Analysis of time-dependent Navier-Stokes flow coupled with Darcy flow

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Abstract

This paper formulates and analyzes a weak solution to the coupling of time-dependent Navier-Stokes flow with Darcy flow under certain boundary conditions, one of them being the Beaver-Joseph-Saffman law on the interface. Existence and a priori estimates for the weak solution are shown under additional regularity assumptions. We introduce a fully discrete scheme with the unknowns being the Navier-Stokes velocity, pressure and the Darcy pressure. The scheme we propose is based on a finite element method in space and a Crank-Nicolson discretization in time where we obtain the solution at the first time step using a first order backward Euler method. Convergence of the scheme is obtained and optimal error estimates with respect to the mesh size are derived.

Keywords: time-dependent, Navier-Stokes, Darcy, Beaver-Joseph-Saffman's condition, Crank-Nicolson, backward Euler

1 Introduction

This work follows a series of papers on the coupling of surface flow with subsurface flow. The domain is divided into two subdomains: in the surface region, flow is characterized by the time-dependent Navier-Stokes equations and in the subsurface region, flow is characterized by the Darcy equations. The coupling of the two types of flow is accomplished through interface conditions. In this work, we define a weak solution and show its existence and uniqueness. We propose a numerical scheme that is second order in time and optimal in space. The underlying space discretization is the classical finite element method. The weak problem of a similar coupling is analyzed in [4], in which an interface problem with Steklov-Poincaré operators is formulated. In [9, 14], we analyze the steady-state problem of Navier-Stokes coupled with Darcy. We show well-posedness of the weak problem and convergence of the numerical algorithms. If the nonlinearity is removed from the Navier-Stokes equations, we obtain the coupling of Stokes and Darcy. This problem has been extensively studied in the literature. The reader can refer to [17, 12] for the analysis of the weak solution and to [21, 6, 11, 10, 20, 16, 2, 19] for a variety of numerical schemes.

We denote by $\Omega \subset \mathbb{R}^2$ a bounded domain decomposed into two disjoint domains Ω_1 and Ω_2. The fluid velocity and pressure in Ω_1 are denoted by $\boldsymbol{u}$ and p_1 respectively. The deformation tensor is

$$D(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T).$$

The flow in Ω_1 over the time interval $(0,T)$ is characterized by the time-dependent Navier-Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nabla \cdot (2\mu D(\boldsymbol{u}) - p_1 \boldsymbol{I}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{f}_1 \quad \text{in} \Omega_1 \times (0,T), \tag{1.1}$$

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$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega_1 \times (0,T). \tag{1.2}$$

The fluid pressure in Ω_2 is denoted by p_2. The flow in Ω_2 over the time interval $(0,T)$ is characterized by the Darcy equation:

$$-\nabla \cdot \boldsymbol{K}\nabla p_2 = f_2 \quad \text{in } \Omega_2 \times (0,T). \tag{1.3}$$

The coefficients in the equations are $\mu > 0$ the fluid viscosity, $\boldsymbol{f}_1$ a body force acting on $\Omega_1 \times [0,T]$, $\boldsymbol{K}$ a positive definite symmetric matrix corresponding to the permeability of Ω_2 and f_2 a body force acting on $\Omega_2 \times [0,T]$. The system of equations is completed by an initial condition $\boldsymbol{u} = \boldsymbol{u}_0$ at time $t = 0$, and a set of boundary conditions. Let $\partial\Omega_i$ denote the boundary of Ω_i with exterior unit normal $\boldsymbol{n}_{\Omega_i}$, let $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$ and let $\Gamma_i = \partial\Omega_i \backslash \Gamma_{12}$ for $i = 1,2$. We decompose the boundary $\Gamma_2 = \Gamma_{2D} \cup \Gamma_{2N}$ and we assume that $|\Gamma_{2D}| > 0$.

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_1 \times (0,T), \tag{1.4}$$

$$p_2 = 0 \text{ on } \Gamma_{2D} \times (0,T), \tag{1.5}$$

$$\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{\Omega_2} = 0 \text{ on } \Gamma_{2N} \times (0,T). \tag{1.6}$$

Let $\boldsymbol{n}_{12}$ be equal to $\boldsymbol{n}_{\Omega_1}$ on Γ_{12} and let $\boldsymbol{\tau}_{12}$ be the tangential unit vector to Γ_{12}. We assume continuity of the normal component of velocity across the interface:

$$\boldsymbol{u} \cdot \boldsymbol{n}_{12} = -\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{12}. \tag{1.7}$$

We assume that the Beaver-Joseph-Saffman law holds [5, 22] with a positive constant $G > 0$ (usually obtained from experimental data.)

$$\boldsymbol{u} \cdot \boldsymbol{\tau}_{12} = -2\mu G(D(\boldsymbol{u})\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12}. \tag{1.8}$$

Finally, we write the balance of forces across the interface by writing

$$((-2\mu D(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12} + \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}) = p_2. \tag{1.9}$$

The balance of forces includes the inertial forces. In [14], this new interface condition was considered in the steady-state coupling of Navier-Stokes with Darcy.

2 Weak Formulation

We define the following Sobolev spaces using the notation in [1]

$$\boldsymbol{X} = \{\boldsymbol{v} \in H^1(\Omega_1)^2 : \boldsymbol{v} = 0 \text{ on } \Gamma_1\},$$

$$M_1 = L^2(\Omega_1),$$

$$M_2 = \{q \in H^1(\Omega_2) : q = 0 \text{ on } \Gamma_{2D}\}.$$

In general, if $\boldsymbol{Z}$ is a Banach space, then the space $L^2(0,T;\boldsymbol{Z})$ denotes the space of square-integrable functions from $[0,T]$ into $\boldsymbol{Z}$. It is a Banach space with the norm $(\int_0^T \|\cdot\|_{\boldsymbol{Z}}^2 dt)^{1/2}$. For any domain $\mathcal{O}$, we denote by $(v,w)_{\mathcal{O}}$ the L^2 inner-product of two functions v, w defined on $\mathcal{O}$. We now define a form γ that takes into account the interface conditions as follows:

$$\forall \boldsymbol{u} \in \boldsymbol{X}, \quad \forall p \in M_2, \quad \gamma(\boldsymbol{u},p;\boldsymbol{v},q) = (p - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}.$$

Consequently, we observe that

$$\gamma(\boldsymbol{v},q;\boldsymbol{v},q) = -(\frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{v}), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}\|\boldsymbol{v}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}. \tag{2.1}$$

We propose the following weak formulation: Find $(\boldsymbol{u}, p_1, p_2) \in (L^2(0,T;\boldsymbol{X}) \cap H^1(0,T;L^2(\Omega_1)^2)) \times L^2(0,T;M_1) \times L^2(0,T;M_2)$ such that

$$(Q) \begin{cases} \forall (\boldsymbol{v}, q) \in \boldsymbol{X} \times M_2, & (\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + 2\mu(D(\boldsymbol{u}), D(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K} \nabla p_2, \nabla q)_{\Omega_2} \\ & + \gamma(\boldsymbol{u}, p_2; \boldsymbol{v}, q) = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}, \\ \forall q \in M_1, & (\nabla \cdot \boldsymbol{u}, q)_{\Omega_1} = 0, \\ \forall \boldsymbol{v} \in \boldsymbol{X}, & (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1} = (\boldsymbol{u}_0, \boldsymbol{v})_{\Omega_1}. \end{cases}$$

Lemma 2.1. *Assume that*

$$\boldsymbol{f}_1 \in L^2(0,T;L^2(\Omega_1)^2), f_2 \in L^2(0,T;L^2(\Omega_2)), K \in L^\infty(\Omega_2)^{2\times 2}, \tag{2.2}$$

and K is uniformly bounded and positive definite in Ω_2: there exist $\lambda_{min}, \lambda_{max} > 0$ such that

$$\lambda_{min}|x|^2 \leq Kx \cdot x \leq \lambda_{max}|x|^2 a.e\ x \in \Omega_2. \tag{2.3}$$

In addition, let $\boldsymbol{u}_0 \in L^2(\Omega_1)^2$. Then any solution $(\boldsymbol{u}, p_1, p_2) \in (L^2(0,T;\boldsymbol{X}) \cap H^1(0,T;L^2(\Omega_1))^2) \times L^2(0,T;M_1) \times L^2(0,T;M_2)$ of (1.1)-(1.9) is also a solution to (Q). Conversely any solution to (Q) satisfies (1.1)-(1.9).

Proof. First, we prove that if $(\boldsymbol{u}, p_1, p_2) \in (L^2(0,T;\boldsymbol{X}) \cap H^1(0,T;L^2(\Omega_1)^2)) \times L^2(0,T;M_1) \times L^2(0,T;M_2)$ is a solution to (1.1)-(1.9), then it satisfies problem (Q). Indeed, let $\boldsymbol{v} \in \boldsymbol{X}$. Then taking the scalar product of (1.1) with $\boldsymbol{v} \in \boldsymbol{X}$ over Ω_1 yields

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} - (\nabla \cdot (2\mu D(\boldsymbol{u}) - p_1 \mathbf{I}), \boldsymbol{v})_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1}.$$

Applying Green's formula to the second term and using the duality pairing $\langle \cdot, \cdot \rangle$, we obtain

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + (2\mu D(\boldsymbol{u}), \nabla \boldsymbol{v})_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v})_{\Omega_1} + \langle (-2\mu D(\boldsymbol{u}) + p_1 I)\boldsymbol{n}_{\Omega_1}, \boldsymbol{v} \rangle_{\partial \Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1}.$$

Since $D(\boldsymbol{u})$ is a symmetric tensor, we have

$$(D(\boldsymbol{u}), \nabla \boldsymbol{v})_{\Omega_1} = (D(\boldsymbol{u}), D(\boldsymbol{v}))_{\Omega_1}. \tag{2.4}$$

This and the assumption that $\boldsymbol{v} = 0$ on Γ_1 gives

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + (2\mu D(\boldsymbol{u}), \nabla \boldsymbol{v})_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v})_{\Omega_1} + \langle (-2\mu D(\boldsymbol{u}) + p_1 I)\boldsymbol{n}_{12}, \boldsymbol{v} \rangle_{\Gamma_{12}} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1}. \tag{2.5}$$

Now let $q \in M_2$. Taking the scalar product of (1.3) with q over Ω_2 yields

$$(-\nabla \cdot \boldsymbol{K} \nabla p_2, q)_{\Omega_2} = (f_2, q)_{\Omega_2}.$$

After applying Green's formula with the boundary condition (1.6) and the fact that $\boldsymbol{n}_{\Omega_2} = -\boldsymbol{n}_{12}$, we get

$$(\boldsymbol{K} \nabla p_2, \nabla q)_{\Omega_2} + \langle (\boldsymbol{K} \nabla p_2) \cdot \boldsymbol{n}_{12}, q \rangle_{\Gamma_{12}} = (f_2, q)_{\Omega_2}. \tag{2.6}$$

Adding (2.5) and (2.6) yields

$$\begin{aligned} &(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + (2\mu D(\boldsymbol{u}), D(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K} \nabla p_2, \nabla q)_{\Omega_2} - (p_1, \nabla \cdot \boldsymbol{v})_{\Omega_1} \\ &+ \langle (-2\mu D(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v} \rangle_{\Gamma_{12}} + \langle (\boldsymbol{K} \nabla p_2) \cdot \boldsymbol{n}_{12}, q \rangle_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}. \end{aligned} \tag{2.7}$$

We write $\boldsymbol{v}$ as a sum of its normal and tangential components, i.e.,

$$\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{n}_{12})\boldsymbol{n}_{12} + (\boldsymbol{v} \cdot \boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}.$$

From [14], we have

$$((2\mu D(\boldsymbol{u}) - p_1\boldsymbol{I})\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12} \in L^2(\Gamma_{12}),$$
$$2\mu(D(\boldsymbol{u})\boldsymbol{n}_{12})\cdot\boldsymbol{\tau}_{12} \in L^4(\Gamma_{12}).$$

So we can write

$$\langle(-2\mu D(\boldsymbol{u})+p_1\boldsymbol{I})\boldsymbol{n}_{12},\boldsymbol{v}\rangle_{\Gamma_{12}} = (((-2\mu D(\boldsymbol{u})+p_1\boldsymbol{I})\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12},\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + (((-2\mu D(\boldsymbol{u}))\boldsymbol{n}_{12})\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}.$$

Then by (1.8) and (1.9),

$$\langle(-2\mu D(\boldsymbol{u})+p_1\boldsymbol{I})\boldsymbol{n}_{12},\boldsymbol{v}\rangle_{\Gamma_{12}} = (p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}.$$

By (1.7), we also have

$$\langle(\boldsymbol{K}\nabla p_2)\cdot\boldsymbol{n}_{12},q\rangle_{\Gamma_{12}} = -(\boldsymbol{u}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}}.$$

So (2.7) becomes,

$$(\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v})_{\Omega_1}+(2\mu D(\boldsymbol{u}),D(\boldsymbol{v}))_{\Omega_1}+(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1}+(\boldsymbol{K}\nabla p_2,\nabla q)_{\Omega_2}-(p_1,\nabla\cdot\boldsymbol{v})_{\Omega_1}+\gamma(\boldsymbol{u},p_2;\boldsymbol{v},q) = (\boldsymbol{f}_1,\boldsymbol{v})_{\Omega_1}+(f_2,q)_{\Omega_2}.$$

Now let $q \in M_1$ and multiply (1.2) by q and integrate over Ω_1 to get $(\nabla\cdot\boldsymbol{u},q)_{\Omega_1} = 0$ which completes the weak formulation (Q). Conversely, assume that $(\boldsymbol{u},p_1,p_2) \in (L^2(0,T;\boldsymbol{X})\cap H^1(0,T;L^2(\Omega_1))^2 \times L^2(0,T;M_1)\times L^2(0,T;M_2)$ is a solution of (Q). As $\boldsymbol{u}(t) \in \boldsymbol{X}$ and $p_2(t) \in M_2$, by definition of the spaces, equations (1.4) and (1.5) are satisfied immediately. The assumption $(\nabla\cdot\boldsymbol{u},q)_{\Omega_1} = 0$ for all $q \in M_1$ gives (1.2). To get (1.1), let $\boldsymbol{v} \in \mathcal{D}(\Omega_1)^2$ and $q = 0$. We recall that $\mathcal{D}(\Omega_1)$ is the space of smooth functions with compact support. With this choice of $(\boldsymbol{v},q)$, the first equation in (Q) becomes

$$(\frac{\partial\boldsymbol{u}}{\partial t} - 2\mu\nabla\cdot D(\boldsymbol{u}) + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p_1,\boldsymbol{v})_{\Omega_1} = (\boldsymbol{f}_1,\boldsymbol{v})_{\Omega_1}.$$

Therefore in the sense of distributions on Ω_1

$$\frac{\partial\boldsymbol{u}}{\partial t} - 2\mu\nabla\cdot D(\boldsymbol{u}) + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p_1 = \boldsymbol{f}_1. \tag{2.8}$$

which gives (1.1). Next, letting $\boldsymbol{v} = 0$ and $q \in \mathcal{D}(\Omega_2)$ in (Q) yields

$$-(\nabla\cdot\boldsymbol{K}\nabla p_2,q)_{\Omega_2} = (f_2,q)_{\Omega_2}.$$

So in the distributional sense on Ω_2, we have

$$-\nabla\cdot\boldsymbol{K}\nabla p_2 = f_2. \tag{2.9}$$

which gives (1.3). Taking the scalar product of (2.8) with $\boldsymbol{v} \in \boldsymbol{X}$ yields

$$(\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v})_{\Omega_1} - (2\mu\nabla\cdot D(\boldsymbol{u}),\boldsymbol{v})_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1} + (\nabla p_1,\boldsymbol{v})_{\Omega_1} = (\boldsymbol{f}_1,\boldsymbol{v})_{\Omega_1}.$$

By Green's formula, we have

$$(\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v})_{\Omega_1}+(2\mu D(\boldsymbol{u}),\nabla\boldsymbol{v})_{\Omega_1}+(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1}-(p_1,\nabla\cdot\boldsymbol{v})_{\Omega_1}+\langle(-2\mu D(\boldsymbol{u})+p_1\mathbf{I})\boldsymbol{n}_{\Omega_1},\boldsymbol{v}\rangle_{\partial\Omega_1} = (\boldsymbol{f}_1,\boldsymbol{v})_{\Omega_1}. \tag{2.10}$$

Multiplying (2.9) by $q \in M_2$ and integrating over Ω_2 gives

$$(-\nabla\cdot\boldsymbol{K}\nabla p_2,q)_{\Omega_2} = (f_2,q)_{\Omega_2}.$$

As $q \in H^1(\Omega_2)$ by Green's formula, we have

$$(\boldsymbol{K}\nabla p_2, \nabla q)_{\Omega_2} - \langle(\boldsymbol{K}\nabla p_2)\cdot \boldsymbol{n}_{\Omega_2}, q\rangle_{\Omega_2} = (f_2, q)_{\Omega_2}. \tag{2.11}$$

Adding (2.10) and (2.11) and using (2.4) gives

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + (2\mu D(\boldsymbol{u}), D(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_{\Omega_1} - (p_1, \nabla\cdot\boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla p_2, \nabla q)_{\Omega_2}$$
$$+\langle(-2\mu D(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{\Omega_1}, \boldsymbol{v}\rangle_{\partial\Omega_1} + \langle-(\boldsymbol{K}\nabla p_2)\cdot\boldsymbol{n}_{\Omega_2}, q\rangle_{\partial\Omega_2} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}.$$

Comparing this with (Q), we end up with

$$\forall(\boldsymbol{v}, q) \in \boldsymbol{X}\times M_2, \quad (p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}$$
$$= \langle(-2\mu D(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{\Omega_1}, \boldsymbol{v}\rangle_{\partial\Omega_1} + \langle-(\boldsymbol{K}\nabla p_2)\cdot\boldsymbol{n}_{\Omega_2}, q\rangle_{\partial\Omega_2}. \tag{2.12}$$

Letting $\boldsymbol{v} = 0$ in (2.12)

$$(\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} = \langle\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{\Omega_2}, q\rangle_{\partial\Omega_2}. \tag{2.13}$$

Choosing $q = 0$ on Γ_{12} and as $q = 0$ on Γ_{2D}

$$\langle\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{\Omega_2}, q\rangle_{\Gamma_{2N}} = 0.$$

which implies (1.6), i.e., $\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{\Omega_2} = 0$ on Γ_{2N}.

Hence, since $\boldsymbol{n}_{\Omega_2} = -\boldsymbol{n}_{12}$ on Γ_{12} and $q = 0$ on Γ_{2D}, equation (2.13) becomes

$$(\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} = -\langle\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{12}, q\rangle_{\Gamma_{12}} \quad \forall q \in M_2.$$

Therefore we obtain (1.7). Next, we take $q = 0$ in (2.12):

$$\forall\boldsymbol{v} \in \boldsymbol{X}, \quad ((p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}))\boldsymbol{n}_{12} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}, \boldsymbol{v})_{\Gamma_{12}} = \langle(-2\mu D(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}}. \tag{2.14}$$

Thus we have

$$(-2\mu D(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12} = (p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}))\boldsymbol{n}_{12} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}. \tag{2.15}$$

in the sense of distributions on Γ_{12}. We obtain immediately:

$$((-2\mu D(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12})\cdot\boldsymbol{\tau}_{12} = p_2 - \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}),$$

and

$$((-2\mu D(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12} = p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}).$$

i.e., (1.8) and (1.9).

□

3 Existence and uniqueness of weak solution

We start this section by recalling Poincaré, Sobolev and trace inequalities. We use the notation $|\boldsymbol{v}|_{H^1(\Omega_1)} = \|\nabla\boldsymbol{v}\|_{L^2(\Omega_1)}$, which is a norm for $\boldsymbol{X}$. There exist constants $P_1, C_0, C_1, C_4, \tilde{C}_4$ which only depend on Ω_1 such that for all $\boldsymbol{v} \in \boldsymbol{X}$,

$$\|\boldsymbol{v}\|_{L^2(\Omega_1)} \le P_1|\boldsymbol{v}|_{H^1(\Omega_1)}, \ \|\boldsymbol{v}\|_{L^4(\Omega_1)} \le \tilde{C}_4|\boldsymbol{v}|_{H^1(\Omega_1)}, \ |\boldsymbol{v}|_{H^1(\Omega_1)} \le C_1\|D(\boldsymbol{v})\|_{L^2(\Omega_1)}, \tag{3.1}$$

$$\|\boldsymbol{v}\|_{L^2(\Gamma_{12})} \le C_0|\boldsymbol{v}|_{H^1(\Omega_1)},\ \|\boldsymbol{v}\|_{L^4(\Gamma_{12})} \le C_4|\boldsymbol{v}|_{H^1(\Omega_1)}. \tag{3.2}$$

There exist constants P_2 and $\tilde{C}_0$ that only depend on Ω_2 such that for all $q \in M_2$

$$\|q\|_{L^2(\Omega_2)} \le P_2|q|_{H^1(\Omega_2)}, \quad \|q\|_{L^2(\Gamma_{12})} \le \tilde{C}_0|q|_{H^1(\Omega_2)}. \tag{3.3}$$

In addition, from the assumption (2.3), we have

$$\frac{1}{\sqrt{\lambda_{max}}}\|\boldsymbol{K}^{1/2}\nabla q\|_{L^2(\Omega_2)} \le |q|_{H^1(\Omega_2)} \le \frac{1}{\sqrt{\lambda_{min}}}\|\boldsymbol{K}^{1/2}\nabla q\|_{L^2(\Omega_2)}. \tag{3.4}$$

Now denote by $\boldsymbol{Y}$ the product space $\boldsymbol{Y} = \boldsymbol{X} \times M_2$ equipped with the norm

$$\forall(\boldsymbol{v}, q) \in \boldsymbol{Y}, \quad \|(\boldsymbol{v}, q)\|_{\boldsymbol{Y}} = (2\mu\|D(\boldsymbol{v})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla q\|^2_{L^2(\Omega_2)})^{1/2}.$$

and the associated scalar product

$$\forall(\boldsymbol{v}, q), (\mathbf{w}, r) \in \boldsymbol{Y}, \quad ((\boldsymbol{v}, q), (\mathbf{w}, r))_{\boldsymbol{Y}} = 2\mu(D(\boldsymbol{v}), D(\boldsymbol{w}))_{\Omega_1} + (\boldsymbol{K}\nabla q, \nabla r)_{\Omega_2}.$$

Because of (3.1) and (3.4) the norm $\|(\cdot,\cdot)\|_{\boldsymbol{Y}}$ is equivalent to the following product norm

$$\forall(\boldsymbol{v}, q) \in \boldsymbol{Y}, \quad \|(\boldsymbol{v}, q)\| = (|\boldsymbol{v}|^2_{H^1(\Omega_1)} + |q|^2_{H^1(\Omega_2)})^{1/2}.$$

So $(\boldsymbol{Y}, \|(\cdot,\cdot)\|_{\boldsymbol{Y}})$ is a Hilbert space. Define the space of divergence free functions by

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{X} : \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega_1\},$$

and the associated subspace $\boldsymbol{W}$ of $\boldsymbol{Y}$ by $\boldsymbol{W} = \boldsymbol{V} \times M_2$. The space $\boldsymbol{W}$ is also a Hilbert space with the norm and scalar product of $\boldsymbol{Y}$. Restricting the test functions $\boldsymbol{v}$ to $\boldsymbol{V}$ in (Q), we obtain a second variational formulation: Find $(\boldsymbol{u}, p_2) \in (L^2(0, T; \boldsymbol{V}) \cap H^1(0, T; L^2(\Omega_1)^2) \times L^2(0, T; M_2)$ such that

$$(P)\begin{cases} \forall(\boldsymbol{v}, q) \in \boldsymbol{W}, & (\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + 2\mu(D(\boldsymbol{u}), D(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla p_2, \nabla q)_{\Omega_2} + \gamma(\boldsymbol{u}, p_2; \boldsymbol{v}, q) \\ & = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}, \\ \forall \boldsymbol{v} \in \boldsymbol{V}, & (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1} = (\boldsymbol{u}_0, \boldsymbol{v})_{\Omega_1}. \end{cases}$$

Clearly if $(\boldsymbol{u}, p_1, p_2)$ is a solution to (Q), then $(\boldsymbol{u}, p_2)$ is a solution to (P). We will now show existence of a solution to problem (P) using the Galerkin method. The spaces $\boldsymbol{V}$ and M_2 are separable Hilbert spaces as they are closed subspaces of separable Hilbert spaces $H^1(\Omega_1)^2$ and $H^1(\Omega_2)$. So we can find a basis $\{\boldsymbol{w}_i, r_i\}_{i\ge1}$ of $\boldsymbol{W}$ such that $\boldsymbol{w}_i \in \boldsymbol{V} \cap H^2(\Omega_1)^2$ and $r_i \in M_2 \cap H^2(\Omega_2)$. Fix $m \in \mathbb{N}$ and let $\boldsymbol{W}_m = \text{span}\{(\boldsymbol{w}_i, r_i), i = 1, \ldots, m\}$. Denote by π_m the orthogonal projection of $\boldsymbol{V}$ onto $\text{span}\{\boldsymbol{w}_i, i = 1, \ldots, m\}$. Then a Galerkin approximation to problem (P) is the finite-dimensional problem (P_m) defined as: Find $(\boldsymbol{u}_m, p_m) \in L^2(0, T; \boldsymbol{W}_m)$ with $\boldsymbol{u}_m \in H^1(0, T; L^2(\Omega_1)^2)$ such that

$$(P_m)\begin{cases} \forall 1 \le i \le m, & (\frac{\partial \boldsymbol{u}_m}{\partial t}, \boldsymbol{w}_i)_{\Omega_1} + 2\mu(D(\boldsymbol{u}_m), D(\boldsymbol{w}_i))_{\Omega_1} + (\boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m, \boldsymbol{w}_i)_{\Omega_1} + (\boldsymbol{K}\nabla p_m, \nabla r_i)_{\Omega_2} \\ & +\gamma(\boldsymbol{u}_m, p_m; \boldsymbol{w}_i, r_i) = (\boldsymbol{f}_1, \boldsymbol{w}_i)_{\Omega_1} + (f_2, r_i)_{\Omega_2}, \\ \forall 1 \le i \le m, & (\boldsymbol{u}_m(0), \boldsymbol{w}_i)_{\Omega_1} = (\pi_m\boldsymbol{u}_0, \boldsymbol{w}_i)_{\Omega_1}. \end{cases}$$

We want to show the existence of a unique solution to (P_m) and also a uniform bound for the solution. We look for a solution $(\boldsymbol{u}_m, p_m)$ of the form

$$\boldsymbol{u}_m(t)(x) = \boldsymbol{u}_m(x, t) = \sum_{j=1}^m a_j^m(t)\boldsymbol{w}_j(x), \quad p_m(t)(x) = p_m(x, t) = \sum_{j=1}^m b_j^m(t)r_j(x).$$

where we wish to select a_j^m and b_j^m so that (P_m) is satisfied. With these $\boldsymbol{u}_m$ and p_m, problem (P_m) becomes

$$\forall 1 \le i \le m, \quad \sum_{j=1}^m \frac{d}{dt}a_j^m(\boldsymbol{w}_j, \boldsymbol{w}_i)_{\Omega_1} + 2\mu\sum_{j=1}^m a_j^m(D(\boldsymbol{w}_j), D(\boldsymbol{w}_i))_{\Omega_1} + \sum_{j=1}^m\sum_{k=1}^m a_j^m a_k^m(\boldsymbol{w}_j\cdot\nabla\boldsymbol{w}_k, \boldsymbol{w}_i)_{\Omega_1}$$

$$+\sum_{j=1}^{m} b_j^m(\boldsymbol{K}\nabla r_j, \nabla r_i)_{\Omega_2} + \sum_{j=1}^{m} b_j^m(r_j, \boldsymbol{w}_i\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} a_j^m a_k^m(\boldsymbol{w}_j\cdot\boldsymbol{w}_k, \boldsymbol{w}_i\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$$

$$+\frac{1}{G}\sum_{j=1}^{m} a_j^m(\boldsymbol{w}_j\cdot\boldsymbol{\tau}_{12}, \boldsymbol{w}_i\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - \sum_{j=1}^{m} a_j^m(\boldsymbol{w}_j\cdot\boldsymbol{n}_{12}, r_i)_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{w}_i)_{\Omega_1} + (f_2, r_i)_{\Omega_2},$$

$$\forall 1\le i\le m, \quad \sum_{j=1}^{m} a_j^m(0)(\boldsymbol{w}_j, \boldsymbol{w}_i)_{\Omega_1} = (\pi_m\boldsymbol{u}_0, \boldsymbol{w}_i)_{\Omega_1}.$$

We rewrite the system in matrix form and define the following mass and stiffness matrices:

$$\boldsymbol{M} = ((\boldsymbol{w}_j, \boldsymbol{w}_i)_{\Omega_1})_{1\le i,j\le m}, \quad \mathbf{A}_1 = (2\mu(D(\boldsymbol{w}_j), D(\boldsymbol{w}_i))_{\Omega_1})_{1\le i,j\le m}, \quad \mathbf{A}_2 = ((\boldsymbol{K}\nabla r_j, \nabla r_i)_{\Omega_2})_{1\le i,j\le m},$$

$$\mathbf{B} = ((r_j, \boldsymbol{w}_i\cdot\boldsymbol{n}_{12})_{\Gamma_{12}})_{1\le i,j\le m}, \quad \mathbf{C} = \frac{1}{G}((\boldsymbol{w}_j\cdot\boldsymbol{\tau}_{12}, \boldsymbol{w}_i\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}})_{1\le i,j\le m},$$

$$\forall 1\le i\le m, \quad \mathbf{N}_i = (\alpha_{ijk})_{1\le j,k\le m}, \quad \text{with} \quad \alpha_{ijk} = (\boldsymbol{w}_j\cdot\nabla\boldsymbol{w}_k, \boldsymbol{w}_i)_{\Omega_1} - \frac{1}{2}(\boldsymbol{w}_j\cdot\boldsymbol{w}_k, \boldsymbol{w}_i\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}.$$

We thus obtain a first order nonhomogeneous nonlinear system of ordinary differential equations

$$\begin{cases} \mathbf{M}\frac{d\boldsymbol{a}}{dt} + (\mathbf{A}_1 + \mathbf{C})\boldsymbol{a} + \mathbf{B}\boldsymbol{b} = \mathbf{g}_1(\boldsymbol{a}), \\ \mathbf{A}_2\boldsymbol{b} - \mathbf{B}^T\boldsymbol{a} = \mathbf{g}_2, \\ \mathbf{M}\boldsymbol{a}(0) = \mathbf{g}_3. \end{cases} \tag{3.5}$$

where

$$\boldsymbol{a} = \begin{pmatrix} a_1^m \\ \vdots \\ a_m^m \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1^m \\ \vdots \\ b_m^m \end{pmatrix},$$

and the right hand side vectors are

$$\mathbf{g}_1(\boldsymbol{a}) = \begin{pmatrix} (\boldsymbol{f}_1, \boldsymbol{w}_1)_{\Omega_1} - \mathbf{N}_1\boldsymbol{a}\cdot\boldsymbol{a} \\ \vdots \\ (\boldsymbol{f}_1, \boldsymbol{w}_m)_{\Omega_1} - \mathbf{N}_m\boldsymbol{a}\cdot\boldsymbol{a} \end{pmatrix}, \quad \mathbf{g}_2 = \begin{pmatrix} (f_2, r_1)_{\Omega_2} \\ \vdots \\ (f_2, r_m)_{\Omega_2} \end{pmatrix}, \quad \mathbf{g}_3 = \begin{pmatrix} (\pi_m\boldsymbol{u}_0, \boldsymbol{w}_1)_{\Omega_2} \\ \vdots \\ (\pi_m\boldsymbol{u}_0, \boldsymbol{w}_m)_{\Omega_2} \end{pmatrix}.$$

As the $\boldsymbol{w}_i$'s are linearly independent, the Gram matrix $\mathbf{M}$ is invertible and positive definite. The matrix $\mathbf{A}_2$ is also invertible as the r_i's are linearly independent. Thus we can solve for $\boldsymbol{b}$ in (3.5) as $\boldsymbol{b} = \mathbf{A}_2^{-1}(\mathbf{B}^T\boldsymbol{a} + \mathbf{g}_2)$, substitute this expression in the first equation and multiply by the inverse of $\mathbf{M}$:

$$\begin{cases} \frac{d\boldsymbol{a}}{dt} + \mathbf{M}^{-1}(\mathbf{A}_1 + \mathbf{C} + \mathbf{B}\mathbf{A}_2^{-1}\mathbf{B}^T)\boldsymbol{a} = \mathbf{M}^{-1}(\mathbf{g}_1(\boldsymbol{a}) - \mathbf{B}\mathbf{A}_2^{-1}\mathbf{g}_2), \\ \boldsymbol{a}(0) = \mathbf{M}^{-1}\mathbf{g}_3. \end{cases} \tag{3.6}$$

By Caratheodory's theorem [8], this system has a maximal solution $\boldsymbol{a}$ defined on some interval $[0, t_m]$. We will show a priori bounds on the solution. This will imply that $t_m = T$.

Once the solution $\boldsymbol{a}$ is obtained, we have a unique solution $\boldsymbol{b} = \mathbf{A}_2^{-1}(\mathbf{B}^T\boldsymbol{a} + \mathbf{g}_2)$.

Choosing $\boldsymbol{w}_i = \boldsymbol{u}_m$ and $r_i = p_m$ in (P_m) yields,

$$\left(\frac{\partial\boldsymbol{u}_m}{\partial t}, \boldsymbol{u}_m\right)_{\Omega_1} + 2\mu(D(\boldsymbol{u}_m), D(\boldsymbol{u}_m))_{\Omega_1} + (\boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m, \boldsymbol{u}_m)_{\Omega_1} + (\boldsymbol{K}\nabla p_m, \nabla p_m)_{\Omega_2}$$
$$+ \gamma(\boldsymbol{u}_m, p_m; \boldsymbol{u}_m, p_m) = (\boldsymbol{f}_1, \boldsymbol{u}_m)_{\Omega_1} + (f_2, p_m)_{\Omega_2}. \tag{3.7}$$

Observe that $\nabla(\boldsymbol{u}_m\cdot\boldsymbol{u}_m) = \nabla\boldsymbol{u}_m\cdot\boldsymbol{u}_m + \boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m = 2\boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m$. By Green's theorem

$$(\nabla\cdot\boldsymbol{u}_m, \boldsymbol{u}_m\cdot\boldsymbol{u}_m)_{\Omega_1} = -(\boldsymbol{u}_m, \nabla(\boldsymbol{u}_m\cdot\boldsymbol{u}_m))_{\Omega_1} + (\boldsymbol{u}_m\cdot\boldsymbol{n}_{\Omega_1}, \boldsymbol{u}_m\cdot\boldsymbol{u}_m)_{\partial\Omega_1}$$

$$= -2(\boldsymbol{u}_m, \boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m)_{\Omega_1} + (\boldsymbol{u}_m \cdot \boldsymbol{n}_{\Omega_1}, \boldsymbol{u}_m \cdot \boldsymbol{u}_m)_{\partial\Omega_1}.$$

for all $\boldsymbol{u}_m \in \boldsymbol{V}$. Therefore as $\nabla \cdot \boldsymbol{u}_m = 0$ and $\boldsymbol{u}_m = 0$ on Γ_1,

$$(\boldsymbol{u}_m, \boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m)_{\Omega_1} = \frac{1}{2}(\boldsymbol{u}_m \cdot \boldsymbol{n}_{12}, \boldsymbol{u}_m \cdot \boldsymbol{u}_m)_{\Gamma_{12}}.$$

From (2.1) and (3.7), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)} + 2\mu\|D(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla p_m\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{u}_m \cdot \boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}$$
$$= (\boldsymbol{f}_1, \boldsymbol{u}_m)_{\Omega_1} + (f_2, p_m)_{\Omega_2}.$$

The terms on the right-hand side are bounded using Cauchy-Schwarz's inequality and the inequalities (3.1)-(3.4)

$$\begin{aligned}(\boldsymbol{f}_1, \boldsymbol{u}_m)_{\Omega_1} + (f_2, p_m)_{\Omega_2} &\leq \|\boldsymbol{f}_1\|_{L^2(\Omega_1)} P_1 |\boldsymbol{u}_m|_{H^1(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)} P_2 |p_m|_{H^1(\Omega_2)} \\ &\leq \|\boldsymbol{f}_1\|_{L^2(\Omega_1)} P_1 C_1 \|D(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)} P_2 \tfrac{1}{\sqrt{\lambda_{min}}} \|\boldsymbol{K}^{1/2}\nabla p_m\|_{L^2(\Omega_2)} \\ &\leq \tfrac{1}{4\mu} P_1^2 C_1^2 \|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + \mu\|D(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \tfrac{1}{2}\tfrac{P_2^2}{\lambda_{min}}\|f_2\|^2_{L^2(\Omega_2)} + \tfrac{1}{2}\|\boldsymbol{K}^{1/2}\nabla p_m\|^2_{L^2(\Omega_2)}.\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)} + \mu\|D(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \frac{1}{2}\|\boldsymbol{K}^{1/2}\nabla p_m\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{u}_m \cdot \boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}$$
$$\leq \frac{1}{4\mu} P_1^2 C_1^2 \|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + \frac{1}{2}\frac{P_2^2}{\lambda_{min}}\|f_2\|^2_{L^2(\Omega_2)}. \quad (3.8)$$

Multiplying (3.8) by 2 and integrating from 0 to t, we conclude that

$$\|\boldsymbol{u}_m(t)\|_{L^2(\Omega_1)} \leq \mathcal{C}_e, \quad (3.9)$$

with

$$\mathcal{C}_e = (\|\boldsymbol{u}_0\|^2_{L^2(\Omega_1)} + \frac{1}{2\mu} P_1^2 C_1^2 \|\boldsymbol{f}_1\|^2_{L^2(0,T;L^2(\Omega_1))} + \frac{P_2^2}{\lambda_{min}}\|f_2\|^2_{L^2(0,T;L^2(\Omega_2))})^{1/2}. \quad (3.10)$$

Again multiplying (3.8) by 2 and integrating this time from 0 to T we obtain $\|(\boldsymbol{u}_m, p_m)\|_{L^2(0,T;\boldsymbol{Y})} \leq \mathcal{C}_e$. This a priori bound implies existence of a solution to (3.6) on the interval $(0, T)$. We summarize what we have so far by the following theorem:

Theorem 3.1. *Under the assumptions of Lemma 2.1 there exists a solution $(\boldsymbol{u}_m, p_m) \in \boldsymbol{W}_m$ to the problem (P_m) satisfying*

$$\sup_{t\in[0,T]} \|\boldsymbol{u}_m(t)\|_{L^2(\Omega_1)} + \|(\boldsymbol{u}_m, p_m)\|_{L^2(0,T;\boldsymbol{Y})} \leq \mathcal{C}_e, \quad (3.11)$$

where $\mathcal{C}_e$ is the constant independent of m defined explicitly by (3.10).

We now pass to the limit to obtain a solution for the problem (P). The sequence $\{(\boldsymbol{u}_m, p_m)\}_{m\in\mathbb{N}}$ is bounded in $L^2(0, T, \boldsymbol{W})$. Since $\boldsymbol{W}$ is a Hilbert space, it is reflexive and so is $L^2(0, T, \boldsymbol{W})$. Hence we can find a subsequence still denoted by $\{(\boldsymbol{u}_m, p_m)\}_{m\in\mathbb{N}}$ and a pair $(\boldsymbol{u}, p_2) \in L^2(0, T; \boldsymbol{W})$ such that

$$\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{weakly in} \quad L^2(0, T; \boldsymbol{V}), \quad (3.12)$$
$$p_m \to p_2 \quad \text{weakly in} \quad L^2(0, T; M_2). \quad (3.13)$$

Also by Banach-Alaoglu Theorem [18] since $\{\boldsymbol{u}_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(0, T; L^2(\Omega_1)^2)$, there exists a further subsequence, still denoted by $\{\boldsymbol{u}_m\}_{m\in\mathbb{N}}$ such that for some $\boldsymbol{u}^* \in L^\infty(0, T; L^2(\Omega_1)^2)$

$$\boldsymbol{u}_m \to \boldsymbol{u}^* \quad \text{in weak}^* \text{ topology of} \quad L^\infty(0, T; L^2(\Omega_1)^2), \quad (3.14)$$

i.e.,

$$\int_0^T (\boldsymbol{u}_m(t) - \boldsymbol{u}^*(t), \boldsymbol{v}(t))_{\Omega_1} dt \to 0, \quad \forall \boldsymbol{v} \in L^1(0,T; L^2(\Omega_1)^2) \supset L^2(0,T; L^2(\Omega_1)^2). \tag{3.15}$$

By (3.12), we have

$$\int_0^T \langle \boldsymbol{u}_m(t) - \boldsymbol{u}(t), \boldsymbol{v}(t)\rangle_{\Omega_1} dt \to 0, \quad \forall \boldsymbol{v} \in L^2(0,T; \boldsymbol{V}') \supset L^2(0,T; L^2(\Omega_1)^2) \tag{3.16}$$

which implies

$$\int_0^T (\boldsymbol{u}_m(t) - \boldsymbol{u}(t), \boldsymbol{v}(t))_{\Omega_1} dt \to 0, \quad \forall \boldsymbol{v} \in L^2(0,T; L^2(\Omega_1)^2) \tag{3.17}$$

Therefore comparing (3.15) and (3.17)

$$\forall \boldsymbol{v} \in L^2(0,T; L^2(\Omega_1)^2), \qquad \int_0^T (\boldsymbol{u}(t) - \boldsymbol{u}^*(t), \boldsymbol{v}(t))_{\Omega_1} \to 0.$$

So

$$\boldsymbol{u} = \boldsymbol{u}^* \in L^2(0,T; \boldsymbol{V}) \cap L^\infty(0,T; L^2(\Omega_1)^2). \tag{3.18}$$

To pass to the limit in (P_m) with the subsequence we extracted consider $\Psi : [0,T] \to \mathbb{R}$ such that $\Psi(T) = 0$ and $\Psi \in \mathcal{C}^1[0,T]$. Multiply the first term in the first equation in (P_m) by $\Psi(t)$ and integrate from 0 to T. Apply integration by parts

$$\int_0^T (\boldsymbol{u}_m'(t), \boldsymbol{w}_j)_{\Omega_1} \Psi(t) dt = -\int_0^T (\boldsymbol{u}_m(t), \boldsymbol{w}_j)_{\Omega_1} \Psi'(t) dt + (\boldsymbol{u}_m(t), \boldsymbol{w}_j)_{\Omega_1} \Psi(t)|_0^T$$

$$= -\int_0^T (\boldsymbol{u}_m(t), \boldsymbol{w}_j)_{\Omega_1} \Psi'(t) dt - (\boldsymbol{u}_m(0), \boldsymbol{w}_j)_{\Omega_1} \Psi(0).$$

So the first equation in (P_m) becomes (as $\boldsymbol{u}_m(0) = \pi_m \boldsymbol{u}_0$)

$$-\int_0^T (\boldsymbol{u}_m(t), \Psi'(t)\boldsymbol{w}_j)_{\Omega_1} dt - (\pi_m \boldsymbol{u}_0, \boldsymbol{w}_j)_{\Omega_1} \Psi(0) + 2\mu \int_0^T (D(\boldsymbol{u}_m), \Psi(t) D(\boldsymbol{w}_j))_{\Omega_1} dt$$

$$+ \int_0^T (\boldsymbol{u}_m(t) \cdot \nabla \boldsymbol{u}_m(t), \Psi(t)\boldsymbol{w}_i)_{\Omega_1} dt + \int_0^T (\boldsymbol{K} \nabla p_m(t), \Psi(t) \nabla r_i)_{\Omega_1} dt$$

$$+ \int_0^T (p_m(t) - \frac{1}{2}(\boldsymbol{u}_m(t) \cdot \boldsymbol{u}_m(t)), \Psi(t)\boldsymbol{w}_j \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} dt + \frac{1}{G} \int_0^T (\boldsymbol{u}_m(t) \cdot \boldsymbol{\tau}_{12}, \Psi(t)\boldsymbol{w}_i \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} dt$$

$$- \int_0^T (\boldsymbol{u}_m(t) \cdot \boldsymbol{n}_{12}, \Psi(t) r_i)_{\Gamma_{12}} dt = \int_0^T (\boldsymbol{f}_1(t), \Psi(t)\boldsymbol{w}_i)_{\Omega_1} dt + \int_0^T (f_2(t), \Psi(t) r_i) \Psi(t) dt.$$

By (3.14), (3.17), (3.18) and as $\boldsymbol{u}_m(0) = \pi_m \boldsymbol{u}_0 \to \boldsymbol{u}_0$ strongly in $L^2(\Omega_1)$, letting $m \to \infty$, for all $j \in \{1, ..., m\}$ we can replace $\boldsymbol{u}_m$ and p_m with $\boldsymbol{u}$ and p_2 in the linear terms and $\pi_m(0)$ with $\boldsymbol{u}_0$. For the nonlinear terms and the interface terms observe that by Sobolev imbeddings for any $1 \le s < \infty$, we can extract another subsequence $(\boldsymbol{u}_m, p_m)$ such that for any $1 \le s < \infty$

$$\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{strongly in} \quad L^2(0,T; L^s(\Omega_1)^2), \tag{3.19}$$

Observe also that for any $\boldsymbol{u} \in \boldsymbol{V}$ and any $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}$ we have

$$(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w}) = -(\boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{v})$$

Indeed,

$$(\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w})=\int_{\Omega_1}\boldsymbol{u}_i\boldsymbol{v}_{j,i}\boldsymbol{w}_j dx=-\int_{\Omega_1}\boldsymbol{u}_{i,i}\boldsymbol{v}_j\boldsymbol{w}_j dx-\int_{\Omega_1}\boldsymbol{u}_i\boldsymbol{v}_j\boldsymbol{w}_{j,i}dx+\int_{\partial\Omega_1}\boldsymbol{u}_i\boldsymbol{n}_i\boldsymbol{v}_j\boldsymbol{w}_j dx$$
$$=-\int_{\Omega_1}\boldsymbol{u}_i\boldsymbol{v}_j\boldsymbol{w}_{j,i}dx+\int_{\Gamma_{12}}\boldsymbol{u}_i\boldsymbol{n}_i\boldsymbol{v}_j\boldsymbol{w}_j dx=\int_{\Omega_1}\boldsymbol{u}_i\boldsymbol{w}_{j,i}\boldsymbol{v}_j dx=-(\boldsymbol{u}\cdot\nabla\boldsymbol{w},\boldsymbol{v})$$

Hence by (3.12) and (3.19) we have

$$\int_0^T(\boldsymbol{u}_m(t)\cdot\nabla\boldsymbol{u}_m(t),\Psi(t)\boldsymbol{w}_i)_{\Omega_1}dt=-\int_0^T(\boldsymbol{u}_m(t)\cdot\Psi(t)\nabla\boldsymbol{w}_i,\boldsymbol{u}_m(t))_{\Omega_1}dt$$
$$\to-\int_0^T(\boldsymbol{u}(t)\cdot\Psi(t)\nabla\boldsymbol{w}_i,\boldsymbol{u}(t))_{\Omega_1}dt=-\int_0^T(\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t),\Psi(t)\boldsymbol{w}_i)_{\Omega_1}dt$$

By the continuity of the trace operator from $H^1(\Omega_i)$ to $H^{1/2}(\partial\Omega_i)$ in the weak topology we have

$$\boldsymbol{u}_m|_{\partial\Omega_1}\to\boldsymbol{u}|_{\partial\Omega_1}\quad\text{weakly in}\quad L^2(0,T;H^{1/2}(\partial\Omega_1))^2,\tag{3.20}$$
$$p_m|_{\partial\Omega_2}\to p_2|_{\partial\Omega_2}\quad\text{weakly in}\quad L^2(0,T;H^{1/2}(\partial\Omega_2)).\tag{3.21}$$

Hence again by Sobolev imbeddings after extracting another subsequence

$$\boldsymbol{u}_m|_{\partial\Omega_1}\to\boldsymbol{u}|_{\partial\Omega_1}\quad\text{strongly in}\quad L^2(0,T;L^4(\partial\Omega_1))^2,\tag{3.22}$$

which will take care of the interface terms.

Finally we have

$$-\int_0^T(\boldsymbol{u}(t),\boldsymbol{w}_j)_{\Omega_1}\Psi'(t)dt+(\boldsymbol{u}_0,\boldsymbol{w}_j)_{\Omega_1}\Psi(0)+2\mu\int_0^T(D(\boldsymbol{u}),D(\boldsymbol{w}_j))_{\Omega_1}\Psi(t)dt+\int_0^T(\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t),\boldsymbol{w}_i)_{\Omega_1}\Psi(t)dt$$
$$+\int_0^T(\boldsymbol{K}\nabla p_2(t),\nabla r_j)_{\Omega_1}\Psi(t)dt+\int_0^T(p_2(t)-\frac{1}{2}(\boldsymbol{u}(t)\cdot\boldsymbol{u}(t)),\boldsymbol{w}_j\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\Psi(t)dt+\frac{1}{G}\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{\tau}_{12},\boldsymbol{w}_j\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}\Psi(t)dt$$
$$-\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{n}_{12},r_j)_{\Gamma_{12}}\Psi(t)dt=\int_0^T(\boldsymbol{f}_1(t),\boldsymbol{w}_j)_{\Omega_1}\Psi(t)dt+\int_0^T(f_2(t),r_j)\Psi(t)dt.\tag{3.23}$$

The second equation in P_m is true for $\boldsymbol{u}$ and $\boldsymbol{u}_0$ as $\pi_m\boldsymbol{u}_0\to\boldsymbol{u}_0$ strongly in $L^2(\Omega_1)^2$, i.e., letting $m\to\infty$ in $(\boldsymbol{u}_m(0),\boldsymbol{w}_j)=(\pi_m\boldsymbol{u}_0,\boldsymbol{w}_j)$ we obtain

$$(\boldsymbol{u}(0),\boldsymbol{w}_j)=(\boldsymbol{u}_0,\boldsymbol{w}_j),\ \forall j\in\{1,...,m\}.$$

(3.23) holds for any $\boldsymbol{v}\in\text{span}\{(\boldsymbol{w}_i,r_i)\}_{i=1}^m$. We have chosen $\{(\boldsymbol{w}_i,r_i)\}_{i\in\mathbb{N}}$ to be total in $\boldsymbol{W}$. So any $(\boldsymbol{v},q)\in\boldsymbol{W}$ can be approximated by elements of $\boldsymbol{W}_m$'s. Therefore, for any $(v,q)\in\boldsymbol{W}$,

$$-\int_0^T(\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\Psi'(t)dt+(\boldsymbol{u}_0,\boldsymbol{v})_{\Omega_1}\Psi(0)+2\mu\int_0^T(D(\boldsymbol{u}),D(\boldsymbol{v}))_{\Omega_1}\Psi(t)dt+\int_0^T(\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt$$
$$+\int_0^T(\boldsymbol{K}\nabla p_2(t),\nabla q)_{\Omega_1}\Psi(t)dt+\int_0^T(p_2(t)-\frac{1}{2}(\boldsymbol{u}(t)\cdot\boldsymbol{u}(t)),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\Psi(t)dt+\frac{1}{G}\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}\Psi(t)dt$$
$$-\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}}\Psi(t)dt=\int_0^T(\boldsymbol{f}_1(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt+\int_0^T(f_2(t),q)_{\Omega_2}\Psi(t)dt.\tag{3.24}$$

As $D(0,T)\subset\mathcal{C}^1[0,T]$ contains functions which vanish at both 0 and T, restricting Ψ to $D(0,T)$ to get rid of the term with $\Psi(0)$, we get

$$-\int_0^T(\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\Psi'(t)dt+2\mu\int_0^T(D(\boldsymbol{u}),D(\boldsymbol{v}))_{\Omega_1}\Psi(t)dt+\int_0^T(\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt$$

$$+\int_0^T(\boldsymbol{K}\nabla p_2(t),\nabla q)_{\Omega_1}\Psi(t)dt+\int_0^T(p_2(t)-\frac{1}{2}(\boldsymbol{u}(t)\cdot\boldsymbol{u}(t)),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\Psi(t)dt+\frac{1}{G}\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}\Psi(t)dt$$

$$-\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}}\Psi(t)dt=\int_0^T(\boldsymbol{f}_1(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt+\int_0^T(f_2(t),q)_{\Omega_2}\Psi(t)dt.$$

By the definition of weak derivatives,

$$-\int_0^T(\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\Psi'(t)dt=\int_0^T(\boldsymbol{u}'(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt.$$

So, for any $\Psi\in D(0,T)$,

$$\int_0^T(\boldsymbol{u}'(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt+2\mu\int_0^T(D(\boldsymbol{u}),D(\boldsymbol{v}))_{\Omega_1}\Psi(t)dt+\int_0^T(\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt$$

$$+\int_0^T(\boldsymbol{K}\nabla p_2(t),\nabla q)_{\Omega_1}\Psi(t)dt+\int_0^T(p_2(t)-\frac{1}{2}(\boldsymbol{u}(t)\cdot\boldsymbol{u}(t)),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\Psi(t)dt+\frac{1}{G}\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}\Psi(t)dt$$

$$-\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}}\Psi(t)dt=\int_0^T(\boldsymbol{f}_1(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt+\int_0^T(f_2(t),q)_{\Omega_2}\Psi(t)dt.$$

Hence for all $(\boldsymbol{v},q)\in\boldsymbol{W}$,

$$(\boldsymbol{u}'(t),\boldsymbol{v})_{\Omega_1}+2\mu(D(\boldsymbol{u}),D(\boldsymbol{v}))_{\Omega_1}+(\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}+(\boldsymbol{K}\nabla p_2(t),\nabla q)_{\Omega_1}+(p_2(t)-\frac{1}{2}(\boldsymbol{u}(t)\cdot\boldsymbol{u}(t)),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$$
$$+\frac{1}{G}(\boldsymbol{u}(t)\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}-(\boldsymbol{u}(t)\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}}=(\boldsymbol{f}_1(t),\boldsymbol{v})_{\Omega_1}+(f_2(t),q)_{\Omega_2}\tag{3.25}$$

in the distributional sense.
To see $\boldsymbol{u}_0=\boldsymbol{u}(0)$ we multiply this with $\Psi\in\mathcal{C}^1[0,T]$ such that $\Psi(T)=0$. Then integration by parts yields

$$\int_0^T(\boldsymbol{u}'(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt=-\int_0^T(\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\Psi'(t)dt-(\boldsymbol{u}(0),\boldsymbol{v})_{\Omega_1}\Psi(0).$$

So,

$$-\int_0^T(\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\Psi'(t)dt-(\boldsymbol{u}(0),\boldsymbol{v})_{\Omega_1}\Psi(0)+2\mu\int_0^T(D(\boldsymbol{u}),D(\boldsymbol{v}))_{\Omega_1}\Psi(t)dt+\int_0^T(\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt$$

$$+\int_0^T(\boldsymbol{K}\nabla p_2(t),\nabla q)_{\Omega_1}\Psi(t)dt+\int_0^T(p_2(t)-\frac{1}{2}(\boldsymbol{u}(t)\cdot\boldsymbol{u}(t)),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\Psi(t)dt+\frac{1}{G}\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}\Psi(t)dt$$

$$-\int_0^T(\boldsymbol{u}(t)\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}}\Psi(t)dt=\int_0^T(\boldsymbol{f}_1(t),\boldsymbol{v})_{\Omega_1}\Psi(t)dt+\int_0^T(f_2(t),q)_{\Omega_2}\Psi(t)dt.$$

Comparing this with (3.24) yields $(\boldsymbol{u}_0,\boldsymbol{v})_{\Omega_1}\Psi(0)=(\boldsymbol{u}(0),\boldsymbol{v})_{\Omega_1}\Psi(0)$. Choosing $\Psi(0)\neq 0$ we get $(\boldsymbol{u}_0-\boldsymbol{u}(0),\boldsymbol{v})_{\Omega_1}=0,\forall\boldsymbol{v}\in\boldsymbol{V}$. Therefore letting $\boldsymbol{v}=\boldsymbol{u}_0-\boldsymbol{u}(0)$ we finally get $\boldsymbol{u}_0=\boldsymbol{u}(0)$. The following a priori estimate follows trivially;

Corollary 3.2. *Under the same assumptions as in Lemma 2.1 every solution $(\boldsymbol{u},p_2)$ of (P) satisfies*

$$\|(\boldsymbol{u},p_2)\|_{L^2(0,T;Y)}\leq\mathcal{C}_e\tag{3.26}$$

where $\mathcal{C}_e$ is defined by (3.10).

Now we will show that the solution for (P) is unique. For that purpose assume that $(\boldsymbol{u}, p_2)$ and $(\tilde{\boldsymbol{u}}, \tilde{p}_2)$ are two solutions. Let $\boldsymbol{w} = \boldsymbol{u} - \tilde{\boldsymbol{u}}$ and $r = p_2 - \tilde{p}_2$. Then $(\boldsymbol{w}, q) \in L^2(0,T;\boldsymbol{W})$ satisfies

$$(\frac{\partial \boldsymbol{w}}{\partial t}, \boldsymbol{v})_{\Omega_1} + 2\mu(D(\boldsymbol{w}), D(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{w}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla r, \nabla q)_{\Omega_2} + (r, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$$

$$+\frac{1}{G}(\boldsymbol{w}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12}) - (\boldsymbol{w}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{u}, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\tilde{\boldsymbol{u}}\cdot\boldsymbol{w}, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} = 0$$

Choose $\boldsymbol{v} = \boldsymbol{w}$ and $q = r$,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + 2\mu\|D(\boldsymbol{w})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla r\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{w}\cdot\boldsymbol{n}_{12}\|^2_{L^2(\Gamma_{12})} + (\boldsymbol{w}\cdot\nabla\boldsymbol{u}, \boldsymbol{w})_{\Omega_1}$$

$$+(\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1} - \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{u}, \boldsymbol{w}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\tilde{\boldsymbol{u}}\cdot\boldsymbol{w}, \boldsymbol{w}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$$

Observing

$$\begin{aligned}(\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1} &= -(\nabla\cdot\tilde{\boldsymbol{u}}, \boldsymbol{w}\cdot\boldsymbol{w})_{\Omega_1} - (\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1} + (\tilde{\boldsymbol{u}}\cdot\boldsymbol{n}_{\partial\Omega_1}, \boldsymbol{w}\cdot\boldsymbol{w})_{\partial\Omega_1}\\ &= -(\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1} + (\tilde{\boldsymbol{u}}\cdot\boldsymbol{n}_{12}, \boldsymbol{w}\cdot\boldsymbol{w})_{\Gamma_{12}}\end{aligned}$$

we have

$$(\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1} = \frac{1}{2}(\tilde{\boldsymbol{u}}\cdot\boldsymbol{n}_{12}, \boldsymbol{w}\cdot\boldsymbol{w})_{\Gamma_{12}}$$

So the equation becomes

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)}+2\mu\|D(\boldsymbol{w})\|^2_{L^2(\Omega_1)}+\|\boldsymbol{K}^{1/2}\nabla r\|^2_{L^2(\Omega_2)} \leq -(\boldsymbol{w}\cdot\nabla\boldsymbol{u}, \boldsymbol{w})_{\Omega_1}-\frac{1}{2}((\boldsymbol{w}\cdot\boldsymbol{w}, \tilde{\boldsymbol{u}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}-(\boldsymbol{w}\cdot(\boldsymbol{u}+\tilde{\boldsymbol{u}}), \boldsymbol{w}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}})$$

The right hand side can be bounded by the virtue of (3.12), (3.13), (3.17) and (3.26) by

$$\begin{aligned}&\leq \|\boldsymbol{w}\|^2_{L^4(\Omega_1)}\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)} + \frac{1}{2}\|\boldsymbol{w}\|^2_{L^4(\Gamma_{12})}(\|\boldsymbol{u}\|_{L^2(\Gamma_{12})} + 2\|\tilde{\boldsymbol{u}}\|_{L^2(\Gamma_{12})})\\ &\leq C_1^3\|D(\boldsymbol{w})\|^2_{L^2(\Omega_1)}\big(\tilde{C}_4^2\|D(\boldsymbol{u})\|_{L^2(\Omega_1)} + \frac{1}{2}C_4^2(C_0\|D(\boldsymbol{u})\|_{L^2(\Omega_1)} + 2C_0\|D(\tilde{\boldsymbol{u}})\|_{L^2(\Omega_1)})\big)\\ &\leq C_1^3\frac{\mathcal{C}_e}{\sqrt{2\mu}}(\tilde{C}_4^2 + \frac{3}{2}C_0C_4^2)\|D(\boldsymbol{w})\|^2_{L^2(\Omega_1)}\end{aligned}$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + (2\mu - C_1^3\frac{\mathcal{C}_e}{\sqrt{2\mu}}(\tilde{C}_4^2 + \frac{3}{2}C_0C_4^2))\|D(\boldsymbol{w})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla r\|^2_{L^2(\Omega_2)} \leq 0.$$

Since $\boldsymbol{w}(0) = 0$ multiplying by 2 and taking the integral from 0 to T we get

$$\|\boldsymbol{w}(T)\|^2_{L^2(\Omega_1)} + 2(2\mu - C_1^3\frac{\mathcal{C}_e}{\sqrt{2\mu}}(\tilde{C}_4^2 + \frac{3}{2}C_0C_4^2))\|D(\boldsymbol{w})\|^2_{L^2(0,T;L^2(\Omega_1))} + \|\boldsymbol{K}^{1/2}\nabla r\|^2_{L^2(0,T;L^2(\Omega_2))} \leq 0.$$

So under the condition

$$(2\mu)^{3/2} > C_1^3\mathcal{C}_e(\tilde{C}_4^2 + \frac{3}{2}C_0C_4^2),$$

we have $(\boldsymbol{w}, r) = (\boldsymbol{0}, 0)$. Now we will show the existence of the pressure p_1 in the distributional sense. We follow the argument in [23] and define

$$\mathbf{U}(t) = \int_0^t \boldsymbol{u}(s)ds, \quad \mathbf{F}_1(t) = \int_0^t \boldsymbol{f}_1(s)ds, \quad \beta(t) = \int_0^t \boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s)ds.$$

Then $\mathbf{U}, \mathbf{F}_1, \boldsymbol{\beta} \in \mathcal{C}(0,T;\boldsymbol{V}')$. Integrating (P) between 0 and t, choosing $\boldsymbol{v} \in \boldsymbol{V}$ with $\boldsymbol{v} = \mathbf{0}$ on Γ_{12} and $q = 0$ yields

$$\forall t \in (0,T), \quad 2\mu(D(\mathbf{U}(t)), D(\boldsymbol{v}))_{\Omega_1} = (\boldsymbol{u}(0) - \boldsymbol{u}(t) - \boldsymbol{\beta}(t) + \mathbf{F}_1(t), \boldsymbol{v})_{\Omega_1}.$$

So for all $t \in [0,T]$ there exists a $P_1(t) \in L^2(\Omega_1)$ such that

$$\forall t \in (0,T), \quad \boldsymbol{u}(t) - \boldsymbol{u}(0) - 2\mu\nabla \cdot D(\mathbf{U}(t)) + \boldsymbol{\beta}(t) + \nabla P_1(t) = \mathbf{F}_1(t). \tag{3.27}$$

Since the gradient operator is an isomorphism from $L^2(\Omega_1) \setminus \mathbb{R}$ into $H^{-1}(\Omega_1)$, we conclude that ∇P_1 belongs to $\mathcal{C}([0,T];H^{-1}(\Omega_1))$ and thus $P_1 \in \mathcal{C}([0,T];L^2(\Omega_1))$. We now differentiate (3.27) in the distributional sense in $\Omega_1 \times (0,T)$ and we obtain

$$\frac{\partial \boldsymbol{u}}{\partial t} - 2\mu\nabla \cdot D(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p_1 = \mathbf{f}_1$$

with

$$p_1 = \frac{\partial P_1}{\partial t}.$$

What we achieved in this section can be stated as follows;

Theorem 3.3. *Let $\boldsymbol{u}_0 \in \boldsymbol{V}$ and suppose that the assumptions of Lemma 2.1 holds. If in addition we assume that*

$$(2\mu)^{3/2} > C_1^3 \mathcal{C}_e\big(\tilde{C}_4^2 + \frac{3}{2}C_0 C_4^2\big),$$

then the problem (P) has a unique solution $(\boldsymbol{u}, p_2) \in (L^2(0,T;\boldsymbol{V}) \cap H^1(0,T;L^2(\Omega_1)^2) \times L^2(0,T;M_2)$ such that

$$\|(\boldsymbol{u}, p_2)\|_{L^2(0,T;\boldsymbol{Y})} \leq \mathcal{C}_e, \tag{3.28}$$

with the constant defined in Theorem 3.1. Moreover, there exists $p_1 \in L^2(0,T;L^2(\Omega_1))$ such that $(\boldsymbol{u}, p_1, p_2)$ is a solution to the problem (Q).

4 Numerical Scheme

We discretize the coupled problem by a finite element method in space and a Crank-Nicolson scheme in time. Let $\boldsymbol{X}_h \subset \boldsymbol{X}$, $M_{1h} \subset M_1$ and $M_{2h} \subset M_2$ be finite element spaces to be specified later. We regroup all the linear terms involving $\boldsymbol{u}$ and p_2 by defining a bilinear form B

$$B([\boldsymbol{u}, p_2];[\boldsymbol{v}, q]) = 2\mu(D(\boldsymbol{u}), D(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{K}\nabla p_2, \nabla q)_{\Omega_2} + (p_2, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}. \tag{4.1}$$

Clearly B is bilinear and so it is bounded since we are in finite dimension. We also have

$$B([\boldsymbol{v}, q];[\boldsymbol{v}, q]) = 2\mu\|D(\boldsymbol{v})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla q\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{v}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} \geq 0. \tag{4.2}$$

The nonlinear reaction term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and the nonlinear term in γ are discretized using the form N

$$N(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{w}, \boldsymbol{v})_{\Omega_1} - \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{w})_{\Omega_1} + \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{v}, \boldsymbol{w}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{w}, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}.$$

Then, the form N is linear with respect to all three arguments, and N satisfies the following property:

$$N(\boldsymbol{u};\boldsymbol{v},\boldsymbol{v}) = 0. \tag{4.3}$$

Lemma 4.1.

$$\forall \boldsymbol{u},\boldsymbol{w},\boldsymbol{v}\in\boldsymbol{X},\quad |N(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})|\le C_N\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega_1)},\tag{4.4}$$

with

$$C_N=\tilde{C}_4^2+C_4^2C_0.$$

Proof. Using Hölder's inequality, we have:

$$|N(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})|\le\frac{1}{2}\|\boldsymbol{u}\|_{L^4(\Omega_1)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega_1)}\|\boldsymbol{v}\|_{L^4(\Omega_1)}+\frac{1}{2}\|\boldsymbol{u}\|_{L^4(\Omega_1)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega_1)}\|\boldsymbol{w}\|_{L^4(\Omega_1)}$$

$$+\frac{1}{2}\|\boldsymbol{u}\|_{L^4(\Gamma_{12})}\|\boldsymbol{w}\|_{L^4(\Gamma_{12})}\|\boldsymbol{v}\cdot\boldsymbol{n}_{12}\|_{L^2(\Gamma_{12})}+\frac{1}{2}\|\boldsymbol{u}\|_{L^4(\Gamma_{12})}\|\boldsymbol{v}\|_{L^4(\Gamma_{12})}\|\boldsymbol{w}\cdot\boldsymbol{n}_{12}\|_{L^2(\Gamma_{12})}.$$

By the bounds (3.1), (3.2) we obtain

$$N(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})|\le\frac{1}{2}\tilde{C}_4^2\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega_1)}+\frac{1}{2}\tilde{C}_4^2\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega_1)}$$

$$+\frac{1}{2}C_4^2C_0\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega_1)}+\frac{1}{2}C_4^2C_0\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega_1)}$$

$$\le C_N\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega_1)}.$$

□

Let $N_T>0$ be the number of time steps, let t^1 be the first timestep and let $\Delta t=(T-t^1)/(N_T-1)$ and let $t^i=t^1+(i-1)\Delta t$ for $i\ge 2$. We use the standard notation

$$\phi^{i+1/2}=\frac{\phi^{i+1}+\phi^i}{2},$$

for a sequence $\{\phi_i\}$ or a function $\phi^i=\phi(t^i)$. We propose the following scheme: Find $\{\boldsymbol{u}_h^i\}_{i\ge0}$ in $\boldsymbol{X}_h$, $\{p_{1h}^i\}_{i\ge1}\in M_{1h}$ and $\{p_{2h}^i\}_{i\ge1}$ in M_{2h} such that

$$\forall\boldsymbol{v}\in\boldsymbol{X}_h,\quad(\boldsymbol{u}_h^0,\boldsymbol{v})=(\boldsymbol{u}(0),\boldsymbol{v}),\tag{4.5}$$

$$\forall\boldsymbol{v}\in\boldsymbol{X}_h,\quad\forall q\in M_{2h},\quad(\frac{\boldsymbol{u}_h^1-\boldsymbol{u}_h^0}{t^1},\boldsymbol{v})_{\Omega_1}+B([\boldsymbol{u}_h^1,p_{2h}^1];[\boldsymbol{v},q])-(p_{1h}^1,\nabla\cdot\boldsymbol{v})_{\Omega_1}$$

$$+N(\boldsymbol{u}_h^1;\boldsymbol{u}_h^1,\boldsymbol{v})=(\boldsymbol{f}_1^1,\boldsymbol{v})_{\Omega_1}+(f_2^1,q)_{\Omega_2},\tag{4.6}$$

$$\forall i\ge1,\quad\forall\boldsymbol{v}\in\boldsymbol{X}_h,\quad\forall q\in M_{2h},\quad(\frac{\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h^i}{\Delta t},\boldsymbol{v})_{\Omega_1}+B([\boldsymbol{u}_h^{i+1/2},p_{2h}^{i+1/2}];[\boldsymbol{v},q])-(p_{1h}^{i+1/2},\nabla\cdot\boldsymbol{v})_{\Omega_1}$$

$$+N(\boldsymbol{u}_h^{i+1/2};\boldsymbol{u}_h^{i+1/2},\boldsymbol{v})=(\boldsymbol{f}_1^{i+1/2},\boldsymbol{v})_{\Omega_1}+(f_2^{i+1/2},q)_{\Omega_2},\tag{4.7}$$

$$\forall i\ge0,\quad\forall q\in M_{1h},\quad(\nabla\cdot\boldsymbol{u}_h^{i+1},q)_{\Omega_1}=0.\tag{4.8}$$

Equation (4.5) represents the initial condition whereas equation (4.6) computes the solution at the first time step using a first order backward Euler scheme. We will choose t^1 small enough so that the resulting scheme is of second order. Equation (4.7) defines the Crank-Nicolson scheme. Finally the incompressibility condition is enforced discretely at each time step by equation (4.8).

Let us now prove existence of the numerical solution. As in the continuous problem, we restrict the discrete problem (4.5)-(4.8) to the space of discretely divergent-free velocities:

$$\boldsymbol{V}_h=\{\boldsymbol{v}\in\boldsymbol{X}_h:\forall q\in M_{1h},\quad(q,\nabla\cdot\boldsymbol{v})_{\Omega_1}=0\}.$$

We show existence of $\{\boldsymbol{u}_h^i\}_{i\geq 0} \in \boldsymbol{V}_h, \{p_{2h}^i\}_{i\geq 1} \in M_{2h}$ satisfying (4.5) and

$$\forall \boldsymbol{v} \in \boldsymbol{V}_h, \quad \forall q \in M_{2h}, \quad (\frac{\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0}{t^1}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{u}_h^1, p_{2h}^1]; [\boldsymbol{v}, q]) + N(\boldsymbol{u}_h^1; \boldsymbol{u}_h^1, \boldsymbol{v}) = (\boldsymbol{f}_1^1, \boldsymbol{v})_{\Omega_1} + (f_2^1, q)_{\Omega_2}, \quad (4.9)$$

$$\forall i \geq 1, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \quad \forall q \in M_{2h}, \quad (\frac{\boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^i}{\Delta t}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{u}_h^{i+1/2}, p_{2h}^{i+1/2}]; [\boldsymbol{v}, q]) + N(\boldsymbol{u}_h^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{v})$$
$$= (\boldsymbol{f}_1^{i+1/2}, \boldsymbol{v})_{\Omega_1} + (f_2^{i+1/2}, q)_{\Omega_2}. \quad (4.10)$$

Clearly $\boldsymbol{u}_h^0$ is uniquely defined. We first give the proof for existence of $(\boldsymbol{u}_h^i, p_h^i)_{i\geq 2}$. The proof of existence of $(\boldsymbol{u}_h^1, p_h^1)$ is simpler and outlined at the end. We assume that $\boldsymbol{u}_h^i$ and p_{2h}^i are given for some $i \geq 1$. We show that the solution $(\boldsymbol{u}^{i+1}, p_{2h}^{i+1})$ satisfying (4.10) exists using a corollary of Brouwer's fixed point theorem. We can modify the argument to show existence of $(\boldsymbol{u}_h^1, p_{2h}^1)$. We introduce a mapping $\mathcal{F}_i : \boldsymbol{V}_h \times M_{2h} \to \boldsymbol{V}_h \times M_{2h}$ defined by

$$\forall (v, q) \in \boldsymbol{V}_h \times M_{2h}, \quad (\mathcal{F}_i(\boldsymbol{z}, t), (\boldsymbol{v}, q))_{\boldsymbol{Y}} = (\frac{2(\boldsymbol{z} - \boldsymbol{u}_h^i)}{\Delta t}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{z}, t]; [\boldsymbol{v}, q]) + N(\boldsymbol{z}; \boldsymbol{z}, \boldsymbol{v})$$
$$-(\boldsymbol{f}_1^{i+1/2}, \boldsymbol{v})_{\Omega_1} - (f_2^{i+1/2}, q)_{\Omega_2}. \quad (4.11)$$

So $\mathcal{F}_i$ is a well-defined map from $\boldsymbol{V}_h \times M_{2h}$ into itself by the Riesz representation theorem. The mapping $\mathcal{F}_i$ is also continuous. Furthermore if $(\boldsymbol{z}^*, t^*)$ is a zero of $\mathcal{F}_i$, then $(2\boldsymbol{z}^* - \boldsymbol{u}_h^i, 2t^* - p_{2h}^i)$ is a solution to (4.10). Compute

$$(\mathcal{F}_i(\boldsymbol{z}, t), (\boldsymbol{z}, t))_{\boldsymbol{Y}} = (\frac{2(\boldsymbol{z} - \boldsymbol{u}_h^i)}{\Delta t}, \boldsymbol{z})_{\Omega_1} + 2\mu \|D(\boldsymbol{z})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla t\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{z} \cdot \boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}$$
$$-(\boldsymbol{f}_1^{i+1/2}, \boldsymbol{z})_{\Omega_1} - (f_2^{i+1/2}, t)_{\Omega_2}$$
$$\geq \frac{1}{2\Delta t}\|\boldsymbol{z}\|^2_{L^2(\Omega_1)} - \frac{1}{2\Delta t}\|\boldsymbol{u}_h^i\|^2_{L^2(\Omega_1)} + 2\mu\|D(\boldsymbol{z})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla t\|^2_{L^2(\Omega_2)}$$
$$-(\boldsymbol{f}_1^{i+1/2}, \boldsymbol{z})_{\Omega_1} - (f_2^{i+1/2}, t)_{\Omega_2}.$$

Using the bound (3), we have

$$(\mathcal{F}_i(\boldsymbol{z}, t), (\boldsymbol{z}, t))_{\boldsymbol{Y}} \geq \frac{1}{2}\|(\boldsymbol{z}, t)\|^2_Y - \frac{1}{4\mu}P_1^2 C_1^2 \|\boldsymbol{f}_1^{i+1/2}\|^2_{L^2(\Omega_1)} - \frac{1}{2}\frac{P_2^2}{\lambda_{min}}\|f_2^{i+1/2}\|^2_{L^2(\Omega_2)} - \frac{1}{2\Delta t}\|\boldsymbol{u}_h^i\|^2_{L^2(\Omega_1)}.$$

We conclude that $(\mathcal{F}_i(\boldsymbol{z}, t), (\boldsymbol{z}, t))_{\boldsymbol{Y}} \geq 0$ for all $(\boldsymbol{z}, t)$ such that

$$\|(\boldsymbol{z}, t)\|_Y = \mathcal{R}_i,$$

with the radius $\mathcal{R}_i$ defined by

$$\mathcal{R}_i = \Big(\frac{1}{2\mu}P_1^2 C_1^2 \|\boldsymbol{f}_1^{i+1/2}\|^2_{L^2(\Omega_1)} + \frac{P_2^2}{\lambda_{min}}\|f_2^{i+1/2}\|^2_{L^2(\Omega_2)} + \frac{1}{\Delta t}\|\boldsymbol{u}_h^i\|^2_{L^2(\Omega_1)}\Big)^{1/2}.$$

This implies that there is a zero of $\mathcal{F}_i$ denoted by $(\boldsymbol{u}_h^{i+1}, p_{2h}^{i+1})$. This zero is a solution to (4.10). To show that $(\boldsymbol{u}_h^1, p_{2h}^1)$ exists, we follow a similar argument with the mapping $\mathcal{F}_1 : \boldsymbol{V}_h \times M_{2h} \to \boldsymbol{V}_h \times M_{2h}$ defined by

$$\forall (v, q) \in \boldsymbol{V}_h \times M_{2h}, \quad (\mathcal{F}_1(\boldsymbol{z}, t), (\boldsymbol{v}, q))_{\boldsymbol{Y}} = (\frac{\boldsymbol{z} - \boldsymbol{u}_h^0}{t^1}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{z}, t]; [\boldsymbol{v}, q]) + N(\boldsymbol{z}; \boldsymbol{z}, \boldsymbol{v})$$
$$-(\boldsymbol{f}_1^1, \boldsymbol{v})_{\Omega_1} - (f_2^1, q)_{\Omega_2}. \quad (4.12)$$

This yields a solution in the ball of radius $\mathcal{R}_1$.

$$\|(\boldsymbol{u}_h^1, p_{2h}^1)\|_Y \leq \mathcal{R}_1,$$

with

$$\mathcal{R}_1 = \Big(\frac{1}{2\mu}P_1^2C_1^2\|\boldsymbol{f}_1^1\|^2_{L^2(\Omega_1)} + \frac{P_2^2}{\lambda_{min}}\|f_2^1\|^2_{L^2(\Omega_2)} + \frac{1}{t^1}\|\boldsymbol{u}(0)\|^2_{L^2(\Omega_1)}\Big)^{1/2}. \tag{4.13}$$

Choosing $(\boldsymbol{v},q) = (\boldsymbol{u}_h^1, p_{2h}^1)$ in (4.9) and $(\boldsymbol{v},q) = (\boldsymbol{u}_h^{i+1/2}, p_{2h}^{i+1/2})$ in (4.10) yields a priori bounds on the solution. We skip the proof as it follows the argument above.

Lemma 4.2. *If $\{(\boldsymbol{u}_h^i, p_{2h}^i)\}_{i\geq 1}$ is a solution of (4.9)-(4.10), it satisfies*

$$\|\boldsymbol{u}_h^1\|^2_{L^2(\Omega_1)} + t^1\|(\boldsymbol{u}_h^1, p_{2h}^1)\|^2_{\boldsymbol{Y}} \leq \|\boldsymbol{u}(0)\|^2_{L^2(\Omega_1)} + \frac{1}{2\mu}P_1^2C_1^2t^1\|\boldsymbol{f}_1^1\|^2_{L^2(\Omega_1)} + \frac{P_2^2}{\lambda_{min}}t^1\|f_2^1\|^2_{L^2(\Omega_2)}, \tag{4.14}$$

$$\forall 2\leq m\leq N_T,\quad \|\boldsymbol{u}_h^m\|^2_{L^2(\Omega_1)} + \Delta t\sum_{i=1}^{m-1}\|(\boldsymbol{u}_h^{i+1/2}, p_{2h}^{i+1/2})\|^2_{\boldsymbol{Y}} \leq \|\boldsymbol{u}_h^1\|^2_{L^2(\Omega_1)}$$
$$+\frac{1}{2\mu}P_1^2C_1^2\Delta t\sum_{i=1}^{N_T-1}\|\boldsymbol{f}_1^{i+1/2}\|^2_{L^2(\Omega_1)} + \frac{P_2^2}{\lambda_{min}}\Delta t\sum_{i=1}^{N_T-1}\|f_2^{i+1/2}\|^2_{L^2(\Omega_2)}. \tag{4.15}$$

The next result states uniqueness of the solution under some condition on the data and on the time step.

Lemma 4.3. *Let $\mathcal{R}_1$ be defined by (4.13). Under the following conditions*

$$(2\mu)^{3/2} > C_1^3C_N\max(\mathcal{R}_1,\mathcal{R}),$$

with

$$\mathcal{R} = \Big(\frac{t^1\mathcal{R}_1^2}{\Delta t} + \frac{1}{2\mu}P_1^2C_1^2\sum_{i=1}^{N_T-1}\|\boldsymbol{f}_1^{i+1/2}\|^2_{L^2(\Omega_1)} + \frac{P_2^2}{\lambda_{min}}\sum_{i=1}^{N_T-1}\|f_2^{i+1/2}\|^2_{L^2(\Omega_2)}\Big)^{1/2}. \tag{4.16}$$

there exists a unique solution $\{(\boldsymbol{u}_h^i, p_{2h}^i)\}_{i\geq 1}$ satisfying (4.9)-(4.10).

Proof. First, we show uniqueness of $(\boldsymbol{u}_h^1, p_{2h}^1)$. Assume that there are two solutions say $(\{\boldsymbol{u}_h^1\},\{p_{2h}^1\})$ and $(\{\tilde{\boldsymbol{u}}_h^1\},\{\tilde{p}_{2h}^1\})$. Let $\boldsymbol{w}^1 = \boldsymbol{u}_h^1 - \tilde{\boldsymbol{u}}_h^1$ and $r^1 = p_{2h}^1 - \tilde{p}_{2h}^1$. From (4.9), we have

$$\forall \boldsymbol{v}\in\boldsymbol{X}_h,\quad \forall q\in M_{2h},\quad (\frac{\boldsymbol{w}^1}{t^1},\boldsymbol{v})_{\Omega_1} + B([\boldsymbol{w}^1,r^1];[\boldsymbol{v},q]) + N(\boldsymbol{u}_h^1;\boldsymbol{u}_h^1,\boldsymbol{v}) - N(\tilde{\boldsymbol{u}}_h^1;\tilde{\boldsymbol{u}}_h^1,\boldsymbol{v}) = 0.$$

Equivalently,

$$\forall \boldsymbol{v}\in\boldsymbol{X}_h,\quad \forall q\in M_{2h},\quad (\frac{\boldsymbol{w}^1}{t^1},\boldsymbol{v})_{\Omega_1} + B([\boldsymbol{w}^1,r^1];[\boldsymbol{v},q]) + N(\boldsymbol{w}^1;\boldsymbol{u}_h^1,\boldsymbol{v}) + N(\tilde{\boldsymbol{u}}_h^1;\boldsymbol{w}^1,\boldsymbol{v}) = 0.$$

Choosing $\boldsymbol{v} = \boldsymbol{w}^1$ and $q = r^1$ yields

$$\frac{1}{t^1}\|\boldsymbol{w}^1\|^2_{L^2(\Omega_1)} + \|(\boldsymbol{w}^1,r^1)\|^2_{\boldsymbol{Y}} \leq |N(\boldsymbol{w}^1;\boldsymbol{u}_h^1,\boldsymbol{w}^1)| \leq C_1^3C_N\|D(\boldsymbol{w}^1)\|^2_{L^2(\Omega_1)}\|D(\boldsymbol{u}_h^1)\|_{L^2(\Omega_1)}.$$

From (4.14) and the definition (4.13), we have

$$\|D(\boldsymbol{u}_h^1)\|_{L^2(\Omega_1)} \leq \frac{\mathcal{R}_1}{(2\mu)^{1/2}}.$$

Therefore, we obtain

$$\frac{1}{t^1}\|\boldsymbol{w}^1\|^2_{L^2(\Omega_1)} + \big(2\mu - \frac{C_1^3C_N\mathcal{R}_1}{(2\mu)^{1/2}}\big)\|D(\boldsymbol{w}^1)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla r^1\|^2_{L^2(\Omega_2)} \leq 0.$$

This means that $\boldsymbol{w}^1$ and r^1 are zero if the following condition is satisfied

$$(2\mu)^{3/2} > C_1^3 C_N \mathcal{R}_1.$$

Next, we fix $i \geq 1$ and show uniqueness of $(\boldsymbol{u}_h^{i+1}, p_{2h}^{i+1})$. We assume that $(\boldsymbol{u}_h^i, p_{2h}^i)$ exists and is unique. As above, we take the difference of two solutions $\boldsymbol{w}^{i+1} = \boldsymbol{u}_h^{i+1} - \tilde{\boldsymbol{u}}_h^{i+1}$ and $r^{i+1} = p_{2h}^{i+1} - \tilde{p}_{2h}^{i+1}$. Then from (4.10) we have

$$\forall \boldsymbol{v} \in \boldsymbol{V}_h,\, \forall q \in M_{2h},\, (\frac{\boldsymbol{w}^{i+1}}{\Delta t}, \boldsymbol{v}) + B([\boldsymbol{w}^{i+1/2}, r^{i+1/2}]; [\boldsymbol{v}, q]) + N(\boldsymbol{u}_h^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{v}) - N(\tilde{\boldsymbol{u}}_h^{i+1/2}; \tilde{\boldsymbol{u}}_h^{i+1/2}, \boldsymbol{v}) = 0.$$

Adding and subtracting the term $N(\boldsymbol{u}_h^{i+1/2}; \tilde{\boldsymbol{u}}_h^{i+1/2}, \boldsymbol{v})$ yields:

$$\frac{1}{\Delta t}(\boldsymbol{w}^{i+1}, \boldsymbol{v}) + B([\boldsymbol{w}^{i+1/2}, r^{i+1/2}]; [\boldsymbol{v}, q]) + N(\boldsymbol{u}_h^{i+1/2}; \boldsymbol{w}^{i+1/2}, \boldsymbol{v}) + N(\boldsymbol{w}^{i+1/2}; \tilde{\boldsymbol{u}}_h^{i+1/2}, \boldsymbol{v}) = 0.$$

Letting $\boldsymbol{v} = \boldsymbol{w}^{i+1/2}$ and $q = r^{i+1/2}$ and using the fact that $N(\boldsymbol{u}_h^{i+1/2}; \boldsymbol{w}^{i+1/2}, \boldsymbol{w}^{i+1/2})$ vanishes, we are left with

$$\frac{1}{\Delta t}\|\boldsymbol{w}^{i+1}\|^2 + 2\mu\|D(\boldsymbol{w}^{i+1/2})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla r^{i+1/2}\|^2_{L^2(\Omega_2)}$$
$$\leq C_N\|\nabla \boldsymbol{w}^{i+1/2}\|^2_{L^2(\Omega_1)}\|\nabla \tilde{\boldsymbol{u}}_h^{i+1/2}\|_{L^2(\Omega_1)} \leq C_N C_1^3 \|D(\boldsymbol{w}^{i+1/2})\|^2_{L^2(\Omega_1)}\|D(\tilde{\boldsymbol{u}}_h^{i+1/2})\|_{L^2(\Omega_1)}.$$

From (4.15) we have

$$\|D(\boldsymbol{u}_h^{i+1/2})\|_{L^2(\Omega_1)} \leq \frac{\mathcal{R}}{(2\mu)^{1/2}},$$

with $\mathcal{R}$ defined by (4.16). Therefore if we assume that $(2\mu)^{3/2} - C_N C_1^3 \mathcal{R} > 0$, the functions $\boldsymbol{w}^{i+1/2}$ and $r^{i+1/2}$ vanish. Since in fact $\boldsymbol{w}^{i+1/2} = \boldsymbol{w}^{i+1}/2$ and $r^{i+1/2} = r^{i+1}/2$, we can conclude. □

We assume that the spaces $\boldsymbol{X}_h$ and M_{1h} are conforming, i.e. they satisfy an inf-sup condition with $\beta > 0$ independent of h.

$$\inf_{q \in M_{1h}} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\nabla \cdot \boldsymbol{v}, q)_{\Omega_1}}{\|D(\boldsymbol{v})\|_{L^2(\Omega_1)}\|q\|_{L^2(\Omega_1)}} \geq \beta. \tag{4.17}$$

A straightforward consequence is the existence and uniqueness of the Navier-Stokes pressure p_{1h}^i for all $i \geq 1$.

5 Error analysis

Before we prove some error estimates, we show that the proposed scheme is consistent.

Lemma 5.1. *The weak solution* $(\boldsymbol{u}, p_1, p_2)$ *of (Q) also satisfies*

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad \forall q \in M_{2h}, \quad (\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{u}, p_2]; [\boldsymbol{v}, q]) - (p_1, \nabla \cdot \boldsymbol{v})_{\Omega_1} + N(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}. \tag{5.1}$$

Proof. It suffices to check that

$$N(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} - \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}.$$

But

$$N(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} - \frac{1}{2}(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{u})_{\Omega_1} + \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{v}, \boldsymbol{u} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}.$$

This is enough to conclude since the second and third term are equal to the first term by using integration by parts and the fact that $\nabla \cdot \boldsymbol{u} = 0$. □

We decompose the errors into approximation and numerical errors. For any $t \geq 0$, let $\tilde{\boldsymbol{u}}(t) \in \boldsymbol{X}_h$ be an approximation of $\boldsymbol{u}$ satisfying $(\nabla \cdot (\boldsymbol{u}(t) - \tilde{\boldsymbol{u}(t)}), q)_{\Omega_1} = 0$ for any q in M_{1h}. Existence of such an approximation can be found for instance in [13]. Also let $\tilde{p}_1 \in M_{1h}$ and $\tilde{p}_2 \in M_{2h}$ be approximations of p_1 and p_2, respectively. We take $\tilde{p}_1$ to be the L^2-projection of p_1, that is, $(p_1 - \tilde{p}_1, q)_{\Omega_1} = 0$ for all $q \in M_{1h}$ and we take $\tilde{p}_2$ to be the Lagrange interpolant. We further assume that the approximation errors are optimal, i.e. , for any $t \geq 0$ and for some positive integers k_1, k_2

$$\|D(\boldsymbol{u}(t)) - D(\tilde{\boldsymbol{u}}(t))\|_{L^2(\Omega_1)} \leq Ch^{k_1}|\boldsymbol{u}(t)|_{H^{k_1+1}(\Omega_1)}, \quad \forall \boldsymbol{u} \in L^2(0,T;H^{k_1+1}(\Omega_1)^2) \cap L^2(0,T;\boldsymbol{X}), \tag{5.2}$$

$$i = 0,1, \quad \|\nabla^i p_1(t) - \nabla^i \tilde{p}_1(t)\|_{L^2(\Omega_1)} \leq Ch^{k_1-i}|p_1(t)|_{H^{k_1}(\Omega_1)}, \quad \forall p_1 \in L^2(0,T;H^{k_1}(\Omega_1)) \cap L^2(0,T;M_1), \tag{5.3}$$

$$i = 0,1, \quad \|\nabla^i p_2(t) - \nabla^i \tilde{p}_2(t)\|_{L^2(\Omega_1)} \leq Ch^{k_2+1-i}|p_2(t)|_{H^{k_2+1}(\Omega_2)}, \quad \forall p_2 \in L^2(0,T;H^{k_2+1}(\Omega_1)) \cap L^2(0,T;M_2). \tag{5.4}$$

By the virtue of triangle inequality and (5.2) we have the following stability condition

$$\|D(\tilde{\boldsymbol{u}}(t))\|_{L^2(\Omega_1)} \leq \|D(\boldsymbol{u}(t)) - D(\tilde{\boldsymbol{u}}(t))\|_{L^2(\Omega_1)} + \|D(\boldsymbol{u}(t))\|_{L^2(\Omega_1)} \leq C_a|\boldsymbol{u}(t)|_{H^1(\Omega_1)}, \quad \forall t \geq 0. \tag{5.5}$$

where $C_a > 0$ is a constant independent of h. Let us give examples of conforming spaces that satisfy the above assumptions [3, 15]. Let $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ be the union of regular triangulations of the subdomains Ω_1 and Ω_2 such that the meshes match at the interface Γ_{12}. For any element T in $\mathcal{T}_h$, let $\mathbb{P}_k(T)$ denote the space of polynomials of degree less than or equal to k and defined on T. The space M_{2h} is chosen to be the usual continuous finite element space of piecewise polynomials of degree k_2 on each mesh element. We give below two examples of Navier-Stokes velocity and pressure spaces.

Example 5.2. $\mathbb{P}_2 - \mathbb{P}_1$ *Taylor-Hood spaces with continuous piecewise quadratic functions in velocity space and continuous piecewise linear functions in pressure space, i.e.,*

$$\boldsymbol{X}_h = \{\boldsymbol{v} \in \mathcal{C}^0(\overline{\Omega_1})^2 : \boldsymbol{v}|_T \in \mathbb{P}_2(T)^2 \quad \forall T \in \mathcal{T}_h^1\} \cap \boldsymbol{X},$$

$$M_{1h} = \{q \in \mathcal{C}^0(\overline{\Omega_1}) : q|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h^1\} \cap M_{1h}.$$

In this case, the estimates (5.2) and (5.3) are satisfied with $k_1 = 2$.

Example 5.3. *MINI element with continuous piecewise linears with bubbles for velocity and continuous piecewise linears for pressure space.*
Let $\mathbb{B}_1(T) = span\{\lambda_1\lambda_2\lambda_3\}$ *where* $\lambda_i \in \mathbb{P}_1(T)$ *with* $\lambda_i(x_j) = \delta_{ij}$ *for each vertex* x_j *of* $T \in \mathcal{T}_h$.

$$\boldsymbol{X}_h = \{\boldsymbol{v} \in \mathcal{C}^0(\overline{\Omega_1})^2 : \boldsymbol{v}|_T \in (\mathbb{P}_1(T) \oplus \mathbb{B}_1(T))^2 \quad \forall T \in \mathcal{T}_h^1\} \cap \boldsymbol{X},$$

$$M_{1h} = \{q \in \mathcal{C}^0(\overline{\Omega_1}) : q|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h^1\} \cap M_{1h}.$$

In that case, the estimates (5.2) and (5.3) are satisfied with $k_1 = 1$.

Next we write

$$\boldsymbol{u}_h - \boldsymbol{u} = \boldsymbol{\chi} - \boldsymbol{\eta}, \quad \boldsymbol{\chi} = \boldsymbol{u}_h - \tilde{\boldsymbol{u}}, \quad \boldsymbol{\eta} = \boldsymbol{u} - \tilde{\boldsymbol{u}},$$

$$p_{2h} - p_2 = \xi - \zeta, \quad \xi = p_{2h} - \tilde{p}_2, \quad \zeta = p_2 - \tilde{p}_2.$$

The following theorem states error bounds of the quantities $\boldsymbol{\chi}$ and ξ.

Theorem 5.4. *Let* $\boldsymbol{u}_{ttt} \in L^2(0,T;X)$. *Assume that the following condition holds*

$$\mu^{3/2} \geq \frac{C_N C_1^3}{\sqrt{2}} \max(\mathcal{R}, \mathcal{R}_1).$$

There exists a constant C independent of $h, t^1, \Delta t$ and μ such that

$$\|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)} + \mu t^1 \|D(\boldsymbol{\chi}^1)\|^2_{L^2(\Omega_1)} + t^1 \|\boldsymbol{K}^{1/2}\nabla \xi^1\|^2_{L^2(\Omega_2)}$$
$$\leq \|\boldsymbol{\chi}^0\|^2_{L^2(\Omega_1)} + C(1+\mu+\mu^{-1} + \frac{\mathcal{R}_1^2+\mathcal{C}_e^2}{\mu^2}) t^1 \|D(\boldsymbol{\eta}^1)\|^2_{L^2(\Omega_1)} + C(1+\mu^{-1}) t^1 |\zeta^1|^2_{H^1(\Omega_2)}$$
$$+ C\mu^{-1} t^1 \|p_1^1 - \tilde{p}_1^1\|^2_{L^2(\Omega_2)} + C\mu^{-1} t^3 \|\boldsymbol{u}_{tt}(\tilde{t}^0)\|^2_{L^2(\Omega_1)} + C\mu^{-1}\frac{1}{t^1}\|\boldsymbol{\eta}^1 - \boldsymbol{\eta}^0\|^2_{L^2(\Omega_1)},$$

and for any $m \geq 1$

$$\|\boldsymbol{\chi}^m\|^2_{L^2(\Omega_1)} + \mu \Delta t \sum_{i=1}^{m-1} \|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} + \Delta t \sum_{i=1}^{m-1} \|\boldsymbol{K}^{1/2}\nabla \xi^{i+1/2}\|^2_{L^2(\Omega_2)}$$
$$\leq \|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)} + C(1+\mu+\mu^{-1} + \frac{\mathcal{R}^2+\mathcal{C}_e^2}{\mu^2}) \Delta t \sum_{i=1}^{m-1} \|D(\boldsymbol{\eta}^{i+1/2})\|^2_{L^2(\Omega_1)} + C(1+\mu^{-1}) \Delta t \sum_{i=1}^{m-1} |\zeta^{i+1/2}|^2_{H^1(\Omega_2)}$$
$$+ C\mu^{-1}\Delta t \sum_{i=1}^{m-1} \|p_1^{i+1/2} - \tilde{p}_1^{i+1/2}\|^2_{L^2(\Omega_2)} + C\mu^{-1}\Delta t^5 \sum_{i=1}^{m-1} (\|\boldsymbol{u}_t(\tilde{t}^i)\|^4_{L^2(\Omega_1)} + \|\boldsymbol{u}_{ttt}(\check{t}^i)\|^2_{L^2(\Omega_1)})$$
$$+ C\mu^{-1}\frac{1}{\Delta t}\sum_{i=1}^{m-1} \|\boldsymbol{\eta}^{i+1} - \boldsymbol{\eta}^i\|^2_{L^2(\Omega_1)}.$$

Proof. First, we take the average of (5.1) at times $t = t^i$ and $t = t^{i+1}$:

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h,\ \forall q \in M_{2h},\ (\boldsymbol{u}_t^{i+1/2}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{u}^{i+1/2}, p_2^{i+1/2}]; [\boldsymbol{v}, q]) + \frac{1}{2}(N(\boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{v}) + N(\boldsymbol{u}^i; \boldsymbol{u}^i, \boldsymbol{v}))$$
$$-(p_1^{i+1/2}, \nabla\cdot\boldsymbol{v}) = (\boldsymbol{f}_1^{i+1/2}, \boldsymbol{v})_{\Omega_1} + (f_2^{i+1/2}, q)_{\Omega_2}. \tag{5.6}$$

From (5.6) and (4.7), we have for any $i \geq 1$:

$$(\frac{\boldsymbol{\chi}^{i+1} - \boldsymbol{\chi}^i}{\Delta t})_{\Omega_1} + B([\boldsymbol{\chi}^{i+1/2}, \xi^{i+1/2}], [\boldsymbol{v}, q]) - (p_{1h}^{i+1/2}, \nabla\cdot\boldsymbol{v})_{\Omega_1}$$
$$+ N(\boldsymbol{u}_h^{i+1/2}, \boldsymbol{u}_h^{i+1/2}, \boldsymbol{v}) = (\boldsymbol{u}_t^{i+1/2}, \boldsymbol{v})_{\Omega_1} - (\frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^i}{\Delta t}, \boldsymbol{v})_{\Omega_1}$$
$$+ B([\boldsymbol{\eta}^{i+1/2}, \zeta^{i+1/2}], [\boldsymbol{v}, q]) - (p_1^{i+1/2}, \nabla\cdot\boldsymbol{v})_{\Omega_1} + \frac{1}{2}N(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{v}) + \frac{1}{2}N(\boldsymbol{u}^i, \boldsymbol{u}^i, \boldsymbol{v}). \tag{5.7}$$

Choose $\boldsymbol{v} = \boldsymbol{\chi}^{i+1/2}$ and $q = \xi^{i+1/2}$ in (5.7). Then

$$\frac{1}{2\Delta t}(\|\boldsymbol{\chi}^{i+1}\|^2_{L^2(\Omega_1)} - \|\boldsymbol{\chi}^i\|^2_{L^2(\Omega_1)}) + 2\mu\|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla\xi^{i+1/2}\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{\chi}^{i+1/2}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}$$
$$\leq |N(\boldsymbol{u}_h^{i+1/2}, \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2}N(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2}N(\boldsymbol{u}^i, \boldsymbol{u}^i, \boldsymbol{\chi}^{i+1/2})|$$
$$+ |(\boldsymbol{u}_t^{i+1/2}, \boldsymbol{\chi}^{i+1/2})_{\Omega_1} - (\frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^i}{\Delta t}, \boldsymbol{\chi}^{i+1/2})_{\Omega_1}| + |B([\boldsymbol{\eta}^{i+1/2}, \zeta^{i+1/2}], [\boldsymbol{\chi}^{i+1/2}, \xi^{i+1/2}])|$$
$$+ |(p_1^{i+1/2} - p_{1h}^{i+1/2}, \nabla\cdot\boldsymbol{\chi}^{i+1/2})_{\Omega_1}|. \tag{5.8}$$

Let us first consider the nonlinear term

$$\mathcal{N} = N(\boldsymbol{u}_h^{i+1/2}, \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2}N(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2}N(\boldsymbol{u}^i, \boldsymbol{u}^i, \boldsymbol{\chi}^{i+1/2}).$$

After adding and subtracting $N(\boldsymbol{u}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2})$, we obtain

$$\mathcal{N} = N((\boldsymbol{u}_h - \boldsymbol{u})^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) + N(\boldsymbol{u}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2} N(\boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2} N(\boldsymbol{u}^{i}; \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+1/2}).$$

Next, separating the approximation error and discrete error and adding and subtracting $N(\boldsymbol{u}^{i+1/2}, \boldsymbol{u}^{i+1/2}, \boldsymbol{\chi}^{i+1/2})$ we have

$$\begin{aligned}\mathcal{N} = {} & N(\boldsymbol{\chi}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - N(\boldsymbol{\eta}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) + N(\boldsymbol{u}^{i+1/2}; \boldsymbol{u}_h^{i+1/2} - \boldsymbol{u}^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) \\ & + N(\boldsymbol{u}^{i+1/2}, \boldsymbol{u}^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2} N(\boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2} N(\boldsymbol{u}^{i}; \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+1/2}).\end{aligned}$$

Again, using the decomposition of $\boldsymbol{u}_h - \boldsymbol{u}$ in the third term gives:

$$\begin{aligned}\mathcal{N} = {} & N(\boldsymbol{\chi}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - N(\boldsymbol{\eta}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) + N(\boldsymbol{u}^{i+1/2}; \boldsymbol{\chi}^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - N(\boldsymbol{u}^{i+1/2}; \boldsymbol{\eta}^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) \\ & + N(\boldsymbol{u}^{i+1/2}, \boldsymbol{u}^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2} N(\boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2} N(\boldsymbol{u}^{i}; \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+1/2}).\end{aligned}$$

From (4.3), the term $N(\boldsymbol{u}^{i+1/2}; \boldsymbol{\chi}^{i+1/2}, \boldsymbol{\chi}^{1+1/2})$ vanishes. We next rewrite the last three terms.

$$\begin{aligned} & N(\boldsymbol{u}^{i+1/2}, \boldsymbol{u}^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2} N(\boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{2} N(\boldsymbol{u}^{i}; \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+1/2}) \\ = {} & \frac{1}{4} N(\boldsymbol{u}^{i}; \boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+1/2}) - \frac{1}{4} N(\boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+1/2}) = -\frac{1}{4} N(\boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}; \boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+1/2}).\end{aligned}$$

Therefore we have

$$\begin{aligned}\mathcal{N} = {} & N(\boldsymbol{\chi}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - N(\boldsymbol{\eta}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) - N(\boldsymbol{u}^{i+1/2}; \boldsymbol{\eta}^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) \\ & - \frac{1}{4} N(\boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}; \boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+1/2}).\end{aligned}$$

Using the bounds (4.4), (3.1) and (4.15) with the definition (4.16), we have

$$\begin{aligned} N(\boldsymbol{\chi}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) & \leq C_N C_1^3 \|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} \|D(\boldsymbol{u}_h^{i+1/2})\|_{L^2(\Omega_1)} \\ & \leq \frac{\mathcal{R}}{(2\mu)^{1/2}} C_N C_1^3 \|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)},\end{aligned}$$

and

$$N(\boldsymbol{\eta}^{i+1/2}; \boldsymbol{u}_h^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) \leq \frac{\mathcal{R}}{(2\mu)^{1/2}} C_N C_1^2 \|D(\boldsymbol{\chi}^{i+1/2})\|_{L^2(\Omega_1)} \|\nabla \boldsymbol{\eta}^{i+1/2}\|_{L^2(\Omega_1)}.$$

Similarly using (4.4), (3.1) and (3.28), we obtain

$$N(\boldsymbol{u}^{i+1/2}; \boldsymbol{\eta}^{i+1/2}, \boldsymbol{\chi}^{i+1/2}) \leq \frac{\mathcal{C}_e}{(2\mu)^{1/2}} C_N C_1^2 \|D(\boldsymbol{\chi}^{i+1/2})\|_{L^2(\Omega_1)} \|\nabla \boldsymbol{\eta}^{i+1/2}\|_{L^2(\Omega_1)}.$$

Finally we have for some $\tilde{t}^i \in (t^i, t^{i+1})$:

$$\begin{aligned}\frac{1}{4} N(\boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}; \boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+1/2}) & \leq \frac{1}{4} C_N C_1 \|D(\boldsymbol{\chi}^{i+1/2})\|_{L^2(\Omega_1)} \|\nabla(\boldsymbol{u}^{i+1} - \boldsymbol{u}^{i})\|^2_{L^2(\Omega_1)} \\ & \leq \frac{1}{4} C_N C_1 \Delta t^2 \|D(\boldsymbol{\chi}^{i+1/2})\|_{L^2(\Omega_1)} \|\nabla \boldsymbol{u}_t(\tilde{t}^i)\|^2_{L^2(\Omega_1)}.\end{aligned}$$

Therefore by Young's inequality, we obtain for any $\delta_0 > 0$:

$$\begin{aligned}\mathcal{N} \leq {} & \frac{\mathcal{R} C_N C_1^3}{(2\mu)^{1/2}} \|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} + 2\mu\delta_0 \|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} \\ & + \frac{C}{\mu^2 \delta_0} (\mathcal{R}^2 + \mathcal{C}_e^2) \|\nabla \boldsymbol{\eta}^{i+1/2}\|^2_{L^2(\Omega_1)} + \frac{C}{\mu\delta_0} \Delta t^4 \|\nabla \boldsymbol{u}_t(\tilde{t}^i)\|^4_{L^2(\Omega_1)}.\end{aligned}$$

Next, we consider the terms

$$\mathcal{D} = (\boldsymbol{u}_t^{i+1/2}, \boldsymbol{\chi}^{i+1/2})_{\Omega_1} - (\frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^i}{\Delta t}, \boldsymbol{\chi}^{i+1/2})_{\Omega_1} = (\boldsymbol{u}_t^{i+1/2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t}, \boldsymbol{\chi}^{i+1/2})_{\Omega_1} + (\frac{\boldsymbol{\eta}^{i+1} - \boldsymbol{\eta}^i}{\Delta t}, \boldsymbol{\chi}^{i+1/2})_{\Omega_1}.$$

From a Taylor expansion, we have for some $t_1^i, t_2^i \in (t^i, t^{i+1})$

$$\boldsymbol{u}_t^{i+1/2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} = \boldsymbol{u}_{ttt}(t_1^i)\frac{\Delta t^2}{8} - \boldsymbol{u}_{ttt}(t_2^i)\frac{\Delta t^2}{24}.$$

Then by (3.1) and Young's inequality, for any $\delta_1 > 0$, we obtain

$$\mathcal{D} \leq 2\mu\delta_1 \|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} + \frac{C}{\mu\delta_1}(\Delta t^4 \sum_{\theta=1}^{2} \|\boldsymbol{u}_{ttt}(t_\theta^i)\|^2_{L^2(\Omega_1)} + \frac{1}{\Delta t^2}\|\boldsymbol{\eta}^{i+1} - \boldsymbol{\eta}^i\|^2_{L^2(\Omega_1)}).$$

Using (3.1)-(3.4), we bound the linear term $B(\cdot,\cdot)$ for any positive numbers δ_2, δ_3

$$\begin{aligned} B([\boldsymbol{\eta}^{i+1/2}, \zeta^{i+1/2}], [\boldsymbol{\chi}^{i+1/2}, \xi^{i+1/2}]) &\leq 2\mu\|D(\boldsymbol{\eta}^{i+1/2})\|_{L^2(\Omega_1)}\|D(\boldsymbol{\chi}^{i+1/2})\|_{L^2(\Omega_1)} \\ &+\|\boldsymbol{K}^{1/2}\nabla\zeta^{i+1/2}\|_{L^2(\Omega_2)}\|\boldsymbol{K}^{1/2}\nabla\xi^{i+1/2}\|_{L^2(\Omega_2)} + C_0\tilde{C}_0C_1\|D(\boldsymbol{\chi}^{i+1/2})\|_{L^2(\Omega_1)}|\zeta^{i+1/2}|_{H^1(\Omega_2)} \\ &+\frac{C_0\tilde{C}_0}{\sqrt{\lambda_{\min}}}\|\boldsymbol{K}^{1/2}\nabla\xi^{i+1/2}\|_{L^2(\Omega_2)}|\boldsymbol{\eta}^{i+1/2}|_{H^1(\Omega_1)} + \frac{1}{G}\|\boldsymbol{\eta}^{i+1/2}\cdot\boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}\|\boldsymbol{\chi}^{i+1/2}\cdot\boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})} \\ &\leq \delta_2(2\mu)\|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} + \delta_3\|\boldsymbol{K}^{1/2}\nabla\xi^{i+1/2}\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{\chi}^{i+1/2}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} \\ &+C(1+\frac{1}{\delta_3}+\frac{\mu}{\delta_2})\|D(\boldsymbol{\eta}^{i+1/2})\|^2_{L^2(\Omega_1)} + C(\frac{1}{\mu\delta_2}+\frac{1}{\delta_3})|\zeta^{i+1/2}|^2_{H^1(\Omega_2)}. \end{aligned}$$

Since $(\nabla\cdot(\boldsymbol{u}^{i+1/2} - \boldsymbol{u}_h^{i+1/2}), q)_{\Omega_1} = 0$ for all $q \in M_{1h}$, we can rewrite the pressure term as

$$(p_1^{i+1/2} - p_{1h}^{i+1/2}, \nabla\cdot\boldsymbol{\chi}^{i+1/2})_{\Omega_1} = (p_1^{i+1/2} - \tilde{p}_1^{i+1/2}, \nabla\cdot\boldsymbol{\chi}^{i+1/2})_{\Omega_1} - (\tilde{p}_1^{i+1/2} - p_{1h}^{i+1/2}, \nabla\cdot\boldsymbol{\chi}^{i+1/2})_{\Omega_1}.$$

The second term vanishes because of the property of the interpolant.

The first term is bounded by Young's inequality for any $\delta_4 > 0$:

$$|(p_1^{i+1/2} - \tilde{p}_1^{i+1/2}, \nabla\cdot\boldsymbol{\chi}^{i+1/2})_{\Omega_1}| \leq 2\mu\delta_4\|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} + \frac{C}{\mu\delta_4}\|p_1^{i+1/2} - \tilde{p}_1^{i+1/2}\|^2_{L^2(\Omega_2)}.$$

We combine the bounds above with (5.8) and choose $\delta_0 = \delta_1 = \delta_2 = \delta_4 = 3/16$ and $\delta_3 = 1/2$. We multiply the resulting inequality by $2\Delta t$ and sum from $i = 1$ to $i = m-1$ for some $m \geq 1$. If the following condition is satisfied

$$\mu^{3/2} \geq \frac{C_N C_1^3 \mathcal{R}}{\sqrt{2}}$$

then we obtain:

$$\begin{aligned} &\|\boldsymbol{\chi}^m\|^2_{L^2(\Omega_1)} + \mu\Delta t\sum_{i=1}^{m-1}\|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} + \Delta t\sum_{i=1}^{m-1}\|\boldsymbol{K}^{1/2}\nabla\xi^{i+1/2}\|^2_{L^2(\Omega_2)} \\ &\leq \|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)} + C(1+\mu+\mu^{-1}+\frac{\mathcal{R}^2+\mathcal{C}_e^2}{\mu^2})\Delta t\sum_{i=1}^{m-1}\|D(\boldsymbol{\eta}^{i+1/2})\|^2_{L^2(\Omega_1)} + C(1+\mu^{-1})\Delta t\sum_{i=1}^{m-1}|\zeta^{i+1/2}|^2_{H^1(\Omega_2)} \\ &+C\mu^{-1}\Delta t\sum_{i=1}^{m-1}\|p_1^{i+1/2} - \tilde{p}_1^{i+1/2}\|^2_{L^2(\Omega_2)} + C\mu^{-1}\Delta t^5\sum_{i=1}^{m-1}(\|\boldsymbol{u}_t(\tilde{t}^i)\|^4_{L^2(\Omega_1)} + \|\boldsymbol{u}_{ttt}(\check{t}^i)\|^2_{L^2(\Omega_1)}) \\ &+C\mu^{-1}\frac{1}{\Delta t}\sum_{i=1}^{m-1}\|\boldsymbol{\eta}^{i+1} - \boldsymbol{\eta}^i\|^2_{L^2(\Omega_1)}. \end{aligned}$$

It remains to find a bound for $\|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)}$. For this, we consider the equation (4.6) and follow a similar derivation as above. We skip the details. Assume that

$$\mu^{3/2} \geq \frac{C_N C_1^3 \mathcal{R}_1}{\sqrt{2}}.$$

Then, we can prove

$$\begin{aligned}
&\|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)} + \mu t^1 \|D(\boldsymbol{\chi}^1)\|^2_{L^2(\Omega_1)} + t^1 \|\boldsymbol{K}^{1/2}\nabla\xi^1\|^2_{L^2(\Omega_2)} \\
\leq\ & \|\boldsymbol{\chi}^0\|^2_{L^2(\Omega_1)} + C(1+\mu+\mu^{-1}+\frac{\mathcal{R}_1^2+\mathcal{C}_e^2}{\mu^2})t^1\|D(\boldsymbol{\eta}^1)\|^2_{L^2(\Omega_1)} + C(1+\mu^{-1})t^1|\zeta^1|^2_{H^1(\Omega_2)} \\
&+ C\mu^{-1}t^1\|p_1^1-\tilde{p}_1^1\|^2_{L^2(\Omega_2)} + C\mu^{-1}t^3\|\boldsymbol{u}_{tt}(\tilde{t}^0)\|^2_{L^2(\Omega_1)} + C\mu^{-1}\frac{1}{t^1}\|\boldsymbol{\eta}^1-\boldsymbol{\eta}^0\|^2_{L^2(\Omega_1)}.
\end{aligned}$$

□

A straightforward corollary is the following result.

Theorem 5.5. *In addition to the assumptions of Theorem 5.4 assume that* $t^1 \leq \Delta t^2$*. Then there exists a constant* C *independent of* h, t^1 *and* Δt *but dependent on* μ *such that for any* $m \geq 2$

$$\begin{aligned}
&\|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)} + \|\boldsymbol{\chi}^m\|^2_{L^2(\Omega_1)} + \mu t^1\|D(\boldsymbol{\chi}^1)\|^2_{L^2(\Omega_1)} + \mu\Delta t\sum_{i=1}^{m-1}\|D(\boldsymbol{\chi}^{i+1/2})\|^2_{L^2(\Omega_1)} \\
&+ t^1\|\boldsymbol{K}^{1/2}\nabla\xi^1\|^2_{L^2(\Omega_2)} + \Delta t\sum_{i=1}^{m-1}\|\boldsymbol{K}^{1/2}\nabla\xi^{i+1/2}\|^2_{L^2(\Omega_2)} \leq C(h^{2k_1}+h^{2k_2}+\Delta t^4).
\end{aligned}$$

Remark 5.6. From the analysis above, it is easy to derive the error estimates for a backward Euler time discretization at each time step. The resulting method is then first order in time. Another extension of this work is to consider non-homogeneous boundary conditions for the Darcy pressure as in [7]. For instance, assume that $p_2 = g_D$ on Γ_{2D} with $g_D \in H^{1/2}_{00}(\Gamma_{2D})$. It suffices to consider a lift of the g_D inside Ω_1, say p_D and the weak solution becomes $(\boldsymbol{u}, p_1, \varphi_2)$ with $\varphi_2 = p_2 + p_D$ and with $(\boldsymbol{u}, p_1, p_2)$ satifying the problem (Q).

6 Conclusions

We formulate a weak problem of the coupling between time-dependent Navier-Stokes and Darcy equations and proved its well-posedness. We approximate the weak solution by a continuous finite element solution. Uniqueness of the solution is obtained under a condition on the data. We show that the scheme is optimal in space and second order in time.

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