

# A weak solution and a multinumerics solution of the coupled Navier-Stokes and Darcy equations

Prince Chidyagwai and Béatrice Rivière \*

## Abstract

This paper introduces and analyzes a weak formulation of the coupling of incompressible Navier-Stokes equations with the porous media flow equations. Using continuous finite elements in the incompressible flow region and discontinuous finite elements in the porous medium, a numerical method is proposed. Existence and uniqueness results under small data condition of the numerical solution are proved. Optimal a priori error estimates are derived.

## 1 Introduction

There is an increasing interest in coupling incompressible flow and porous media flow. Applications of such complex phenomena can be found in geosciences (modeling of the interaction of rivers with groundwater) and in health sciences (modeling of blood flow and organs). In this work, we consider the coupling of the nonlinear Navier-Stokes equations with the Darcy equations. Non-homogeneous boundary conditions are imposed on the boundary of the porous medium. We first prove the well-posedness of a weak formulation. This generalizes the problem defined in [14] where homogeneous boundary conditions were assumed. We also propose a numerical scheme that couples the continuous finite element method with the Discontinuous Galerkin (DG) method. Because of legacy codes, multinumerics approaches are attractive. In addition, one can take advantage of the benefits of the different methods used in the subdomains. On one hand, classical finite elements are popular for computational fluid dynamics. On the other hand, the advantages of DG methods include the flexible use of mesh adaptivity and high order of approximation. The DG methods we consider here are called primal DG methods and they are variations of interior penalty methods. These methods encompass the non-symmetric interior penalty Galerkin method [23, 24, 18], the incomplete interior penalty Galerkin method [8] and the symmetric interior penalty Galerkin method [27, 2]. In [14], the coupled problem is approximated by totally discontinuous elements. In the linear case of Stokes coupled with Darcy, there exist in the literature analysis and implementation of several algorithms (a non-exhaustive list is [10, 9, 20, 25, 21, 22, 11, 16, 5]).

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , that is subdivided into two disjoint subdomains  $\Omega_1$  and  $\Omega_2$ . Let  $\Gamma_{12}$  denote the interface between the subdomains:  $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$ . We assume that  $\Gamma_{12}$  is a polygonal line. The flow in  $\Omega_1$  is incompressible and characterized by the Navier-Stokes equations:

$$-\nabla \cdot (2\mu D(\mathbf{u}_1) - p_1 \mathbf{I}) + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1 = \mathbf{f}_1, \quad \text{in } \Omega_1, \quad (1)$$

$$\nabla \cdot \mathbf{u}_1 = 0, \quad \text{in } \Omega_1, \quad (2)$$

$$\mathbf{u}_1 = 0, \quad \text{on } \partial\Omega_1 \setminus \Gamma_{12} = \Gamma_1. \quad (3)$$

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\*Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA, 15260. The authors acknowledge the support of NSF through the grant DMS 0506039 and of NIH through the grant P50-GM-53789-08.

The fluid velocity and pressure in  $\Omega_1$  are denoted by  $\mathbf{u}_1$  and  $p_1$  respectively. The coefficient  $\mu > 0$  is the fluid viscosity, the function  $\mathbf{f}_1$  is an external force acting on the fluid,  $\mathbf{I}$  is the identity matrix and the matrix  $\mathbf{D}(\mathbf{u}_1)$  is the stress tensor:

$$\mathbf{D}(\mathbf{u}_1) = \frac{1}{2}(\nabla \mathbf{u}_1 + \nabla \mathbf{u}_1^T). \quad (4)$$

The flow in  $\Omega_2$  is of Darcy type. We assume that the boundary  $\Gamma_2 = \partial\Omega_2 \setminus \Gamma_{12}$  is the union of two disjoint sets  $\Gamma_{2D}$  and  $\Gamma_{2N}$  on which Dirichlet and Neumann boundary conditions are imposed.

$$-\nabla \cdot \mathbf{K}\nabla p_2 = f_2, \text{ in } \Omega_2, \quad (5)$$

$$-\mathbf{K}\nabla p_2 = \mathbf{u}_2, \text{ in } \Omega_2, \quad (6)$$

$$p_2 = g_D, \text{ on } \Gamma_{2D}, \quad (7)$$

$$\mathbf{K}\nabla p_2 \cdot \mathbf{n}_2 = g_N, \text{ on } \Gamma_{2N}. \quad (8)$$

Similarly, the fluid velocity and pressure in  $\Omega_2$  are denoted by  $\mathbf{u}_2$  and  $p_2$  respectively. The function  $f_2$  is an external force acting on the fluid, the functions  $g_D$  and  $g_N$  are the prescribed value and flux respectively, the vector  $\mathbf{n}_2$  denotes the unit vector normal to  $\Gamma_2$  and the coefficient  $\mathbf{K}$  is a symmetric positive definite matrix uniformly bounded above and below. There exist constants  $\lambda_{\min} > 0$  and  $\lambda_{\max} > 0$  such that:

$$\text{a.e. } \mathbf{x} \in \Omega_2, \lambda_{\min} \mathbf{x} \cdot \mathbf{x} \leq \mathbf{K} \mathbf{x} \cdot \mathbf{x} \leq \lambda_{\max} \mathbf{x} \cdot \mathbf{x}. \quad (9)$$

The system of equations (1)-(8) is completed by interface conditions, corresponding to the continuity of the normal component of velocity, the balance of forces across the interface and the Beaver-Joseph-Saffman law. More details on the meaning of these conditions can be found in [4, 26, 19, 14]. Let  $\mathbf{n}_{12}$  be the unit normal vector to  $\Gamma_{12}$  directed from  $\Omega_1$  to  $\Omega_2$  and let  $\boldsymbol{\tau}_{12}$  be the unit tangent vector on  $\Gamma_{12}$ .

$$\mathbf{u}_1 \cdot \mathbf{n}_{12} = \mathbf{u}_2 \cdot \mathbf{n}_{12}, \quad (10)$$

$$((-2\mu\mathbf{D}(\mathbf{u}_1) + p_1\mathbf{I})\mathbf{n}_{12}) \cdot \mathbf{n}_{12} + \frac{1}{2}(\mathbf{u}_1 \cdot \mathbf{u}_1) = p_2, \quad (11)$$

$$\mathbf{u}_1 \cdot \boldsymbol{\tau}_{12} = -2\mu G(\mathbf{D}(\mathbf{u}_1)\mathbf{n}_{12}) \cdot \boldsymbol{\tau}_{12}. \quad (12)$$

We point out that in the coupling of the linear equations of Stokes with Darcy, the balance of forces (11) reduces to:

$$((-2\mu\mathbf{D}(\mathbf{u}_1) + p_1\mathbf{I})\mathbf{n}_{12}) \cdot \mathbf{n}_{12} = p_2.$$

Here in (11), the term  $\frac{1}{2}\mathbf{u}_1 \cdot \mathbf{u}_1$  corresponds to inertial forces and plays an important role in the mathematical analysis of the coupled problem.

The rest of the paper is as follows. A weak problem is defined and analyzed in Section 2. A multinumercs approach is proposed in Section 3. Theoretical error estimates are derived in Section 4. Conclusions are given in the last section.

## 2 Variational Formulation

Let  $H^s(\mathcal{O})$  be the usual Sobolev space of order  $s$  (see [1]) with norm  $\|\cdot\|_{H^s(\mathcal{O})}$ . We first lift the Dirichlet boundary condition (7). If  $g_D \in H^{1/2}(\Gamma_{2D})$ , there exists a function  $p_D \in H^1(\Omega_2)$  satisfying:

$$p_D = g_D, \text{ on } \Gamma_{2D}, \quad (13)$$

$$p_D = 0, \text{ on } \Gamma_{12}, \quad (14)$$

$$\|p_D\|_{H^1(\Omega_2)} \leq C_0 \|g_D\|_{H^{1/2}(\Gamma_{2D})}, \quad (15)$$

where  $C_0$  is a constant that only depends on  $\Omega_2$ . We now define the standard Sobolev spaces:

$$\begin{aligned}\mathbf{X}_1 &= \{\mathbf{v}_1 \in (H^1(\Omega_1))^2 : \mathbf{v}_1 = 0 \text{ on } \Gamma_1\}, \\ M_1 &= L^2(\Omega_1), \\ M_2 &= \{q_2 \in H^1(\Omega_2) : q_2 = 0 \text{ on } \Gamma_{2D}\}.\end{aligned}$$

We propose the following variational formulation:

$$(W) \begin{cases} \text{Find } \mathbf{u}_1 \in \mathbf{X}_1, p_1 \in M_1, p_2 = \varphi_2 + p_D, \text{ with } \varphi_2 \in M_2, \text{ s.t.} \\ \forall \mathbf{v}_1 \in \mathbf{X}_1, \forall q_2 \in M_2, \quad 2\mu(\mathbf{D}(\mathbf{u}_1), \mathbf{D}(\mathbf{v}_1))_{\Omega_1} + (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1, \mathbf{v}_1)_{\Omega_1} - (p_1, \nabla \cdot \mathbf{v}_1)_{\Omega_1} \\ + (\varphi_2 - \frac{1}{2} \mathbf{u}_1 \cdot \mathbf{u}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{G} (\mathbf{u}_1 \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{u}_1 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} + (\mathbf{K} \nabla \varphi_2, \nabla q_2)_{\Omega_2} \\ = (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} + (f_2, q_2)_{\Omega_2} - (\mathbf{K} \nabla p_D, \nabla q_2)_{\Omega_2} + (g_N, q_2)_{\Gamma_{2N}}, \forall q_1 \in M_1, \quad (\nabla \cdot \mathbf{u}_1, q_1)_{\Omega_1} = 0. \end{cases}$$

Here, we have used the notation  $(\cdot, \cdot)_{\mathcal{O}}$  for the  $L^2$  inner-product on a region  $\mathcal{O}$ . We recall the usual Cauchy-Schwarz and Young's inequalities:

$$\forall v, w \in L^2(\mathcal{O}), \quad |(v, w)_{\mathcal{O}}| \leq \|v\|_{L^2(\mathcal{O})} \|w\|_{L^2(\mathcal{O})}, \quad (16)$$

$$\forall a, b \in \mathbb{R}, \forall \delta > 0, \quad ab \leq \frac{\delta}{2} a^2 + \frac{1}{2\delta} b^2. \quad (17)$$

We also recall Poincaré and Korn's inequalities and trace and Sobolev inequalities: there exist constants  $\mathcal{P}_1, C_1, C_2, C_4$  and  $\mathcal{P}_4$ , that only depend on  $\Omega_1$ , and  $\mathcal{P}_2, C_3$  that only depend on  $\Omega_2$ , such that for all  $\mathbf{v} \in \mathbf{X}_1$ ,

$$\|\mathbf{v}\|_{L^2(\Omega_1)} \leq \mathcal{P}_1 \|\nabla \mathbf{v}\|_{L^2(\Omega_1)}, \quad \|\mathbf{v}\|_{L^4(\Omega_1)} \leq \mathcal{P}_4 \|\nabla \mathbf{v}\|_{L^2(\Omega_1)}, \quad (18)$$

$$\|\nabla \mathbf{v}\|_{L^2(\Omega_1)} \leq C_1 \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega_1)}, \quad (19)$$

$$\|\mathbf{v}\|_{L^2(\Gamma_{12})} \leq C_2 \|\nabla \mathbf{v}\|_{L^2(\Omega_1)}, \quad \|\mathbf{v}\|_{L^4(\Gamma_{12})} \leq C_4 \|\nabla \mathbf{v}\|_{L^2(\Omega_1)}, \quad (20)$$

and for all  $q \in M_2$ ,

$$\|q\|_{L^2(\Omega_2)} \leq \mathcal{P}_2 \|\nabla q\|_{L^2(\Omega_2)}, \quad (21)$$

$$\|q\|_{L^2(\Gamma_{2N})} \leq C_3 \|\nabla q\|_{L^2(\Omega_2)}; \quad (22)$$

moreover, owing to (9), for all  $q \in H^1(\Omega_2)$ :

$$\frac{1}{\sqrt{\lambda_{\max}}} \|\mathbf{K}^{1/2} \nabla q\|_{L^2(\Omega_2)} \leq \|\nabla q\|_{L^2(\Omega_2)} \leq \frac{1}{\sqrt{\lambda_{\min}}} \|\mathbf{K}^{1/2} \nabla q\|_{L^2(\Omega_2)}. \quad (23)$$

We first show that the variational formulation and the model problem are equivalent.

**Lemma 1.** *If  $(\mathbf{u}_1, p_1, p_2) \in \mathbf{X}_1 \times M_1 \times H^1(\Omega_2)$  satisfies (1)-(12), then it is also a solution to problem (W). The converse is also true.*

*Proof.* Let  $(\mathbf{u}_1, p_1, p_2) \in \mathbf{X}_1 \times M_1 \times H^1(\Omega_2)$  be a solution to (1)-(12). Multiply (1), (2) and (5) by test functions  $\mathbf{v}_1 \in \mathbf{X}_1, q_1 \in M_1$  and  $q_2 \in M_2$  respectively and use Green's theorem and boundary conditions:

$$\begin{aligned} & 2\mu(\mathbf{D}(\mathbf{u}_1), \mathbf{D}(\mathbf{v}_1))_{\Omega_1} - (p_1, \nabla \cdot \mathbf{v}_1)_{\Omega_1} + (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1, \mathbf{v}_1)_{\Omega_1} \\ & + ((-2\mu \mathbf{D}(\mathbf{u}_1) + p_1 \mathbf{I}) \mathbf{n}_{12}, \mathbf{v}_1)_{\Gamma_{12}} = (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1}, \end{aligned} \quad (24)$$

$$(\nabla \cdot \mathbf{u}_1, q_1) = 0, \quad (25)$$

$$(\mathbf{K} \nabla p_2, \nabla q_2)_{\Omega_2} + (\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} = (f_2, q_2)_{\Omega_2} + (g_N, q_2)_{\Gamma_{2N}}. \quad (26)$$

Rewriting  $\mathbf{v}_1 = (\mathbf{v}_1 \cdot \mathbf{n}_{12})\mathbf{n}_{12} + (\mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}$ , adding (24) and (26) and using the interface conditions, we obtain:

$$\begin{aligned} & 2\mu(\mathbf{D}(\mathbf{u}_1), \mathbf{D}(\mathbf{v}_1))_{\Omega_1} - (p_1, \nabla \cdot \mathbf{v}_1)_{\Omega_1} + (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1, \mathbf{v}_1)_{\Omega_1} + (\mathbf{K}\nabla p_2, \nabla q_2)_{\Omega_2} + (p_2 - \frac{1}{2}\mathbf{u}_1 \cdot \mathbf{u}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ & + \frac{1}{G}(\mathbf{u}_1 \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{u}_1 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} = (\mathbf{f}_1, \mathbf{v})_{\Omega_1} + (f_2, q_2)_{\Omega_2} + (g_N, q_2)_{\Gamma_{2N}}, \\ & (\nabla \cdot \mathbf{u}_1, q_1) = 0. \end{aligned}$$

We now define  $\varphi_2 = p_2 - p_D$  and remark that the trace  $p_2 = \varphi_2$  on  $\Gamma_{12}$  due to (14). We obtain the resulting equations:

$$\begin{aligned} & 2\mu(\mathbf{D}(\mathbf{u}_1), \mathbf{D}(\mathbf{v}_1))_{\Omega_1} - (p_1, \nabla \cdot \mathbf{v}_1)_{\Omega_1} + (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1, \mathbf{v}_1)_{\Omega_1} + (\mathbf{K}\nabla \varphi_2, \nabla q_2)_{\Omega_2} + (\varphi_2 - \frac{1}{2}\mathbf{u}_1 \cdot \mathbf{u}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ & + \frac{1}{G}(\mathbf{u}_1 \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{u}_1 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} = (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} + (f_2, q_2)_{\Omega_2} + (g_N, q_2)_{\Gamma_{2N}} - (\mathbf{K}\nabla p_D, \nabla q_2)_{\Omega_2}, \\ & (\nabla \cdot \mathbf{u}_1, q_1) = 0, \end{aligned}$$

which correspond to problem (W). Conversely, assume that  $(\mathbf{u}_1, p_1, p_2)$  is a solution to (W). By choosing appropriate test functions, we recover the equations (1), (2) and (5) in a distributional sense. First, take  $\mathbf{v}_1 \in \mathcal{D}(\Omega_1)$ ,  $q_1 = q_2 = 0$ . We recall that for any domain  $\mathcal{O}$ , the space  $\mathcal{D}(\mathcal{O})$  is the space of  $\mathcal{C}^\infty$  functions with compact support in  $\mathcal{O}$  (see [1]). We obtain in the sense of distributions:

$$-\nabla \cdot (2\mu \mathbf{D}(\mathbf{u}_1) - p_1 \mathbf{I}) + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1 = \mathbf{f}_1. \quad (27)$$

Second, take  $q_1 \in \mathcal{D}(\Omega_1)$ ,  $\mathbf{v}_1 = \mathbf{0}$ ,  $q_2 = 0$ :

$$\nabla \cdot \mathbf{u}_1 = 0. \quad (28)$$

Third, take  $q_2 \in \mathcal{D}(\Omega_2)$ ,  $\mathbf{v}_1 = \mathbf{0}$ ,  $q_1 = 0$ :

$$-\nabla \cdot \mathbf{K}\nabla(\varphi_2 + p_D) = f_2. \quad (29)$$

Next, multiply (27), (29) by functions  $\mathbf{v}_1 \in \mathbf{X}_1$  and  $q_2 \in M_2$  respectively, use Green's theorem, add the two equations and compare with (W):

$$\begin{aligned} & (\varphi_2 - \frac{1}{2}(\mathbf{u}_1 \cdot \mathbf{u}_1), \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - (\mathbf{u}_1 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} + \frac{1}{G}(\mathbf{u}_1 \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (g_N, q_2)_{\Gamma_{2N}} \\ & = ((-2\mu \mathbf{D}(\mathbf{u}_1) + p_1 \mathbf{I})\mathbf{n}_{12}, \mathbf{v}_1)_{\Gamma_{12}} + (\mathbf{K}\nabla p_2 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} - (\mathbf{K}\nabla p_2 \cdot \mathbf{n}_2, q_2)_{\Gamma_{2N}}. \end{aligned} \quad (30)$$

By choosing  $\mathbf{v}_1 = \mathbf{0}$  and either  $q_2|_{\Gamma_{12}} = 0$  or  $q_2|_{\Gamma_{2N}} = 0$ , we recover the Neumann boundary condition (8) and the interface condition (10). Next, by choosing  $q_2 = 0$  and  $\mathbf{v}_1 = v_1 \mathbf{n}_{12}$  where  $v_1$  is a smooth function defined on each curvilinear segment of  $\Gamma_{12}$  and vanishing in a neighborhood of  $\partial\Omega_1 \setminus \Gamma_{12}$ , we recover the interface condition (11) by noting that  $p_2 = \varphi_2$  on  $\Gamma_{12}$  due to (14). Finally, choosing  $q_2 = 0$  and  $\mathbf{v}_1 = v_1 \boldsymbol{\tau}_{12}$  where  $v_1$  is a smooth function defined on each curvilinear segment of  $\Gamma_{12}$  and vanishing in a neighborhood of  $\partial\Omega_1 \setminus \Gamma_{12}$ , we recover the interface condition (12).  $\square$

We now prove existence and uniqueness of the weak solution  $(\mathbf{u}_1, p_1, p_2)$ . For this, we restrict the test functions  $\mathbf{v}_1$  to the subspace of divergence free functions:

$$\mathbf{V}_1 = \{\mathbf{v}_1 \in \mathbf{X}_1, \quad \nabla \cdot \mathbf{v}_1 = 0\}.$$

The variational formulation then becomes:

$$(\tilde{W}) \left\{ \begin{array}{l} \text{Find } \mathbf{u}_1 \in \mathbf{V}_1, p_2 = \varphi_2 + p_D, \text{ with } \varphi_2 \in M_2, \text{ s.t.} \\ \forall \mathbf{v}_1 \in \mathbf{V}_1, \forall q_2 \in M_2, \quad 2\mu(\mathbf{D}(\mathbf{u}_1), \mathbf{D}(\mathbf{v}_1))_{\Omega_1} + (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1, \mathbf{v}_1)_{\Omega_1} + (\varphi_2 - \frac{1}{2}\mathbf{u}_1 \cdot \mathbf{u}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ + \frac{1}{G}(\mathbf{u}_1 \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{u}_1 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} + (\mathbf{K}\nabla \varphi_2, \nabla q_2)_{\Omega_2} \\ = (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} + (f_2, q_2)_{\Omega_2} - (\mathbf{K}\nabla p_D, \nabla q_2)_{\Omega_2} + (g_N, q_2)_{\Gamma_{2N}}. \end{array} \right.$$

Problems  $(W)$  and  $(\tilde{W})$  are equivalent in the sense that if  $(\mathbf{u}_1, p_1, p_2)$  is a solution to  $(W)$  then clearly  $(\mathbf{u}_1, p_2)$  is also a solution to  $(\tilde{W})$ . Conversely, if  $(\mathbf{u}_1, p_2)$  is a solution to  $(\tilde{W})$ , there is a unique  $p_1 \in M_1$  such that  $(\mathbf{u}_1, p_1, p_2)$  is a solution to  $(W)$ . This result is a consequence of the following inf-sup condition proved in [14].

$$\inf_{q_1 \in M_1} \sup_{(\mathbf{v}_1, q_2) \in \mathbf{X}_1 \times M_2} \frac{|(\nabla \cdot \mathbf{v}, q_1)_{\Omega_1}|}{(\|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)}^2 + \|\nabla q_2\|_{L^2(\Omega_2)}^2)^{1/2} \|q_1\|_{L^2(\Omega_1)}} \geq \beta > 0.$$

Therefore, we now focus on the existence and uniqueness of the solution to  $(\tilde{W})$ .

## 2.1 Existence of solution to problem $(\tilde{W})$

We use the technique of the Galerkin method. Since the spaces  $\mathbf{V}_1$  and  $M_2$  are separable, let  $\{(\mathbf{w}_m, t_m)\}_{m \geq 1}$  be a sequence of smooth functions that form a basis of  $\mathbf{V}_1 \times M_2$ . Consider the finite dimensional space  $Y_m = \text{span}\{(\mathbf{w}_i, t_i) : 1 \leq i \leq m\}$  equipped with the inner-product:

$$((\mathbf{v}, q), (\mathbf{w}, t))_Y = 2\mu(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{w}))_{\Omega_1} + (\mathbf{K}\nabla q, \nabla t)_{\Omega_2}.$$

We restrict problem  $(\tilde{W})$  to  $Y_m$  and obtain a finite dimensional problem:

$$(\tilde{W}_m) \begin{cases} \text{Find } (\mathbf{u}_m, \varphi_m) \in Y_m \text{ s.t.} \\ \forall 1 \leq i \leq m, \quad 2\mu(\mathbf{D}(\mathbf{u}_m), \mathbf{D}(\mathbf{w}_i))_{\Omega_1} + (\mathbf{u}_m \cdot \nabla \mathbf{u}_m, \mathbf{w}_i)_{\Omega_1} + (\varphi_m - \frac{1}{2} \mathbf{u}_m \cdot \mathbf{u}_m, \mathbf{w}_i \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ + \frac{1}{G} (\mathbf{u}_m \cdot \boldsymbol{\tau}_{12}, \mathbf{w}_i \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{u}_m \cdot \mathbf{n}_{12}, t_i)_{\Gamma_{12}} + (\mathbf{K}\nabla \varphi_m, \nabla t_i)_{\Omega_2} \\ = (\mathbf{f}_1, \mathbf{w}_i)_{\Omega_1} + (f_2, t_i)_{\Omega_2} - (\mathbf{K}\nabla p_D, \nabla t_i)_{\Omega_2} + (g_N, t_i)_{\Gamma_{2N}}. \end{cases}$$

We then define a continuous mapping  $\Psi_m : Y_m \rightarrow Y_m$ :

$$(\Psi_m(\mathbf{v}, q), (\mathbf{w}, t))_Y = 2\mu(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{w}))_{\Omega_1} + (\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{w})_{\Omega_1} + (q - \frac{1}{2} \mathbf{v} \cdot \mathbf{v}, \mathbf{w} \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{G} (\mathbf{v} \cdot \boldsymbol{\tau}_{12}, \mathbf{w} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \\ - (\mathbf{v} \cdot \mathbf{n}_{12}, t)_{\Gamma_{12}} + (\mathbf{K}\nabla q, \nabla t)_{\Omega_2} - (\mathbf{f}_1, \mathbf{w})_{\Omega_1} - (f_2, t)_{\Omega_2} + (\mathbf{K}\nabla p_D, \nabla t)_{\Omega_2} - (g_N, t)_{\Gamma_{2N}}.$$

Clearly a zero of  $\Psi_m$  is a solution to problem  $(\tilde{W}_m)$ . We will apply a corollary of Brouwer's fixed point theorem to conclude that there is at least one zero of  $\Psi_m$  in a certain ball centered at the origin. For completeness, the result is recalled below [13].

**Lemma 2.** *Let  $H$  be a finite dimensional Hilbert space with inner-product  $(\cdot, \cdot)_H$  and norm  $\|\cdot\|_H$ . Let  $\mathcal{F}$  be a continuous mapping from  $H$  into  $H$ . Assume there is a constant  $\mathcal{R}$  such that*

$$\forall v \in H \text{ with } \|v\|_H = \mathcal{R}, \quad (\mathcal{F}(v), v)_H \geq 0.$$

*Then, there exists an element  $v_0 \in H$  such that*

$$\mathcal{F}(v_0) = 0, \quad \|v_0\|_H \leq \mathcal{R}.$$

Therefore, we evaluate

$$(\Psi_m(\mathbf{v}, q), (\mathbf{v}, q))_Y = 2\mu(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{v}))_{\Omega_1} + (\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{v})_{\Omega_1} + (q - \frac{1}{2} \mathbf{v} \cdot \mathbf{v}, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{G} (\mathbf{v} \cdot \boldsymbol{\tau}_{12}, \mathbf{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \\ - (\mathbf{v} \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} + (\mathbf{K}\nabla q, \nabla q)_{\Omega_2} - (\mathbf{f}_1, \mathbf{v})_{\Omega_1} - (f_2, q)_{\Omega_2} + (\mathbf{K}\nabla p_D, \nabla q)_{\Omega_2} - (g_N, q)_{\Gamma_{2N}}.$$

We remark that for  $\mathbf{v} \in \mathbf{V}_1$

$$(\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{v})_{\Omega_1} = -\frac{1}{2} (\nabla \cdot \mathbf{v}, \mathbf{v} \cdot \mathbf{v})_{\Omega_1} + \frac{1}{2} (\mathbf{v} \cdot \mathbf{n}_1, \mathbf{v} \cdot \mathbf{v})_{\partial\Omega_1} = \frac{1}{2} (\mathbf{v} \cdot \mathbf{n}_1, \mathbf{v} \cdot \mathbf{v})_{\partial\Omega_1}$$

Therefore,

$$(\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{v})_{\Omega_1} + (q - \frac{1}{2} \mathbf{v} \cdot \mathbf{v}, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} - (\mathbf{v} \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} = 0,$$

because  $\mathbf{v} = 0$  on  $\Gamma_1$ . We are left with

$$\begin{aligned} (\Psi_m(\mathbf{v}, q), (\mathbf{v}, q))_Y &= 2\mu \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega_1)}^2 + \frac{1}{G} \|\mathbf{v} \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 + \|\mathbf{K}^{1/2} \nabla q\|_{L^2(\Omega_2)}^2 \\ &\quad - (\mathbf{f}_1, \mathbf{v})_{\Omega_1} - (f_2, q)_{\Omega_2} + (\mathbf{K} \nabla p_D, \nabla q)_{\Omega_2} - (g_N, q)_{\Gamma_{2N}}. \end{aligned} \quad (31)$$

We now bound the terms in the second line of (31). Using (16), (18), (19) and (17), we obtain

$$(\mathbf{f}_1, \mathbf{v})_{\Omega_1} \leq \|\mathbf{f}_1\|_{L^2(\Omega_1)} \|\mathbf{v}\|_{L^2(\Omega_1)} \leq \mathcal{P}_1 C_1 \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega_1)} \|\mathbf{f}_1\|_{L^2(\Omega_1)} \leq \frac{\mu}{2} \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega_1)}^2 + \frac{\mathcal{P}_1^2 C_1^2}{2\mu} \|\mathbf{f}_1\|_{L^2(\Omega_1)}^2. \quad (32)$$

Similarly, using (16), (21), (23) and (17), we have

$$(f_2, q)_{\Omega_2} \leq \frac{1}{4} \|\mathbf{K}^{1/2} \nabla q\|_{L^2(\Omega_2)}^2 + \frac{1}{\lambda_{\min}} \mathcal{P}_2^2 \|f_2\|_{L^2(\Omega_2)}^2.$$

Using the bounds (15), (16) and (23), we have

$$(\mathbf{K} \nabla p_D, \nabla q)_{\Omega_2} \leq \frac{1}{4} \|\mathbf{K}^{1/2} \nabla q\|_{L^2(\Omega_2)}^2 + C_0^2 \lambda_{\max} \|g_D\|_{H^{1/2}(\Gamma_{2D})}^2.$$

Finally, using (16), (22), (23) and (17), we obtain

$$(g_N, q)_{\Gamma_{2N}} \leq \frac{1}{4} \|\mathbf{K}^{1/2} \nabla q\|_{L^2(\Omega_2)}^2 + \frac{C_3^2}{\lambda_{\min}} \|g_N\|_{L^2(\Gamma_{2N})}^2.$$

Therefore

$$\begin{aligned} (\Psi_m(\mathbf{v}, q), (\mathbf{v}, q))_Y &\geq \frac{1}{4} \left( 2\mu \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega_1)}^2 + \|\mathbf{K}^{1/2} \nabla q\|_{L^2(\Omega_2)}^2 \right) - \left( \frac{\mathcal{P}_1^2 C_1^2}{2\mu} \|\mathbf{f}_1\|_{L^2(\Omega_1)}^2 + \frac{\mathcal{P}_2^2}{\lambda_{\min}} \|f_2\|_{L^2(\Omega_2)}^2 \right) \\ &\quad + C_0^2 \lambda_{\max} \|g_D\|_{H^{1/2}(\Gamma_{2D})}^2 + \frac{C_3^2}{\lambda_{\min}} \|g_N\|_{L^2(\Gamma_{2N})}^2, \end{aligned}$$

so  $(\Psi_m(\mathbf{v}, q), (\mathbf{v}, q))_Y \geq 0$  provided  $\|(\mathbf{v}, q)\|_Y = ((\mathbf{v}, q), (\mathbf{v}, q))_Y^{1/2} = \mathcal{R}_0$  with

$$\mathcal{R}_0 = 2 \left( \frac{\mathcal{P}_1^2 C_1^2}{2\mu} \|\mathbf{f}_1\|_{L^2(\Omega_1)}^2 + \frac{\mathcal{P}_2^2}{\lambda_{\min}} \|f_2\|_{L^2(\Omega_2)}^2 + C_0^2 \lambda_{\max} \|g_D\|_{H^{1/2}(\Gamma_{2D})}^2 + \frac{C_3^2}{\lambda_{\min}} \|g_N\|_{L^2(\Gamma_{2N})}^2 \right)^{1/2}.$$

Therefore, for any  $m$ , there is a solution  $(\mathbf{u}_m, \varphi_m)$  of problem  $(\tilde{W}_m)$  satisfying:

$$\|(\mathbf{u}_m, \varphi_m)\|_Y \leq \mathcal{R}_0.$$

We have thus constructed a bounded sequence in the Hilbert space  $\mathbf{V}_1 \times M_2$ . Therefore, there exists a subsequence, still denoted by  $\{(\mathbf{u}_m, \varphi_m)\}_m$ , that converges weakly to an element  $(\mathbf{u}, \varphi) \in \mathbf{V}_1 \times M_2$ . Using a standard argument and Sobolev imbeddings, we can pass to the limit in the equation of problem  $(\tilde{W}_m)$  as  $m$  tends to infinity. Denoting  $p = \varphi + p_D$ , we then obtain that  $(\mathbf{u}, p)$  is a solution to problem  $(\tilde{W})$ . Using the same argument as above, we can show that any solution  $(\mathbf{u}, \varphi)$  to problem  $(\tilde{W})$  is bounded:

$$2\mu \|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega_1)}^2 + \|\mathbf{K}^{1/2} \nabla \varphi\|_{L^2(\Omega_2)}^2 \leq \mathcal{R}_0^2. \quad (33)$$

This yields the bound:

$$2\mu \|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega_1)}^2 + \|\mathbf{K}^{1/2} \nabla p\|_{L^2(\Omega_2)}^2 \leq \mathcal{R}_1^2, \quad (34)$$

where

$$\mathcal{R}_1^2 = \mathcal{R}_0^2 + 2\|\mathbf{K}^{1/2} \nabla p_D\|_{L^2(\Omega_2)}^2. \quad (35)$$

## 2.2 Uniqueness of solution to problem $(\tilde{W})$

**Lemma 3.** *Assume that the data satisfies:*

$$\begin{aligned} \frac{8\mu^3}{C_1^6(\mathcal{P}_4^2 + \frac{3}{2}C_4^2C_2)^2} &> \frac{2\mathcal{P}_1^2C_1^2}{\mu}\|\mathbf{f}_1\|_{L^2(\Omega_1)}^2 + \frac{4\mathcal{P}_2^2}{\lambda_{\min}}\|f_2\|_{L^2(\Omega_2)}^2 \\ &+ 4C_0^2\lambda_{\max}\|g_D\|_{H^{1/2}(\Gamma_{2D})}^2 + \frac{4C_3^2}{\lambda_{\min}}\|g_N\|_{L^2(\Gamma_{2N})}^2 + 2\|\mathbf{K}^{1/2}\nabla p_D\|_{L^2(\Omega_2)}^2. \end{aligned}$$

Then problem  $(\tilde{W})$  has a unique weak solution.

*Proof.* The proof is very similar to the proof of Theorem 2.4 in [14]. We repeat it here for completeness. Assume that  $(\mathbf{u}_1^1, p_2^1)$  and  $(\mathbf{u}_1^2, p_2^2)$  are two solutions of problem  $(\tilde{W})$ . Their difference, say  $(\mathbf{w}_1, z_2)$ , belongs to the space  $\mathbf{V}_1 \times M_2$  and satisfies:

$$\begin{aligned} \forall (\mathbf{v}_1, q_2) \in \mathbf{V}_1 \times M_2, \quad &2\mu(\mathbf{D}(\mathbf{w}_1), \mathbf{D}(\mathbf{v}_1))_{\Omega_1} + (\mathbf{K}\nabla z_2, \nabla q_2)_{\Omega_2} + (\mathbf{w}_1 \cdot \nabla \mathbf{u}_1^1, \mathbf{v}_1)_{\Omega_1} \\ &+ (\mathbf{u}_1^2 \cdot \nabla \mathbf{w}_1, \mathbf{v}_1)_{\Omega_1} + \frac{1}{G}(\mathbf{w}_1 \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} + (z_2 - \frac{1}{2}(\mathbf{w}_1 \cdot \mathbf{u}_1^1), \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ &- (\mathbf{w}_1 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} - \frac{1}{2}(\mathbf{u}_1^2 \cdot \mathbf{w}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} = 0. \end{aligned}$$

By choosing  $(\mathbf{v}_1, q_2) = (\mathbf{w}_1, z_2) \in \mathbf{V}_1 \times M_2$  and applying Green's formula and the boundary condition on the functions of  $\mathbf{X}_1$ , this equation becomes

$$\begin{aligned} 2\mu\|\mathbf{D}(\mathbf{w}_1)\|_{L^2(\Omega_1)}^2 + \|\mathbf{K}^{1/2}\nabla z_2\|_{L^2(\Omega_2)}^2 + \frac{1}{G}\|\mathbf{w}_1 \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 \\ + (\mathbf{w}_1 \cdot \nabla \mathbf{u}_1^1, \mathbf{w}_1)_{\Omega_1} + \frac{1}{2}\left((\mathbf{w}_1 \cdot \mathbf{w}_1, \mathbf{u}_1^2 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - (\mathbf{w}_1 \cdot (\mathbf{u}_1^1 + \mathbf{u}_1^2), \mathbf{w}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}}\right) = 0. \end{aligned} \quad (36)$$

Applying (18) and (19), the first non-linear term in the second line of (36) is bounded above by

$$\|\mathbf{w}_1\|_{L^4(\Omega_1)}^2 \|\nabla \mathbf{u}_1^1\|_{L^2(\Omega_1)} \leq C_1^3 \mathcal{P}_4^2 \frac{1}{\sqrt{\mu}} \|\mathbf{D}(\mathbf{w}_1)\|_{L^2(\Omega_1)}^2 (\sqrt{\mu} \|\mathbf{D}(\mathbf{u}_1^1)\|_{L^2(\Omega_1)}).$$

Similarly, applying formulas (18)–(20), the second term in the second line of (36) is bounded above by

$$\begin{aligned} \frac{1}{2}\|\mathbf{w}_1\|_{L^4(\Gamma_{12})}^2 (\|\mathbf{u}_1^1\|_{L^2(\Gamma_{12})} + 2\|\mathbf{u}_1^2\|_{L^2(\Gamma_{12})}) \\ \leq \frac{1}{2}C_4^2C_2C_1^3 \frac{1}{\sqrt{\mu}} \|\mathbf{D}(\mathbf{w}_1)\|_{L^2(\Omega_1)}^2 (\sqrt{\mu} \|\mathbf{D}(\mathbf{u}_1^1)\|_{L^2(\Omega_1)} + 2\sqrt{\mu} \|\mathbf{D}(\mathbf{u}_1^2)\|_{L^2(\Omega_1)}). \end{aligned}$$

Hence, using the a priori estimate (34), the second line in (36) is bounded above by

$$\frac{C_1^3}{\sqrt{2\mu}} \left( \mathcal{P}_4^2 + \frac{3}{2}C_4^2C_2 \right) \mathcal{R}_1 \|\mathbf{D}(\mathbf{w}_1)\|_{L^2(\Omega_1)}^2.$$

Thus if

$$(2\mu)^{3/2} > C_1^3 \left( \mathcal{P}_4^2 + \frac{3}{2}C_4^2C_2 \right) \mathcal{R}_1,$$

then  $(\mathbf{w}_1, z_2) = (\mathbf{0}, 0)$ . □

A straightforward consequence due to Lemma 1 is the existence and uniqueness of a solution to problem  $(W)$ .

In the next section, we propose a numerical scheme for solving (1)–(12) that couples the continuous finite element method with the discontinuous Galerkin method.

### 3 A Multinumerics Scheme

Let  $\mathcal{E}_1^h$  be a conforming triangulation of  $\Omega_1$  and let  $\mathcal{E}_2^h$  be a general subdivision of  $\Omega_2$  consisting of triangular elements. The mesh  $\mathcal{E}_2^h$  may contain hanging nodes. As usual, the parameter  $h$  denotes the maximum diameter of the elements. We assume that the resulting mesh  $\mathcal{E}^h = \mathcal{E}_1^h \cup \mathcal{E}_2^h$  is regular [6]. In addition, we assume that the vertices of the polygonal line  $\Gamma_{12}$  are vertices in the mesh  $\mathcal{E}^h$ . However, the meshes  $\mathcal{E}_1^h$  and  $\mathcal{E}_2^h$  do not have to match on the interface  $\Gamma_{12}$ . In our numerical scheme, we propose to approximate the Navier-Stokes velocity and pressure in conforming finite element spaces  $\mathbf{X}_1^h \subset \mathbf{X}_1$  and  $M_1^h \subset M_1$  satisfying the discrete inf-sup condition with  $\beta_*$  independent of  $h$ :

$$\inf_{q_1 \in M_1^h} \sup_{\mathbf{v}_1 \in \mathbf{X}_1^h} \frac{|(\nabla \cdot \mathbf{v}_1, q_1)_{\Omega_1}|}{\|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)} \|q_1\|_{L^2(\Omega_1)}} \geq \beta_* > 0. \quad (37)$$

Examples of such conforming finite elements are the Crouzeix-Raviart elements [7], the mini elements [3] and the Taylor-Hood elements [17]. We also propose to approximate the Darcy pressure in totally discontinuous finite element spaces. In order to define the discontinuous Galerkin method, we introduce further notation. We denote by  $\Gamma_h^2$  the set of interior edges in  $\Omega_2$ . To each edge  $e$  of  $\mathcal{E}_2^h$  we associate once and for all a unit normal vector  $\mathbf{n}_e$ . For  $e \in \Gamma_{12}$ , we set  $\mathbf{n}_e = \mathbf{n}_{12}$ , i.e.  $\mathbf{n}_e$  is the exterior normal to  $\Omega_1$ . If  $\mathbf{n}_e$  points from the element  $E^1$  to the element  $E^2$ , the jump  $[\![\varphi]\!]$  and average  $\{\!\{ \varphi \}\!\}$  of a function  $\varphi$  are given by:

$$[\![\varphi]\!] = \varphi|_{E^1} - \varphi|_{E^2}, \quad \{\!\{ \varphi \}\!\} = \frac{1}{2}\varphi|_{E^1} + \frac{1}{2}\varphi|_{E^2}.$$

For an integer  $k_2 \geq 1$ , we define

$$M_2^h = \{q_2 \in L^2(\Omega_2); q_2|_{\Gamma_{2D}} = 0 \quad \text{and} \quad \forall E \in \mathcal{E}_2^h, q_2|_E \in \mathcal{P}_{k_2}(E)\},$$

equipped with the usual DG norm:

$$\forall q_2 \in M_2^h, \quad \|q_2\|_{\Omega_2} = \left( \sum_{E \in \mathcal{E}_2^h} \|\mathbf{K}^{1/2} \nabla q_2\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h^2} \frac{1}{|e|} \|[\![q_2]\!]\|_{L^2(e)}^2 \right)^{1/2}. \quad (38)$$

**Lemma 4.** *Assume that  $p_D \in H^{k_2+1}(\Omega_2)$  is the lift defined in (13)-(15). Then, there exists  $P_D \in M_2^h$  and a constant  $C$  independent of  $h$  satisfying:*

$$P_D = 0, \quad \text{on} \quad \Gamma_{12}, \quad (39)$$

$$\|p_D - P_D\|_{\Omega_2} \leq Ch^{k_2} \|p_D\|_{H^{k_2+1}(\Omega_2)}. \quad (40)$$

In the rest of the text, we denote by  $C$  a generic constant independent of  $h$  and  $\mu$ , that takes different values at different places. Next, we define several bilinear forms:  $a_{\text{NS}}, b_{\text{NS}}, c_{\text{NS}}$  are the discretizations of the viscous term, pressure term and nonlinear term respectively in the Navier-Stokes equations;  $a_D$  is the discretization



of the diffusion term in the Darcy equations; and  $\gamma$  is the form containing terms related to the interface  $\Gamma_{12}$ .

$$\begin{aligned}
& \forall \mathbf{v}_1, \mathbf{w}_1 \in \mathbf{X}_1^h, \quad a_{\text{NS}}(\mathbf{v}_1, \mathbf{w}_1) = 2\mu(\mathbf{D}(\mathbf{v}_1), \mathbf{D}(\mathbf{w}_1))_{\Omega_1}, \\
& \forall \mathbf{v}_1 \in \mathbf{X}_1^h, \forall q_1 \in M_1^h, \quad b_{\text{NS}}(\mathbf{v}_1, q_1) = -(q_1, \nabla \cdot \mathbf{v}_1)_{\Omega_1}, \\
& \forall \mathbf{z}_1, \mathbf{v}_1, \mathbf{w}_1 \in \mathbf{X}_1^h, \quad c_{\text{NS}}(\mathbf{z}_1, \mathbf{v}_1, \mathbf{w}_1) = \frac{1}{2}(\mathbf{z}_1 \cdot \nabla \mathbf{v}_1, \mathbf{w}_1)_{\Omega_1} - \frac{1}{2}(\mathbf{z}_1 \cdot \nabla \mathbf{w}_1, \mathbf{v}_1)_{\Omega_1} + \frac{1}{2}(\mathbf{z}_1 \cdot \mathbf{n}_{12}, \mathbf{v}_1 \cdot \mathbf{w}_1)_{\Gamma_{12}}, \\
& \forall q_2, t_2 \in M_2^h, \quad a_{\text{D}}(q_2, t_2) = \sum_{E \in \mathcal{E}_2^h} (\mathbf{K} \nabla q_2, \nabla t_2)_E - \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla q_2 \cdot \mathbf{n}_e\} [t_2])_e \\
& \quad + \epsilon \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla t_2 \cdot \mathbf{n}_e\}, [q_2])_e + \sum_{e \in \Gamma_2^h} \frac{\sigma_e}{|e|} ([q_2], [t_2])_e \\
& \forall \mathbf{v}_1, \mathbf{w}_1 \in \mathbf{X}_1^h, \quad \forall q_2, t_2 \in M_2^h, \quad \gamma(\mathbf{v}_1, q_2; \mathbf{w}_1, t_2) = (q_2, \mathbf{w}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\mathbf{v}_1 \cdot \mathbf{v}_1, \mathbf{w}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\
& \quad + \frac{1}{G}(\mathbf{v}_1 \cdot \boldsymbol{\tau}_{12}, \mathbf{w}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{v}_1 \cdot \mathbf{n}_{12}, t_2)_{\Gamma_{12}}.
\end{aligned}$$

In the definition of  $a_{\text{D}}$  the parameter  $\epsilon$  yields a symmetric bilinear form if  $\epsilon = -1$  and a non-symmetric bilinear form if  $\epsilon = 0$  or  $\epsilon = 1$ . The parameter  $\sigma_e$  is a penalty parameter that varies with respect to the edge in  $\mathcal{E}_2^h$ . We recall that  $a_{\text{D}}$  is coercive [27, 15]. There exists a constant  $\kappa > 0$  independent of  $h$  such that:

$$\forall q_2 \in M_2^h, \quad \kappa \| \|q_2\| \|_{\Omega_2}^2 \leq a_{\text{D}}(q_2, q_2). \quad (41)$$

It has been shown that if  $\epsilon \in \{-1, 0\}$ , property (41) is valid if the penalty parameter is large enough. From [12], the lower bound for the penalty parameter is:

$$\forall e = \partial E_e^1 \cap \partial E_e^2, \quad \sigma_e \geq \frac{3\lambda_{\max}^2}{2\lambda_{\min}} k_2(k_2 + 1)(\cot \theta_{E_e^1} + \cot \theta_{E_e^2}),$$

where  $\theta_{E_e^i}$  denotes the smallest angle in the triangle  $E_e^i$ . We also define the form  $L$ :

$$\begin{aligned}
\forall \mathbf{v}_1 \in \mathbf{X}_1^h, \quad \forall q_2 \in M_2^h, \quad L(\mathbf{v}_1, q_2) &= (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} + (f_2, q_2)_{\Omega_2} + (g_{\text{N}}, q_2)_{\Gamma_{2\text{N}}} \\
&\quad - \sum_{E \in \mathcal{E}_2^h} (\mathbf{K} \nabla p_{\text{D}}, \nabla q_2)_E + \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla p_{\text{D}} \cdot \mathbf{n}_e\}, [q_2])_e.
\end{aligned}$$

We can now introduce our numerical scheme: find  $\mathbf{U}_1 \in \mathbf{X}_1^h, P_1 \in M_1^h, P_2 = \Phi_2 + P_{\text{D}}$  with  $\Phi_2 \in M_2^h$ , such that for all  $\mathbf{v}_1 \in \mathbf{X}_1^h, q_2 \in M_2^h, q_1 \in M_1^h$ :

$$a_{\text{NS}}(\mathbf{U}_1, \mathbf{v}_1) + b_{\text{NS}}(\mathbf{v}_1, P_1) + c_{\text{NS}}(\mathbf{U}_1; \mathbf{U}_1, \mathbf{v}_1) + a_{\text{D}}(\Phi_2, q_2) + \gamma(\mathbf{U}_1, \Phi_2; \mathbf{v}_1, q_2) = L(\mathbf{v}_1, q_2), \quad (42)$$

$$b_{\text{NS}}(\mathbf{U}_1, q_1) = 0. \quad (43)$$

We end this section by giving important properties of the discrete spaces and the continuity property of the bilinear form  $c_{\text{NS}}$ .

*Approximation properties.* Assume that  $(\mathbf{v}_1, p_1, p_2) \in \mathbf{X}_1 \times M_1 \times M_2$  is smooth enough, i.e.  $\mathbf{v}_1 \in H^{k_1+1}(\Omega_1)$ ,  $p_1 \in H^{k_1}(\Omega_1)$  and  $p_2 \in H^{k_2+1}(\Omega_2)$  for integers  $k_1, k_2$ . Then, there exists an approximation  $(\tilde{\mathbf{v}}_1, \tilde{p}_1, \tilde{p}_2) \in \mathbf{X}_1^h \times M_1^h \times M_2^h$  such that

$$\|\nabla(\mathbf{v}_1 - \tilde{\mathbf{v}}_1)\|_{L^2(\Omega_1)} \leq Ch^{k_1} \|\mathbf{v}_1\|_{H^{k_1+1}(\Omega_1)}, \quad (44)$$

$$\forall q_1 \in M_1^h, \quad (\nabla \cdot (\mathbf{v}_1 - \tilde{\mathbf{v}}_1), q_1)_{\Omega_1} = 0, \quad (45)$$

$$\|p_1 - \tilde{p}_1\|_{L^2(\Omega_1)} \leq Ch^{k_1} \|p_1\|_{H^{k_1}(\Omega_1)}, \quad (46)$$

$$i = 0, 1, \quad \|\nabla^i(p_2 - \tilde{p}_2)\|_{L^2(\Omega_2)} \leq Ch^{k_2+1-i} \|p_2\|_{H^{k_2+1}(\Omega_2)}. \quad (47)$$

It is easy to check that (47) implies

$$\|p_2 - \tilde{p}_2\|_{\Omega_2} \leq Ch^{k_2} \|p_2\|_{H^{k_2+1}(\Omega_2)}. \quad (48)$$

$L^2$  bound. There exists a constant  $C_5 > 0$  independent of  $h$  such that

$$\forall q_2 \in M_2^h, \quad \|q_2\|_{L^2(\Omega_2)} \leq C_5 \|q_2\|_{\Omega_2}. \quad (49)$$

*Trace theorem.* There exists a constant  $C_6 > 0$  independent of  $h$  such that

$$\forall q_2 \in M_2^h, \quad \|q_2\|_{L^2(\Gamma_{12})} \leq C_6 \|q_2\|_{\Omega_2}. \quad (50)$$

The proof of (49) is given in Lemma 6.2 of [15] and the proof of (50) is given in Theorem 4.4 of [14]. We next show that the form  $c_{\text{NS}}$  is continuous.

**Lemma 5.** *There exists a constant  $C_7$  such that*

$$\forall \mathbf{z}_1, \mathbf{v}_1, \mathbf{w}_1 \in \mathbf{X}_1, \quad c_{\text{NS}}(\mathbf{z}_1; \mathbf{v}_1, \mathbf{w}_1) \leq C_7 \|\mathbf{D}(\mathbf{z}_1)\|_{L^2(\Omega_1)} \|\mathbf{D}(\mathbf{v}_1)\|_{L^2(\Omega_1)} \|\mathbf{D}(\mathbf{w}_1)\|_{L^2(\Omega_1)}. \quad (51)$$

An expression for the constant  $C_7$  is

$$C_7 = C_1^3 (\mathcal{P}_4^2 + \frac{1}{2} C_2 C_4^2).$$

*Proof.* Using (16), we have

$$\begin{aligned} c_{\text{NS}}(\mathbf{z}_1; \mathbf{v}_1, \mathbf{w}_1) &= \frac{1}{2} (\mathbf{z}_1 \cdot \nabla \mathbf{v}_1, \mathbf{w}_1)_{\Omega_1} - \frac{1}{2} (\mathbf{z}_1 \cdot \nabla \mathbf{w}_1, \mathbf{v}_1)_{\Omega_1} + \frac{1}{2} (\mathbf{z}_1 \cdot \mathbf{n}_{12}, \mathbf{v}_1 \cdot \mathbf{w}_1)_{\Gamma_{12}} \\ &\leq \frac{1}{2} \|\mathbf{z}_1\|_{L^4(\Omega_1)} (\|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)} \|\mathbf{w}_1\|_{L^4(\Omega_1)} + \|\nabla \mathbf{w}_1\|_{L^2(\Omega_1)} \|\mathbf{v}_1\|_{L^4(\Omega_1)}) \\ &\quad + \frac{1}{2} \|\mathbf{z}_1\|_{L^2(\Gamma_{12})} \|\mathbf{v}_1\|_{L^4(\Gamma_{12})} \|\mathbf{w}_1\|_{L^4(\Gamma_{12})}. \end{aligned}$$

Using (18), (20) and (19) we have

$$\begin{aligned} c_{\text{NS}}(\mathbf{z}_1; \mathbf{v}_1, \mathbf{w}_1) &\leq (\mathcal{P}_4^2 + \frac{1}{2} C_2 C_4^2) \|\nabla \mathbf{z}_1\|_{L^2(\Omega_1)} \|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)} \|\nabla \mathbf{w}_1\|_{L^2(\Omega_1)} \\ &\leq C_1^3 (\mathcal{P}_4^2 + \frac{1}{2} C_2 C_4^2) \|\mathbf{D}(\mathbf{z}_1)\|_{L^2(\Omega_1)} \|\mathbf{D}(\mathbf{v}_1)\|_{L^2(\Omega_1)} \|\mathbf{D}(\mathbf{w}_1)\|_{L^2(\Omega_1)}. \end{aligned}$$

□

### 3.1 Consistency

Let  $(\mathbf{u}_1, p_1, p_2)$  be the solution to (1)-(12) that is smooth enough. Define  $\varphi_2 = p_2 - p_D$ . Then, we have for all  $\mathbf{v}_1 \in \mathbf{X}_1^h, q_2 \in \mathbf{M}_2^h, q_1 \in \mathbf{M}_1^h$ :

$$a_{\text{NS}}(\mathbf{u}_1, \mathbf{v}_1) + b_{\text{NS}}(\mathbf{v}_1, p_1) + c_{\text{NS}}(\mathbf{u}_1; \mathbf{u}_1, \mathbf{v}_1) + a_D(\varphi_2, q_2) + \gamma(\mathbf{u}_1, \varphi_2; \mathbf{v}_1, q_2) = L(\mathbf{v}_1, q_2), \quad (52)$$

$$b_{\text{NS}}(\mathbf{u}_1, q_1) = 0. \quad (53)$$

*Proof.* Equation (53) is simply obtained by multiplying (2) by  $q_1 \in M_1^h$  and integrating over  $\Omega_1$ . Next, we multiply (1) by a test function  $\mathbf{v}_1 \in \mathbf{X}_1^h$ , integrate over  $\Omega_1$  and use Green's theorem. The resulting equation is exactly (24). Finally, we multiply (5) by a test function  $q_2 \in M_2^h$ , integrate over one element  $E$ , apply Green's theorem and sum over all elements in  $\mathcal{E}_2^h$ .

$$\sum_{E \in \mathcal{E}_2^h} (\mathbf{K} \nabla p_2, \nabla q_2)_E - \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla p_2 \cdot \mathbf{n}_e\}, [q_2])_e + \sum_{e \in \Gamma_{12}} (\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} = (f_2, q_2)_{\Omega_2} + (g_N, q_2)_{\Gamma_{12}}.$$

Using the splitting  $p_2 = \varphi_2 + p_D$ , we obtain:

$$\sum_{E \in \mathcal{E}_2^h} (\mathbf{K} \nabla \varphi_2, \nabla q_2)_E - \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla \varphi_2 \cdot \mathbf{n}_e\}, [q_2])_e + \sum_{e \in \Gamma_{12}} (\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} \quad (54)$$

$$= (f_2, q_2)_{\Omega_2} + (g_N, q_2)_{\Gamma_{12}} - \sum_{E \in \mathcal{E}_2^h} (\mathbf{K} \nabla p_D, \nabla q_2)_E + \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla p_D \cdot \mathbf{n}_e\}, [q_2])_e. \quad (55)$$

We then add (24) and (55), and use the fact that  $[\varphi_2]|_e = 0$  in  $L^2(e)$  for all  $e \in \Gamma_2^h$ .

$$\begin{aligned} & 2\mu(\mathbf{D}(\mathbf{u}_1), \mathbf{D}(\mathbf{v}_1))_{\Omega_1} - (p_1, \nabla \cdot \mathbf{v}_1)_{\Omega_1} + (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1, \mathbf{v}_1)_{\Omega_1} \\ & + \sum_{E \in \mathcal{E}_2^h} (\mathbf{K} \nabla \varphi_2, \nabla q_2)_E - \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla \varphi_2 \cdot \mathbf{n}_e\}, [q_2])_e + \epsilon \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla q_2 \cdot \mathbf{n}_e\}, [\varphi_2])_e \\ & + \sum_{e \in \Gamma_{12}} (\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} + ((-2\mu \mathbf{D}(\mathbf{u}_1) + p_1 \mathbf{I}) \mathbf{n}_{12}, \mathbf{v}_1)_{\Gamma_{12}} \\ & = (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} + (f_2, q_2)_{\Omega_2} + (g_N, q_2)_{\Gamma_{12}} - \sum_{E \in \mathcal{E}_2^h} (\mathbf{K} \nabla p_D, \nabla q_2)_E + \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla p_D \cdot \mathbf{n}_e\}, [q_2])_e. \end{aligned} \quad (56)$$

In this equation, the terms  $\sum_{e \in \Gamma_{12}} (\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} + ((-2\mu \mathbf{D}(\mathbf{u}_1) + p_1 \mathbf{I}) \mathbf{n}_{12}, \mathbf{v}_1)_{\Gamma_{12}}$  are handled exactly as in the proof of Lemma 1. We remark that  $\mathbf{u}_1 \in \mathbf{V}_1$  and thus we have

$$(\mathbf{u}_1 \cdot \nabla \mathbf{u}_1, \mathbf{v}_1)_{\Omega_1} = -(\mathbf{u}_1 \cdot \nabla \mathbf{v}_1, \mathbf{u}_1)_{\Omega_1} + (\mathbf{u}_1 \cdot \mathbf{n}_{12}, \mathbf{v}_1 \cdot \mathbf{u}_1)_{\Gamma_{12}},$$

which yields easily:

$$(\mathbf{u}_1 \cdot \nabla \mathbf{u}_1, \mathbf{v}_1)_{\Omega_1} = c_{\text{NS}}(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}_1).$$

Combining this result with (56), we obtain equation (52).  $\square$

### 3.2 Existence of solution

We now proceed to show that there exists a unique solution to (42)-(43). We define the space of weakly divergence-free functions:

$$\mathbf{V}_1^h = \{\mathbf{v}_1 \in \mathbf{X}_1^h : \forall q_1 \in M_1^h, \quad b_{\text{NS}}(\mathbf{v}_1, q_1) = 0\}.$$

We note that  $\mathbf{U}_1 \in \mathbf{V}_1^h$  so that the scheme reduces to:

$$\begin{aligned} & \forall \mathbf{v}_1 \in \mathbf{X}_1^h, \forall q_2 \in M_2^h, \quad a_{\text{NS}}(\mathbf{U}_1, \mathbf{v}_1) + b_{\text{NS}}(\mathbf{v}_1, P_1) + c_{\text{NS}}(\mathbf{U}_1; \mathbf{U}_1, \mathbf{v}_1) \\ & + a_D(P_2, q_2) + \gamma(\mathbf{U}_1, P_2; \mathbf{v}_1, q_2) = L(\mathbf{v}_1, q_2). \end{aligned} \quad (57)$$

Clearly, if  $(\mathbf{U}_1, P_1, P_2)$  is a solution to (42)-(43), then  $(\mathbf{U}_1, P_2)$  is a solution to (57). Conversely, assume that  $(\mathbf{U}_1, P_2)$  is a solution to (57). Then, the discrete inf-sup (37) implies that there exists a unique  $P_1 \in M_1^h$  such that  $(\mathbf{U}_1, P_1, P_2)$  is a solution to (42)-(43). Based on this equivalence between the two problems, it suffices to show that there exists a solution  $(\mathbf{U}_1, P_2) \in \mathbf{V}_1^h \times M_2^h$  of (57). We will use Lemma 2 and we define the inner-product on  $Y^h = \mathbf{V}_1^h \times M_2^h$ :

$$((\mathbf{v}_1, q_2), (\mathbf{w}_1, t_2))_{Y^h} = 2\mu(\mathbf{D}(\mathbf{v}_1), \mathbf{D}(\mathbf{w}_1))_{\Omega_1} + \sum_{E \in \mathcal{E}_2^h} (\mathbf{K} \nabla q_2, \nabla t_2)_E + \sum_{e \in \Gamma_2^h} \frac{1}{|e|} ([q_2], [t_2])_e. \quad (58)$$

Next define  $\Psi^h : Y^h \rightarrow Y^h$  such that:

$$(\Psi^h(\mathbf{v}_1, q_2), (\mathbf{w}_1, t_2))_{Y^h} = a_{\text{NS}}(\mathbf{v}_1, \mathbf{w}_1) + c_{\text{NS}}(\mathbf{v}_1; \mathbf{v}_1, \mathbf{w}_1) + a_D(q_2, t_2) + \gamma(\mathbf{v}_1, q_2; \mathbf{w}_1, t_2) - L(\mathbf{w}_1, t_2).$$

Using (41) and the definitions of the bilinear forms, we obtain a lower bound of  $(\Psi^h(\mathbf{v}_1, q_2), (\mathbf{v}_1, q_2))_{Y^h}$ :

$$(\Psi^h(\mathbf{v}_1, q_2), (\mathbf{v}_1, q_2))_{Y^h} \geq 2\mu \|\mathbf{D}(\mathbf{v}_1)\|_{L^2(\Omega_1)}^2 + \kappa \|q_2\|_{\Omega_2}^2 + \frac{1}{G} \|\mathbf{v}_1 \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 - L(\mathbf{v}_1, q_2).$$

From (16), (17) and (49), we have for any  $\delta > 0$ :

$$(f_2, q_2)_{\Omega_2} \leq \frac{\delta}{2} \|q_2\|_{\Omega_2}^2 + \frac{C_5^2}{2\delta} \|f_2\|_{L^2(\Omega_2)}^2. \quad (59)$$

Similarly, from (16), (17) and (50), we have for any  $\delta > 0$ :

$$(g_N, q_2)_{\Gamma_{12}} \leq \frac{\delta}{2} \|q_2\|_{\Omega_2}^2 + \frac{C_6^2}{2\delta} \|g_N\|_{L^2(\Gamma_{12})}^2. \quad (60)$$

Using a trace theorem [24], (16), (17) and (23), we have for any  $\delta > 0$ :

$$\left| - \sum_{E \in \mathcal{E}_2^h} (\mathbf{K} \nabla p_D, \nabla q_2)_E + \sum_{e \in \Gamma_2^h} (\{\mathbf{K} \nabla p_D \cdot \mathbf{n}_e\}, [q_2])_e \right| \leq \delta \|q_2\|_{\Omega_2}^2 + \frac{\lambda_{\max}}{2\delta} \|p_D\|_{H^1(\Omega_2)}^2 + \frac{C_t^2}{2\delta} \sum_{E \in \mathcal{E}_2^h} \|p_D\|_{H^2(E)}^2. \quad (61)$$

Combining the bounds (59), (60), (61) and (32), we obtain:

$$\begin{aligned} (\Psi^h(\mathbf{v}_1, q_2), (\mathbf{v}_1, q_2))_{Y^h} &\geq \frac{3\mu}{2} \|\mathbf{D}(\mathbf{v}_1)\|_{L^2(\Omega_1)}^2 + \frac{\kappa}{2} \|q_2\|_{\Omega_2}^2 + \frac{1}{G} \|\mathbf{v}_1 \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 \\ &- \left( \frac{\mathcal{P}_1^2 C_1^2}{2\mu} \|\mathbf{f}_1\|_{L^2(\Omega_1)}^2 + \frac{2C_5^2}{\kappa} \|f_2\|_{L^2(\Omega_2)}^2 + \frac{2C_6^2}{\kappa} \|g_N\|_{L^2(\Gamma_{2N})}^2 + \frac{2\lambda_{\max}}{\kappa} \|p_D\|_{H^1(\Omega_1)}^2 + \frac{2C_t^2}{\kappa} \sum_{E \in \mathcal{E}_2^h} \|p_D\|_{H^2(E)}^2 \right). \end{aligned}$$

Therefore,  $(\Psi^h(\mathbf{v}_1, q_2), (\mathbf{v}_1, q_2))_{Y^h} \geq 0$  provided that  $\|(\mathbf{v}_1, q_2)\|_{Y^h} = \mathcal{R}_2$  with

$$\mathcal{R}_2 = \left( \max\left(\frac{3}{4}, \frac{\kappa}{2}\right) \right)^{1/2} \left( \frac{\mathcal{P}_1^2 C_1^2}{2\mu} \|\mathbf{f}_1\|_{L^2(\Omega_1)}^2 + \frac{2C_5^2}{\kappa} \|f_2\|_{L^2(\Omega_2)}^2 + \frac{2C_6^2}{\kappa} \|g_N\|_{L^2(\Gamma_{2N})}^2 + \frac{2\lambda_{\max}}{\kappa} \|p_D\|_{H^1(\Omega_1)}^2 + \frac{2C_t^2}{\kappa} \sum_{E \in \mathcal{E}_2^h} \|p_D\|_{H^2(E)}^2 \right)^{1/2}. \quad (62)$$

This concludes the proof of existence of a solution  $(\mathbf{U}_1, P_2)$  of (57). The same argument can be used to show that any solution  $(\mathbf{U}_1, P_2)$  of (57) is bounded as follows:

$$2\mu \|\mathbf{D}(\mathbf{U}_1)\|_{L^2(\Omega_1)}^2 + \|P_2\|_{\Omega_2}^2 \leq \mathcal{R}_2^2. \quad (63)$$

### 3.3 Uniqueness of Solution

**Theorem 6.** *Let  $\mathcal{R}_2$  be defined by (62). Under the condition*

$$\mu^{3/2} > \frac{C_1^3}{\sqrt{2}} (\mathcal{P}_4^2 + C_2 C_4^2) \mathcal{R}_2 \quad (64)$$

*problem (57) admits a unique solution.*

*Proof.* To prove uniqueness, we assume that  $(\mathbf{U}_1^1, P_2^1)$  and  $(\mathbf{U}_1^2, P_2^2)$  are two solutions of problem (57), and let  $\mathbf{W}_1 = \mathbf{U}_1^1 - \mathbf{U}_1^2$  and  $\chi_2 = P_2^1 - P_2^2$ .

$$\begin{aligned} a_{\text{NS}}(\mathbf{W}_1, \mathbf{v}_1) + c_{\text{NS}}(\mathbf{U}_1^1, \mathbf{U}_1^1, \mathbf{v}_1) - c_{\text{NS}}(\mathbf{U}_1^2, \mathbf{U}_1^2, \mathbf{v}_1) + a_D(\chi_2, q_2) + (\chi_2, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - \frac{1}{2} (\mathbf{U}_1^1 \cdot \mathbf{U}_1^1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ + \frac{1}{2} (\mathbf{U}_1^2 \cdot \mathbf{U}_1^2, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{G} (\mathbf{W}_1 \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\mathbf{W}_1 \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} = 0. \end{aligned}$$

In particular, we choose  $\mathbf{v}_1 = \mathbf{W}_1$  and  $q_2 = \chi_2$ .

$$a_{NS}(\mathbf{W}_1, \mathbf{W}_1) + a_D(\chi_2, \chi_2) + \frac{1}{G} \|\mathbf{W}_1 \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 + c_{NS}(\mathbf{U}_1^1, \mathbf{U}_1^1, \mathbf{W}_1) - c_{NS}(\mathbf{U}_1^2, \mathbf{U}_1^2, \mathbf{W}_1) + (\chi_2, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\mathbf{U}_1^1 \cdot \mathbf{U}_1^1, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\mathbf{U}_1^2 \cdot \mathbf{U}_1^2, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - (\mathbf{W}_1 \cdot \mathbf{n}_{12}, \chi_2)_{\Gamma_{12}} = 0.$$

Using (41) and rewriting the nonlinear terms as

$$c_{NS}(\mathbf{U}_1^1, \mathbf{U}_1^1, \mathbf{W}_1) - c_{NS}(\mathbf{U}_1^2, \mathbf{U}_1^2, \mathbf{W}_1) = c_{NS}(\mathbf{W}_1, \mathbf{U}_1^1, \mathbf{W}_1) + c_{NS}(\mathbf{U}_1^2, \mathbf{W}_1, \mathbf{W}_1),$$

$$-\frac{1}{2}(\mathbf{U}_1^1 \cdot \mathbf{U}_1^1, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\mathbf{U}_1^2 \cdot \mathbf{U}_1^2, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} = -\frac{1}{2}(\mathbf{W}_1 \cdot \mathbf{U}_1^1, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\mathbf{W}_1 \cdot \mathbf{U}_1^2, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}},$$

we obtain

$$2\mu \|\mathbf{D}(\mathbf{W}_1)\|_{L^2(\Omega_1)}^2 + \kappa \|\chi_2\|_{\Omega_2}^2 + \frac{1}{G} \|\mathbf{W}_1 \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 + c_{NS}(\mathbf{W}_1, \mathbf{U}_1^1, \mathbf{W}_1) + c_{NS}(\mathbf{U}_1^2, \mathbf{W}_1, \mathbf{W}_1) - \frac{1}{2}(\mathbf{W}_1 \cdot \mathbf{U}_1^1, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\mathbf{W}_1 \cdot \mathbf{U}_1^2, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \leq 0.$$

From Lemma 5, we have

$$c_{NS}(\mathbf{W}_1; \mathbf{U}_1^1, \mathbf{W}_1) + c_{NS}(\mathbf{U}_1^2; \mathbf{W}_1, \mathbf{W}_1) \leq C_7 \|\mathbf{D}(\mathbf{W}_1)\|_{L^2(\Omega_1)} (\|\mathbf{D}(\mathbf{U}_1^1)\|_{L^2(\Omega_1)} + \|\mathbf{D}(\mathbf{U}_1^2)\|_{L^2(\Omega_1)}).$$

Similarly, using (20) and (19), we have

$$\frac{1}{2}(\mathbf{W}_1 \cdot \mathbf{U}_1^1, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\mathbf{W}_1 \cdot \mathbf{U}_1^2, \mathbf{W}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \leq \frac{1}{2} C_4^2 C_2 C_1^3 \|\mathbf{D}(\mathbf{W}_1)\|_{L^2(\Omega_1)}^2 (\|\mathbf{D}(\mathbf{U}_1^1)\|_{L^2(\Omega_1)} + \|\mathbf{D}(\mathbf{U}_1^2)\|_{L^2(\Omega_1)}).$$

Combining the two bounds above with (63), we obtain:

$$(2\mu - \frac{\mathcal{R}_2}{\sqrt{\mu}} (\sqrt{2}C_7 + \frac{1}{\sqrt{2}}C_1^3C_2C_4^2)) \|\mathbf{D}(\mathbf{W}_1)\|_{L^2(\Omega_1)}^2 + \kappa \|\chi_2\|_{\Omega_2}^2 + \frac{1}{G} \|\mathbf{W}_1 \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 \leq 0.$$

This clearly implies that  $\mathbf{W}_1 = \mathbf{0}$  and  $\chi_2 = 0$  if the condition

$$2\mu > \frac{\mathcal{R}_2}{\sqrt{\mu}} (\sqrt{2}C_7 + \frac{1}{\sqrt{2}}C_1^3C_2C_4^2)$$

is satisfied. This condition is equivalent to (64).  $\square$

## 4 A Priori Error Estimates

**Theorem 7.** *Assume that the solution to problem (W) is smooth enough, i.e.  $\mathbf{u}_1 \in (H^{k_1+1}(\Omega_1))^2$ ,  $p_1 \in H^{k_1}(\Omega_1)$  and  $p_2 = \varphi_2 + p_D$  with  $\varphi_2 \in H^{k_2+1}(\Omega_2)$ . Let  $\mathcal{R}_1$  be defined by (35) and let  $\mathcal{R}_2$  be defined by (62). Assume that the data satisfies:*

$$\mu^{3/2} > \frac{C_1^3}{\sqrt{2}} (\mathcal{P}_4^2 + C_2C_4^2) (\mathcal{R}_1 + \mathcal{R}_2).$$

*Then, there exists a constant  $C$  independent of  $h$  and  $\mu$  such that*

$$\mu \|\mathbf{D}(\mathbf{u}_1 - \mathbf{U}_1)\|_{L^2(\Omega_1)}^2 + \|\varphi_2 - \Phi_2\|_{\Omega_2}^2 + \|(\mathbf{u}_1 - \mathbf{U}_1) \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 \leq C \left(1 + \frac{(\mathcal{R}_1 + \mathcal{R}_2)^2}{\mu^2}\right) h^{2k_1} \|\mathbf{u}_1\|_{H^{k_1+1}(\Omega_1)}^2 + C \left(1 + \frac{1}{\mu}\right) h^{2k_2} \|\varphi_2\|_{H^{k_2+1}(\Omega_2)}^2 + C \frac{1}{\mu} h^{2k_1} \|p_1\|_{H^{k_1}(\Omega_1)}^2.$$

*Proof.* Let  $\tilde{\mathbf{u}}_1, \tilde{p}_1, \tilde{\varphi}_2$  be approximations to  $\mathbf{u}_1, p_1, \varphi_2$  in the spaces  $\mathbf{X}_1^h, M_1^h$  and  $M_2^h$  respectively. Assume that the error bounds (44), (46) and (47) hold. Let

$$\begin{aligned}\chi_1 &= \mathbf{U}_1 - \tilde{\mathbf{u}}_1, & \xi_1 &= P_1 - \tilde{p}_1, & \xi_2 &= \Phi_2 - \tilde{\varphi}_2, \\ \zeta_1 &= \mathbf{u}_1 - \tilde{\mathbf{u}}_1, & \eta_1 &= p_1 - \tilde{p}_1, & \eta_2 &= \varphi_2 - \tilde{\varphi}_2.\end{aligned}$$

Subtracting (52)-(53) from (42)-(43), we obtain the error equations:

$$\begin{aligned}\forall \mathbf{v}_1 \in \mathbf{X}_1^h, \forall q_2 \in M_2^h, & \quad a_{\text{NS}}(\chi_1, \mathbf{v}_1) + a_{\text{D}}(\xi_2, q_2) + b_{\text{NS}}(\mathbf{v}_1, \xi_1) + c_{\text{NS}}(\mathbf{U}_1; \mathbf{U}_1, \mathbf{v}_1) - c_{\text{NS}}(\mathbf{u}_1; \mathbf{u}_1, \mathbf{v}_1) \\ & + \gamma(\mathbf{U}_1, \Phi_2; \mathbf{v}_1, q_2) - \gamma(\mathbf{u}_1, \varphi_2; \mathbf{v}_1, q_2) = a_{\text{NS}}(\zeta_1, \mathbf{v}_1) + a_{\text{D}}(\eta_2, q_2) + b_{\text{NS}}(\mathbf{v}_1, \eta_1), \\ \forall q_1 \in M_1^h, & \quad b_{\text{NS}}(\chi_1, q_1) = b_{\text{NS}}(\zeta_1, q_1).\end{aligned}$$

Let  $\mathbf{v}_1 = \chi_1, q_1 = \xi_1, q_2 = \xi_2$ , then from (41), we have

$$\begin{aligned}2\mu \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_1)}^2 + \kappa \|\xi_2\|_{\Omega_2}^2 + c_{\text{NS}}(\mathbf{U}_1; \mathbf{U}_1, \chi_1) - c_{\text{NS}}(\mathbf{u}_1; \mathbf{u}_1, \chi_1) \\ + \gamma(\mathbf{U}_1, \Phi_2; \chi_1, \xi_2) - \gamma(\mathbf{u}_1, \varphi_2; \chi_1, \xi_2) \leq a_{\text{NS}}(\zeta_1, \chi_1) + a_{\text{D}}(\eta_2, \xi_2) + b_{\text{NS}}(\chi_1, \eta_1) - b_{\text{NS}}(\zeta_1, \xi_1).\end{aligned}\quad (65)$$

We first expand the terms involving the linear form  $\gamma$ :

$$\begin{aligned}\gamma(\mathbf{U}_1, \Phi_2; \chi_1, \xi_2) - \gamma(\mathbf{u}_1, \varphi_2; \chi_1, \xi_2) &= -\frac{1}{2}(\mathbf{U}_1 \cdot \mathbf{U}_1, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\mathbf{u}_1 \cdot \mathbf{u}_1, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{G} \|\chi_1 \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 \\ &\quad - \frac{1}{G} (\zeta_1 \cdot \boldsymbol{\tau}_{12}, \chi_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\eta_2, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + (\xi_2, \zeta_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}}.\end{aligned}\quad (66)$$

The nonlinear terms are rewritten as

$$\begin{aligned}A_1 &= -\frac{1}{2}(\mathbf{U}_1 \cdot \mathbf{U}_1, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\mathbf{u}_1 \cdot \mathbf{u}_1, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} = \frac{1}{2}(\mathbf{U}_1 \cdot \chi_1, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\chi_1 \cdot \mathbf{u}_1, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ &\quad - \frac{1}{2}(\mathbf{U}_1 \cdot \zeta_1, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\zeta_1 \cdot \mathbf{u}_1, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}},\end{aligned}\quad (67)$$

and bounded by using (16), (17), (20), (19), (34) and (63)

$$\begin{aligned}A_1 &\leq \frac{1}{2}C_1^3 C_2 C_4^2 \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_1)}^2 (\|\mathbf{D}(\mathbf{U}_1)\|_{L^2(\Omega_1)} + \|\mathbf{D}(\mathbf{u}_1)\|_{L^2(\Omega_1)}) \\ &\quad + C \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_1)} \|\nabla \zeta_1\|_{L^2(\Omega_1)} (\|\mathbf{D}(\mathbf{U}_1)\|_{L^2(\Omega_1)} + \|\mathbf{D}(\mathbf{u}_1)\|_{L^2(\Omega_1)}) \\ &\leq \frac{\mu}{5} \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_1)}^2 + \frac{1}{2}C_1^3 C_2 C_4^2 \frac{\mathcal{R}_1 + \mathcal{R}_2}{\sqrt{2\mu}} \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_1)}^2 \\ &\quad + \frac{C(\mathcal{R}_1 + \mathcal{R}_2)^2}{\mu^2} \|\nabla \zeta_1\|_{L^2(\Omega_1)}^2.\end{aligned}$$

The linear terms in (66) are bounded by (16), (17), (20), (19) and (50)

$$\begin{aligned}\frac{1}{G} (\zeta_1 \cdot \boldsymbol{\tau}_{12}, \chi_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} &\leq \frac{1}{2G} \|\chi_1 \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 + C \|\nabla \zeta_1\|_{L^2(\Omega_1)}^2 \\ (\eta_2, \chi_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} &\leq \frac{\mu}{5} \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_1)}^2 + \frac{C}{\mu} \|\eta_2\|_{\Omega_2}^2, \\ (\xi_2, \zeta_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} &\leq \frac{\kappa}{5} \|\xi_2\|_{\Omega_2}^2 + C \|\nabla \zeta_1\|_{L^2(\Omega_1)}^2.\end{aligned}$$

We rewrite the nonlinear terms involving  $c_{\text{NS}}$  in (65) in a similar way as with the term  $A_1$  defined in (67). We obtain a bound by using Lemma 5.

$$\begin{aligned}c_{\text{NS}}(\mathbf{U}_1; \mathbf{U}_1, \chi_1) - c_{\text{NS}}(\mathbf{u}_1; \mathbf{u}_1, \chi_1) &= c_{\text{NS}}(\mathbf{U}_1; \chi_1, \chi_1) + c_{\text{NS}}(\chi_1; \mathbf{u}_1, \chi_1) \\ &\quad - c_{\text{NS}}(\mathbf{U}_1; \zeta_1, \chi_1) - c_{\text{NS}}(\zeta_1; \mathbf{u}, \chi_1) \\ &\leq \frac{\mu}{5} \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_1)}^2 + C_7 \frac{\mathcal{R}_1 + \mathcal{R}_2}{\sqrt{2\mu}} \|\mathbf{D}(\chi_1)\|_{L^2(\Omega_1)}^2 \\ &\quad + C \frac{(\mathcal{R}_1 + \mathcal{R}_2)^2}{\mu^2} \|\nabla \zeta_1\|_{L^2(\Omega_1)}^2.\end{aligned}$$

The term  $a_{\text{NS}}(\boldsymbol{\zeta}_1, \boldsymbol{\chi}_1)$  is simply bounded using Cauchy-Schwarz and Young's inequalities.

$$a_{\text{NS}}(\boldsymbol{\zeta}_1, \boldsymbol{\chi}_1) \leq \frac{\mu}{5} \|\mathbf{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)}^2 + C\mu \|\mathbf{D}(\boldsymbol{\zeta}_1)\|_{L^2(\Omega_1)}^2.$$

The term  $b_{\text{NS}}(\boldsymbol{\zeta}_1, \xi_1)$  vanishes because of property (45). The term  $a_{\text{D}}(\eta_2, \xi_2)$  is bounded using standard DG techniques (see [24]) and the approximation property (47).

$$a_{\text{D}}(\eta_2, \xi_2) \leq \frac{\kappa}{4} \|\xi_2\|_{\Omega_2}^2 + Ch^{2k_2} \|\varphi_2\|_{H^{k_2+1}(\Omega_2)}^2.$$

Finally, the term  $b_{\text{NS}}(\boldsymbol{\chi}_1, \eta_1)$  is bounded as:

$$b_{\text{NS}}(\boldsymbol{\chi}_1, \eta_1) \leq \frac{\mu}{5} \|\mathbf{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)}^2 + \frac{C}{\mu} \|\eta_1\|_{L^2(\Omega_1)}^2.$$

Combining the results above, the error equation (65) becomes:

$$\begin{aligned} \left( \mu - \left( \frac{1}{2} C_1^3 C_2 C_4^2 + C_7 \right) \frac{\mathcal{R}_1 + \mathcal{R}_2}{\sqrt{2\mu}} \right) \|\mathbf{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)}^2 + \frac{\kappa}{2} \|\xi_2\|_{\Omega_2}^2 + \frac{1}{2G} \|\boldsymbol{\chi}_1 \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 \leq C \left( 1 + \frac{(\mathcal{R}_1 + \mathcal{R}_2)^2}{\mu^2} \right) \|\nabla \boldsymbol{\zeta}_1\|_{L^2(\Omega_1)}^2 \\ + C \frac{1}{\mu} \|\eta_2\|_{\Omega_2}^2 + Ch^{2k_2} \|\varphi_2\|_{H^{k_2+1}(\Omega_2)}^2 + C \frac{1}{\mu} \|\eta_1\|_{L^2(\Omega_1)}^2. \end{aligned}$$

The final result is obtained by using the approximation properties (44), (46), (48), a trace theorem and the inequalities:

$$\begin{aligned} \|\mathbf{D}(\mathbf{u}_1 - \mathbf{U}_1)\|_{L^2(\Omega_1)}^2 &\leq C \|\mathbf{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)}^2 + C \|\mathbf{D}(\boldsymbol{\zeta}_1)\|_{L^2(\Omega_1)}^2, \\ \|(\mathbf{u}_1 - \mathbf{U}_1) \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 &\leq C \|(\boldsymbol{\chi}_1) \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 + C \|(\boldsymbol{\zeta}_1) \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2, \\ \|p_2 - P_2\|_{\Omega_2}^2 &\leq C \|\xi_2\|_{\Omega_2}^2 + C \|\eta_2\|_{\Omega_2}^2. \end{aligned}$$

□

A straight consequence of Lemma 4 and Theorem 7 is a bound on the pressure error.

**Corollary 8.** *Under the assumptions of Theorem 7 and if the function  $p_{\text{D}}$  belongs to  $H^{k_2+1}(\Omega_2)$ , there exists a constant  $C$  independent of  $h$  and  $\mu$  such that*

$$\begin{aligned} \|p_2 - P_2\|_{\Omega_2}^2 &\leq C \left( 1 + \frac{(\mathcal{R}_1 + \mathcal{R}_2)^2}{\mu^2} \right) h^{2k_1} \|\mathbf{u}_1\|_{H^{k_1+1}(\Omega_1)}^2 + C \left( 1 + \frac{1}{\mu} \right) h^{2k_2} \|\varphi_2\|_{H^{k_2+1}(\Omega_2)}^2 \\ &\quad + Ch^{2k_2} \|p_{\text{D}}\|_{H^{k_2+1}(\Omega_2)}^2 + C \frac{1}{\mu} h^{2k_1} \|p_1\|_{H^{k_1}(\Omega_1)}^2. \end{aligned}$$

**Theorem 9.** *Under the assumptions of Theorem 7 and Corollary 8, there exists a constant  $C$  independent of  $h$  such that*

$$\|p_1 - P_1\|_{L^2(\Omega_1)} \leq Ch^{k_1} \|p_1\|_{H^{k_1}(\Omega_1)} + Ch^{k_1} \|\mathbf{u}_1\|_{H^{k_1+1}(\Omega_1)} + Ch^{k_2} (\|\varphi_2\|_{H^{k_2+1}(\Omega_2)} + \|p_{\text{D}}\|_{H^{k_2+1}(\Omega_2)}).$$

*Proof.* Using the same notation as in the proof of Theorem 7, we can rewrite the error equation by taking  $q_2 = 0$ :

$$\begin{aligned} b_{\text{NS}}(\mathbf{v}_1, \xi_1) &= b_{\text{NS}}(\mathbf{v}_1, \eta_1) + a_{\text{NS}}(\mathbf{u}_1 - \mathbf{U}_1, \mathbf{v}_1) - \frac{1}{2} (\mathbf{u}_1 \cdot \mathbf{u}_1 - \mathbf{U}_1 \cdot \mathbf{U}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ &\quad + c_{\text{NS}}(\mathbf{u}_1; \mathbf{u}_1, \mathbf{v}_1) - c_{\text{NS}}(\mathbf{U}_1; \mathbf{U}_1, \mathbf{v}_1) + (\varphi_2 - \Phi_2, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{G} ((\mathbf{u}_1 - \mathbf{U}_1) \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}}. \end{aligned}$$

We now bound all terms in the right-hand side. Cauchy-Schwarz's inequality yields simply

$$\begin{aligned} b_{\text{NS}}(\mathbf{v}_1, \eta_1) &\leq C \|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)} \|\eta_1\|_{L^2(\Omega_1)}, \\ a_{\text{NS}}(\mathbf{u}_1 - \mathbf{U}_1, \mathbf{v}_1) &\leq C \mu \|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)} \|\mathbf{D}(\mathbf{u}_1 - \mathbf{U}_1)\|_{L^2(\Omega_1)}. \end{aligned}$$

The nonlinear terms are handled like the term  $A_1$  in (67).

$$\begin{aligned} \frac{1}{2}(\mathbf{u}_1 \cdot \mathbf{u}_1 - \mathbf{U}_1 \cdot \mathbf{U}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} &= \frac{1}{2}(\mathbf{U}_1 \cdot \boldsymbol{\chi}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\boldsymbol{\chi}_1 \cdot \mathbf{u}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ &\quad - \frac{1}{2}(\mathbf{U}_1 \cdot \boldsymbol{\zeta}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{\zeta}_1 \cdot \mathbf{u}_1, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} \\ &\leq \frac{C(\mathcal{R}_1 + \mathcal{R}_2)}{\sqrt{\mu}} \|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)} (\|\mathbf{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)} + \|\nabla \boldsymbol{\zeta}_1\|_{L^2(\Omega_1)}), \\ c_{\text{NS}}(\mathbf{u}_1; \mathbf{u}_1, \mathbf{v}_1) - c_{\text{NS}}(\mathbf{U}_1; \mathbf{U}_1, \mathbf{v}_1) &= c_{\text{NS}}(\mathbf{U}_1; \boldsymbol{\chi}_1, \mathbf{v}_1) + c_{\text{NS}}(\boldsymbol{\chi}_1; \mathbf{u}_1, \mathbf{v}_1) \\ &\quad - c_{\text{NS}}(\mathbf{U}_1; \boldsymbol{\zeta}_1, \mathbf{v}_1) - c_{\text{NS}}(\boldsymbol{\zeta}_1; \mathbf{u}, \mathbf{v}_1) \\ &\leq \frac{C(\mathcal{R}_1 + \mathcal{R}_2)}{\sqrt{\mu}} \|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)} (\|\mathbf{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)} + \|\nabla \boldsymbol{\zeta}_1\|_{L^2(\Omega_1)}). \end{aligned}$$

Finally, the last two terms are bounded as:

$$\begin{aligned} (\varphi_2 - \Phi_2, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} &\leq C(\|\xi_2\|_{\Omega_2} + \|\eta_2\|_{L^2(\Gamma_{12})}) \|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)}, \\ \frac{1}{G}((\mathbf{u}_1 - \mathbf{U}_1) \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} &\leq C\|(\mathbf{u}_1 - \mathbf{U}_1) \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Omega_1)} \|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)}. \end{aligned}$$

Therefore, we obtain:

$$b_{\text{NS}}(\mathbf{v}_1, \xi_1) \leq C\Theta \|\nabla \mathbf{v}_1\|_{L^2(\Omega_1)},$$

with

$$\begin{aligned} \Theta &= \|\eta_1\|_{L^2(\Omega_1)} + \mu \|\mathbf{D}(\mathbf{u}_1 - \mathbf{U}_1)\|_{L^2(\Omega_1)} + \frac{\mathcal{R}_1 + \mathcal{R}_2}{\sqrt{\mu}} (\|\mathbf{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)} + \|\nabla \boldsymbol{\zeta}_1\|_{L^2(\Omega_1)}) \\ &\quad + \|\xi_2\|_{\Omega_2} + \|\eta_2\|_{L^2(\Gamma_{12})} + \|(\mathbf{u}_1 - \mathbf{U}_1) \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Omega_1)}. \end{aligned}$$

The inf-sup condition (37) then yields

$$\|\xi_1\|_{L^2(\Omega_1)} \leq \frac{C}{\beta_*} \Theta.$$

Using the approximation results (44), (46), (47) and Theorem 7, we can conclude.  $\square$

## 5 Conclusions

We prove existence and uniqueness of the solution under small data condition for the coupled system of Navier-Stokes and Darcy equations. We formulate a method that combines the classical conforming finite element method for Navier-Stokes with the discontinuous Galerkin method for Darcy. We obtain optimal error estimates with respect to the mesh size. The meshes on the interface can be non-matching. This is an attractive feature if one wants to implement the method using a domain decomposition approach.

## References

- [1] R. Adams. *Sobolev Spaces*. Academic Press, New-York, 1975.



- [2] D.N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.
- [3] D.N. Arnold, F. Brezzi, and M. Fortin. A stable finite element for the Stokes equations. *Calcolo*, 21:337–344, 1982.
- [4] G.S. Beavers and D.D. Joseph. Boundary conditions at a naturally impermeable wall. *J. Fluid. Mech.*, 30:197–207, 1967.
- [5] E. Burman and P. Hansbo. A unified stabilized method for Stokes and Darcy’s equations. *J. Computational and Applied Mathematics*, 198(1):35–51, 2007.
- [6] P. Ciarlet. *The finite element method for elliptic problems*. North-Holland, Amsterdam, 1978.
- [7] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *RAIRO Numerical Analysis*, 193(R-3):33–75, 1973.
- [8] C. Dawson, S. Sun, and M.F. Wheeler. Compatible algorithms for coupled flow and transport. *Comput. Meth. Appl. Mech. Eng.*, 193:2565–2580, 2004.
- [9] M. Discacciati, E. Miglio, and A. Quarteroni. Mathematical and numerical models for coupling surface and groundwater flows. *Appl. Numer. Math.*, 43:57–74, 2001.
- [10] M. Discacciati and A. Quarteroni. Analysis of a domain decomposition method for the coupling of Stokes and Darcy equations. In Brezzi et al, editor, *Numerical Analysis and Advanced Applications - ENUMATH 2001*, pages 3–20. Springer, Milan, 2003.
- [11] M. Discacciati, A. Quarteroni, and A. Valli. Robin-robin domain decomposition methods for the Stokes-Darcy coupling. *SIAM J. Numer. Anal.*, 45(3):1246–1268, 2007.
- [12] Y. Epshteyn and B. Rivière. Estimation of penalty parameters for symmetric interior penalty Galerkin methods. *Journal of Computational and Applied Mathematics*, 2006. Published online doi:10.1016/j.cam.2006.08.029.
- [13] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*, volume 5. Springer-Verlag, 1986.
- [14] V. Girault and B. Rivière. DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition. *SIAM Journal on Numerical Analysis*, 2007. Submitted.
- [15] V. Girault, B. Rivière, and M. Wheeler. A discontinuous Galerkin method with non-overlapping domain decomposition for the Stokes and Navier-Stokes problems. *Mathematics of Computation*, 74:53–84, 2004.
- [16] N.S. Hanspal, A.N. Waghode, V. Nassehi, and R.J. Wakeman. Numerical analysis of coupled Stokes/Darcy flows in industrial filtrations. *Transport in Porous Media*, 64(1):1573–1634, 2006.
- [17] P. Hood and C. Taylor. A numerical solution of the Navier-Stokes equations using the finite element technique. *Comp. and Fluids*, 1:73–100, 1973.
- [18] P. Houston, C. Schwab, and E. Süli. Discontinuous hp-finite element methods for advection-diffusion problems. *SIAM J. Numer. Anal.*, 39(6):2133–2163, 2002.
- [19] W. Jäger and A. Mikelić. On the interface boundary condition of Beavers, Joseph, and Saffman. *SIAM J. Appl. Math.*, 60:1111–1127, 2000.
- [20] W.J. Layton, F. Schieweck, and I. Yotov. Coupling fluid flow with porous media flow. *SIAM J. Numer. Anal.*, 40(6):2195–2218, 2003.

- [21] B. Rivière. Analysis of a discontinuous finite element method for the coupled Stokes and Darcy problems. *Journal of Scientific Computing*, 22:479–500, 2005.
- [22] B. Rivière. Analysis of a multi-numeric/multi-physics problem. *Numerical Mathematics and Advanced Applications*, pages 726–735, 2005.
- [23] B. Rivière, M.F. Wheeler, and V. Girault. Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. Part I. *Computational Geosciences*, 3:337–360, April 1999.
- [24] B. Rivière, M.F. Wheeler, and V. Girault. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *SIAM Journal on Numerical Analysis*, 39(3):902–931, 2001.
- [25] B. Rivière and I. Yotov. Locally conservative coupling of Stokes and Darcy flow. *SIAM J. Numer. Anal.*, 42:1959–1977, 2005.
- [26] P. Saffman. On the boundary condition at the surface of a porous media. *Stud. Appl. Math.*, 50:292–315, 1971.
- [27] M.F. Wheeler. An elliptic collocation-finite element method with interior penalties. *SIAM Journal on Numerical Analysis*, 15(1):152–161, 1978.