Stability of the IMEX methods, CNLF and BDF2-AB2, for uncoupling systems of evolution equations

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Abstract

Stability is proven for two second order, two step methods for uncoupling a system of two evolution equations with exactly skew symmetric coupling: the Crank-Nicolson Leap Frog (CNLF) combination and the BDF2-AB2 combination. The form of the coupling studied arises in spatial discretizations of the Stokes-Darcy problem. For CNLF we prove stability for the coupled system under the time step condition suggested by linear stability theory for the Leap-Frog scheme. This seems to be a first proof of a widely believed result. For BDF2-AB2 we prove stability under a condition that is better than the one suggested by linear stability theory for the individual methods. This report is an expended version of the one submitted for publication.

Key words: partitioned methods, IMEX methods, CNLF, Stokes-Darcy coupling

1 Introduction

This is an expanded version, containing supplementary material, of a report with the same title.

In this report we prove stability of two, second order IMEX methods for uncoupling two evolution equations with exactly skew symmetric coupling:

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$$\frac{du}{dt} + A_1 u + C\phi = f(t), \text{ for } t > 0 \text{ and } u(0) = u_0$$
$$\frac{d\phi}{dt} + A_2\phi - C^T u = g(t), \text{ for } t > 0 \text{ and } \phi(0) = \phi_0.$$

This problem occurs, for example, after spatial discretization of the evolutionary Stokes-Darcy problem, e.g., [16,12,18,17]. Here

$$u : [0,\infty) \to \mathbb{R}^N, \phi : [0,\infty) \to \mathbb{R}^M,$$

and f, g, u_0, ϕ_0 and the matrices $A_{1/2}, C$ have compatible dimensions (and in particular C is $N \times M$). Note especially the exactly skew symmetric coupling linking the two equations. We assume that **the A_i are SPD**. Our analysis extends to the case of A_i positive real or even nonlinear with $\langle A(v), v\rangle \geq Const.|v|^2$. With superscript denoting the time step number, the first method is CNLF, the combination of Crank-Nicolson and Leap Frog given by: for $n \geq 2$

$$\frac{u^{n+1} - u^{n-1}}{2\triangle t} + A_1\frac{u^{n+1} + u^{n-1}}{2} + C\phi^n = f^n, \qquad \text{(CNLF)}$$
$$\frac{\phi^{n+1} - \phi^{n-1}}{2\triangle t} + A_2\frac{\phi^{n+1} + \phi^{n-1}}{2} - C^T u^n = g^n.$$

Since the stability region of LF is the interval $-1 < \mathrm{Im}(z) < +1$, from the scalar case we expect a stability restriction of the form $\triangle t\sqrt{\lambda_{\max}(C^TC)} \leq 1$. Interestingly, it seems that sufficiency in the non-commutative case is not yet proven Verwer [22], remark 3.1, page 6. We prove in Section 2 that CNLF is indeed stable under (1), exactly the condition suggested by the linear stability theory.

For vectors of the same length, denote the usual euclidean inner product and norm by $\langle u, v\rangle := u^T v$, $|\phi|^2 := \langle \phi, \phi\rangle$. We denote the weighted norms by

$$|u|^2_{A_1} := u^T A_1 u, \text{ and } |\phi|^2_{A_2} := \phi^T A_2 \phi.$$

Theorem 1 (Stability of CNLF) *Consider CNLF. Suppose the time step restriction holds:*

$$\triangle t\sqrt{\lambda_{\max}(C^TC)} \leq \alpha < 1, \textit{ for some } \alpha < 1. \qquad (1)$$

Then for any $n \geq 2$

$$
\begin{aligned}
&\frac{1-\alpha}{2}\left[|u^{n+1}|^2+|\phi^{n+1}|^2+|u^n|^2+|\phi^n|^2\right] \\
&+\triangle t\sum_{\ell=1}^{n}\frac{1}{4}\left(|u^{\ell+1}+u^{\ell-1}|^2_{A_1}+|\phi^{\ell+1}+\phi^{\ell-1}|^2_{A_2}\right) \\
&\leq \frac{1}{2}\left[|u^1|^2+|\phi^1|^2+|u^0|^2+|\phi^0|^2\right]+\triangle t\left[\langle C\phi^0,u^1\rangle-\langle C\phi^1,u^0\rangle\right] \\
&\quad +\triangle t\sum_{\ell=1}^{n}\left(\lambda_{\min}^{-1}(A_1)|f^\ell|^2+\lambda_{\min}^{-1}(A_2)|g^\ell|^2\right).
\end{aligned}
$$

Next we establish the stability of BDF2 with explicit AB2 coupling

$$
\begin{aligned}
&\frac{3u^{n+1}-4u^n+u^{n-1}}{2\triangle t}+A_1u^{n+1}+C(2\phi^n-\phi^{n-1})=f^{n+1}, \\
&\frac{3\phi^{n+1}-4\phi^n+\phi^{n-1}}{2\triangle t}+A_2\phi^{n+1}-C^T(2u^n-u^{n-1})=g^{n+1}.
\end{aligned}
\tag{BDF2-AB2}
$$

The stability region of AB2 suggests that this combination is strictly worse than CNLF. However, we prove that the combination inherits enough stability from BDF2 to be stable under a time step condition that in many cases is better than the one for CNLF.

Theorem 2 (Stability of BDF2-AB2) *Consider BDF2-AB2. Suppose that the time step restriction holds*

$$
\Delta t\max\{\lambda_{\max}(A_1^{-1}CC^T),\lambda_{\max}(A_2^{-1}C^TC)\}\leq\alpha<1,\quad \textit{for some } \alpha>0, \tag{2}
$$

then BDF2-AB2 is stable:

$$
|u^n|^2+|\phi^n|^2\leq C(initial\ data, forcing\ terms),\ for\ any\ n\geq 2.
$$

More precisely, for all $n \geq 1$, *we have that*

$$
\begin{aligned}
&\frac{1}{2}\Big(|u^{n+1}|^2+|\phi^{n+1}|^2\Big)+\frac{1}{2}\Big(|2u^{n+1}-u^n|^2+|2\phi^{n+1}-\phi^n|^2\Big)+\Delta t\sum_{\ell=1}^{n}\frac{1}{2}\left(\mathcal{R}^{\ell+1}+\mathfrak{R}^{\ell+1}\right) \\
&\leq\frac{1}{2}\Big(|u^1|^2+|\phi^1|^2\Big)+\frac{1}{2}\Big(|2u^1-u^0|^2+|2\phi^1-\phi^0|^2\Big) \\
&\quad +\Delta t\sum_{\ell=1}^{n}\frac{1}{2(1-\alpha)}\Big(\frac{|f^{\ell+1}|^2}{\lambda_{\min}(A_1)}+\frac{|g^{\ell+1}|^2}{\lambda_{\min}(A_2)}\Big),
\end{aligned}
$$

where we have denoted

$$
\begin{aligned}
\mathcal{R}^{\ell+1} =& \left|\sqrt{\Delta t} C^T u^{\ell+1} - \frac{1}{2\sqrt{\Delta t}}(\phi^{\ell+1} - 2\phi^{\ell} + \phi^{\ell-1})\right|^2 \\
&+ \left|\sqrt{\Delta t} C \phi^{\ell+1} + \frac{1}{2\sqrt{\Delta t}}(u^{\ell+1} - 2u^{\ell} + u^{\ell-1})\right|^2, \\
\mathfrak{R}^{\ell+1} =& \left|\lambda_{\min}^{1/2}(A_1 - \Delta t C C^T) u^{\ell+1} - \frac{1}{2\lambda_{\min}^{1/2}(A_1 - \Delta t C C^T)} f^{\ell+1}\right|^2 \\
&+ \left|\lambda_{\min}^{1/2}(A_2 - \Delta t C^T C) \phi^{\ell+1} - \frac{1}{2\lambda_{\min}^{1/2}(A_2 - \Delta t C^T C)} g^{\ell+1}\right|^2.
\end{aligned}
$$

Note that (2) implies that $A_1 - \Delta t C^T C, A_2 - \Delta t C C^T$ are SPD.

Both methods use 3 levels; approximations are needed at the first two time steps to begin. We suppose these are computed to appropriate accuracy, Verwer [22].

Because the problem and methods are linear, stability immediately implies that the error is bounded by its consistency error.

1.1 Connection to the coupled Stokes-Darcy problem

To specify the motivating problem leading to the system of evolution equations considered, let two domains be denoted by Ω_f, Ω_p and lie across an interface I from each other. The fluid velocity and porous media piezometric head (Darcy pressure) satisfy

$$
\begin{gathered}
u_t - \nu \triangle u + \nabla p = \mathbf{f}_f(x,t), \nabla \cdot u = 0, \text{ in } \Omega_f, \\
S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) = f_p, \text{ in } \Omega_p, \\
\phi(x,0) = \phi_0, \text{ in } \Omega_p \text{ and } u(x,0) = u_0, \text{ in } \Omega_f, \\
\phi(x,t) = 0, \text{ in } \partial\Omega_p \backslash I \text{ and } u(x,t) = 0, \text{ in } \partial\Omega_f \backslash I, \\
+ \text{ coupling conditions across } I.
\end{gathered}
\tag{3}
$$

Let $\widehat{n}_{f/p}$ denote the indicated, outward pointing, unit normal vector on I. The coupling conditions are conservation of mass and balance of forces on I

$$
\begin{gathered}
u \cdot \widehat{n}_f + \mathbf{u}_p \cdot \widehat{n}_p = 0, \text{ on I } \Leftrightarrow u \cdot \widehat{n}_f - \frac{1}{\eta} \mathcal{K} \nabla \phi \cdot \widehat{n}_p = 0, \text{ on I}, \\
p - \nu\, \widehat{n}_f \cdot \nabla u \cdot \widehat{n}_f = \rho g \phi \text{ on I.}
\end{gathered}
$$

The last condition needed is a tangential condition on the fluid region's velocity on the interface. The most correct condition is not completely understood (possibly due to matching a pointwise velocity in the fluid region with an averaged or homogenized velocity in the porous region). We take the Beavers-Joseph-Saffman (-Jones) interfacial coupling

$$-\nu \ \nabla u \cdot \widehat{n}_f = \frac{\alpha}{\sqrt{\widehat{\tau}_i \cdot \mathcal{K} \cdot \widehat{\tau}_i}} u \cdot \widehat{\tau}_i, \text{ on } I \text{ for any } \widehat{\tau}_i \text{ tangent vector on } I.$$

This is a simplification of the original and more physically realistic Beavers-Joseph conditions (in $u \cdot \widehat{\tau}_i$ which is replaced by $(u - \mathbf{u}_p) \cdot \widehat{\tau}_i$). Here:

$$\begin{aligned}
\phi &= \text{Darcy pressure + elevation induced pressure = piezometric head}\\
\mathbf{q} &= \text{ volume discharge,}\\
\mathbf{u}_p &= \text{ fluid velocity in porous media region, } \Omega_p,\\
\mathbf{u} &= \text{ fluid velocity in Stokes region, } \Omega_f,\\
\mathbf{f}_f, f_p &= \text{body forces in fluid region and source in porous region,}\\
\mathcal{K} &= \text{ hydraulic conductivity tensor,}\\
\nu &= \text{ kinematic viscosity of fluid,}\\
S_0 &= \text{ specific mass storativity coefficient,}\\
\eta &= \text{volumetric porosity,}\\
\rho &= \text{ density,}\\
g &= \text{ gravitational acceleration constant.}
\end{aligned}$$

We shall assume that the boundary conditions are simple Dirichlet conditions on the exterior boundaries (not including the interface I).

We denote the $L^2(I)$ norm by $|| \cdot ||_I$ and the $L^2(\Omega_{f/p})$ norms by $|| \cdot ||_{f/p}$, respectively; the corresponding inner products are denoted by $(\cdot, \cdot)_{f/p}$. Define

$$\begin{aligned}
X_f \ &: = \{v \in \left(H^1(\Omega_f)\right)^d : v = 0 \text{ on } \ \partial\Omega_f \backslash I\},\\
X_p \ &: = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \ \partial\Omega_p \backslash I\},\\
Q &= L_0^2(\Omega_f).
\end{aligned}$$

Define the bilinear forms

$$\begin{aligned}
a_f(u, v) &= (\nu \nabla u, \nabla v)_f + \sum_i \int_I \frac{\alpha}{\sqrt{\widehat{\tau}_i \cdot \mathcal{K} \cdot \widehat{\tau}_i}} (u \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i) ds,\\
a_p(\phi, \psi) &= (K \nabla \phi, \nabla \psi)_p,\\
c_I(u, \phi) &= n\rho g \int_I \phi u \cdot \widehat{n}_f ds.
\end{aligned}$$

A (monolithic) variational formulation of the coupled problem is to find $(u, p, \phi) : [0, \infty) \to X_f \times Q_f \times X_p$ satisfying the given initial conditions and, for all

$v \in X_f, q \in Q_f, \psi \in X_p$

$$\begin{aligned}(u_t, v)_f + a_f(u, v) - (p, \nabla \cdot v)_f + c_I(v, \phi) &= (\mathbf{f}_f, v)_f,\\ (q, \nabla \cdot u)_f &= 0,\\ S_0(\phi_t, \psi)_p + a_p(\phi, \psi) - c_I(u, \psi) &= (f_p, \psi)_p.\end{aligned} \tag{4}$$

Note that, setting $v = u, \psi = \phi$ and adding, the coupling terms exactly cancel in the monolithic sum yielding the energy estimate for the coupled system.

To discretize the Stokes-Darcy problem in space by the finite element method, we select finite element spaces

$$\text{velocity: } X_f^h \subset X_f, \text{ Darcy pressure: } X_p^h \subset X_p, \text{ Stokes pressure: } Q_f^h \subset Q_f$$

based on a conforming FEM triangulation with maximum triangle diameter denoted "h". No mesh compatibility at the interface I between the FEM meshes in the two subdomains is assumed. The Stokes velocity-pressure FEM spaces are assumed to satisfy the usual discrete inf-sup condition for stability of the discrete pressure. We denote the discretely divergence free velocities by

$$V^h := X_f^h \cap \{v_h : (q_h, \nabla \cdot v_h)_f = 0, \text{ for all } q_h \in Q_f^h\}$$

The semi-discrete approximations are maps $(u_h, p_h, \phi_h) : [0, \infty) \to X_f^h \times Q_f^h \times X_p^h$ satisfying the given initial conditions and, for all $v_h \in X_f^h$, $q_h \in Q_f^h, \psi_h \in V_p^h$

$$\begin{aligned}(u_{h,t}, v_h)_f + a_f(u_h, v_h) - (p_h, \nabla \cdot v_h)_f + c_I(v_h, \phi_h) &= (\mathbf{f}_f, v_h)_f,\\ (q_h, \nabla \cdot u_h)_f &= 0,\\ S_0(\phi_{h,t}, \psi_h)_p + a_p(\phi_h, \psi_h) - c_I(u_h, \psi_h) &= (f_p, \psi_h)_p.\end{aligned} \tag{5}$$

Note in particular the exactly skew symmetric coupling between the Stokes and the Darcy sub-problems. If the velocity is restricted to the discretely divergence free subspace we obtain The semi-discrete approximations are maps $(u_h, \phi_h) : [0, \infty) \to V_f^h \times X_p^h$ satisfying the given initial conditions and, for all $v_h \in V_f^h, \psi_h \in V_p^h$

$$\begin{aligned}(u_{h,t}, v_h)_f + a_f(u_h, v_h) + c_I(v_h, \phi_h) &= (\mathbf{f}_f, v_h)_f,\\ S_0(\phi_{h,t}, \psi_h)_p + a_p(\phi_h, \psi_h) - c_I(u_h, \psi_h) &= (f_p, \psi_h)_p.\end{aligned}$$

The exactly skew symmetric coupling between the Stokes and the Darcy sub-problems is retained. Picking a basis for the FEM spaces in the above, this leads to a system:

$$\begin{aligned}M_f \frac{du}{dt} + A_1 u + C\phi &= f(t), \text{ for } t > 0 \text{ and } u(0) = u_0\\ S_0 M_p \frac{d\phi}{dt} + A_2 \phi - C^T u &= g(t), \text{ for } t > 0 \text{ and } \phi(0) = \phi_0.\end{aligned}$$

Here the respective FEM mass matrices are denoted $M_{f/p}$. These are often spectrally equivalent to the identity. The above system can also be reduced to the one studied by a further change of variable. The Stokes-Darcy problem has experienced a rapid development of numerical methods. We end the paper with a list of some additional papers that, while not relevant to the precise problem considered herein, are on numerical methods for the Stokes -Darcy problem.

1.2 Previous work

When A_i are SPD, IMEX methods, like CNLF and BDF2-AB2 require the solution of two, smaller SPD systems per time step (which can be done by legacy codes for the independent sub-problems) as compared to one larger, nonsymmetric system for monolithically coupled methods. Given this potentially large simplification, it is not surprising that IMEX methods (and associated partitioned schemes) have been used extensively in the computational practice of multi-domain, multiphysics applications. The theory of IMEX methods is also developing; see [11,21,2] and [13] for early papers and [1,10] and particularly [22] and the book [14] for recent work. CNLF is itself a classic (e.g. [15]) combination of methods in computational fluid dynamics with wide practical use, including in the dynamic core of the NCAR climate model, [19].

Partitioned methods are often motivated by available codes for subproblems [20] and tend to be application specific. Examples of partitioned methods include ones designed for fluid-structure interaction [3,4,7], Maxwell's equations [23] and atmosphere-ocean coupling [8,10,9]. The block system we study arises in evolutionary groundwater-surface water coupling, e.g., [6,5,12,16]. Mu and Zhu [17] gave the first (in 2010) numerical analysis of a partitioned method based on the backward Euler-forward Euler IMEX scheme; this has been extended to, so-called, asynchronous time stepping (different time steps for different system components) in [18]. Our work herein is motivated by the search for partitioned methods for the Stokes-Darcy problem with higher accuracy and better stability.

2 Proof of stability of CNLF

This section gives a complete proof of Theorem 1.

Lemma 3 *We estimate*

$$\langle C\phi, u\rangle = \frac{1}{2}|C\phi|^2 + \frac{1}{2}|u|^2 - \frac{1}{2}|u - C\phi|^2$$

and, if A_i are SPD

$$
\begin{aligned}
&|u| \leq \lambda_{\min}^{-1/2}(A_1)|u|_{A_1}, |\phi| \leq \lambda_{\min}^{-1/2}(A_2)|\phi|_{A_2},\\
&|C\phi| \leq \sqrt{\lambda_{\max}(C^T C)}|\phi|.
\end{aligned}
$$

Thus

$$
|\langle C\phi, u\rangle| \leq \frac{1}{2}\sqrt{\lambda_{\max}(C^T C)}|\phi|^2 + \frac{1}{2}\sqrt{\lambda_{\max}(C^T C)}|u|^2.
$$

Proof. The first claim is the polarization identity. The second inequality is elementary while the fourth follows by inserting the third into the first. For the third, we have

$$
|C\phi| = \langle C\phi, C\phi\rangle^{1/2} = \langle C^T C\phi, \phi\rangle^{1/2} \leq \lambda_{\max}^{1/2}(C^T C)|\phi|.
$$

■

The **first** of three main steps in the proof of Theorem 1 is to take the inner product of CNLF with $u^{n+1} + u^{n-1}$ and $\phi^{n+1} + \phi^{n-1}$ and add:

$$
\begin{aligned}
&\frac{1}{2\triangle t}\left[|u^{n+1}|^2 + |\phi^{n+1}|^2\right] - \frac{1}{2\triangle t}\left[|u^{n-1}|^2 + |\phi^{n-1}|^2\right]\\
&\quad + \frac{1}{2}\left[|u^{n+1} + u^{n-1}|_{A_1}^2 + |\phi^{n+1} + \phi^{n-1}|_{A_2}^2\right]\\
&\quad + \langle C\phi^n, u^{n+1} + u^{n-1}\rangle - \langle C^T u^n, \phi^{n+1} + \phi^{n-1}\rangle\\
&\quad = \langle f^n, u^{n+1} + u^{n-1}\rangle + \langle g^n, u^{n+1} + u^{n-1}\rangle.
\end{aligned} \tag{6}
$$

The **second** step is to rearrange the coupling terms as an exact difference between two time levels: *Coupling* $= \langle C\phi^n, u^{n+1} - u^{n-1}\rangle - \langle C^T u^n, \phi^{n+1} - \phi^{n-1}\rangle = C^{n+1/2} - C^{n-1/2}$, where

$$
\begin{aligned}
C^{n+1/2} &:= \langle C\phi^n, u^{n+1}\rangle - \langle C\phi^{n+1}, u^n\rangle,\\
C^{n-1/2} &:= \langle C\phi^{n-1}, u^n\rangle - \langle C\phi^n, u^{n-1}\rangle.
\end{aligned}
$$

The **third** step is to add and subtract $|\boldsymbol{u^n}|^2 + |\boldsymbol{\phi^n}|^2$ to the control the energy at level t_n:

$$
\begin{aligned}
&\frac{1}{2\triangle t}\left[|u^{n+1}|^2 + |\phi^{n+1}|^2 + |\boldsymbol{u^n}|^2 + |\boldsymbol{\phi^n}|^2\right] - \frac{1}{2\triangle t}\left[|\boldsymbol{u^n}|^2 + |\boldsymbol{\phi^n}|^2 + |u^{n-1}|^2 + |\phi^{n-1}|^2\right]\\
&\quad + \frac{1}{2}\left[|u^{n+1} + u^{n-1}|_{A_1}^2 + |\phi^{n+1} + \phi^{n-1}|_{A_2}^2\right] + C^{n+1/2} - C^{n-1/2}\\
&= \langle f^n, u^{n+1} + u^{n-1}\rangle + \langle g^n, u^{n+1} + u^{n-1}\rangle \equiv \textbf{RHS}.
\end{aligned}
$$

Using Lemma 3 we treat **RHS** in a standard way:

$$\begin{aligned}\mathbf{RHS} &\leq |f^n|\lambda_{\min}^{-1/2}(A_1)|u^{n+1}+u^{n-1}|_{A_1}+|g^n|\lambda_{\min}^{-1/2}(A_2)|\phi^{n+1}+\phi^{n-1}|_{A_2}\\ &\leq \left(\lambda_{\min}^{-1}(A_1)|f^n|^2+\lambda_{\min}^{-1}(A_2)|g^n|^2\right)+\frac{1}{4}(|u^{n+1}+u^{n-1}|_{A_1}^2+|\phi^{n+1}+\phi^{n-1}|_{A_2}^2).\end{aligned}$$

Thus, define the system energy

$$E^{n+1/2}:=\frac{1}{2}\left[|u^{n+1}|^2+|\phi^{n+1}|^2+|u^n|^2+|\phi^n|^2\right]+\triangle tC^{n+1/2}.$$

Collecting terms we obtain

$$\begin{aligned}E^{n+1/2}-E^{n+1/2}+\triangle t\left(|u^{n+1}+u^{n-1}|_{A_1}^2+|\phi^{n+1}+\phi^{n-1}|_{A_2}^2\right)\\ \leq \triangle t(\lambda_{\min}^{-1}(A_1)|f^n|^2+\lambda_{\min}^{-1}(A_2)|g^n|^2).\end{aligned}$$

Obviously, $E^{n+1/2}-E^{n-1/2}+\{\text{positive\_terms}\}\leq RHS$ immediately implies stability provided only that $E^{n+1/2}>0$ for every n. We have (using Lemma 3 to bound the coupling terms)

$$\begin{aligned}E^{n+1/2}\geq &\frac{1}{2}\left[|u^{n+1}|^2+|\phi^{n+1}|^2+|u^n|^2+|\phi^n|^2\right]\\ &-\frac{\triangle t}{2}\sqrt{\lambda_{\max}(C^TC)}\left[|u^{n+1}|^2+|u^n|^2+|\phi^{n+1}|^2+|\phi^n|^2\right].\end{aligned}$$

This is positive (completing the proof) provided

$$\triangle t\sqrt{\lambda_{\max}(C^TC)}<1.$$

3 Proof of stability of BDF2-AB2

We proceed to prove Theorem 2. Take the inner product of BDF2-AB2 with u^{n+1}, ϕ^{n+1}, respectively, and add. There are two keys to the proof of stability. The **first key** is the treatment of the *BDF2 term*. Apply the identity

$$\left[\frac{a^2}{4}+\frac{(2a-b)^2}{4}\right]-\left[\frac{b^2}{4}+\frac{(2b-c)^2}{4}\right]+\frac{(a-2b+c)^2}{4}=\frac{1}{2}(3a-4b+c)a$$

with $a = u^{n+1}, b = u^n, c = u^{n-1}$, and once with $a = \phi^{n+1}, b = \phi^n, c = \phi^{n-1}$. This gives

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(|u^{n+1}|^2 + |2u^{n+1} - u^n|^2\Big) - \frac{1}{4\Delta t}\Big(|u^n|^2 + |2u^n - u^{n-1}|^2\Big) \\
&\qquad + \frac{1}{4\Delta t}|u^{n+1} - 2u^n + u^{n-1}|^2 \\
&+ \frac{1}{4\Delta t}\Big(|\phi^{n+1}|^2 + |2\phi^{n+1} - \phi^n|^2\Big) - \frac{1}{4\Delta t}\Big(|\phi^n|^2 + |2\phi^n - \phi^{n-1}|^2\Big) \\
&\qquad + \frac{1}{4\Delta t}|\phi^{n+1} - 2\phi^n + \phi^{n-1}|^2 \\
&+ |u^{n+1}|^2_{A_1} + |\phi^{n+1}|^2_{A_2} + \langle C(2\phi^n - \phi^{n-1}), u^{n+1}\rangle - \langle C^T(2u^n - u^{n-1}), \phi^{n+1}\rangle \\
&= \langle f^{n+1}, u^{n+1}\rangle + \langle g^{n+1}, \phi^{n+1}\rangle.
\end{aligned} \tag{7}
$$

The **second key** is to *rearrange the coupling terms.* We use the skew-symmetry of the coupling term and the polarization identity (Lemma 3) to write it as follows:

$$
\begin{aligned}
&Coupling = \langle C(2\phi^n - \phi^{n-1}), u^{n+1}\rangle - \langle C^T(2u^n - u^{n-1}), \phi^{n+1}\rangle \\
&= -\langle C(\phi^{n+1} - 2\phi^n + \phi^{n-1}), u^{n+1}\rangle + \langle C^T(u^{n+1} - 2u^n + u^{n-1}), \phi^{n+1}\rangle \\
&= -\frac{1}{4\Delta t}|\phi^{n+1} - 2\phi^n + \phi^{n-1}|^2 - \Delta t|u^{n+1}|^2_{CC^T} \\
&\qquad - \frac{1}{4\Delta t}|u^{n+1} - 2u^n + u^{n-1}|^2 - \Delta t|\phi^{n+1}|^2_{C^TC} + \mathcal{R}^{n+1}.
\end{aligned} \tag{8}
$$

Then (7) and (8) give

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(|u^{n+1}|^2 + |2u^{n+1} - u^n|^2\Big) - \frac{1}{4\Delta t}\Big(|u^n|^2 + |2u^n - u^{n-1}|^2\Big) \\
&\quad + \frac{1}{4\Delta t}\Big(|\phi^{n+1}|^2 + |2\phi^{n+1} - \phi^n|^2\Big) - \frac{1}{4\Delta t}\Big(|\phi^n|^2 + |2\phi^n - \phi^{n-1}|^2\Big) \\
&\quad + |u^{n+1}|^2_{A_1} + |\phi^{n+1}|^2_{A_2} - \Delta t|u^{n+1}|^2_{CC^T} - \Delta t|\phi^{n+1}|^2_{C^TC} + \mathcal{R}^{n+1} \\
&= \langle f^{n+1}, u^{n+1}\rangle + \langle g^{n+1}, \phi^{n+1}\rangle.
\end{aligned}
$$

Using again the polarization identity yields

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(|u^{n+1}|^2 + |2u^{n+1} - u^n|^2\Big) - \frac{1}{4\Delta t}\Big(|u^n|^2 + |2u^n - u^{n-1}|^2\Big) \\
&\quad + \frac{1}{4\Delta t}\Big(|\phi^{n+1}|^2 + |2\phi^{n+1} - \phi^n|^2\Big) - \frac{1}{4\Delta t}\Big(|\phi^n|^2 + |2\phi^n - \phi^{n-1}|^2\Big) \\
&\quad + |u^{n+1}|^2_{A_1} + |\phi^{n+1}|^2_{A_2} - \Delta t|u^{n+1}|^2_{CC^T} - \Delta t|\phi^{n+1}|^2_{C^TC} + \mathcal{R}^{n+1} \\
&= \lambda_{\min}(A_1 - \Delta t CC^T)|u^{n+1}|^2 + \frac{1}{4\lambda_{\min}(A_1 - \Delta t CC^T)}|f^{n+1}|^2 \\
&\qquad + \lambda_{\min}(A_2 - \Delta t C^TC)|\phi^{n+1}|^2 + \frac{1}{4\lambda_{\min}(A_2 - \Delta t C^TC)}|g^{n+1}|^2 - \mathfrak{R}^{n+1},
\end{aligned}
$$

which by summation implies the stability result

$$\frac{|u^{n+1}|^2}{4\Delta t}+\frac{1}{4\Delta t}|2u^{n+1}-u^n|^2+\frac{|\phi^{n+1}|^2}{4\Delta t}+\frac{1}{4\Delta t}|2\phi^{n+1}-\phi^n|^2+\sum_{\ell=1}^{n}(\mathcal{R}^{\ell+1}+\mathfrak{R}^{\ell+1})$$
$$\leq \frac{|u^1|^2}{4\Delta t}+\frac{1}{4\Delta t}|2u^1-u^0|^2+\frac{|\phi^1|^2}{4\Delta t}+\frac{1}{4\Delta t}|2\phi^1-\phi^0|^2$$
$$+\sum_{\ell=1}^{n}\Big(\frac{1}{4(1-\alpha)\lambda_{\min}(A_1)}|f^{n+1}|^2+\frac{1}{4(1-\alpha)\lambda_{\min}(A_2)}|g^{n+1}|^2\Big).$$

4 Numerical verification of the Theorems

We give two numerical tests that confirm the theory (showing in particular that the restriction (1) is sharp). The examples also illustrate that there are cases where each method's time step restriction is better than the other method.

In all test cases, the initial conditions are

$$u^0=\begin{pmatrix}1\\1\end{pmatrix},\quad \phi^0=\begin{pmatrix}1\\1\end{pmatrix}$$

and u^1,ϕ^1 are computed using the implicit backward Euler. We take $f=g=0$, so that *any growth* in the energy is an instability.

Test 1. In the first case the matrices are

$$A_1=\begin{pmatrix}10&0\\0&20\end{pmatrix},\quad A_2=\begin{pmatrix}30&0\\0&50\end{pmatrix},\quad C=\begin{pmatrix}2&3\\4&5\end{pmatrix}$$

yielding the following time step restrictions

$$\Delta t_{\text{CNLF}}=0.1361,\qquad \Delta t_{\text{BDFAB}}=0.2990.$$

With the time step

$$\Delta t=0.99*\Delta t_{\text{CNLF}}$$

both methods are observed to be stable, Figure 1). With the time step $\Delta t=1.01*\Delta t_{\text{CNLF}}$ the CNLF approximations exhibit growth and thus are unstable Since $1.01*\Delta t_{\text{CNLF}}<\Delta t_{BDFAB}$ the theory predicts BDF2-AB2 to be stable and this is indeed seen in Figure 2.

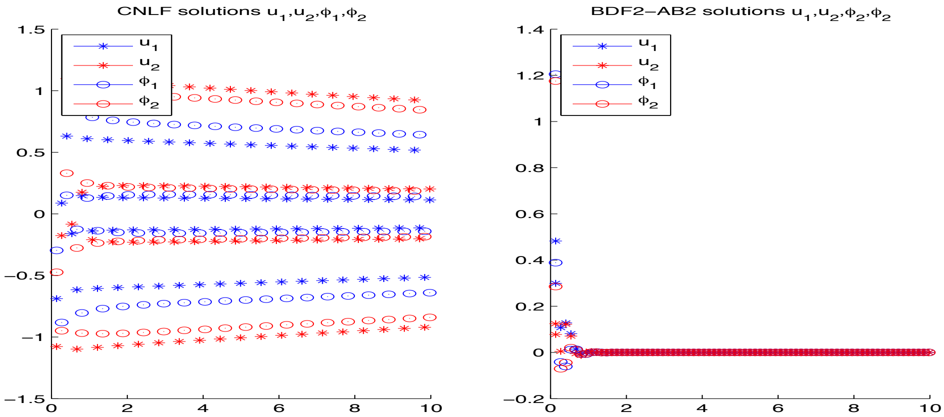

Fig. 1. Both methods stable, as predicted.

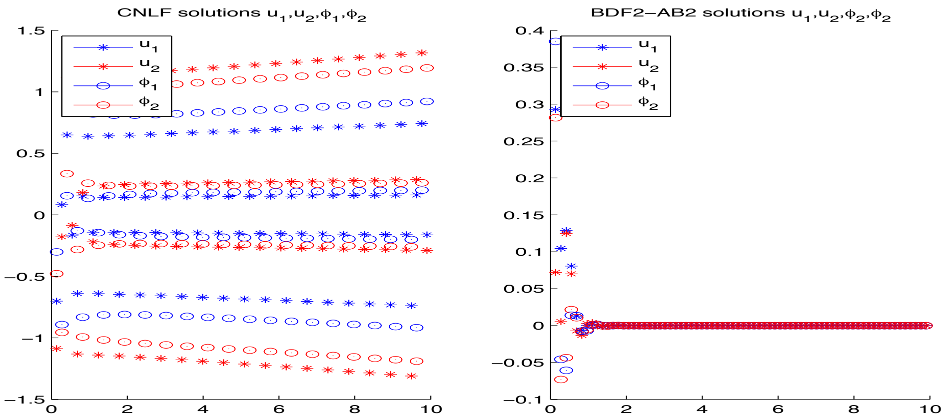

Fig. 2. CNLF unstable, BDF2-AB2 stable, as predicted.

<u>Test 2.</u> With matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

the time step restrictions are

$$\Delta t_{\text{CNLF}} = 0.1361, \qquad \Delta t_{\text{BDFAB}} = 0.0299.$$

With time step $\Delta t = .99 * \Delta t_{\text{CNLF}}$ the CNLF converges, while with BDF2-AB2 the solution is unstable, Figure 3.

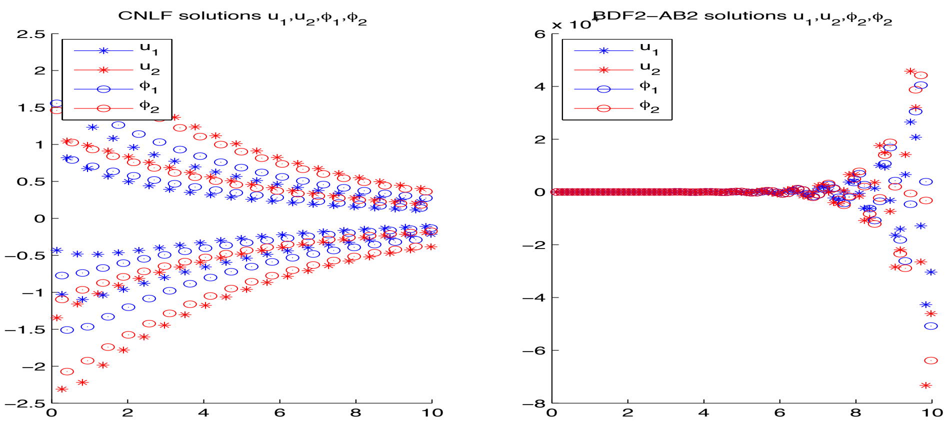

Fig. 3. CNLF stable, BDF2-AB2 unstable, as predicted.

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