STABILITY ANALYSIS OF THE CRANK-NICOLSON-LEAP-FROG METHOD WITH THE ROBERT-ASSELIN-WILLIAMS TIME FILTER *

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Abstract. Geophysical flow simulations have evolved sophisticated Implicit-Explicit time stepping methods (based on fast slow wave splittings) followed by time filters to control any unstable models that result. These time filters are wonderfully elegant as algorithms, modular and embarrassingly parallel. Their effect on stability of the overall process has been tested in numerous simulations but never studied analytically. We do so herein.

Key words. Conditional stability, IMEX methods, Crank-Nicolson, Leap-Frog, Robert-Asselin filter

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1. Introduction. The fundamental method for time stepping in most current geophysical fluid dynamics (GFD) codes consists of one step of the Crank-Nicolson-Leap-Frog (CNLF) method (based on a fast-slow wave decoupling strategy) followed by one step of the Robert-Asselin (RA) [17, 2] time filter to control CNLF's computational mode followed by one step of the Williams [22] time filter to restore lost accuracy and anti-diffuse the RA step. This combination is algorithmically elegant, subtle and effective. Williams [22] lists 25 major GFD codes using this approach including the Community Climate Systems Model, e.g., [20]. Our goal herein is to complement the numerous numerical tests with a first rigorous numerical analysis of the combination for systems, complementing root condition analysis and numerical tests in, e.g., [5, 22]. Thus, given $N \times N$ matrices A, Λ, we consider the system for $u : [0,\infty) \to \mathbb{R}^N$

$$u_t + Au = \Lambda u, \;\; u(0) = u_0, \text{ where} \tag{1.1}$$
$$A + A^T \geq 0, \text{ and } \Lambda = -\Lambda^T. \tag{1.2}$$

When $A + A^T = 0$ so $A = -A^T$ the system is exactly conservative, for the euclidean norm: $||u(t)|| = ||u(0)||$. When $A + A^T > 0$ it is dissipative, $||u(t)|| \to 0$ as $t \to \infty$. CNLF followed by the Robert-Asselin-Williams (**RAW**) filter [1, 22, 23] is: given timestep $\triangle t$, filter parameters $\nu > 0, \alpha \in [\frac{1}{2}, 1]$ and starting values u_0, v_1, (of sufficient accuracy, [21]) find u_n, w_{n+1}, v_{n+1}

$$(\textbf{CNLF}\text{ step}) \qquad \frac{w_{n+1} - u_{n-1}}{2\Delta t} = -A\frac{w_{n+1} + u_{n-1}}{2} + \Lambda v_n \tag{1.3}$$

$$(\textbf{RA}\text{ step}) \qquad u_n = v_n + \frac{\nu\alpha}{2}(w_{n+1} - 2v_n + u_{n-1}) \tag{1.4}$$

$$(\textbf{W}\text{ step}) \qquad v_{n+1} = w_{n+1} + \frac{\nu(\alpha-1)}{2}(w_{n+1} - 2v_n + u_{n-1}). \tag{1.5}$$

The parameter ν is the Robert-Asselin filter parameter, usually $\mathscr{O}(0.01 \text{ to } 0.2)$, and α is the Williams filter parameter, around $\frac{1}{2}$. If $\alpha = 1$ then the third, W filter, step, drops out and the method reduces to CNLF + RA filter (1.4), and if $\nu = 0$, CNLF is recovered. Unfiltered CNLF is stable in the absence of roundoff error [14] under the CFL condition

$$\triangle t||\Lambda|| < 1. \tag{1.6}$$

In the presence of roundoff error, CNLF possesses an undamped (since $u_{n+1} + u_{n-1} \equiv 0$ for the mode) and slightly unstable mode, that can take a long time to appear but is inevitable in long time simulations without time filters, e.g., [5].

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1.1. A summary of results. There have been many numerical studies of CNLF with the RAW filter giving evidence of control of CNLF's unstable mode. Our analysis supports this conclusion two ways: (i) the stability regions we calculate for CNLF+RAW in Section 5 for $\frac{1}{2} < \alpha < 1$ contain a section of the imaginary axis strictly inside them, and (ii) the stability estimates for the system in Section 4 exhibit damping in all modes (see Corollary 1.4 below).

There have been reports in [4], see also [11, section 13.5], of stable CNLF simulations that were destabilized when RAW steps were added. These instabilities (not related to the nonlinear instabilities studied in [6]) were resolved in [4] by cutting the timestep severely. We give two analytic explanations. First, the stability regions of CNLF+RAW in Section 5 and the system energy analysis in Theorem 1.3 lead to CFL conditions

$$\textbf{(Necessary)}\ \triangle t||\Lambda|| \leq CFL_{scalar}(\alpha,\nu),$$
$$\textbf{(Sufficient)}\quad \triangle t||\Lambda|| \leq CFL_{system}(\alpha,\nu),$$
$$\text{where } CFL_{system}(\alpha,\nu) := \sqrt{2\alpha-1}\sqrt{\frac{2-\nu}{\alpha^2(2+\nu-2\alpha\nu)}}\left(1-\frac{\nu}{2}(2\alpha-1)\right)$$
$$CFL_{scalar}(\alpha,\nu) := CFL_{system}(\alpha,\nu)\frac{1}{\sqrt{1-\nu^2(\frac{1}{2}-\alpha)^2}}.$$

The first condition is necessary (and sufficient for scalar problems) while the second is sufficient. The RHS' are related by

$$CFL_{system}(\alpha,\nu) \leq CFL_{scalar}(\alpha,\nu) < 1 = CFL_{CNLF}.$$

For $\alpha > 1/2$, $CFL_{scalar}(\alpha,\nu) \to 0$ as $\alpha \to 1/2$. Thus, a stable CNLF simulation at or near its CFL limit would be exponentially destabilized by an added RAW step. Second, in Section 4 we show that the parameter choice $\alpha = 1/2$ for RAW results in a method that is *unconditionally unstable* for wave propagation problems.

COROLLARY 1.1. *Let $A = 0, \Lambda \neq 0$. Then, CNLF+RAW is unconditionally unstable if $\alpha = \frac{1}{2}$.*

Proof. Two different proofs: First, the stability region of CNLF+RAW does not intersect the imaginary axis if $\alpha = \frac{1}{2}$, see Section 5, Figure 5.1. Second, in the above necessary condition for stability $CFL_{scalar}(\alpha,\nu) \to 0$ if $\alpha \to \frac{1}{2}$. □

This corollary is one explanation of the common practice* of taking $\alpha > 1/2$ but close to $\alpha = 1/2$. The natural, but so far unsuccessful, strategy to analyze stability of CNLF + RAW for systems is to track the evolution of a system energy through the individual steps of (1.3), (1.4), (1.5). Instead, our stability analysis of CNLF+RAW is based on reduction of CNLF+RAW to an equivalent multistep method followed by application of the tools of stability regions and G-stability theory [3, 8]. Thus, the first main result is the equivalence to a linear multistep method and analysis of its consistency error, proven in Section 3.

THEOREM 1.2. *The approximation u_n of CNLF plus RAW satisfies*

$$u_n - \nu u_{n-1} - (1-\nu)u_{n-2} = -\Delta t A(u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2}) + \Delta t\Lambda\Big(\big(2+\nu(\alpha-1)\big)u_{n-1} - \nu\alpha u_{n-2}\Big). \tag{1.7}$$

For $\nu > 0, \alpha \in (1/2, 1]$ the CNLF+RAW has second order consistency error. For the special value $\alpha = \frac{1}{2}$, CNLF + RAW has third order consistency error.

Utility of CNLF+RAW depends on stability. For scalar problems, the stability regions of (1.7) are presented in Section 5. To analyze stability for systems, in Section 4 we construct a 2×2 matrix G

$$G_{11} = 1 - \frac{\nu}{4}(\nu + 2\alpha - 2\alpha\nu) - \frac{\nu}{4}\sqrt{(2+\nu-2\alpha\nu)(2\alpha-1)(2-\nu)},$$

*Thus, the recommendation $\alpha = 1/2$ is shorthand for, e.g., $\alpha = 1/2 + \triangle t$, which also preserves the higher consistency error of $\alpha = 1/2$.

$$G_{12} = G_{21} = \frac{\nu}{4}(2\alpha+\nu-2-2\alpha\nu) + \frac{\nu}{4}\sqrt{(2+\nu-2\alpha\nu)(2\alpha-1)(2-\nu)},$$
$$G_{22} = 1 - \frac{\nu}{4}(4+2\alpha-\nu-2\alpha\nu) - \frac{\nu}{4}\sqrt{(2+\nu-2\alpha\nu)(2\alpha-1)(2-\nu)}.$$

In Lemma 4.4 we show G is SPD for $\frac{1}{2} \leq \alpha \leq 1$ and $0 \leq \nu \leq 0.3$. Thus, we can define the method's G-norm

$$\|(u_n, u_{n-1})\|_G^2 := (u_n, u_{n-1})G(u_n, u_{n-1})^T.$$

Denote γ, r, s, τ by

$$\gamma = \nu\sqrt{(1-\alpha)^2 + \alpha^2\Delta t^2\|\Lambda\|^2}, \;\; r = 1 - \frac{\gamma}{2}, \quad (1.8)$$
$$s = 1 - \frac{\gamma}{2} - \nu\left(1 - \frac{\nu}{2}\right), \quad \tau = \sqrt{(\gamma-\nu)^2 + ((2-\nu)\Delta t\|\Lambda\|)^2},$$

and remark that $r \geq \tau^2/(4s)$. Our main result (Theorem 4.1 of Section 4) is as follows.

THEOREM 1.3. *Consider the CNLF method with RAW time filter (1.3)-(1.5) with* $\alpha \in (\frac{1}{2}, 1]$ *and* $\nu \in [0, 0.3]$. *Suppose that the time step condition holds*

$$\Delta t\|\Lambda\| \leq CFL_{system}(\alpha, \nu).$$

Then CNLF with RAW time filter is stable. More precisely, for each $N \geq 2$ *we have*

$$\begin{aligned}
&\left(r - \frac{\tau^2}{4s}\right)|u_N|^2 + \left(\frac{\tau}{2\sqrt{s}}|u_N| - \sqrt{s}|u_{N-1}|\right)^2 \quad (1.9)\\
&\quad + \frac{\gamma}{2}\sum_{n=2}^{N}\left(|u_n - u_{n-1}| - |u_{n-1} - u_{n-2}|\right)^2 + \left(\frac{\nu}{2}(2\alpha - \nu(2\alpha-1)) - \gamma\right)\sum_{n=2}^{N-1}|u_n - u_{n-1}|^2\\
&\quad + \Delta t\sum_{n=2}^{N}\|u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2}\|_A^2\\
&\leq |u_1|^2 + |u_0|^2 - (2-\nu)\Delta t\langle\Lambda u_0, u_1\rangle + \frac{\gamma}{2}|u_1 - u_0|^2.
\end{aligned}$$

This theorem implies control of the unstable mode when A is SPD.

COROLLARY 1.4. *Assume* $\Delta t\|\Lambda\| \leq CFL_{system}(\alpha, \nu)$. *Let* $A + A^T > 0$. *Then, the approximations generated by CNLF+RAW satisfy* $u_n \to 0$ *as* $n \to \infty$.

Proof. The RHS of (1.9) is independent of N. Thus , letting $N \to \infty$, the infinite series coming from terms 2 through 4 in (1.9) converge. Hence, their n^{th} terms must approach zero. Since A has positive definite symmetric part, as $n \to \infty$ this implies

$$\begin{aligned}
a_n &= u_n - u_{n-1} \to 0\\
b_n &= u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2} \to 0.
\end{aligned}$$

From these two sequences, form the sequence that must also approach zero: $b_n - \nu\alpha a_{n-1} - (a_n - a_{n-1}) = (2 - \nu)u_{n-1} \to 0$. □

The above stability bound (1.9) and the proof of the corollary shows the mechanism by which RAW controls the unstable mode of CNLF. The second dissipative term controls all modes except those for which $u_n - u_{n-1} \equiv 0$ while the third dissipative term controls all modes except those for which $u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2} \equiv 0$. In combination, all modes are controlled.

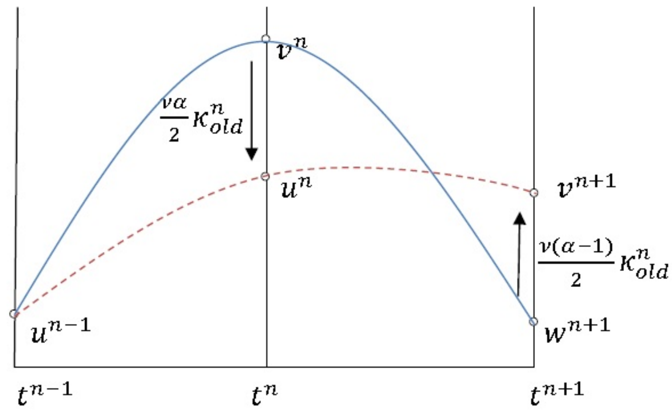


FIG. 2.1. ***RA** moves v^n down to decrease κ; **W** moves w^{n+1} to preserve means.*

2. Background on the RAW time filter. Detailed background for CNLF+RAW is given in, e.g., [18, 17, 2, 22, 23, 1]. Thus, we give only a brief outline in this section. In GFD simulations the choice of the terms A, Λ in (1.1) is to associate Λu with Coriolis terms yielding waves that are high energy and slow and Au with low energy but fast waves plus other effects. Thus the CFL condition associated with explicit treatment of Λu is modest. Durran [5, page 412] shows that accuracy degrades rapidly when fully implicit treatment of all terms is used in conjunction with a larger time step violating the CFL condition for the high energy components, see also [15, 13, 9].

The method CNLF has a long history. Stability (in the absence of roundoff) was proven by Fourier methods in 1963 [12] and in 2012 by energy methods for systems, [14]. In the presence of roundoff error, CNLF has an unstable mode that can take a long time to appear before growing disastrously. This has been long noted (e.g., [10] and [7, page 242] note that LF being marginally stable is often "slightly unstable") and is explained many ways. For example, even when $A = A^T > 0$ is a dissipative term in (1.1) the CN discretization of it, $A(u_{n+1} + u_{n-1})$, does not damp the energy in modes like $u_n \sim +1, +1, -1, -1, +1, +1, \cdots$ which can grow as a result. The added RA filter controls the unstable mode. It also reduces the accuracy of CNLF+RA from second order (for CNLF) to first order and over damps: $u_n \to 0$, even when $||u(t)|| \equiv ||u(0)||$. The W step anti-diffuses the RA step and restores accuracy, namely, CNLF + RAW is second order if $\alpha = \frac{1}{2} + \mathscr{O}(\Delta t)$.

2.1. Curvature Evolution. This section reviews a beautiful geometric interpretation of the RAW filter in terms of curvature evolution by Robert, Asselin and Williams.

DEFINITION 2.1. *The discrete curvature of ϕ_n is*

$$\kappa(\phi_n) = \phi_{n+1} - 2\phi_n + \phi_{n-1}.$$

Denote a curvature in time before and after the time filter, (1.4) and (1.5), as, respectively,

$$\kappa^n_{old} = w_{n+1} - 2v_n + u_{n-1}$$
$$\kappa^n_{new} = v_{n+1} - 2u_n + u_{n-1}.$$

Figure 2.1 illustrates how the time filter reduces the discrete curvature of the solution. After solving for w_{n+1} in the CNLF step (1.3) the first solution curve is the continuous line. The curvature obtained is κ^n_{old}. Next, performing the filter (1.4) and (1.5), leads to the new solution curve (the dashed line of Figure 2.1), with curvature κ^n_{new}.

PROPOSITION 2.2. *For $n \geq 1$ we have*

$$\kappa^n_{new} = (1 - \nu\frac{\alpha+1}{2})\kappa^n_{old},$$

$$|\kappa_{new}^n| < |\kappa_{old}^n| \ \ for\ 0 < \nu < 1, \frac{1}{2} < \alpha \leq 1.$$

When $\alpha = \frac{1}{2}$ CNLF + RAW preserves the mean of the solution curves:

$$\frac{v_{n+1} + u_n + u_{n-1}}{3} = \frac{w_{n+1} + v_n + u_{n-1}}{3}.$$

Proof. The first step of the filter (1.4) is $u_n = v_n + \frac{\nu\alpha}{2}\kappa_{old}^n$ and the second step (1.5) is $v_{n+1} = w_{n+1} + \frac{\nu(\alpha-1)}{2}\kappa_{old}^n$. A calculation with (1.4) and (1.5) shows

$$\begin{aligned}\kappa_{new}^n &= v_{n+1} - 2u_n + u_{n-1} = (w_{n+1} - 2v_n + u_{n-1}) + (v_{n+1} - w_{n+1}) + 2(v_n - u_n) \\ &= \kappa_{old}^n + \frac{\nu(\alpha-1)}{2}\kappa_{old}^n + 2(\frac{-\nu\alpha}{2})\kappa_{old}^n = (1 - \frac{\nu(\alpha+1)}{2})\kappa_{old}^n.\end{aligned} \tag{2.1}$$

When $\alpha = \frac{1}{2}$ the means are:

$$\frac{v_{n+1} + u_n + u_{n-1}}{3} = \frac{w_{n+1} + v_n + u_{n-1} + \frac{(2\alpha-1)\nu}{2}\kappa_{old}^n}{3} = \frac{w_{n+1} + v_n + u_{n-1}}{3}.$$

∎

3. The Equivalent Linear Multistep Method. The identification of the equivalent linear multistep method is one key in the stability analysis of CNLF + RAW. After this, powerful tools from the theory of such methods will be applied.

THEOREM 3.1. *The approximation u_n of CNLF plus RAW satisfies*

$$\begin{aligned}u_n - \nu u_{n-1} - (1-\nu)u_{n-2} = &-\Delta t A(u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2}) \\ &+ \Delta t \Lambda\Big(\big(2 + \nu(\alpha-1)\big)u_{n-1} - \nu\alpha u_{n-2}\Big),\end{aligned} \tag{3.1}$$

which can be rewritten as

$$\begin{aligned}&\frac{u^{n+1} - u^{n-1}}{2\Delta t} + A\Big(\frac{u^{n+1} + u^{n-1}}{2}\Big) - \Lambda u^n \\ &-\frac{\nu}{2}\left(\frac{u^n - u^{n-1}}{\Delta t} + A\left((1-\alpha)u^n + \alpha u^{n-1}\right) - \Lambda\left((1-\alpha)u^n + \alpha u^{n-1}\right)\right) = 0.\end{aligned}$$

For $\alpha = 1$, (CNLF + RA) (3.1) is first order accurate (in the sense of having second order local truncation error) for fixed $\nu > 0$, and second order if $\nu = \mathcal{O}(\Delta t)$. For fixed $\nu > 0, \alpha \in (1/2, 1]$, (3.1) is first order accurate and is second order if $\nu(2\alpha - 1) = \mathcal{O}(\Delta t)$. For the special value $\alpha = \frac{1}{2}$, (3.1) is formally second order accurate.

Proof. [Theorem 3.1] **Step 1: Equivalence to a linear multistep method:** This equivalence is proven by algebraic elimination of intermediate variables. Indeed, the first two steps, CNLF+RA can be rewritten as a block 2x2 system

$$\begin{bmatrix} (I + \triangle t A) & -2\triangle t\Lambda \\ \frac{\nu\alpha}{2}I & (1-\nu\alpha)I \end{bmatrix}\begin{bmatrix} w_{n+1} \\ v_n \end{bmatrix} = \begin{bmatrix} (I - \triangle t A)\,u_{n-1} \\ u_n - \frac{\nu\alpha}{2}u_{n-1} \end{bmatrix}.$$

We solve this system expressing w_{n+1}, v_n in terms of u_n, u_{n-1}. Next these are inserted into the third step yielding the claimed linear multistep method.

Step 2: The local truncation orders. Let u be the exact solution of $u'(t) = -Au + \Lambda u$ and u_n be the numerical approximation. By Taylor expansion,

$$u(t_n) = u(t_{n-1}) + \Delta t u'(t_{n-1}) + \frac{\Delta t^2}{2}u''(t_{n-1}) + O(\Delta t^3)$$

$$= u(t_{n-1}) + \Delta t(-A+\Lambda)u(t_{n-1}) + \frac{\Delta t^2}{2}(-A+\Lambda)^2 u(t_{n-1}) + O(\Delta t^3).$$

Similarly, we have

$$\begin{aligned}
u(t_{n-2}) &= u(t_{n-1}) - \Delta t(-A+\Lambda)u(t_{n-1}) + \frac{\Delta t^2}{2}(-A+\Lambda)^2 u(t_{n-1}) + O(\Delta t^3),\\
Au(t_n) &= Au(t_{n-1}) + \Delta t A(-A+\Lambda)u(t_{n-1}) + O(\Delta t^2),\\
Au(t_{n-2}) &= Au(t_{n-1}) - \Delta t A(-A+\Lambda)u(t_{n-1}) + O(\Delta t^2),\\
\Lambda u(t_{n-2}) &= \Lambda u(t_{n-1}) - \Delta t \Lambda(-A+\Lambda)u(t_{n-1}) + O(\Delta t^2).
\end{aligned}$$

Using this, the local truncation error of the scheme (1.7) is

$$\begin{aligned}
\tau_n =& \Big| \frac{u(t_n) - \nu u(t_{n-1}) - (1-\nu)u(t_{n-2})}{\Delta t}\\
&+ A\Big(u(t_n) + \nu(\alpha-1)u(t_{n-1}) + (1-\nu\alpha)u(t_{n-2})\Big)\\
&- \Lambda\Big((2+\nu(\alpha-1))u(t_{n-1}) - \nu\alpha u(t_{n-2})\Big)\Big|\\
\le& \frac{\nu}{2}(2\alpha-1)\|-A+\Lambda\|^2\|u\|_\infty \Delta t + O(\Delta t^2),
\end{aligned}$$

where $\|u\|_\infty = \sup\{|u(t)| : t \ge 0\}$. Thus, the method is second order for $\nu(\alpha - \frac{1}{2}) = \mathscr{O}(\Delta t)$. ☐

REMARK 1 (On parameter values). *The question arises if similar effects can be obtained from using only the RA filter (so no W step) but taking smaller parameter values. This does not seen to be the case. Typical values for the RA parameter* ν *vary from* $\nu = 0.01$ *in quasi-geostrophic models [22, 19], to* $\nu = 0.12$ *for global atmospheric models, to* $\nu = 0.5-0.6$ *for convective cloud models [5, 16]. RA is more stable than RAW, see e.g. Figure 5.2, and RA and RAW are both first order (for* $\alpha > \frac{1}{2}$*). However, RAW has a smaller error coefficient. For example, suppose the RA parameter* $\nu := 0.2$, $\alpha := 0.53$*. If we then apply only one RA step with* $\nu_{RA-new} = \nu(\alpha - \frac{1}{2}) = 0.006$*, then this RA-new and RAW have the same order of accuracy and the same coefficient of* Δt *in the local truncation error. However, the RA step with this new value* 0.006 *does not dampen enough the leapfrog computational mode since* 0.006 *lies outside the range observed in practice of acceptable RA filter values.*

4. G-Stability Analysis of CNLF Method with the RAW Filter. For vectors of the same length, denote the usual euclidean inner product and norm by $\langle u, v\rangle := u^T v$, $|u|^2 := \langle u, u\rangle$, the weighted norm of the symmetric positive definite matrix A by $\|u\|_A^2 := u^T A u$, and by $\|\Lambda\|$ the matrix norm of Λ. Recall from Section 1.1 that

$$CFL_{system}(\alpha,\nu) := \sqrt{2\alpha-1}\sqrt{\frac{2-\nu}{\alpha^2(2+\nu-2\alpha\nu)}}\left(1-\frac{\nu}{2}(2\alpha-1)\right)$$

$$CFL_{scalar}(\alpha,\nu) := \sqrt{2\alpha-1}\sqrt{\frac{2-\nu}{\alpha^2(2+\nu-2\alpha\nu)}}\left(1-\frac{\nu}{2}(2\alpha-1)\right)\frac{1}{\sqrt{1-\nu^2(\frac{1}{2}-\alpha)^2}}.$$

Let us define

$$\Delta t_{RAW} := \frac{1}{\|\Lambda\|}CFL_{system}(\alpha,\nu), \tag{4.1}$$

$$\Delta t'_{RAW} := \frac{1}{\|\Lambda\|}CFL_{scalar}(\alpha,\nu). \tag{4.2}$$

THEOREM 4.1. *For CNLF method with RAW time filter with* $\alpha \in (\frac{1}{2}, 1]$ *and* $\nu \in [0, 0.3]$*. Suppose that the time step condition holds*

$$\Delta t \le \Delta t_{RAW}, \text{ equivalently } \Delta t\|\Lambda\| \le CFL_{system}(\alpha,\nu). \tag{4.3}$$

Then CNLF + RAW is stable for (1.1): For each $N \geq 2$ we have

$$\begin{aligned}
&\left(r-\frac{\tau^2}{4s}\right)|u_N|^2+\left(\frac{\tau}{2\sqrt{s}}|u_N|-\sqrt{s}|u_{N-1}|\right)^2 \\
&\quad+\frac{\gamma}{2}\sum_{n=2}^{N}\left(|u_n-u_{n-1}|-|u_{n-1}-u_{n-2}|\right)^2+\left(\frac{\nu}{2}(2\alpha-\nu(2\alpha-1))-\gamma\right)\sum_{n=2}^{N-1}|u_n-u_{n-1}|^2 \\
&\quad+\Delta t\sum_{n=2}^{N}\|u_n+\nu(\alpha-1)u_{n-1}+(1-\nu\alpha)u_{n-2}\|_A^2 \\
&\leq |u_1|^2+|u_0|^2-(2-\nu)\Delta t\langle\Lambda u_0,u_1\rangle+\frac{\gamma}{2}|u_1-u_0|^2,
\end{aligned}$$

where $\gamma,\ r,\ s,\ \tau>0$ are defined in (1.8) and satisfy $r\geq\frac{\tau^2}{4s}$.

COROLLARY 4.2. *When $\alpha=1$ the resulting CNLF method with RA time filter with $\nu\in[0,0.3]$ is stable when*

$$\Delta t\|\Lambda\|\leq 1-\frac{\nu}{2}.$$

REMARK 2. *The sufficient time step restriction Δt_{RAW} in (4.1) is derived using an energy estimate. Stability region analysis for scalar problems in (Section 5.1) gives the necessary condition $\Delta t\leq\Delta t'_{RAW}$, (4.2).*

COROLLARY 4.3. *The leapfrog method with RAW time filter for $\alpha=\frac{1}{2}$ is unstable for all $\nu\in[0,0.3]$.* Before proving these results, we provide necessary supporting lemmas.

LEMMA 4.4. *The 2×2 matrix G given below is symmetric positive definite for $\alpha\in[\frac{1}{2},1]$ and $\nu\in[0,0.3]$:*

$$\begin{aligned}
G_{11}&=1-\frac{\nu}{4}(\nu+2\alpha-2\alpha\nu)-\frac{\nu}{4}\sqrt{(2+\nu-2\alpha\nu)(2\alpha-1)(2-\nu)},\\
G_{12}=G_{21}&=\frac{\nu}{4}(2\alpha+\nu-2-2\alpha\nu)+\frac{\nu}{4}\sqrt{(2+\nu-2\alpha\nu)(2\alpha-1)(2-\nu)},\\
G_{22}&=1-\frac{\nu}{4}(4+2\alpha-\nu-2\alpha\nu)-\frac{\nu}{4}\sqrt{(2+\nu-2\alpha\nu)(2\alpha-1)(2-\nu)}.
\end{aligned}$$

Proof. First we note that for arbitrary $\nu\in[0,0.3]$ the following terms

$$\nu+2\alpha-2\alpha\nu,\ (2+\nu-2\alpha\nu)(2\alpha-1)(2-\nu)\text{ and }4+2\alpha-\nu-2\alpha\nu$$

are monotone increasing functions of α. Then the main diagonal terms satisfy

$$\begin{aligned}
G_{11}&\geq G_{11}\Big|_{\alpha=1}=1-\nu+\frac{\nu^2}{2}\geq 1-\nu>0,\\
G_{22}&\geq G_{22}\Big|_{\alpha=1}=(1-\nu)^2,
\end{aligned}$$

while for the secondary diagonal we have

$$\begin{aligned}
|G_{12}|&=\frac{\nu}{4}|2(\alpha-1)+\nu(1-2\alpha)|+\frac{\nu}{4}\sqrt{(2+\nu-2\alpha\nu)(2\alpha-1)(2-\nu)}\\
&\leq\frac{\nu}{4}(2|\alpha-1|+\nu|1-2\alpha|)+\frac{\nu}{4}(2-\nu)\leq\frac{\nu}{4}\left(2\left|\frac{1}{2}-1\right|+\nu|1-2|\right)+\frac{\nu}{4}(2-\nu)\\
&=\frac{3\nu}{4}.
\end{aligned}$$

Therefore the determinant of G is positive

$$\det(G)=G_{11}G_{22}-G_{12}^2\geq(1-\nu)^3-\frac{9}{16}\nu^2>0\text{ for }\nu\in[0,0.3],$$

which completes the proof. □

LEMMA 4.5. *Suppose $p,\ q \in \mathbb{R}^N$, $v \in \mathbb{C}^N$ and skew symmetric matrix $\Lambda \in \mathbb{R}^{N\times N}$. Then we have*

$$|pv + q\Lambda v| \le \sqrt{p^2 + q^2\|\Lambda\|^2}|v|.$$

Proof. Note that

$$|pv + q\Lambda v|^2 = p^2|v|^2 + q^2|\Lambda v|^2 \le p^2|v|^2 + q^2\|\Lambda\|^2|v|^2 = (p^2 + q^2\|\Lambda\|^2)|v|^2.$$

□ To simplify the proofs, in them we will denote by x, y and η

$$x = \frac{1}{2}\sqrt{\frac{\nu}{2}(2+\nu-2\alpha\nu)}, \quad y = \frac{1}{2}\sqrt{\frac{\nu}{2}(2-\nu)(2\alpha-1)}, \text{and } \eta_n = u_n - u_{n-1}. \tag{4.4}$$

Then (1.8) takes the form

$$\gamma = \sqrt{4(x^2-y^2)^2 + (\alpha\nu\Delta t\|\Lambda\|)^2}, \qquad r = 1 - \frac{\gamma}{2}, \qquad s = 1 - \frac{\gamma}{2} - 4x^2 - \nu^2(\alpha-1),$$
$$\tau = \sqrt{\left(\nu(\alpha-2) + 2(x^2-y^2) + \gamma\right)^2 + \left((2-\nu)\Delta t\|\Lambda\|\right)^2},$$

and

$$G = \begin{pmatrix} 1-(x+y)^2 & \frac{\nu}{2}(\alpha-2) + 2x(x+y) \\ \frac{\nu}{2}(\alpha-2) + 2x(x+y) & 1-(x+y)^2 - 4x^2 - \nu^2(\alpha-1) \end{pmatrix}. \tag{4.5}$$

LEMMA 4.6. *For each $n \ge 2$ we have*

$$\begin{aligned}&|(x+y)u_n - 2xu_{n-1} + (x-y)u_{n-2}|^2 - \alpha\nu\Delta t\Big(\langle\Lambda u_{n-1}, u_n\rangle - \langle\Lambda u_{n-2}, u_n\rangle + \langle\Lambda u_{n-2}, u_{n-1}\rangle\Big)\\ &\ge \Big(2(x^2+y^2) - \frac{\gamma}{2}\Big)|\eta_n|^2 - \frac{\gamma}{2}|\eta_{n-1}|^2 + \frac{\gamma}{2}\big(|\eta_n| - |\eta_{n-1}|\big)^2 - (x-y)^2\big(|\eta_n|^2 - |\eta_{n-1}|^2\big).\end{aligned}$$

Proof. Using Lemma 4.5, we have

$$\begin{aligned}
&|(x+y)u_n - 2xu_{n-1} + (x-y)u_{n-2}|^2\\
&\quad - \alpha\nu\Delta t\Big(\langle\Lambda u_{n-1}, u_n\rangle - \langle\Lambda u_{n-2}, u_n\rangle + \langle\Lambda u_{n-2}, u_{n-1}\rangle\Big)\\
=&|(x+y)\eta_n - (x-y)\eta_{n-1}|^2 - \alpha\nu\Delta t\langle\Lambda\eta_{n-1}, \eta_n\rangle\\
=&(x+y)^2|\eta_n|^2 + (x-y)^2|\eta_{n-1}|^2 - 2(x^2-y^2)\langle\eta_{n-1}, \eta_n\rangle - \alpha\nu\Delta t\langle\Lambda\eta_{n-1}, \eta_n\rangle\\
=&\big((x+y)^2 + (x-y)^2\big)|\eta_n|^2 - 2(x^2-y^2)\langle\eta_{n-1}, \eta_n\rangle - \alpha\nu\Delta t\langle\Lambda\eta_{n-1}, \eta_n\rangle\\
&\quad - (x-y)^2|\eta_n|^2 + (x-y)^2|\eta_{n-1}|^2\\
=&2(x^2+y^2)|\eta_n|^2 - \langle 2(x^2-y^2)\eta_{n-1} + \alpha\nu\Delta t\Lambda\eta_{n-1}, \eta_n\rangle - (x-y)^2(|\eta_n|^2 - |\eta_{n-1}|^2)\\
\ge&2(x^2+y^2)|\eta_n|^2 - |2(x^2-y^2)\eta_{n-1} + \alpha\nu\Delta t\Lambda\eta_{n-1}||\eta_n| - (x-y)^2(|\eta_n|^2 - |\eta_{n-1}|^2)\\
\ge&2(x^2+y^2)|\eta_n|^2 - \sqrt{4(x^2-y^2)^2 + (\alpha\nu\Delta t\|\Lambda\|)^2}|\eta_{n-1}||\eta_n| - (x-y)^2(|\eta_n|^2 - |\eta_{n-1}|^2)\\
=&2(x^2+y^2)|\eta_n|^2 - \gamma|\eta_{n-1}||\eta_n| - (x-y)^2(|\eta_n|^2 - |\eta_{n-1}|^2)\\
=&2(x^2+y^2)|\eta_n|^2 - \frac{\gamma}{2}(|\eta_{n-1}|^2 + |\eta_n|^2 - (|\eta_n| - |\eta_{n-1}|)^2) - (x-y)^2(|\eta_n|^2 - |\eta_{n-1}|^2)
\end{aligned}$$

$$= \left(2(x^2+y^2) - \frac{\gamma}{2}\right)|\eta_n|^2 - \frac{\gamma}{2}|\eta_{n-1}|^2 + \frac{\gamma}{2}(|\eta_n| - |\eta_{n-1}|)^2 - (x-y)^2(|\eta_n|^2 - |\eta_{n-1}|^2).$$

□

LEMMA 4.7. *For all $n \geq 2$, the following inequality holds*

$$\begin{aligned}
&\left\| \begin{array}{c} u_n \\ u_{n-1} \end{array} \right\|_G^2 - (2-\nu)\Delta t \langle \Lambda u_{n-1}, u_n \rangle + \left((x+y)^2 - \frac{\gamma}{2}\right)|u_n - u_{n-1}|^2 \\
&\geq \left(r - \frac{\tau^2}{4s}\right)|u_n|^2 + \left(\frac{\tau}{2\sqrt{s}}|u_n| - \sqrt{s}|u_{n-1}|\right)^2.
\end{aligned}$$

Proof. By (4.5) we have

$$\begin{aligned}
&\left\| \begin{array}{c} u_n \\ u_{n-1} \end{array} \right\|_G^2 - (2-\nu)\Delta t \langle \Lambda u_{n-1}, u_n \rangle + \left((x+y)^2 - \frac{\gamma}{2}\right)|u_n - u_{n-1}|^2 \\
=&(1-(x+y)^2)|u_n|^2 + (1-(x+y)^2 - 4x^2 - \nu^2(\alpha-1))|u_{n-1}|^2 \\
&+ 2\left(\frac{\nu}{2}(\alpha-2) + 2x(x+y)\right)\langle u_n, u_{n-1}\rangle - (2-\nu)\Delta t\langle \Lambda u_{n-1}, u_n\rangle \\
&+ \left((x+y)^2 - \frac{\gamma}{2}\right)|u_n - u_{n-1}|^2 \\
=&(1-(x+y)^2)|u_n|^2 + (1-(x+y)^2 - 4x^2 - \nu^2(\alpha-1))|u_{n-1}|^2 \\
&+ (\nu(\alpha-2) + 4x(x+y))\langle u_n, u_{n-1}\rangle \\
&- (2-\nu)\Delta t\langle \Lambda u_{n-1}, u_n\rangle + \left((x+y)^2 - \frac{\gamma}{2}\right)\left(|u_n|^2 + |u_{n-1}|^2 - 2\langle u_n, u_{n-1}\rangle\right) \\
=&(1-\frac{\gamma}{2})|u_n|^2 + (1-\frac{\gamma}{2} - 4x^2 - \nu^2(\alpha-1))|u_{n-1}|^2 \\
&+ \left(\nu(\alpha-2) + 4x(x+y) - 2(x+y)^2 + \gamma\right)\langle u_n, u_{n-1}\rangle - (2-\nu)\Delta t\langle \Lambda u_{n-1}, u_n\rangle \\
=&(1-\frac{\gamma}{2})|u_n|^2 + (1-\frac{\gamma}{2} - 4x^2 - \nu^2(\alpha-1))|u_{n-1}|^2 \\
&+ \langle \left(\nu(\alpha-2) + 2(x^2-y^2) + \gamma\right) u_{n-1} - (2-\nu)\Delta t \Lambda u_{n-1}, u_n\rangle \\
\geq&(1-\frac{\gamma}{2})|u_n|^2 + (1-\frac{\gamma}{2} - 4x^2 - \nu^2(\alpha-1))|u_{n-1}|^2 \\
&- \sqrt{\left(\nu(\alpha-2) + 2(x^2-y^2) + \gamma\right)^2 + ((2-\nu)\Delta t\|\Lambda\|)^2}\,|u_{n-1}||u_n| \\
=&r|u_n|^2 + s|u_{n-1}|^2 - \tau|u_n||u_{n-1}| \\
=&\left(r - \frac{\tau^2}{4s}\right)|u_n|^2 + \left(\frac{\tau}{2\sqrt{s}}|u_n| - \sqrt{s}|u_{n-1}|\right)^2,
\end{aligned}$$

where we applied Lemma 4.5 again. □ Now we are ready to give the proof of Theorem 4.1.

Proof. [Proof of Theorem 4.1] If we take the inner product of (1.7) with $(u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2})$, then the left hand side can be written as

$$\begin{aligned}
LHS =& \langle u_n - \nu u_{n-1} - (1-\nu)u_{n-2}, u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2}\rangle \\
=& \left\| \begin{array}{c} u_n \\ u_{n-1} \end{array} \right\|_G^2 - \left\| \begin{array}{c} u_{n-1} \\ u_{n-2} \end{array} \right\|_G^2 + |(x+y)u_n - 2xu_{n-1} + (x-y)u_{n-2}|^2, \qquad (4.6)
\end{aligned}$$

while the right hand side is

$$RHS = -\Delta t\|u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2}\|_A^2$$

$$
\begin{aligned}
&+\Delta t\langle\Lambda((2+\nu(\alpha-1))u_{n-1}-\nu\alpha u_{n-2}),u_n+\nu(\alpha-1)u_{n-1}+(1-\nu\alpha)u_{n-2}\rangle\\
=&-\Delta t\|u_n+\nu(\alpha-1)u_{n-1}+(1-\nu\alpha)u_{n-2}\|_A^2\\
&+\Delta t(2+\nu(\alpha-1))\langle\Lambda u_{n-1},u_n\rangle+(2+\nu(\alpha-1))(1-\nu\alpha)\Delta t\langle\Lambda u_{n-1},u_{n-2}\rangle\\
&-\alpha\nu\Delta t\langle\Lambda u_{n-2},u_n\rangle-\alpha\nu\Delta t\langle\Lambda u_{n-2},\nu(\alpha-1)u_{n-1}\rangle\\
=&-\Delta t\|u_n+\nu(\alpha-1)u_{n-1}+(1-\nu\alpha)u_{n-2}\|_A^2\\
&+(2-\nu)\Delta t\left(\langle\Lambda u_{n-1},u_n\rangle-\langle\Lambda u_{n-2},u_{n-1}\rangle\right)\\
&+\alpha\nu\Delta t(\langle\Lambda u_{n-1},u_n\rangle-\langle\Lambda u_{n-2},u_n\rangle+\langle\Lambda u_{n-2},u_{n-1}\rangle).
\end{aligned}\tag{4.7}
$$

According to Lemma 4.6, equations (4.6) and (4.7) can be written as follows.

$$
\begin{aligned}
0=&\left(\left\|\begin{matrix}u_n\\u_{n-1}\end{matrix}\right\|_G^2-(2-\nu)\Delta t\langle\Lambda u_{n-1},u_n\rangle\right)-\left(\left\|\begin{matrix}u_{n-1}\\u_{n-2}\end{matrix}\right\|_G^2-(2-\nu)\Delta t\langle\Lambda u_{n-2},u_{n-1}\rangle\right)\\
&+\Delta t\|u_n+\nu(\alpha-1)u_{n-1}+(1-\nu\alpha)u_{n-2}\|_A^2+|(x+y)u_n-2xu_{n-1}+(x-y)u_{n-2}|^2\\
&-\alpha\nu\Delta t(\langle\Lambda u_{n-1},u_n\rangle-\langle\Lambda u_{n-2},u_n\rangle+\langle\Lambda u_{n-2},u_{n-1}\rangle)\\
\geq&\left(\left\|\begin{matrix}u_n\\u_{n-1}\end{matrix}\right\|_G^2-(2-\nu)\Delta t\langle\Lambda u_{n-1},u_n\rangle-(x-y)^2|\eta_n|^2\right)\\
&\qquad-\left(\left\|\begin{matrix}u_{n-1}\\u_{n-2}\end{matrix}\right\|_G^2-(2-\nu)\Delta t\langle\Lambda u_{n-2},u_{n-1}\rangle-(x-y)^2|\eta_{n-1}|^2\right)\\
&+\Delta t\|u_n+\nu(\alpha-1)u_{n-1}+(1-\nu\alpha)u_{n-2}\|_A^2\\
&+\left(2(x^2+y^2)-\frac{\gamma}{2}\right)|\eta_n|^2-\frac{\gamma}{2}|\eta_{n-1}|^2+\frac{\gamma}{2}\left(|\eta_n|-|\eta_{n-1}|\right)^2.
\end{aligned}\tag{4.8}
$$

Then sum up (4.8) from $n=2$ to N,

$$
\begin{aligned}
0\geq&\sum_{n=2}^{N}\left(\left\|\begin{matrix}u_n\\u_{n-1}\end{matrix}\right\|_G^2-(2-\nu)\Delta t\langle\Lambda u_{n-1},u_n\rangle-(x-y)^2|\eta_n|^2\right)\\
&-\sum_{n=2}^{N}\left(\left\|\begin{matrix}u_{n-1}\\u_{n-2}\end{matrix}\right\|_G^2-(2-\nu)\Delta t\langle\Lambda u_{n-2},u_{n-1}\rangle-(x-y)^2|\eta_{n-1}|^2\right)\\
&+\Delta t\sum_{n=2}^{N}\|u_n+\nu(\alpha-1)u_{n-1}+(1-\nu\alpha)u_{n-2}\|_A^2\\
&+\sum_{n=2}^{N}\left(\left(2(x^2+y^2)-\frac{\gamma}{2}\right)|\eta_n|^2-\frac{\gamma}{2}|\eta_{n-1}|^2+\frac{\gamma}{2}\left(|\eta_n|-|\eta_{n-1}|\right)^2\right)\\
=&\left(\left\|\begin{matrix}u_N\\u_{N-1}\end{matrix}\right\|_G^2-(2-\nu)\Delta t\langle\Lambda u_{N-1},u_N\rangle-(x-y)^2|\eta_N|^2\right)\\
&-\left(\left\|\begin{matrix}u_1\\u_0\end{matrix}\right\|_G^2-(2-\nu)\Delta t\langle\Lambda u_0,u_1\rangle-(x-y)^2|\eta_1|^2\right)\\
&+\Delta t\sum_{n=2}^{N}\|u_n+\nu(\alpha-1)u_{n-1}+(1-\nu\alpha)u_{n-2}\|_A^2\\
&+\sum_{n=2}^{N}\left(\left(2(x^2+y^2)-\frac{\gamma}{2}\right)|\eta_n|^2-\frac{\gamma}{2}|\eta_{n-1}|^2+\frac{\gamma}{2}\left(|\eta_n|-|\eta_{n-1}|\right)^2\right)\\
=&\left(\left\|\begin{matrix}u_N\\u_{N-1}\end{matrix}\right\|_G^2-(2-\nu)\Delta t\langle\Lambda u_{N-1},u_N\rangle-(x-y)^2|\eta_N|^2\right)
\end{aligned}\tag{4.9}
$$

$$-\left(\left\| \begin{array}{c} u_1 \\ u_0 \end{array} \right\|_G^2 - (2-\nu)\Delta t\langle \Lambda u_0, u_1\rangle - (x-y)^2|\eta_1|^2\right)$$
$$+\Delta t \sum_{n=2}^{N} \|u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2}\|_A^2$$
$$+\left(2(x^2+y^2) - \frac{\gamma}{2}\right)|\eta_N|^2 - \frac{\gamma}{2}|\eta_1|^2$$
$$+\frac{\gamma}{2}\sum_{n=2}^{N}(|\eta_n| - |\eta_{n-1}|)^2 + \big(2(x^2+y^2)-\gamma\big)\sum_{n=2}^{N-1}|\eta_n|^2.$$

By simplifying (4.9), we have

$$\begin{aligned}
&\left(\left\| \begin{array}{c} u_N \\ u_{N-1} \end{array} \right\|_G^2 - (2-\nu)\Delta t\langle \Lambda u_{N-1}, u_N\rangle + \left((x+y)^2 - \frac{\gamma}{2}\right)|\eta_N|^2\right) \\
&\quad + \Delta t \sum_{n=2}^{N} \|u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2}\|_A^2 + \frac{\gamma}{2}\sum_{n=2}^{N}\big(|\eta_n| - |\eta_{n-1}|\big)^2 \\
&\quad + \big(2(x^2+y^2)-\gamma\big)\sum_{n=2}^{N-1}|\eta_n|^2 \\
&\leq \left\| \begin{array}{c} u_1 \\ u_0 \end{array} \right\|_G^2 - (2-\nu)\Delta t\langle \Lambda u_0, u_1\rangle + \left(\frac{\gamma}{2} - (x-y)^2\right)|\eta_1|^2 \\
&\leq \left\| \begin{array}{c} u_1 \\ u_0 \end{array} \right\|_G^2 - (2-\nu)\Delta t\langle \Lambda u_0, u_1\rangle + \frac{\gamma}{2}|\eta_1|^2 \\
&\leq |u_1|^2 + |u_0|^2 - (2-\nu)\Delta t\langle \Lambda u_0, u_1\rangle + \frac{\gamma}{2}|\eta_1|^2.
\end{aligned} \tag{4.10}$$

Under the time step condition (4.3), all terms on the left hand side of (4.10) are positive. Then we apply Lemma 4.7 to (4.10) and obtain

$$\begin{aligned}
&\left(r - \frac{\tau^2}{4s}\right)|u_N|^2 + \left(\frac{\tau}{2\sqrt{s}}|u_N| - \sqrt{s}|u_{N-1}|\right)^2 \\
&\quad + \Delta t \sum_{n=2}^{N} \|u_n + \nu(\alpha-1)u_{n-1} + (1-\nu\alpha)u_{n-2}\|_A^2 + \frac{\gamma}{2}\sum_{n=2}^{N}(|\eta_n| - |\eta_{n-1}|)^2 \\
&\quad + \big(2(x^2+y^2)-\gamma\big)\sum_{n=2}^{N-1}|\eta_n|^2 \\
&\leq |u_1|^2 + |u_0|^2 - (2-\nu)\Delta t\langle \Lambda u_0, u_1\rangle + \frac{\gamma}{2}|\eta_1|^2.
\end{aligned} \tag{4.11}$$

The coefficient of the energy term is positive since

$$4rs - \tau^2 \geq \frac{(2-\nu)^2}{\alpha^2}\Big((1-\alpha)^2(1-\alpha\nu+\alpha\nu^2) + \frac{\nu^2}{4}(2\alpha-1)\big(1-2\alpha(1-\alpha)\big)\Big) \geq 0$$

for $\alpha \in [\frac{1}{2}, 1]$ and $\nu \in [0, 0.3]$, with equality only when $\alpha = 1$ and $\nu = 0$.
Finally, using (4.4), we write (4.11) in terms of α, ν and u_n to obtain the energy bound (1.9), which concludes the proof. □

5. Absolute Stability Region and Root Locus Curve of Leapfrog with RAW. Stability can be analyzed exactly for scalar problems in terms of the root locus curve. The results of this analysis is organized in a compact and useful manner by the methods stability regions, derived in this section. This analysis gives necessary conditions for stability in the system case and allows us to test sharpness of the filter-dependent CFL timestep conditions derived in the Section 4.

5.1. Root Locus Curve. Taking $A = 0$, we consider leapfrog with RAW time filter, which is

$$u_n - \nu u_{n-1} - (1-\nu)u_{n-2} = \Delta t \Lambda\Big(\big(2+\nu(\alpha-1)\big)u_{n-1} - \nu\alpha u_{n-2}\Big). \tag{5.1}$$

Then the characteristic polynomials of (5.1) are

$$\begin{aligned}\rho(\zeta) &= \zeta^2 - \nu\zeta - (1-\nu),\\ \sigma(\zeta) &= (2+\nu(\alpha-1))\zeta - \nu\alpha.\end{aligned}$$

Denote $w = \Delta t\lambda$, where λ is an arbitrary non-zero eigenvalue of Λ, thus, w is a purely imaginary number. Let ζ be on the unit circle, i.e., $\zeta = e^{i\theta},\ \ \theta \in [0, 2\pi]$, then the root locus curve [8, section V.1] for w is determined by

$$\rho(\zeta) - w\sigma(\zeta) = 0 \ \text{ or } \ w = \frac{\rho(\zeta)}{\sigma(\zeta)}.$$

Thus,

$$\begin{aligned} w = \frac{\rho(\zeta)}{\sigma(\zeta)} &= \frac{\zeta^2 - \nu\zeta - (1-\nu)}{(2+\nu(\alpha-1))\zeta - \nu\alpha}\\ &= \frac{(\cos 2\theta + i\sin 2\theta) - \nu(\cos\theta + i\sin\theta) - (1-\nu)}{(2+\nu(\alpha-1))(\cos\theta + i\sin\theta) - \nu\alpha}\\ &= Re(w) + iIm(w),\end{aligned}$$

where

$$\begin{aligned} Re(w) &= \frac{\nu(1-\cos\theta)(2\alpha + \nu - 2\alpha\nu - 2 + 2\alpha\cos\theta)}{\left((2+\nu(\alpha-1))\cos\theta - \nu\alpha\right)^2 + \left((2+\nu(\alpha-1))\sin\theta\right)^2},\\ Im(w) &= \frac{\sin\theta\left((2-\nu)^2 + 2\alpha\nu(1-\cos\theta)\right)}{\left((2+\nu(\alpha-1))\cos\theta - \nu\alpha\right)^2 + \left((2+\nu(\alpha-1))\sin\theta\right)^2}.\end{aligned}$$

Since w is purely imaginary, set $Re(w) = 0$. Then this gives

$$\cos\theta = 1 \text{ or } \cos\theta = \nu - 1 + \frac{2-\nu}{2\alpha},$$

and in this case $Im(w) = 0$ or

$$Im(w) = \Delta t\lambda = \pm\frac{\sqrt{(2-\nu)(2\alpha-1)}}{\alpha\sqrt{2-\nu+2\alpha\nu}} = \pm\frac{\Delta t_{RAW}}{\sqrt{1-\nu^2(\frac{1}{2}-\alpha)^2}} = \pm\Delta t'_{RAW}.$$

Therefore we have proven the following result on the absolute stability of Leapfrog method ($A = 0$) with the RAW time filter (1.4)-(1.5).

PROPOSITION 5.1. *The Leapfrog method with RAW time filter is stable if and only if the time-step satisfies*

$$\Delta t \leq \Delta t'_{RAW} \text{ equivalently } \Delta t||\Lambda|| \leq CFL_{scalar}(\alpha,\nu).$$

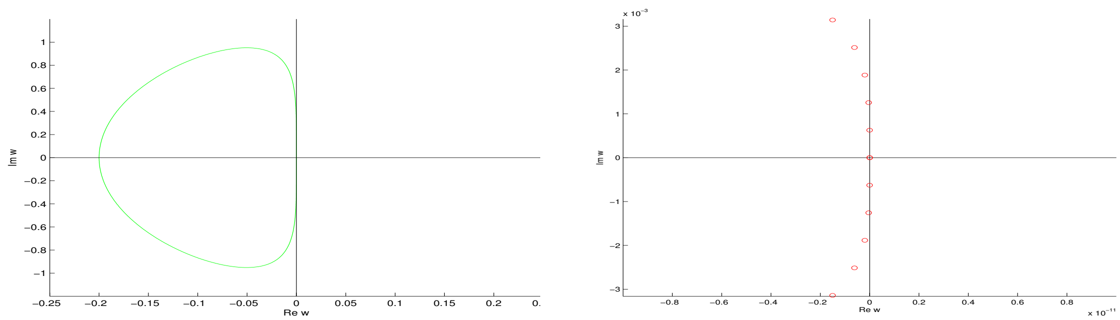
REMARK 3. *The time step (4.1) yielding the CFL_{system} condition also gives the largest time interval on which the amplification factor function is differentiable [22].*

5.2. Unconditional instability of Leapfrog + RAW for $\alpha = \frac{1}{2}$. Now we give the proof of Corollary 4.3.

Proof. [Corollary 4.3] From (4.2) we note that $\Delta t'_{RAW} = 0$ when $\alpha = \frac{1}{2}$, which implies that the intersection of the root locus curve with the imaginary axis is the origin, concluding the argument. □

This instability is visualized through the root locus curve for LF+RAW with $\alpha = \frac{1}{2}$ in Figure 5.1. As predicted, the zoom of Figure 5.1(b) shows that CNLF-RAW with $\alpha = \frac{1}{2}$ is unconditionally unstable, confirming geometrically the above algebraic proof. The real axis is rescaled of order 10^{-11} around imaginary axis. The only intersection of root locus curve with the imaginary axis is the origin, which implies the unconditional instability of leapfrog with RAW time filter for $\alpha = \frac{1}{2}$.

FIG. 5.1. *(a) The root locus curve / region of absolute stability for Leapfrog with RAW, for $\alpha = \frac{1}{2}$ and $\nu = 0.2$. (b) Points on the root locus curve of leapfrog with RAW time filter .*



(a) LF-RAW (b) LF-RAW-zoom

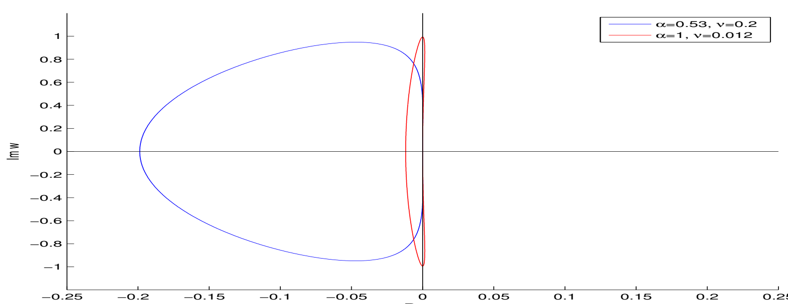


FIG. 5.2. *Absolute stability of leapfrog with RAW vs RA.*

6. Numerical Tests. We will give an example of energy obtained by both Δt_{RAW} and $\Delta t'_{RAW}$. Consider

$$u'(t) = \Lambda u(t), \quad \Lambda = \begin{pmatrix} 0 & 50 \\ -50 & 0 \end{pmatrix}, \; u(0) = [1,1]^T.$$

Take $\alpha = 0.53$ and $\nu = 0.2$, then we calculate the sufficient and necessary timestep restrictions for stability to be, respectively,

$$\Delta t_{RAW} = 0.00874267 \;\text{ and }\; \Delta t'_{RAW} = 0.00874283.$$

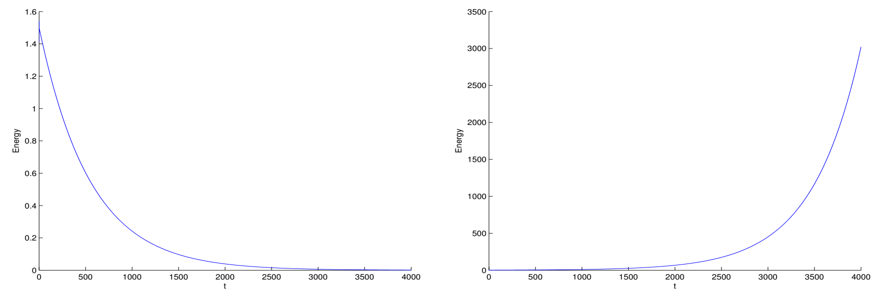


FIG. 6.1. *Energy obtained with step size* $\Delta t = 0.99\Delta t'_{RAW}$ *on the left, and* $\Delta t = 1.01\Delta t'_{RAW}$ *on the right.*

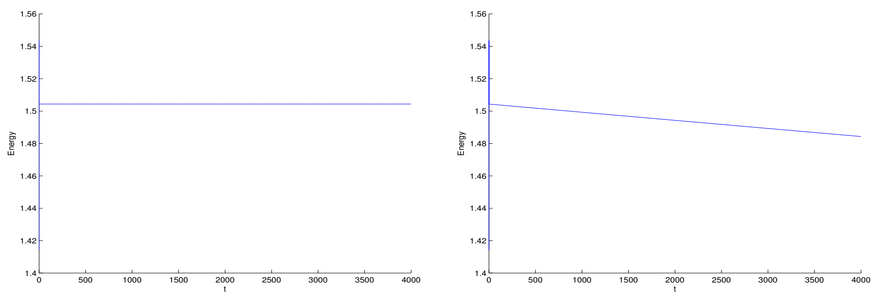


FIG. 6.2. *Energy obtained with step size* $\Delta t = \Delta t'_{RAW}$ *on the left, and* $\Delta t = \Delta t_{RAW}$ *on the right.*

Finally, the predicted instability leapfrog with RAW time filter for $\alpha = \frac{1}{2}$ is verified in Figure 6.3.

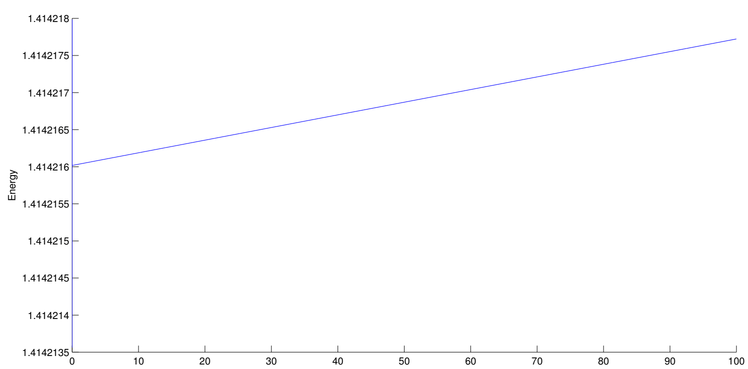


FIG. 6.3. *Energy obtained by using step size* $\Delta t = 0.00005$*, with* $\alpha = 0.5$*,* $\nu = 0.2$*, for which the scheme is unconditionally unstable. The figure implies that no matter how small the step size is, the energy eventually blows up.*

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