\phi))\end{aligned}$$

The first three terms on the right side are transformed according to (2.7), yielding

$$\begin{aligned}\mathbf{curl}^{2N+2}\phi \times \mathbf{curl}\phi = &\mathbf{curl}^{2N}(\mathbf{curl}\phi) \times (\mathbf{curl}(\mathbf{curl}\phi))\\ &- \nabla\cdot\big((\mathbf{curl}^{2N+1}\phi - \mathbf{curl}\phi)(\mathbf{curl}^{2N+1}\phi - \mathbf{curl}\phi)\big)\\ &+ \frac{1}{2}\nabla(|\mathbf{curl}^{2N+1}\phi - \mathbf{curl}\phi|^2)\\ &+ \nabla\cdot((\mathbf{curl}\phi)(\mathbf{curl}\phi)) - \frac{1}{2}\nabla(|\mathbf{curl}\phi|^2)\\ &+ \nabla\cdot\big((\mathbf{curl}^{2N+1}\phi)(\mathbf{curl}^{2N+1}\phi)\big) - \frac{1}{2}\nabla(|\mathbf{curl}^{2N+1}\phi|^2).\end{aligned}$$

The binomial dyadic products can be expanded, as can the norm when regarded as an inner product. Expanding and collecting like terms simplifies the expression.

$$\begin{aligned}\mathbf{curl}^{2N+2}\phi \times \mathbf{curl}\phi = &\mathbf{curl}^{2N}(\mathbf{curl}\phi) \times (\mathbf{curl}(\mathbf{curl}\phi))\\ &+ \nabla\cdot\big((\mathbf{curl}^{2N+1}\phi)(\mathbf{curl}\phi) + (\mathbf{curl}\phi)(\mathbf{curl}^{2N+1}\phi)\big)\\ &- \nabla\big((\mathbf{curl}^{2N+1}\phi)\cdot(\mathbf{curl}\phi)\big)\end{aligned}$$

The first term on the right can be transformed by the induction hypothesis applied to $\mathbf{curl}\phi$, yielding the following expression.

$$\begin{aligned}\mathbf{curl}^{2N+2}\phi \times \mathbf{curl}\phi = &\sum_{i=1}^{N}\nabla\cdot\big((\mathbf{curl}^{2N-i+1}\phi)(\mathbf{curl}^{i+1}\phi)+\\ &\qquad + (\mathbf{curl}^{i+1}\phi)(\mathbf{curl}^{2N-i+1}\phi)\big)\\ &- \sum_{i=1}^{N}\nabla\big((\mathbf{curl}^{2N-i+1}\phi)\cdot(\mathbf{curl}^{i+1}\phi)\big)\\ &- \nabla\cdot\big((\mathbf{curl}^{N+1}\phi)(\mathbf{curl}^{N+1}\phi)\big) + \frac{1}{2}\nabla(|\mathbf{curl}^{N+1}\phi|^2)\\ &+ \nabla\cdot\big((\mathbf{curl}^{2N+1}\phi)(\mathbf{curl}\phi) + (\mathbf{curl}\phi)(\mathbf{curl}^{2N+1}\phi)\big)\\ &- \nabla\big((\mathbf{curl}^{2N+1}\phi)\cdot(\mathbf{curl}\phi)\big)\end{aligned}$$

The dummy variable i in the sums can be replaced with $i-1$, and combining the final two terms, yields the following expression.

$$\begin{aligned}\mathbf{curl}^{2N+2}\phi \times \mathbf{curl}\phi = &\sum_{i=1}^{N+1} \nabla \cdot \big((\mathbf{curl}^{2N+2-i}\phi)(\mathbf{curl}^{i}\phi) + (\mathbf{curl}^{i}\phi)(\mathbf{curl}^{2N+2-i}\phi)\big) \\ &- \sum_{i=1}^{N+1} \nabla \big((\mathbf{curl}^{2N+2-i}\phi)\cdot(\mathbf{curl}^{i}\phi)\big) \\ &- \nabla \cdot \big((\mathbf{curl}^{N+1}\phi)(\mathbf{curl}^{N+1}\phi)\big) + \frac{1}{2}\nabla(|\mathbf{curl}^{N+1}\phi|^2)\end{aligned}$$

and the lemma is proved. ▯

REMARK 2.1. ***Existence of solutions.*** *It is known that strong solutions of the NS-alpha model exist uniquely and are as smooth as the problem data [FHT02]. A corresponding theory can be developed for the NS-alpha-deconvolution family [MS08]. Since the data for (1.1) are regular, we may proceed formally in our estimates without loss of generality.*

3. Kinetic energy balance of the NS-alpha-deconvolution family. The mathematical key to the estimates of energy and helicity dissipation rates is the following energy estimate, recalled from [R08].

PROPOSITION 3.1. *If w is a solution of (1.2), w satisfies both a differential form of conservation of energy*

$$\frac{d}{dt}\frac{1}{2}(w(t), \overline{w}(t))_{D_N} + \nu(\nabla w(t), \nabla \overline{w}(t))_{D_N} = (f, \overline{w}(t))_{D_N}, \tag{3.1}$$

and an integral form of conservation of energy

$$\begin{aligned}\frac{1}{2}\{\|\overline{w}(T)\|_{D_N}^2 + \alpha^2 \|\nabla \overline{w}(T)\|_{D_N}^2\} + \int_0^T \nu\{\|\nabla \overline{w}(t)\|_{D_N}^2 + \alpha^2 \|\triangle \overline{w}(t)\|_{D_N}^2\}dt = \\ = \frac{1}{2}\{\|\overline{w}(0)\|_{D_N}^2 + \alpha^2 \|\nabla \overline{w}(0)\|_{D_N}^2\} + \int_0^T (f(t), \overline{w}(t))_{D_N} dt.\end{aligned}$$

From Proposition 3.1, the model kinetic energy, $E_{\alpha,\mathrm{N}}(w)$, and energy dissipation rate, $\varepsilon_{\alpha,\mathrm{N}}(w)$ are clearly identified.

$$E_{\alpha,\mathrm{N}}(w) := \frac{1}{2}(w, \overline{w})_{D_N} = \frac{1}{2}(\|\overline{w}\|_{D_N}^2 + \alpha^2 \|\nabla \overline{w}\|_{D_N}), \tag{3.2}$$

$$\varepsilon_{\alpha,\mathrm{N}}(w) := \nu(\nabla w, \nabla \overline{w})_{D_N} = \nu(\|\nabla \overline{w}\|_{D_N}^2 + \alpha^2 \|\triangle \overline{w}\|_{D_N}^2). \tag{3.3}$$

The following corollary shows that $E_{\alpha,\mathrm{N}}$ is bounded for all time and that $\langle \varepsilon_{\alpha,\mathrm{N}} \rangle$ is finite. Thus, $w \in L^\infty(0,\infty; H^1(\Omega))$.

COROLLARY 3.2. *Let $f, u_0 \in L_0^2(\Omega)$ and w be an L_Ω-periodic solution of (1.2) then*

$$\sup_{t\in(0,\infty)} E_{\alpha,N}(w(t)) \le \frac{\|f\|_{D_N}^2}{\nu^2(L_\Omega/\pi)^2} + E_{\alpha,N}(w(0)) < \infty$$

$$\frac{1}{T}\int_0^T \varepsilon_{\alpha,N}(w(t))dt \le \frac{1}{T}E_{\alpha,N}(w(0)) + \frac{\|f\|_{D_N}^2}{\nu(L_\Omega/\pi)} + \|f\|_{D_N}\sqrt{E_{\alpha,N}(w(0))} < \infty$$

Proof. We begin with the conservation of energy expression (3.1). Lemma 2.3 applied to $\overline{w}$ and $\nabla\overline{w}$ yields

$$\frac{d}{dt}E_{\alpha,\mathrm{N}} + \nu(L_\Omega/\pi)E_{\alpha,\mathrm{N}} \le (f,\overline{w})_{D_N}.$$

Because of the Cauchy-Schwarz and Young inequalities,

$$(f,\overline{w})_{D_N} \le \|f\|_{D_N}\|\overline{w}\|_{D_N} \le \frac{\|f\|_{D_N}^2}{2\nu(L_\Omega/\pi)} + \frac{\nu(L_\Omega/\pi)}{2}\|\overline{w}\|_{D_N}^2,$$

so that, since $\|\overline{w}\|_{D_N}^2 \le E_{\alpha,\mathrm{N}}$,

$$\frac{d}{dt}E_{\alpha,\mathrm{N}}(t) + \frac{\nu(L_\Omega/\pi)}{2}E_{\alpha,\mathrm{N}}(t) \le \frac{\|f\|_{D_N}^2}{2\nu(L_\Omega/\pi)}.$$

This inequality admits an integrating factor to find

$$E_{\alpha,\mathrm{N}}(w(t)) \le \frac{\|f\|_{D_N}^2}{\nu^2(L_\Omega/\pi)^2} + E_{\alpha,\mathrm{N}}(w(0))e^{-\nu(L_\Omega/\pi)t/2}$$

and the first conclusion follows immediately.

For boundedness of the time averaged dissipation rate, divide the energy estimate from Proposition 3.1 by T, and note that $\|w\|_{D_N}^2 \le \sup_t E_{\alpha,\mathrm{N}}(w(t))$:

$$\begin{aligned}\frac{1}{T}E_{\alpha,\mathrm{N}}(w(T)) + \frac{1}{T}\int_0^T \varepsilon_{\alpha,\mathrm{N}}(w(t))dt &= \frac{1}{T}E_{\alpha,\mathrm{N}}(w(0)) + \frac{1}{T}\int_0^T (f,\overline{w}(t))_{D_N}dt \\ &\le \frac{1}{T}E_{\alpha,\mathrm{N}}(w(0)) + \|f\|_{D_N}\sup_t\sqrt{E_{\alpha,\mathrm{N}}(w(t))} \\ &\le \frac{1}{T}E_{\alpha,\mathrm{N}}(w(0)) + \frac{\|f\|_{D_N}^2}{\nu(L_\Omega/\pi)} + \|f\|_{D_N}\sqrt{E_{\alpha,\mathrm{N}}(w(0))}\end{aligned}$$

□

REMARK 3.1. *In the next section, we will need the second estimate in Corollary 3.2 modified by integration over the interval $[s,t]$.*

$$\nu\alpha^2\int_s^t \|\triangle\overline{w}(\tau)\|_{D_N}^2\,d\tau \le E_{\alpha,N}(w(0)) + (t-s)\left(\frac{\|f\|_{D_N}^2}{\nu(L_\Omega/\pi)} + \|f\|_{D_N}\sqrt{E_{\alpha,N}(w(0))}\right) \tag{3.4}$$

4. Bounds on energy dissipation rates. In this section, bounds for energy dissipation rates are derived, based on precise estimates of the model's energy balance. These bounds are inspired by arguments of Foias [F97], Doering and Foias [DF02] for the NSE, and Foias, Holm and Titi, [FHT02] for the NS-alpha regularization. The following estimate on the model's time averaged energy dissipation rates is the major result of this section.

THEOREM 4.1. *If w is a solution of (1.2) for some fixed $N \ge 0$, the average energy dissipation rate for w satisfies*

$$\langle\varepsilon_{\alpha,N}(w)\rangle \le 6\frac{1+3(N+1)}{L_N}U_N^3 + \nu U_N^2\left(\frac{1}{L_N^2} + \alpha^2\frac{1}{L_N^4}\right), \tag{4.1}$$

or in terms of Re_N

$$\langle \varepsilon_{\alpha,N}(w)\rangle \leq \frac{U_N^3}{L_N}\left(24+18N+\frac{1}{\mathrm{Re}_N}(1+\frac{\alpha^2}{L_N^2})\right). \tag{4.2}$$

Proof. The first of two key bounds is obtained by time averaging the energy equality of Proposition 3.1. From the expression for $\varepsilon_{\alpha,\mathrm{N}}$ in (3.3)

$$\langle \varepsilon_{\alpha,\mathrm{N}}\rangle = \nu\langle(\nabla w, \nabla \overline{w})_{D_N}\rangle = \langle (f, \overline{w}(t))_{D_N}\rangle = \langle (\overline{f}, w(t))_{D_N}\rangle.$$

The Cauchy-Schwarz inequality and Corollary 3.2 imply the following expression.

$$\langle\ \varepsilon_{\alpha,\mathrm{N}}\rangle \leq \overline{F}_N U_N \tag{4.3}$$

This expression will be used later.

Next, time averaging the model (1.2) gives

$$\langle \frac{\partial}{\partial t} w\rangle_{[0,T]} - \langle D_N(\overline{w}) \times (\nabla \times w)\rangle_{[0,T]} - \nu\triangle\langle w\rangle_{[0,T]} + \nabla\langle Q\rangle_{[0,T]} = \langle f\rangle_{[0,T]}$$

Taking the weighted inner product of the averaged model with the $\overline{f}$, which is divergence-free because f is assumed divergence-free, yields

$$(\overline{f}, f)_{D_N} = (\overline{f}, \langle\frac{\partial}{\partial t} w\rangle_{[0,T]})_{D_N} - (\overline{f}, \langle D_N\overline{w} \times (\nabla \times w)\rangle_{[0,T]})_{D_N} + \nu(\nabla\overline{f}, \nabla\langle w\rangle_{[0,T]})_{D_N}. \tag{4.4}$$

Each term is considered separately below.

Time derivative term. The first term on the right of (4.4) vanishes because $E_{\alpha,\mathrm{N}}$ is bounded by Corollary 3.2.

Viscous term. The viscous term in (4.4) satisfies

$$\begin{aligned}
\nu(\nabla\overline{f}, \nabla\langle w\rangle_{[0,T]})_{D_N} &= \nu(A^{1/2}\nabla\overline{f}, \nabla\langle A^{-1/2}w\rangle_{[0,T]})_{D_N}\\
&\leq \nu\|A^{1/2}\nabla\overline{f}\|_{D_N}\|\nabla\langle A^{-1/2}w\rangle_{[0,T]}\|_{D_N}\\
&\leq \sqrt{\nu\|A^{1/2}\nabla\overline{f}\|^2_{D_N}}\sqrt{\varepsilon_{\alpha,0}(\langle w\rangle_{[0,T]})_{D_N}}.
\end{aligned}$$

Further,

$$\begin{aligned}
\|A^{1/2}\nabla\overline{f}\|^2_{D_N} &= (A^{1/2}\nabla\overline{f}, A^{1/2}\nabla\overline{f})_{D_N} = (\nabla\overline{f}, A\nabla\overline{f})_{D_N}\\
&= \|\nabla\overline{f}\|^2_{D_N} + \alpha^2\|\Delta\overline{f}\|^2_{D_N}\\
&\leq \frac{\overline{F}_N^2}{L^2} + \alpha^2\frac{\overline{F}_N^2}{L^4}
\end{aligned}$$

In summary,

$$\nu(\nabla\overline{f}, \nabla\langle w\rangle_{[0,T]}) \leq \sqrt{\nu\left(\frac{\overline{F}^2}{L^2}+\alpha^2\frac{\overline{F}^2}{L^4}\right)}\sqrt{\varepsilon_{\alpha,\mathrm{N}}(\langle w\rangle_{[0,T]})}$$

Convection term. The discussion below provides estimates of the convection term in terms of $\overline{F}U_N^2/L_N$. Eliminating gradients from the estimate requires application of Lemma 2.4. Because the notation is intricate and because these estimates do not depend on temporal averaging, it will be delayed until the final step.

Using (2.1), D_N is written as $D_N = A - A(I - A^{-1})^{N+1}$ so that the convection term can be decomposed in the following way.

$$(\overline{f}, D_N\overline{w}\times(\nabla\times w))_{D_N} = (\overline{f}, w\times(\nabla\times w))_{D_N} - (\overline{f}, (I-A^{-1})^{N+1}w\times(\nabla\times w))_{D_N} \quad (4.5)$$

We consider the second term first. Estimating this term requires repeated application of Lemmas 2.2 and 2.4. Begin by writing

$$(\overline{f}, (I - A^{-1})^{N+1}w \times (\nabla \times w))_{D_N} = (\overline{f}, (A - I)^{N+1}(A^{-1})^{N+1}w \times (\nabla \times w))_{D_N}. \quad (4.6)$$

Temporarily denoting $\widetilde{w} = (A^{-1})^{N+1}w$, or $w = A^{N+1}\widetilde{w}$, and $\phi_n = (-\alpha^2\triangle)^n\widetilde{w} = (-\alpha^2\triangle)^n(A^{-1})^{N+1}w$, (4.6) can be rewritten using the binomial theorem.

$$\begin{aligned}(\overline{f}, (I - A^{-1})^{N+1}w \times (\nabla \times w))_{D_N} &= (\overline{f}, (-\alpha^2\triangle)^{N+1}\widetilde{w} \times (\nabla \times A^{N+1}\widetilde{w}))_{D_N} \\ &= \sum_{n=0}^{N+1}\binom{N+1}{n}\left(\overline{f}, (-\alpha^2\triangle)^{N-n+1}\phi_n \times (\nabla\times\phi_n)\right)_{D_N}. \end{aligned} \quad (4.7)$$

Each of the terms, denoted T_n, in the sum can be estimated using Lemma 2.4.

$$\begin{aligned} T_n &= \left(\overline{f}, (-\alpha^2\triangle)^{N-n+1}\phi_n \times (\nabla\times\phi_n)\right)_{D_N} \\ &= (+\alpha^2)^{N-n+1}\sum_{k=1}^{N-n+1}\Big[\left(\overline{f}, \nabla\cdot[(\mathbf{curl}^{2N-2n+2-k}\phi_n)(\mathbf{curl}^k\phi_n)]\right)_{D_N} + \\ &\qquad\qquad \left(\overline{f}, [(\mathbf{curl}^k\phi_n)(\mathbf{curl}^{2N-2n+2-k}\phi_n)]\right)_{D_N}\Big] \\ &\quad - (\alpha^2)^{N-n+1}\left(\overline{f}, \nabla\cdot\left[(\mathbf{curl}^{N-n+1}\phi_n)(\mathbf{curl}^{N-n+1}\phi_n)\right]\right)_{D_N} \\ &\quad - (\alpha^2)^{N-n+1}\left(\overline{f}, \nabla S\right)_{D_N} \end{aligned}$$

where $S = \frac{1}{2}|\mathbf{curl}^N\phi_n|^2 - \sum_{k=1}^N(\mathbf{curl}^{2N-k}\phi_n)\cdot(\mathbf{curl}^k\phi_n)$ is the scalar from Lemma 2.4. Integrating by parts in each inner product and noticing that the term involving S is trivial because $\overline{f}$ is divergence-free, yields the following expression.

$$\begin{aligned} T_n &= (+\alpha^2)^{N-n+1}\sum_{k=1}^{N-n+1}\Big[\left(\nabla\overline{f}, [(\mathbf{curl}^{2N-2n+2-k}\phi_n)(\mathbf{curl}^k\phi_n)]\right)_{D_N} + \\ &\qquad\qquad \left(\nabla\overline{f}, [(\mathbf{curl}^k\phi_n)(\mathbf{curl}^{2N-2n+2-k}\phi_n)]\right)_{D_N}\Big] \\ &\quad - (\alpha^2)^{N-n+1}\left(\nabla\overline{f}, [(\mathbf{curl}^{N-n+1}\phi_n)(\mathbf{curl}^{N-n+1}\phi_n)]\right)_{D_N} \end{aligned}$$

To see how to estimate these terms, write, for example, with ℓ and m integers,

$$\begin{aligned}
&\left(\nabla\overline{f}, \left[(\mathbf{curl}^{\ell}\phi_n)\,(\mathbf{curl}^{m}\phi_n)\right]\right)_{D_N}\\
&\qquad = \int \nabla\overline{f}\cdot D_N\left((\mathbf{curl}^{\ell}\phi_n)(\mathbf{curl}^{m}\phi_n)\right)\\
&\qquad = \int D_N^{1/2}\nabla\overline{f}\cdot D_N^{1/2}\left((\mathbf{curl}^{\ell}\phi_n)(\mathbf{curl}^{m}\phi_n)\right)\\
&\qquad \le \left\|D_N^{1/2}\nabla\overline{f}\right\|_\infty \sum_{i,j}\int |D_N^{1/2}(\mathbf{curl}^{\ell}\phi_n)_i(\mathbf{curl}^{m}\phi_n)_j|\\
&\qquad \le 9\left\|D_N^{1/2}\nabla\overline{f}\right\|_\infty \left\|\mathbf{curl}^{\ell}\phi_n\right\|_{D_N}\left\|\mathbf{curl}^{m}\phi_n\right\|
\end{aligned}$$

and $\|\cdot\| \le \|\cdot\|_{D_N}$, so that

$$|T_n| \le 18(\alpha^2)^{N-n+1}\left\|{D_N}^{1/2}\nabla\overline{f}\right\|_\infty \sum_{k=1}^{N-n+1}\left\|\mathbf{curl}^{2N-2n+2-k}\phi_n\right\|_{D_N}\left\|\mathbf{curl}^{k}\phi_n\right\|_{D_N}$$

where a factor of two arises because there is one term when $k = N-n+1$ and two terms otherwise. Distributing α into the sum and recalling that $\phi_n = (-\alpha^2\triangle)^n(A^{-1})^{N+1}w$ and $\triangle w = -\mathbf{curl}^2 w$, since w is divergence-free, produces the two expressions

$$\begin{aligned}
\left\|(\alpha\,\mathbf{curl})^{2N-2n+2-k}\phi_n\right\|_{D_N} &= \left\|(\alpha\,\mathbf{curl})^{2N-2n+2-k}(A^{-1})^{N+1-n}(-\alpha^2\triangle A^{-1})^n w\right\|_{D_N},\\
\left\|(\alpha\,\mathbf{curl})^{k}\phi_n\right\|_{D_N} &= \left\|(\alpha\,\mathbf{curl})^{k}(A^{-1})^{N+1-n}(-\alpha^2\triangle A^{-1})^n w\right\|_{D_N}.
\end{aligned}$$

Applying Lemma 2.2 to each of these expressions yields the estimate

$$|T_n| \le 18(N+1-n)\left\|{D_N}^{1/2}\nabla\overline{f}\right\|_\infty \left(\frac{1}{2}\right)^{N+1-n}\|w\|_{D_N}^2. \tag{4.8}$$

Combining (4.8) and (4.7) yields the estimate

$$|(\overline{f}, (I-A^{-1})^{N+1}w\times(\nabla\times w))_{D_N}| \le 18(N+1)\left\|{D_N}^{1/2}\nabla\overline{f}\right\|_\infty \|w\|_{D_N}^2,$$

since the sum of the binomial coefficients is a power of 2.

The first term in (4.5) is similarly estimated[1] as

$$|(\overline{f}, w\times(\nabla\times w))_{D_N}| \le 3\left\|{D_N}^{1/2}\nabla\overline{f}\right\|_\infty \|w\|_{D_N}^2$$

so that the convection term can be estimated as

$$|(\overline{f}, D_N\overline{w}\times(\nabla\times w))_{D_N}| \le 3(1+6(N+1))\left\|{D_N}^{1/2}\nabla\overline{f}\right\|_\infty \|w\|_{D_N}^2$$

Finally, taking time averages yields the following estimate.

$$|(\overline{f}, \langle D_N\overline{w}\times(\nabla\times w)\rangle_{[0,T]})_{D_N}| \le 3(1+6(N+1))\frac{\overline{F}}{L_N}U_N^2 \tag{4.9}$$

[1] For a 3×3 matrix M and vector v, $|M_{ij}v_iv_j| \le \|M\|_\infty \sum_{i,j}|v_i|\,|v_j| \le \|M\|_\infty (1/2)\sum_{i,j}(|v_i|^2 + |v_j|^2) \le 3\|M\|_\infty \|v\|^2$.

Estimates of the viscous and convection terms can be put into (4.4). Taking the limit superior as $T \to \infty$ of both sides gives

$$(f,\overline{f})_{D_N} \le 3(1+6(N+1)\frac{\overline{F}_N}{L_N}U^2 + \sqrt{\nu\left(\frac{\overline{F}_N^2}{L_N^2} + \alpha^2\frac{\overline{F}_N^2}{L_N^4}\right)}\sqrt{\varepsilon_{\alpha,\mathrm{N}}(\langle w\rangle)}\,. \tag{4.10}$$

To interchange the time averaging and square roots, the generalized triangle inequality gives

$$\|\nabla\langle w\rangle\|_{D_N}^2 \le \langle\|\nabla w\|_{D_N}^2\rangle \text{ , and } \|\triangle\langle w\rangle\|_{D_N}^2 \le \langle\|\triangle w\|_{D_N}^2\rangle. \tag{4.11}$$

so that $\sqrt{\varepsilon_{\alpha,\mathrm{N}}(\langle w\rangle)} \le \sqrt{\langle\varepsilon_{\alpha,\mathrm{N}}(w)\rangle}$. The definitions of $\overline{F}_N$ and L yield $\nu\left\|\nabla\overline{f}\right\|_{D_N}^2 \le \nu\frac{\overline{F}_N^2}{L_N^2}$, and $\nu\alpha^2\left\|\triangle\overline{f}\right\|_{D_N}^2 \le \nu\alpha^2\frac{\overline{F}_N^2}{L_N^4}$.

Using these bounds in (4.10) gives

$$(f,\overline{f})_{D_N} \le 3(1+6(N+1))\frac{\overline{F}_N}{L_N}U_N^2 + \sqrt{\nu\left(\frac{\overline{F}_N^2}{L_N^2} + \alpha^2\frac{\overline{F}_N^2}{L_N^4}\right)}\sqrt{\langle\varepsilon_{\alpha,\mathrm{N}}(w)\rangle}\,.$$

Noting that

$$\begin{aligned}(f,\overline{f})_{D_N} &= \left\|\overline{f}\right\|_{D_N}^2 + \alpha^2\left\|\nabla\overline{f}\right\|_{D_N}^2 \\ &\ge \left\|\overline{f}\right\|_{D_N}^2 = \overline{F}_N^2\end{aligned}$$

so that $\overline{F}_N^2$ can replace the expression $(f,\overline{f})_{D_N}$ on the left, yields

$$\overline{F}_N^2 \le 3(1+6(N+1))\frac{\overline{F}_N}{L_N}U_N^2 + \sqrt{\nu\left(\frac{\overline{F}_N^2}{L_N^2} + \alpha^2\frac{\overline{F}_N^2}{L_N^4}\right)}\sqrt{\langle\varepsilon_{\alpha,\mathrm{N}}(w)\rangle}\,. \tag{4.12}$$

Dividing (4.12) through by $\overline{F}_N$, multiplying by U_N and inserting the first basic estimate (4.3) $\langle\varepsilon_{\alpha,\mathrm{N}}(w)\rangle \le \overline{F}_N U_N$ on the left side gives

$$\begin{aligned}\langle\varepsilon_{\alpha,\mathrm{N}}(w)\rangle &\le \overline{F}_N U_N \\ &\le 3(1+6(N+1))\frac{1}{L_N}U_N^3 + U_N\sqrt{\nu\left(\frac{1}{L_N^2} + \alpha^2\frac{1}{L_N^4}\right)}\sqrt{\langle\varepsilon_{\alpha,\mathrm{N}}(w)\rangle}\end{aligned} \tag{4.13}$$

Thus, by Young's inequality

$$\langle\varepsilon_{\alpha,\mathrm{N}}(w)\rangle \le 6\frac{1+6(N+1)}{L_N}U_N^3 + U_N^2\nu\left(\frac{1}{L_N^2} + \alpha^2\frac{1}{L_N^4}\right) \tag{4.14}$$

and Theorem 4.1 is proved. □

Remark 4.1. *(4.12) and (4.14) can be combined to relate $\overline{F}_N$ to U_N.*

$$\overline{F}_N \le 2\frac{U_N^2}{L_N}\left((18N+21) + \frac{1}{\mathrm{Re}_N}(1+\frac{\alpha^2}{L_N^2})\right). \tag{4.15}$$

5. Bounds on helicity dissipation rates. This section considers bounds on the time averaged helicity dissipation rate for the NS-alpha-deconvolution family. The keys that make the proof work are the fact that the solution and its derivatives are bounded in time and the estimate (4.15).

Foias, Holm and Titi [FHT01] present helicity conservation for NS-alpha and [R08], [MR08] discuss the helicity balance equation (5.1) in the deconvolution case. This equation is the direct consequence of taking the L^2 inner product of the model equation (1.2) with the vorticity $\nabla\times w$ and taking account of the vector identity (2.8).

$$\frac{1}{2}H(w)(T)+\int_0^T \nu(\nabla\times w,\nabla\times(\nabla\times w))dt=\frac{1}{2}H(w_0)+\int_0^T(\nabla\times f,w)dt. \tag{5.1}$$

The second term is thus the helicity dissipation rate, given by

$$\gamma(w):=\nu(\nabla\times w,\nabla\times(\nabla\times w)). \tag{5.2}$$

Dividing (5.1) by $T\to\infty$ will yield an estimate for $\langle\gamma\rangle$ if $H(w)$ and w are bounded in time. A careful reading of the proof of Theorem 3 in Foias, Holm and Titi, [FHT02], where the quantity $k_2(t)$ is uniformly bounded for large time, shows uniform boundedness in time of w and $\nabla\times w$, and hence helicity, for the NS-alpha case. These bounds are extended to the $N>0$ case in [MS08], with the following result.

Proposition 5.1. *For all $N=0,1,\dots,$ solutions w to (1.2) with $\alpha>0$, with sufficiently smooth forcing, and with finite initial energy and helicity have helicity bounded uniformly in time.*

We are now in a position to present the main theorem in this section.

Theorem 5.2. *If w is a solution of (1.2) for some fixed $N\geq 0$ and $\alpha>0$,*

$$|\langle\gamma(w)\rangle|\leq 2(1+\frac{\alpha^2}{L_N^2})\left((18N+21)+\frac{1}{\mathrm{Re}_N}(1+\frac{\alpha^2}{L_N^2})\right)\frac{U_N^3}{L_N^2}. \tag{5.3}$$

Proof. Divide the helicity balance relation (5.1) by $T\to\infty$. Because helicity is bounded, both helicity terms drop out, leaving

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\gamma(w)(t)dt=\lim_{T\to\infty}\frac{1}{T}\int_0^T(\nabla\times f,w)dt.$$

We will employ the estimate (4.15). To do so write $(\nabla\times f,w)=(\nabla\times\overline{f},Aw)=(\nabla\times\overline{f},w)-\alpha^2(\nabla\times\triangle\overline{f},w)$, and recall that the L^2 norm is dominated by the weighted norm. Hence

$$|\langle(\nabla\times f,w)\rangle|\leq(\left\|\nabla\times\overline{f}\right\|_{D_N}+\alpha^2\left\|\nabla\times\triangle\overline{f}\right\|_{D_N})\left\|w\right\|_{D_N}$$

so that

$$|\langle\gamma(w)\rangle|\leq\frac{\overline{F}_N}{L_N}U_N(1+\frac{\alpha^2}{L_N^2}).$$

Inserting the bound (4.15) yields (5.3). ∎

6. Conclusions. Similarity theories of cascades in homogeneous, isotropic turbulence are based on several assumptions which have yet to be verified directly from the NSE. Nevertheless, the predictions of these theories have been observed in many turbulent flows in nature. As analytic understanding advances, many of these predictions have also been proven directly from the Navier-Stokes equations.

The correctness of the predictions of turbulence models, however, can be unclear and simulations based on those models can be even more so. We have considered the energy and helicity dissipation rates of general solutions of the NS-alpha-deconvolution models. Rigorous upper bounds of the time averaged energy and helicity dissipation rates are derived which agree with those proven for energy for the NSE and estimated by similarity theories in homogeneous, isotropic turbulence. This analysis is based on a rigorous understanding of physical integral invariants of flow models and their corresponding dissipation rates. It gives important analytic insight into the reliability of models and the predictions coming from them. NS-alpha-deconvolution models have a systematic mathematical derivation which is reflected in their high accuracy, exact conservation ($\nu = 0$) of helicity and a model energy and the verified correctness of their predictions of time averaged energy and helicity dissipation rates ($\nu > 0$).

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