SUPPLEMENTARY MATERIALS: COMPETITION BETWEEN TRANSIENTS IN THE RATE OF APPROACH TO A FIXED POINT

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1. Eigenvector Configurations and Regions of Tolerance in the Planar Linear Case. Consider the linear system

$$\dot{x} = Ax, \tag{1.1}$$

where $A \in M^{2\times 2}$, $x \in \mathbb{R}^{2+} = [0,\infty)\times[0,\infty)$. We will assume as in the main text:

(A1) Assume that there exists an asymptotically stable fixed point of (1.1) at the origin $(0,0)$, the eigenvalues of which are real and negative (to eliminate spirals and center directions).

From $(A1)$ the basin of attraction of $(0,0)$ is the entire space. Let $\Gamma_0 = \mathbb{R}^{2+}$, the positive quadrant. Let $\Gamma_0^+ = \{x(0) \in \Gamma_0 | x(t) \in \mathbb{R}^{2+}$ for all $t \geq 0\}$, namely the collection of trajectories that converge to the origin without leaving $\mathbb{R}^{2+}$. Let $\phi(t) = (\phi_1(t), \phi_2(t))$ and $\psi(t) = (\psi_1(t), \psi_2(t))$ be two solutions to the initial value problem of (1.1).

(A2) Assume that $\phi(t), \psi(t) \in \Gamma_0^+$.

(A3) Assume that $\psi_1(0) \geq \phi_1(0)$.

We consider eigenvector configurations that accommodate solutions that satisfy the nonnegativity requirement (A2). There are four such configurations, and each is displayed in a panel of Figure 1.1. For each configuration, we subdivide the positive quadrant into regions and then, for (x_r, y_r) in each region, determine precisely which locations for (x_p, y_p) will lead to tolerance and which will not. The results for all the eigenvector configurations shown in Figure 1.1 are summarized in Table 1.1 and are illustrated in the figures referenced in the table. We now explain how to identify the regions of tolerance given an initial condition (x_r, y_r), using eigenvector configuration (a) seen in the top left panel of Figure 1.1 as an example.

For eigenvector configuration (a), there are three regions in which to consider initial conditions:

- **REGION 1a**: (x_r, y_r) on the x-axis
- **REGION 2a**: (x_r, y_r) in the first quadrant below the slow eigenvector v and above the x-axis
- **REGION 3a**: (x_r, y_r) in the first quadrant above the eigenvector v

REGION 1a: First, we look at the case when the initial condition is on the x-axis. In the top left panel of Figure 1.2, an arbitrary point on the x-axis is shown in the context of eigenvector configuration (a), with lines drawn (portions dashed), showing the addition of scalar multiples of the two eigenvectors to attain the point

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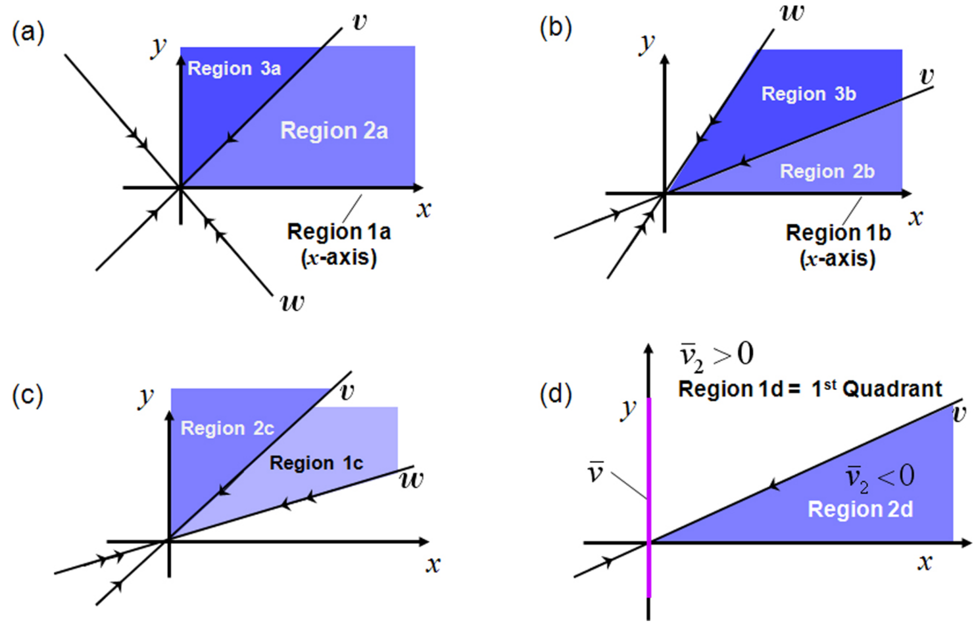

FIG. 1.1. *Regions of initial conditions (x_r, y_r) in the first quadrant for relevant eigenvector configurations, (a)-(d). Note that we label the slow eigenvector v with one arrow and the fast eigenvector w with two arrows. In (d), a configuration when $\bar{v}_1 = 0$ is shown. The figure displays two regions of interest for the configuration, depending on the sign of $\bar{v}_2$. For $\bar{v}_2 > 0$, the entire first quadrant is relevant. When $\bar{v}_2 < 0$, only the slice of the first quadrant between the x-axis and v is relevant.*

(x_r, y_r).We refer to these lines as the c_1-line and c_2-line. In this case, they divide the first quadrant into three different subregions. We consider the portions of these regions that lie to the right of the line $x = x_r$, as shown in the top right panel of Figure 1.2.

Recall from Section 3.1 of the main text that the P trajectory's initial condition (x_p, y_p) was expressed as $(x_p, y_p) = d_1 v + d_2 w$. For all (x_p, y_p) in a given subregion, there is a corresponding relationship between d_1, d_2 and c_1, c_2. Using this relationship, we determine if there exists a region where the criteria $c_1 > d_1$ and $c_2 < d_2$ of Proposition 3.4 (in the main text) and the initial condition criterion $(x_p \geq x_r)$ are all satisfied. For any (x_p, y_p) in such a region, tolerance will occur, while for (x_p, y_p) not in such a region, tolerance will not occur. In fact, for eigenvector configuration (a), if (x_r, y_r) is on the x-axis, then there are no subregions in the first quadrant where both $c_1 > d_1$ and $c_2 < d_2$ hold. In particular, in $\mathbb{I}_{1a}$, $c_1 < d_1$ and $c_2 > d_2$, and in $\mathbb{III}_{1a}$, $c_1 < d_1$ and $c_2 < d_2$; see Figure 1.2. Thus, there exist no (x_p, y_p) that produce tolerance.

Next, consider an initial condition (x_r, y_r) in **REGION 2a**. The middle left panel of Figure 1.2 shows an arbitrary point in this region, with lines drawn (portions dashed), showing the addition of the two eigenvectors to attain the point (x_r, y_r). The middle right panel of Figure 1.2 shows the subregions formed in the first quadrant by the c_1-line and c_2-line. Note again that these subregions only include points to the right of the line $x = x_r$. In general, we follow the convention of truncating these subregions to ensure that (A3) is satisfied. In this case, if $(x_p, y_p) \notin \mathbb{I}_{2a}$, then the conditions of Proposition 4.1 fail and tolerance will not occur. In contrast, for

Eigenvector Configuration:	If (x_r, y_r) is in Region:	Then, tolerance is produced by (x_p, y_p) in:	Figure Reference:
(a) Figure 1.1a	1a	None	Figure 1.2 (top)
	2a	Region $\mathbb{I}_{2a}$	Figure 1.2 (middle)
	3a	Region $\mathbb{I}_{3a}$	Figure 1.2 (bottom)
(b) Figure 1.1b	1b	Region $\mathbb{I}_{1b}$	Figure 1.3 (top)
	2b	Region $\mathbb{I}_{2b}$	Figure 1.3 (middle)
	3b	Region $\mathbb{I}_{3b}$	Figure 1.3 (bottom)
(c) Figure 1.1c	1c	Region $\mathbb{I}_{1c}$	Figure 1.4 (top)
	2c	Region $\mathbb{I}_{2c}$	Figure 1.4 (bottom)
(d) Figure 1.1d	1d if $(\bar{v}_2 > 0)$	Region $\mathbb{I}_{1d}$	Figure 1.5 (top)
	2d if $(\bar{v}_2 < 0)$	Region $\mathbb{I}_{2d}$	Figure 1.5 (bottom)

TABLE 1.1
Summary of tolerance results for eigenvector configurations shown in Figure 1.1

$(x_p, y_p) \in \mathbb{I}_{2a}$, we have that $c_1 > d_1$ and $c_2 < d_2$, satisfying the conditions of Proposition 4.1. Hence, for eigenvector configuration (a), if (x_r, y_r) is in the first quadrant below the slow eigenvector v (but not on the x-axis), then tolerance will be exhibited precisely for all (x_p, y_p) in the green region labeled $\mathbb{I}_{2a}$.

The same strategy demonstrated above can be applied to Region 3a as well as Regions 1b-3b and 1c-2c in eigenvector configurations (b) (Figure 1.1(b)) and (c) (Figure 1.1(c)), respectively. As before, in each case, the c_1-line, c_2-line, and the line $x = x_r$ partition the first quadrant into subregions, as shown in Figures 1.2-1.4. In general, green subregions are tolerance regions, where both $c_1 > d_1$ and $c_2 < d_2$, satisfying Proposition 4.1. Nontolerance subregions are shaded red or blue, where red subregions denote the area where $c_1 < d_1$ and $c_2 > d_2$ and blue subregions denote the area where $c_1 < d_1$ and $c_2 < d_2$.

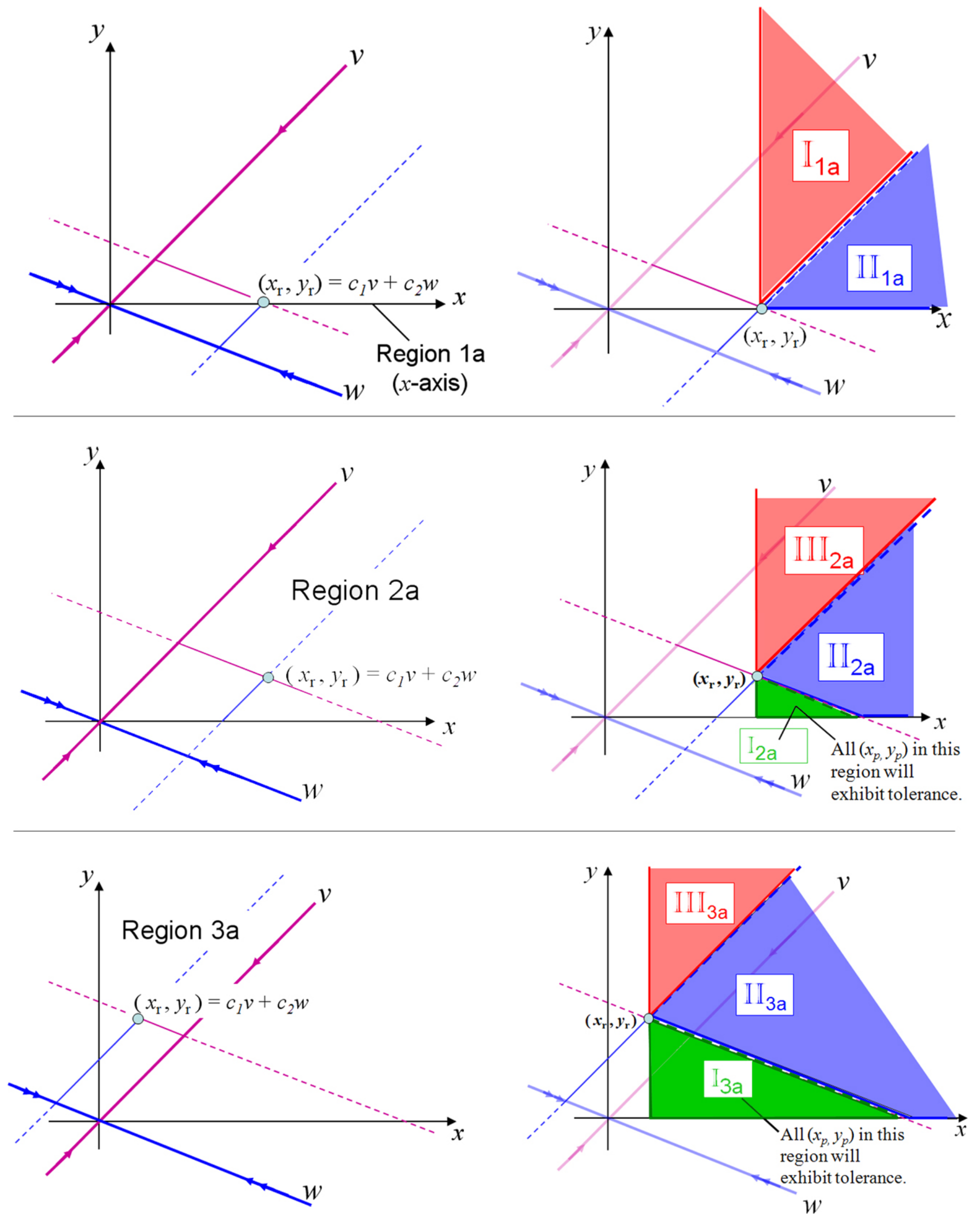

FIG. 1.2. *Left Side: Eigenvector configuration (a) with an arbitrary initial condition (x_r, y_r) labeled in Region 1a-3a. Right Side: The first quadrant partitioned into several different subregions by the line $x = x_r$ and the c_1-and c_2-lines associated with the point $(x_r, y_r) = c_1v + c_2w$ lying in one of the initial regions 1a-3a. Nontolerance subregions are shaded red or blue, where red subregions denote the area where $c_1 < d_1$ and $c_2 > d_2$ and blue subregions denote the area where $c_1 < d_1$ and $c_2 < d_2$.*

To finish our analysis, we examine eigenvector configuration (d), shown in Figure 1.1(d), corresponding to Case 2 in Section 3.1 of the main text. Because $\bar{v}$ is a generalized eigenvector, we can choose the first component, $\bar{v}_1$, to be zero, as noted in the main text, in the proof of Proposition 3.5. However, the sign of $\bar{v}_2$ may be negative or positive. In each case, there is one relevant region in which to consider initial conditions to explore the existence of tolerance: (x_r, y_r) in the first quadrant if $\bar{v}_2 > 0$ (Region 1d) or in the first quadrant below v if $\bar{v}_2 < 0$ (Region 2d). The conditions for tolerance to exist are different from those of eigenvector configurations (a)-(c), so we handle this case separately. The conclusion regarding tolerance for this case was given by Proposition 3.5 (in the main text), which shows that it is necessary and sufficient that $c_1 \leq d_1$ and $c_2 > d_2$ for $(A3)$ to hold and tolerance to be exhibited in (1.1). These conditions are satisfied precisely for those (x_p, y_p) in $\mathbb{I}_{1d}$ (if $\bar{v}_2 > 0$) and in $\mathbb{I}_{2d}$ (if $\bar{v}_2 < 0$). These are the green subregions labeled in the top and bottom right panels of Figure 1.5, whereas the blue subregions indicate where $c_2 \leq d_2$.

2. Proof of Tolerance in 3D Linear Case. Here we formulate conditions for the existence of tolerance in a three dimensional linear system, given by

$$\dot{x} = Ax, \tag{2.1}$$

where $A \in M^{3\times 3}$, $x \in \mathbb{R}^{3+} = [0, \infty) \times [0, \infty) \times [0, \infty)$. Throughout this section, we will assume, similarly to before:

(A1) $(0, 0, 0)$ is an asymptotically stable fixed point of (2.1), the eigenvalues of which are real and negative.
(A2) Assume that $\phi(t), \psi(t) \in \Gamma_0^+$.
(A3) Assume that $\psi_1(0) \geq \phi_1(0)$.

Assume that A has three distinct negative eigenvalues (and hence, three linearly independent eigenvectors). Without loss of generality assume

$$\lambda_3 < \lambda_2 < \lambda_1 < 0 \text{ and let } V = \{v_1, v_2, v_3\}$$

be the set of linearly independent eigenvectors, so that (λ_i, v_i) is an eigenpair. We may write

$$\begin{aligned} \phi_1(t) &= c_1 v_{11} e^{\lambda_1 t} + c_2 v_{21} e^{\lambda_2 t} + c_3 v_{31} e^{\lambda_3 t} \text{ and} \\ \psi_1(t) &= d_1 v_{11} e^{\lambda_1 t} + d_2 v_{21} e^{\lambda_2 t} + d_3 v_{31} e^{\lambda_3 t}, \end{aligned}$$

where v_{i1} is the first component of the i^{th} eigenvector. We assume each v_{i1}, $i = 1, 2, 3$, is nonzero; otherwise, the problem reduces to a dimension less than three. Consequently, without loss of generality, we may arrange to have $v_{i1} > 0$ for $i = 1, 2, 3$. Consider the difference

$$\phi_1(t) - \psi_1(t) = (c_1 - d_1) v_{11} e^{\lambda_1 t} + (c_2 - d_2) v_{21} e^{\lambda_2 t} + (c_3 - d_3) v_{31} e^{\lambda_3 t}. \tag{2.2}$$

In Section 3.1 of the main text, based on inspection of equation (3.11), we established that $c_1 - d_1 > 0$ was a sufficient condition for eventual tolerance. Now, we consider the case in three dimensions when $c_1 \leq d_1$. Obviously, if $c_2 \leq d_2$ and $c_3 \leq d_3$

as well, then there will be no tolerance, since $\phi_1(t) - \psi_1(t) \le 0$ for all $t \ge 0$ in such a case. Also note that if $c_1 = d_1$, then the 3-D case reduces to the 2-D case. So, for the 3-D case, we explore what happens when $c_1 < d_1$, with at least one $c_i > d_i$ for $i = 2, 3$. Factoring out $e^{\lambda_1 t}$ from the right hand side of (2.2) gives

$$\begin{aligned}\phi_1(t) - \psi_1(t) = e^{\lambda_1 t}\big[v_{11}(c_1 - d_1) &+ v_{21}(c_2 - d_2)e^{(\lambda_2 - \lambda_1)t} \\ &+ v_{31}(c_3 - d_3)e^{(\lambda_3 - \lambda_1)t}\big].\end{aligned} \tag{2.3}$$

Note that $\phi_1(t) - \psi_1(t) > 0$ if and only if

$$\frac{\phi_1(t) - \psi_1(t)}{e^{\lambda_1 t}} > 0. \tag{2.4}$$

Hence, we attempt to find out when there exists a maximum for the left hand side of (2.4), over $t \in [0, \infty)$, for which (2.4) holds. Note that $\phi_1(0) - \psi_1(0) < 0$, by assumption $(A3)$, that $\phi_1(t) - \psi_1(t) \to 0$ as $t \to \infty$, and that $(\phi_1(t) - \psi_1(t))/e^{\lambda_1 t} \to v_{11}(c_1 - d_1) < 0$ as $t \to \infty$. Now, we take the derivative of the left hand side of (2.4) with respect to t (denoted by $'$) to obtain

$$\left[\frac{\phi_1(t) - \psi_1(t)}{e^{\lambda_1 t}}\right]' = v_{21}(\lambda_2 - \lambda_1)(c_2 - d_2)e^{(\lambda_2 - \lambda_1)t} + v_{31}(\lambda_3 - \lambda_1)(c_3 - d_3)e^{(\lambda_3 - \lambda_1)t}.$$

At most, there is one unique solution, say at $t = \hat{t}$, where $\left[\frac{\phi_1(t) - \psi_1(t)}{e^{\lambda_1 t}}\right]' = 0$. Specifically,

$$-\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1} v_{21} e^{(\lambda_2 - \lambda_1)\hat{t}}(c_2 - d_2) = v_{31} e^{(\lambda_3 - \lambda_1)\hat{t}}(c_3 - d_3). \tag{2.5}$$

Using (2.5) we can now rewrite (2.3):

$$\frac{\phi_1(\hat{t}) - \psi_1(\hat{t})}{e^{\lambda_1 \hat{t}}} = v_{11}(c_1 - d_1) + v_{21}(c_2 - d_2)e^{(\lambda_2 - \lambda_1)\hat{t}}\left[1 - \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}\right]. \tag{2.6}$$

Thus, if $\hat{t}$ satisfies the inequality

$$v_{21}(c_2 - d_2)e^{(\lambda_2 - \lambda_1)\hat{t}}\left[1 - \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}\right] > -v_{11}(c_1 - d_1) \tag{2.7}$$

and assumption $(A3)$ is satisfied, then tolerance is exhibited at $\hat{t}$.

From (2.5), $\hat{t}$ satisfies

$$-\left(\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}\right)\left(\frac{v_{21}}{v_{31}}\right)\left(\frac{c_2 - d_2}{c_3 - d_3}\right) = e^{(\lambda_3 - \lambda_2)\hat{t}},$$

implying that

$$\hat{t} = \frac{\ln\left[-\left(\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right)\left(\frac{v_{21}}{v_{31}}\right)\left(\frac{c_2-d_2}{c_3-d_3}\right)\right]}{\lambda_3-\lambda_2} \equiv T_3. \tag{2.8}$$

Note that $T_3 > 0$ when

$$0 < -\left(\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right)\left(\frac{v_{21}}{v_{31}}\right)\left(\frac{c_2-d_2}{c_3-d_3}\right) < 1, \tag{2.9}$$

since $(\lambda_3 - \lambda_2)$ (the denominator of (2.8)) is negative. Further note

$$-\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1} < 0 \text{ and } \frac{v_{21}}{v_{31}} > 0,$$

implying $\frac{c_2-d_2}{c_3-d_3} < 0$ is needed in order for T_3 to be positive and further implying that either

(A) $c_2 - d_2 < 0$ and $c_3 - d_3 > 0$, or
(B) $c_2 - d_2 > 0$ and $c_3 - d_3 < 0$.

If $c_2 - d_2 < 0$ then $\frac{\phi_1(\hat{t})-\psi_1(\hat{t})}{e^{\lambda_1 \hat{t}}} < 0$, (see (2.6)), implying no tolerance. Thus, the only hope of tolerance at T_3 in the case where $c_1 < d_1$ is if $c_2 - d_2 > 0$, which is case B above, giving that $c_3 - d_3 < 0$ is also necessary. In summary,

Proposition 2.1. *Assume (A1), (A2), (A3), $\lambda_3 < \lambda_2 < \lambda_1 < 0$, $v_{i1} > 0$ for $i = 1, 2, 3$ of eigenvectors v_i, and $c_1 < d_1$. There exists a time value, $T_3 > 0$ at which (2.1) produces tolerance if and only if*
(i) *T_3 is given by*

$$T_3 = \frac{\ln\left[-\left(\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right)\left(\frac{v_{21}}{v_{31}}\right)\left(\frac{c_2-d_2}{c_3-d_3}\right)\right]}{\lambda_3-\lambda_2},$$

(ii) *$c_2 - d_2 > 0$ and $c_3 - d_3 < 0$, and*
(iii) *T_3 satisfies (2.7):*

$$v_{21}(c_2-d_2)\left[-\left(\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right)\left(\frac{v_{21}}{v_{31}}\right)\left(\frac{c_2-d_2}{c_3-d_3}\right)\right]^{\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_2}}\left[1-\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right] > -v_{11}(c_1-d_1)$$

Proof. Let the assumptions in the statement of Proposition 2.1 be given.
(Sufficiency) Assume (i), (ii), and (iii) above. Then, the calculations prior to the statement of the proposition show sufficiency for tolerance to occur at T_3.

(Necessity) Assume there exists an $S > 0$ such that (2.1) produces tolerance. Then

$$\phi_1(S) - \psi_1(S) > 0 \text{ implying that}$$

$$\frac{\phi_1(S) - \psi_1(S)}{e^{\lambda_1 S}} > 0.$$

Consequently, since

1. $\frac{\phi_1(t)-\psi_1(t)}{e^{\lambda_1 t}}$ is continuous and differentiable on $[0,\infty)$,
2. $\left.\frac{\phi_1(t)-\psi_1(t)}{e^{\lambda_1 t}}\right|_{t=0} < 0$, and
3. $\lim_{t\to\infty} \frac{\phi_1(t)-\psi_1(t)}{e^{\lambda_1 t}} < 0$

there exists a $T^* > 0$ such that

$$\left.\left[\frac{\phi_1(t)-\psi_1(t)}{e^{\lambda_1 t}}\right]'\right|_{t=T^*} = 0$$

($\prime \equiv$ derivative with respect to t) and

$$\frac{\phi_1(T^*)-\psi_1(T^*)}{e^{\lambda_1 T^*}} > 0. \tag{2.10}$$

Moreover,

$$\frac{\phi_1(T^*)-\psi_1(T^*)}{e^{\lambda_1 T^*}} \geq \frac{\phi_1(S)-\psi_1(S)}{e^{\lambda_1 S}}.$$

The only solution of

$$\left[\frac{\phi_1(t)-\psi_1(t)}{e^{\lambda_1 t}}\right]' = 0$$

is at

$$t = T^* = \ln\left[-\left(\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right)\left(\frac{v_{21}}{v_{31}}\right)\left(\frac{c_2-d_2}{c_3-d_3}\right)\right]\lambda_3-\lambda_2,$$

which is the formula for T_3 given by (i). Since $T_3 = T^* > 0$, (2.9) holds, implying (ii) must hold. In addition, since $T_3 = T^*$, (2.10) gives

$$\frac{\phi_1(T_3)-\psi_1(T_3)}{e^{\lambda_1 T_3}} > 0,$$

implying that (iii) (i.e. (2.7)) holds for $\hat{t} = T_3$.

□

Hence, we have found necessary and sufficient conditions for the existence of tolerance in a three dimensional linear system in the case when $c_1 < d_1$.

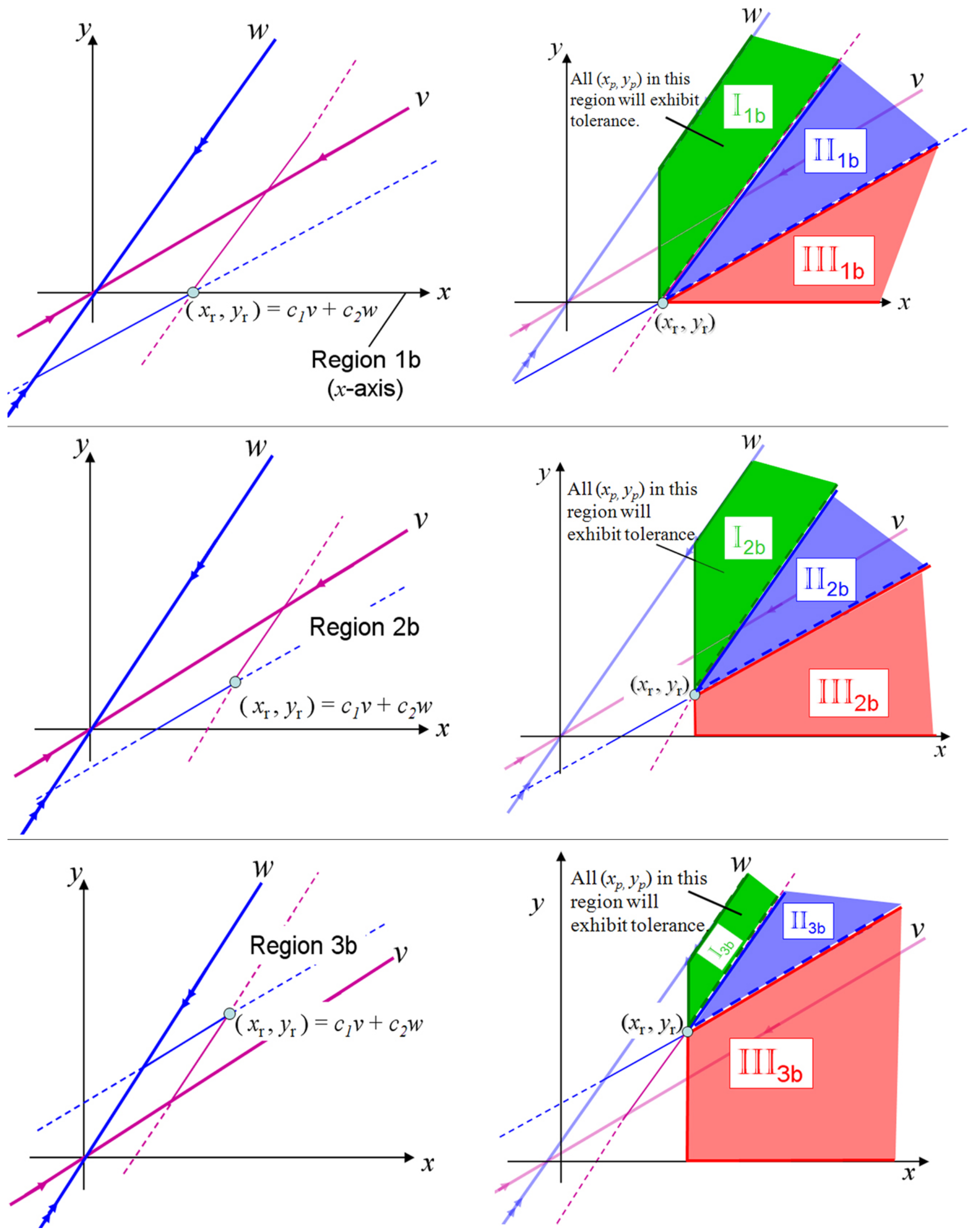

FIG. 1.3. *Left Side: Eigenvector configuration (b) with an arbitrary initial condition (x_r, y_r) labeled in Regions 1b-3b. Right Side: The first quadrant partitioned into several different regions by the line $x = x_r$ and the c_1-and c_2-lines associated with the point $(x_r, y_r) = c_1v + c_2w$ lying in one of the initial regions 1b-3b. Nontolerance subregions are shaded red or blue, where red subregions denote the area where $c_1 < d_1$ and $c_2 > d_2$ and blue subregions denote the area where $c_1 < d_1$ and $c_2 < d_2$.*

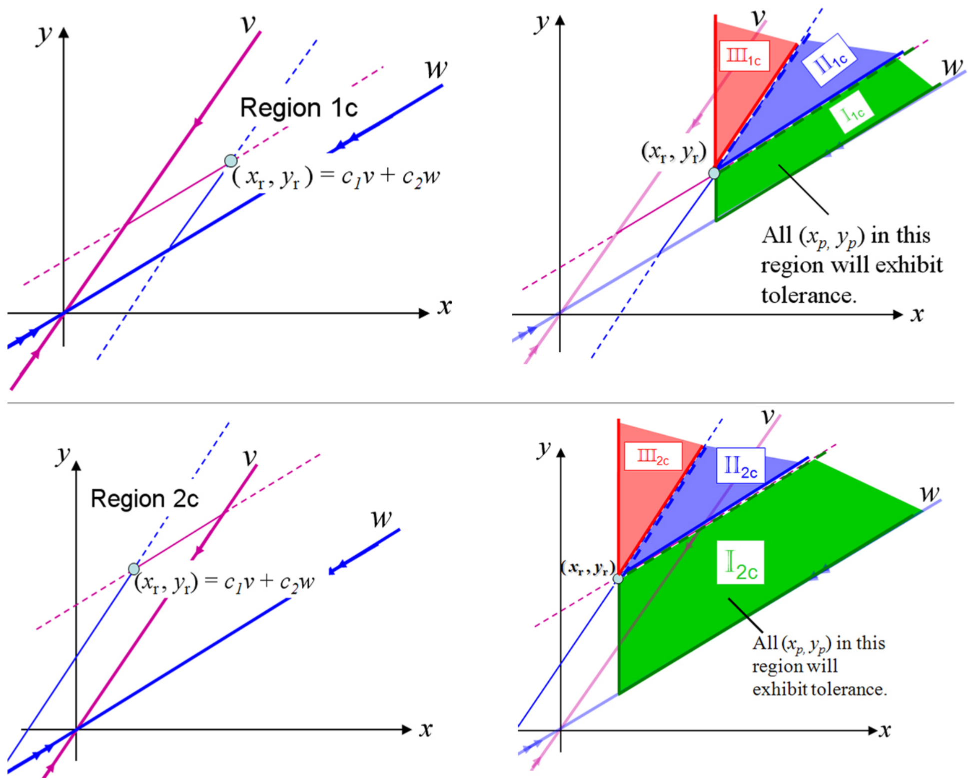

FIG. 1.4. *Left Side: Eigenvector configuration (c) with an arbitrary initial condition* (x_r, y_r) *labeled in Regions 1c-2c. Right Side: The first quadrant partitioned into several different regions by the line* $x = x_r$ *and the* c_1*-and* c_2*-lines associated with the point* $(x_r, y_r) = c_1v + c_2w$ *lying in one of the initial regions 1c-2c. Nontolerance subregions are shaded red or blue, where red subregions denote the area where* $c_1 < d_1$ *and* $c_2 > d_2$ *and blue subregions denote the area where* $c_1 < d_1$ *and* $c_2 < d_2$.

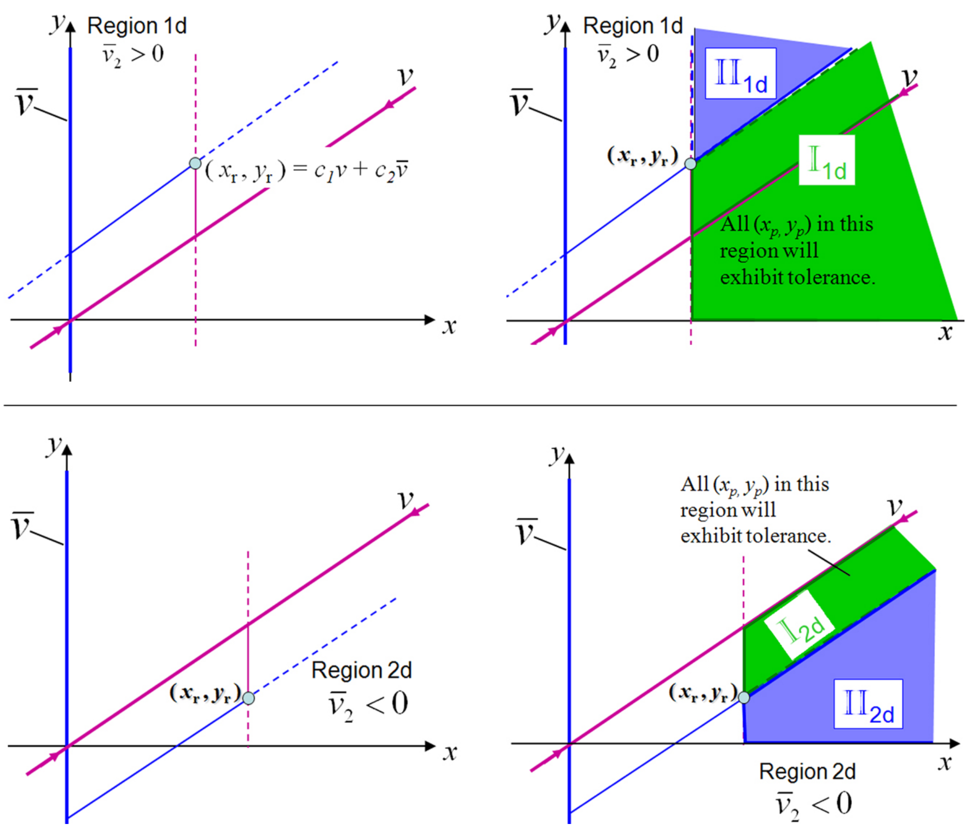

FIG. 1.5. *Top Left Panel: Eigenvector configuration (d) with an arbitrary initial condition (x_r, y_r) labeled in Region 1d (i.e. the first quadrant), which is the relevant region when $\bar{v}_2 > 0$. Top Right Panel: The first quadrant of eigenvector configuration (d) partitioned into four subregions by the line $x = x_r$ and the c_1- and c_2-lines associated with the point $(x_r, y_r) = c_1 v + c_2 \bar{v}$ lying in Region 1d, when $\bar{v}_2 > 0$. Bottom Left Panel: Eigenvector configuration (d) with an arbitrary initial condition (x_r, y_r) labeled in Region 2d, which is the relevant region when $\bar{v}_2 < 0$. Bottom Right Panel: Similar to the top right panel, except now the point $(x_r, y_r) = c_1 v + c_2 \bar{v}$ is in Region 2d, the relevant region when $\bar{v}_2 < 0$. Tolerance conditions ($c_1 \leq d_1$ and $c_2 > d_2$) are satisfied precisely for those (x_p, y_p) in $\mathbb{I}_{1d}$ (if $\bar{v}_2 > 0$) and in $\mathbb{I}_{2d}$ (if $\bar{v}_2 < 0$). These are the green subregions labeled in the top and bottom right panels, whereas the blue subregions indicate where $c_2 \leq d_2$.*