UNCOUPLING EVOLUTIONARY GROUNDWATER-SURFACE WATER FLOWS USING THE CRANK-NICOLSON LEAPFROG METHOD

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Abstract. Consider an incompressible fluid in a region Ω_f flowing both ways across an interface, I, into a porous media domain Ω_p saturated with the same fluid. The physical processes in each domain have been well studied and are described by the Stokes equations in the fluid region and the Darcy equations in the porous media region. Taking the interfacial conditions into account produces a system with an exactly skew symmetric coupling. Spatial discretization by finite element method and time discretization by Crank-Nicolson LeapFrog gives a second order partitioned method requiring only one Stokes and one Darcy sub-physics and sub-domain solver per time step for the fully evolutionary Stokes-Darcy problem. Analysis of this method leads to a time-step condition sufficient for stability and convergence. Numerical tests verify predicted rates of convergence, however stability tests reveal the problem of growth of numerical noise in unstable modes in some cases. In such instances, the addition of time filters adds stability.

1. Introduction. Many important problems in environmental science seek an accurate solution for the coupling of groundwater flows with surface flows. For example, one serious problem concerns how pollution discharged into lakes, streams, and rivers, permeates ground water supply. Descriptions of the interaction of groundwater and surface water can be found in many places, see for example, Pinder and Celia [1]. One difficulty in solving such problems comes from the coupling of two domains, a fluid region and a porous media region, along with the two physical processes happening in each region, described by the Stokes or Navier-Stokes and Darcy or Brinkman equations respectively. In both sub-regions, the problem is time-dependent and since different physical processes are taking place in each region, this suggests the efficiency of utilizing different codes for each sub-process.

This paper examines the effectiveness of the Crank-Nicolson LeapFrog to uncouple the problem into two sub problems requiring only one Stokes and one Darcy sub-physics and sub-domain solver per time step. We analyze the stability of such a method in relation to the physical parameters and mesh width and conduct an error analysis over long time intervals. Finally, we present numerical experiments and compare and contrast these results with the analysis of the method.

Denote the fluid region by Ω_f and the porous media region by Ω_p. Assume both domains are bounded and regular. Let I represent the interface between the two domains. Assume that the fluid motion in Ω_f is governed by the Stokes equations and let the Darcy equations govern the fluid motion in Ω_p. Then the fluid velocity, u, fluid pressure, p, and porous media piezometric head (Darcy pressure), ϕ, satisfy

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$$
\begin{aligned}
u_t - \nu\Delta u + \nabla p = f_f \text{ in } \Omega_f, \\
\nabla \cdot u = 0, \text{ in } \Omega_f, \\
S_0\phi_t - \nabla\cdot(\kappa\nabla\phi) = f_p \text{ in } \Omega_p, \\
u(x,0) = u_0 \text{ in } \Omega_f \text{ and } \phi(x,0) = \phi_0 \text{ in } \Omega_p, \\
u(x,t) = 0 \text{ on } \partial\Omega_f \setminus I \text{ and } \phi(x,t) = 0 \text{ on } \partial\Omega_p \setminus I, \\
+ \text{ coupling conditions across } I.
\end{aligned}
$$

We assume no-slip along the exterior boundary of the coupled region. Let the dimension, d, be 2 or 3. Below is a list of the variables in the above equations and many of the variables used throughout this paper.

$$
\begin{aligned}
u &= \text{fluid velocity in porous media region } \Omega_f, \\
\nu &= \text{kinematic viscosity of fluid}, \\
p &= \text{fluid pressure in } \Omega_f, \text{ rescaled by density}, \\
f_f, f_p &= \text{body forces in fluid region rescaled by density, source in porous region} \\
\mathbf{u}_p &= \text{fluid velocity in porous media region } \Omega_p, \\
\phi &= \text{Darcy pressure + elevation induced pressure } = \text{piezometric head}, \\
S_0 &= \text{specific mass storativity coefficient}, \\
\kappa &= \text{hydraulic conductivity tensor}, \\
g &= \text{gravitational acceleration}.
\end{aligned}
$$

All functions depend on both space, $x = (x_1, x_2, ..., x_d)$ and time, t. Note that u, $\mathbf{u}_p$, and f_f are vector valued functions. We assume all material and fluid parameters above are positive, and also that the eigenvalues of the hydraulic conductivity tensor satisfy $0 < k_{min} = \lambda_{min}(\kappa) \leq \lambda_{max}(\kappa) = k_{max} < \infty$. Denote the outward unit normal vectors of $\Omega_{f,p}$ by $\hat{n}_{f,p}$. Coupling conditions along the interface I for this problem include conservation of mass and the balance of normal forces across the interface:

$$
\begin{aligned}
u\cdot\hat{n}_f + \mathbf{u}_p\cdot\hat{n}_p = 0 \text{ on } I \Leftrightarrow u\cdot\hat{n}_f - \kappa\nabla\phi\cdot\hat{n}_p = 0 \text{ on } I, \ \textit{(conservation of mass)} \\
p - \nu\hat{n}_f\cdot\nabla u\cdot\hat{n}_f = g\phi \text{ on } I. \ \textit{(balance of normal forces)}
\end{aligned}
$$

In addition to these coupling conditions, we need a tangential condition on the fluid region's velocity along the interface. For our analysis we utilize the Beavers-Joseph-Saffman coupling condition (see, e.g. [2], [3]),

$$
-\nu\hat{\tau}_j\cdot\nabla u\cdot\hat{n}_f = \frac{\alpha}{\sqrt{\hat{\tau}_j\cdot\kappa\cdot\hat{\tau}_j}}\, u\cdot\hat{\tau}_j \text{ on } I \text{ for any } \hat{\tau}_j \text{ tangent vector on } I,
$$

where $\hat{\tau}_j, j = 1, ..., d-1$ denotes the orthonormal system of tangent vectors on I, and $\alpha > 0$ is a constant that must be experimentally determined and depends on properties of the porous medium. This condition is a simplification of the more physically realistic Beavers-Joseph coupling condition in [4].

1.1. Previous Work. There has been considerable growth on the numerical analysis of methods for the Stokes-Darcy coupled problems. The study of the equilibrium problem is advanced, see for example, Layton, Schieweck, and Yotov in [5], Discacciati, Miglio, and Quarteroni in [6], and Payne and Straughan in [7]. Domain decomposition for the equilibrium problem has been studied in Discacciati in [8] and Discacciati, Quarteroni, and Valli in [9]. Cao, Gunzberger et. all in [10] provide an analysis of the finite element method for the time-dependent problem using the Beavers-Joseph interface conditions, and [11] analyzes parallel domain decomposition methods. In this paper we focus on a specific partitioned method which allows us to uncouple the fully evolutionary problem and use one (SPD) Stokes and one (SPD) Darcy solver per time step. The first study on partitioned methods for this problem was presented by Mu and Zhu [12] in 2010. Partitioned methods were also studied by Layton and Trenchea in [13], and by Layton, Trenchea, and Tran in [14]. Partitioned methods utilizing different time steps in each domain have been examined by Layton, Shan, and Zheng in [15] and [16].

2. The Continuous Problem and Semi-Discrete Approximation. For simplicity, denote the $L^2(I)$ norm by $\|\cdot\|_I$ and the $L^2(\Omega_{f,p})$ norms by $\|\cdot\|_{f,p}$ respectively. Likewise, we represent their corresponding inner products by $(\cdot,\cdot)_{I,f,p}$. Define

$$
\begin{aligned}
X_f &= \{v \in (H^1(\Omega_f))^d : v = 0 \text{ on } \partial\Omega_f \setminus I\},\\
X_p &= \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p \setminus I\},\\
Q_f &= L_0^2(\Omega_f).
\end{aligned}
$$

Define the following norms on the dual spaces $(X_f)^*$ and $(X_p)^*$.

$$
\|f\|_{-1,f,p} = \sup_{0\neq w\in X_{f,p}} \frac{(f,w)_{f,p}}{\|\nabla w\|_{f,p}} \tag{2.1}
$$

Our analysis will make use of the following inequalities.

LEMMA 2.1. *(A Trace Inequality) Let Ω be a bounded regular domain, $u \in H^1(\Omega)$. Then there exists a constant $C_\Omega > 0$ depending on the domain Ω such that the following inequality holds.*

$$
\|u\|_{L^2(\partial\Omega)} \leq C_\Omega \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}}.
$$

Proof. See, for example Brenner and Scott in [17], Ch. 1.6 p.36-38. ☐

LEMMA 2.2. *(Poincaré Inequality) Let $v \in X_f$, $\psi \in X_p$. Then there exists a constant $C_P > 0$ such that the following holds for $w = v$ or ψ.*

$$
\|w\| \leq C_P \|\nabla w\|. \qquad \textit{(Poincaré inequality)} \tag{2.2}
$$

Define the bilinear forms

$$a_f(u,v) = \nu(\nabla u, \nabla v)_f + \sum_{i=1}^{d-1} \int_I \frac{\alpha}{\sqrt{\hat{\tau}_i \cdot \kappa \cdot \hat{\tau}_i}} (u \cdot \hat{\tau}_i)(v \cdot \hat{\tau}_i)\, ds,$$
$$a_p(\phi, \psi) = g(\kappa \nabla \phi, \nabla \psi)_p,$$
$$c_I(u, \phi) = g \int_I \phi u \cdot \hat{n}_f \, ds.$$

Note that the bilinear forms $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$ are both continuous and coercive on their respective domains, Ω_f and Ω_p, as given below in the following lemma.

Lemma 2.3. *(Continuity and Coercivity of the Bilinear Forms) The following inequalities hold.*

$$\begin{aligned} a_f(u,v) &\le \left(\nu + \frac{\alpha C}{\sqrt{k_{min}}}\right) \|\nabla u\|_f \|\nabla v\|_f, \\ a_p(\phi,\psi) &\le g k_{max} \|\nabla \phi\|_p \|\nabla \phi\|_p, \\ a_f(u,u) &\ge \nu \|\nabla u\|_f^2, \\ a_p(\phi,\phi) &\ge g k_{min} \|\nabla \phi\|_p^2, \end{aligned} \tag{2.3}$$

where $C > 0$ is a positive constant depending on the domain of Ω_f arising from the Trace and Poincaré Inequalities.

The variational formulation for the coupled Stokes-Darcy problem is as follows.

$$\begin{aligned} &\text{Find } (u,p,\phi) : [0,\infty) \to (X_f, Q_f, X_p) \text{ satisfying} \\ &(u_t, v)_f + a_f(u,v) - (p, \nabla \cdot v)_f + c_I(v,\phi) = (f_f, v)_f, \\ &\qquad (q, \nabla \cdot u)_f = 0, \\ &gS_0(\phi_t, \psi)_p + a_p(\phi,\psi) - c_I(u,\psi) = g(f_p,\psi)_p, \\ &\qquad \text{for all } (v,q,\psi) \in (X_f, Q_f, X_p). \end{aligned} \tag{2.4}$$

It is interesting to note that setting $v = u$ and $\psi = \phi$ and adding the first and third equations exactly cancels the coupling terms. In other words, the sub-Stokes and sub-Darcy problems are exactly skew symmetric. Applying coercivity of the two bilinear forms, using the Cauchy-Schwartz and Young inequalities and integrating in time yields the energy estimate for the coupled system.

$$\begin{aligned} \|u(t)\|_f^2 + gS_0\|\phi(t)\|_p^2 + \nu \int_0^t \|\nabla u(s)\|_f^2 \, ds + g k_{min} \int_0^2 \|\nabla \phi(s)\|_p^2 \, ds \\ \le \frac{1}{2\nu} \int_0^t \|f_f(s)\|_f^2 \, ds + \frac{g}{2k_{min}} \int_0^t \|f_p(s)\|_p^2 \, ds + \|u_0\|_f^2 + gS_0\|\phi_0\|_p^2. \end{aligned}$$

Select a mesh for Ω_f and Ω_p. Let $\mathcal{T}_h$ denote the combined mesh of $\Omega_f \cup \Omega_p$, with the maximum triangle diameter denoted by h. Assume that $\mathcal{T}_h$ is quasi-uniform. Select finite element spaces,

$$\begin{aligned} \text{fluid velocity: } & X_f^h \subset X_f, \\ \text{Darcy pressure: } & X_p^h \subset X_p, \\ \text{Stokes pressure: } & Q_f^h \subset Q_f, \end{aligned}$$

based on a conforming FEM triangulation. We assume that the Stokes velocity-pressure FEM spaces, X_f^h and Q_f^h satisfy the usual discrete inf-sup condition for stability of the discrete pressure, denoted by (LBB^h) [18], as stated below.

$$\exists\, \beta_h > 0 \text{ such that } \inf_{q_h \in Q_f^h,\, q_h \neq 0} \sup_{v_h \in X_f^h,\, v_h \neq 0} \frac{(q_h, \nabla \cdot v_h)_f}{\|\nabla v_h\|_f \|q_h\|_f} > \beta_h \quad (LBB^h)$$

No assumption is made on the mesh compatibility or inter-domain continuity on the interface I between the two FEM meshes in the two subdomains. Assume that X_f^h, X_p^h, and Q_f^h satisfy approximation properties of piecewise polynomials on quasi-uniform meshes of local degrees $r-1, r$, and $r+1$. That is,

$$\begin{aligned}
\inf_{u_h \in X_f^h} \|u - u_h\|_f &\leq Ch^{r+1}\|u\|_{H^{r+1}(\Omega_f)}, \\
\inf_{u_h \in X_f^h} \|u - u_h\|_{H^1(\Omega_f)} &\leq Ch^{r}\|u\|_{H^{r+1}(\Omega_f)}, \\
\inf_{\phi_h \in X_p^h} \|\phi - \phi_h\|_p &\leq Ch^{r+1}\|\phi\|_{H^{r+1}(\Omega_p)}, \\
\inf_{\phi_h \in X_p^h} \|\phi - \phi_h\|_{H^1(\Omega_p)} &\leq Ch^{r}\|\phi\|_{H^{r+1}(\Omega_p)}, \\
\inf_{p_h \in Q_f^h} \|p - p_h\|_f &\leq Ch^{r+1}\|p\|_{H^{r+1}(\Omega_f)}.
\end{aligned} \tag{2.5}$$

Further, assume that the following inverse inequality holds for our choice of $\mathcal{T}_h$ and finite element spaces. Note that this assumption implies an angle condition. See Brenner and Scott, [17], chapter 4 for more on inverse inequalities.

LEMMA 2.4. *(An Inverse Inequality) Let $w_h \in X_f^h$ or X_p^h, then*

$$h\|\nabla w_h\| \leq C_{(inv)}\|w_h\|. \qquad \textit{(Inverse Inequality)}$$

The semi-discretization for the coupled Stokes-Darcy problem is as follows.

$$\begin{aligned}
\text{Find } (u_h(\cdot,t), p_h(\cdot,t), \phi_h(\cdot,t)) &: [0,\infty) \to (X_f^h, Q_f^h, X_p^h) \\
\text{satisfying for all } &(v_h, q_h, \psi_h) \in (X_f^h, Q_f^h, X_p^h), \\
(u_{h,t}, v_h)_f + a_f(u_h, v_h) - (p_h, \nabla \cdot v_h)_f + c_I(u_h, \phi_h) &= (f_f, v_h)_f, \\
(q_h, \nabla \cdot u_h) &= 0, \\
gS_0(\phi_{h,t}, \psi_h)_p + a_p(\phi_h, \psi_h) - c_I(u_h, \psi_h) &= g(f_p, \psi_h)_p.
\end{aligned} \tag{2.6}$$

Throughout this paper, C represents a positive constant independent of the time step and mesh width and will vary from situation to situation. Analysis of the method will require special treatment of the coupling term, $c_I(\cdot,\cdot)$.

LEMMA 2.5. *(Coupling Inequalities) For all $v_h \in X_f^h$ and $\psi_h \in X_p^h$, with $\mathcal{C} = C_{\Omega_f} C_{\Omega_p} C_{(inv)} g$ we have*

$$c_I(v_h, \psi_h) \leq \frac{1}{2}\mathcal{C}h^{-2}\|v_h\|_f^2 + \frac{1}{2}\mathcal{C}\|\psi_h\|_p^2 \tag{2.7}$$

$$c_I\left(v_h, \psi_h\right) \le \frac{1}{2}\mathcal{C}h^{-1}\|v_h\|_f^2 + \frac{1}{2}\mathcal{C}h^{-1}\|\psi_h\|_p^2. \tag{2.8}$$

Proof. We make use of the Cauchy-Schwarz, Trace, Inverse, and Young inequalities in that order, picking up the corresponding constants which depend on the geometry of the spaces Ω_f or Ω_p.

$$\begin{aligned} c_I(v_h, \psi_h) &= g\int_I \psi_h v_h \cdot \hat{n}_f ds \le \left| g\int_I \psi_h v_h \cdot \hat{n}_f ds \right| \\ &\le g\|\psi_h\|_I\|v_h\|_I \le C_{\Omega_f}C_{\Omega_p} g\|\psi_h\|_p^{\frac{1}{2}}\|\nabla\psi_h\|_p^{\frac{1}{2}}\|v_h\|_f^{\frac{1}{2}}\|\nabla v_h\|_f^{\frac{1}{2}} \\ &\le C_{\Omega_f}C_{\Omega_p}C_{(inv)}h^{-1}g\|\psi_h\|_p\|v_h\|_f \\ &\le \frac{1}{2}C_{\Omega_f}C_{\Omega_p}C_{(inv)}h^{-2}g\|v_h\|_f^2 + \frac{1}{2}C_{\Omega_f}C_{\Omega_p}C_{(inv)}g\|\psi_h\|_p^2 \end{aligned}$$

Note that we can replace the last line with

$$c_I\left(v_h, \psi_h\right) \le \frac{1}{2}C_{\Omega_f}C_{\Omega_p}C_{(inv)}h^{-1}\rho g\|v_h\|_f^2 + \frac{1}{2}C_{\Omega_f}C_{\Omega_p}C_{(inv)}h^{-1}\rho g\|\psi_h\|_p^2.$$

□

3. CNLF for the coupled Stokes Darcy Equations. One of the difficulties in solving the coupled Stokes-Darcy equations arises from the desire to uncouple the equations in order to implement existing codes optimized to solve the physical processes in each sub domain. By treating the coupling terms explicitly with Leapfrog, we successfully uncouple the two equations.

Definition 1. *(Crank-Nicolson LeapFrog Method) Let $t^n := n\Delta t$ and $w^n := w(x, t^n)$ for any function $w(x,t)$. CNLF for the evolutionary Stokes-Darcy equations is as follows.*

Given $\left(u_h^k, p_h^k, \phi_h^k\right)$, $\left(u_h^{k-1}, p_h^{k-1}, \phi_h^{k-1}\right) \in \left(X_f^h, Q_f^h, X_p^h\right)$, find $\left(u_h^{k+1}, p_h^{k+1}, \phi_h^{k+1}\right) \in \left(X_f^h, Q_f^h, X_p^h\right)$ satisfying for all $(v_h, q_h, \psi_h) \in \left(X_f^h, Q_f^h, X_p^h\right)$:

$$\begin{aligned} &\left(\tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(\tfrac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) - \left(\tfrac{p_h^{k+1}+p_h^{k-1}}{2}, \nabla\cdot v_h\right)_f \\ &\qquad\qquad\qquad\qquad + c_I(v_h, \phi_h^k) = (f_f^k, v_h)_f, \\ &\left(q_h, \nabla\cdot\left(\tfrac{u_h^{k+1}+u_h^{k-1}}{2}\right)\right)_f = 0, \\ &gS_0\left(\tfrac{\phi_h^{k+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\tfrac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) - c_I(u_h^k, \psi_h) = g(f_p^k, \psi_h)_p. \end{aligned} \tag{3.1}$$

CNLF is a 3 level method. The first terms, (u_h^0, p_h^0, ϕ_h^0), arise from the initial conditions of the problem. To obtain (u_h^1, p_h^1, ϕ_h^1) one must use another method. Note that errors in this first step will affect the overall convergence rate of the method.

3.1. Stability of CNLF. We derive a CFL-type time step condition for the stability of CNLF. Under this condition, approximate solutions to the Stokes-Darcy coupled problem are uniform in time stable and convergent.

THEOREM 3.1. *(Stability for CNLF Method) Suppose Δt satisfies*

$$\Delta t \le \mathcal{C}^{-1} \max\left\{\min\left\{h^2, gS_0\right\}, \min\left\{h, gS_0 h\right\}\right\}, \tag{3.2}$$

where $\mathcal{C} = C_{\Omega_f} C_{\Omega_p} C_{(inv)} g$. Let $\alpha = \min\{1 - \Delta t \mathcal{C}^{-1} h^{-1}, 1 - \Delta t \mathcal{C}^{-1} h^{-2}\} > 0$ and $\beta = \min\{gS_0 - \Delta t \mathcal{C}^{-1} h^{-1}, gS_0 - \Delta t \mathcal{C}^{-1}\} > 0$. Then, for $N = 1, 2, 3, \ldots$ CNLF stability holds:

$$\begin{aligned}
&\alpha(\|u_h^{N+1}\|_f^2 + \|u_h^N\|_f^2) + \beta(\|\phi_h^{N+1}\|_p^2 + \|\phi_h^N\|_p^2) \\
&\qquad + \Delta t \sum_{k=1}^{N-1} \left[\frac{\nu}{2}\|\nabla\left(u_h^{k+1} + u_h^{k-1}\right)\|_f^2 + \frac{g k_{min}}{2}\|\nabla\left(\phi_h^{k+1} + \phi_h^{k-1}\right)\|_p^2\right] \\
&\le (\|u_h^1\|_f^2 + \|u_h^0\|_f^2) + gS_0(\|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2) + 2\Delta t\left(c_I(u_h^1, \phi_h^0) - c_I(u_h^0, \phi_h^1)\right) \\
&\qquad + 2\Delta t \sum_{k=1}^{N-1}\left[(\nu)^{-1}\|f_f^k\|_{-1}^2 + g(k_{min})^{-1}\|f_p^k\|_{-1}^2\right].
\end{aligned}$$

Proof. Choose $v_h = u_h^{k+1} + u_h^{k-1}$ and $\psi_h = \phi_h^{k+1} + \phi_h^{k-1}$. Then the second equation drops out and the first and third equations of the method added together become

$$\begin{aligned}
&\tfrac{1}{2\Delta t}\left(\|u_h^{k+1}\|_f^2 + gS_0\|\phi_h^{k+1}\|_g^2 - \|u_h^{k-1}\|_f^2 - gS_0\|\phi_h^{k-1}\|_g^2\right) \\
&\qquad + a_f\left(\tfrac{u_h^{k+1} + u_h^{k-1}}{2}, u_h^{k+1} + u_h^{k-1}\right) + a_p\left(\tfrac{\phi_h^{k+1} + \phi_h^{k-1}}{2}, \phi_h^{k+1} + \phi_h^{k-1}\right) \\
&\qquad + c_I(u_h^{k+1} + u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1} + \phi_h^{k-1}) \\
&\qquad\qquad = (f_f^k, u_h^{k+1} + u_h^{k-1})_f + g(f_p^k, \phi_h^{k+1} + \phi_h^{k-1})_p.
\end{aligned}$$

Consider the right-hand-side of the above equation. Using the Cauchy-Schwarz and Young inequalities, one obtains the following bound.

$$\begin{aligned}
&(f_f^k, u_h^{k+1} + u_h^{k-1})_f + g(f_p^k, \phi_h^{k+1} + \phi_h^{k-1})_p \\
&\qquad \le \|f_f^k\|_{-1,f}\|\nabla(u_h^{k+1} + u_h^{k-1})\|_f + g\|f_p^k\|_{-1,p}\|\nabla(\phi_h^{k+1} + \phi_h^{k-1})\|_p \\
&\qquad \le \frac{\nu}{4}\|\nabla(u_h^{k+1} + u_h^{k-1})\|_f^2 + (\nu)^{-1}\|f_f^k\|_{-1,f}^2 \\
&\qquad\qquad + \frac{g k_{min}}{4}\|\nabla(\phi_h^{k+1} + \phi_h^{k-1})\|_p + g(k_{min})^{-1}\|f_p^k\|_{-1,p}^2.
\end{aligned}$$

This bound along with coercivity of $a_f(.,.)$ and $a_p(.,.)$ gives,

$$\begin{aligned}
&\tfrac{1}{2\Delta t}\left(\|u_h^{k+1}\|_f^2 + gS_0\|\phi_h^{k+1}\|_p^2 - \|u_h^{k-1}\|_f^2 - gS_0\|\phi_h^{k-1}\|_p^2\right) + \frac{\nu}{4}\|\nabla(u_h^{k+1} + u_h^{k-1})\|_f^2 \\
&\qquad + \frac{g k_{min}}{4}\|\nabla(\phi_h^{k+1} + \phi_h^{k-1})\|_p + c_I(u_h^{k+1} + u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1} + \phi_h^{k-1}) \\
&\qquad\qquad \le (\nu)^{-1}\|f_f^k\|_{-1}^2 + g(k_{min})^{-1}\|f_p^k\|_{-1}^2.
\end{aligned}$$

Consider the coupling terms $c_I(u_h^{k+1} + u_h^{k-1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1} + \phi_h^{k-1})$. Define

$$C^{k+\frac{1}{2}} = c_I(u_h^{k+1}, \phi_h^k) - c_I(u_h^k, \phi_h^{k+1}).$$

The coupling terms equal $C^{k+\frac{1}{2}} - C^{k-\frac{1}{2}}$. Multiplying by $2\Delta t$ and adding and subtracting $\|u_h^k\|_f^2$ and $gS_0\|\phi_h^k\|_p^2$ yields the following.

$$\begin{aligned}
\big(\|u_h^{k+1}\|_f^2 &+ \|u_h^k\|_f^2 + gS_0\|\phi_h^{k+1}\|_g^2 + gS_0\|\phi_h^k\|_p^2\big) \\
&- \big(\|u_h^k\|_f^2 + \|u_h^{k-1}\|_f^2 + gS_0\|\phi_h^k\|_p^2 + gS_0\|\phi_h^{k-1}\|_p^2\big) \\
&+ 2\Delta t \left(\frac{\nu}{4}\|\nabla(u_h^{k+1} + u_h^{k-1})\|_f^2 + \frac{gk_{min}}{4}\|\nabla(\phi_h^{k+1} + \phi_h^{k-1})\|_p\right) \\
&+ 2\Delta t \left(C^{k+\frac{1}{2}} - C^{k-\frac{1}{2}}\right) \\
&\qquad\qquad \le 2\Delta t \left((\nu)^{-1}\|f_f^k\|_{-1,f}^2 + g(k_{min})^{-1}\|f_p^k\|_{-1,p}^2\right).
\end{aligned}$$

Define the energy terms.

$$E^{k+\frac{1}{2}} = \|u_h^{k+1}\|_f^2 + \|u_h^k\|_f^2 + gS_0\|\phi_h^{k+1}\|_p^2 + gS_0\|\phi_h^k\|_p^2.$$

Using this notation, sum the previous inequality from $k = 1$ to $N-1$.

$$\begin{aligned}
E^{N-\frac{1}{2}} &- E^{\frac{1}{2}} + 2\Delta t C^{N-\frac{1}{2}} - 2\Delta t C^{\frac{1}{2}} \\
&\qquad + 2\Delta t \sum_{k=1}^{N-1} \left[\frac{\nu}{4}\|\nabla(u_h^{k+1} + u_h^{k-1})\|_f^2 + \frac{gk_{min}}{4}\|\nabla(\phi_h^{k+1} + \phi_h^{k-1})\|_p\right] \\
&\le 2\Delta t \sum_{k=1}^{N-1} \left[(\nu)^{-1}\|f_f^k\|_{-1,f}^2 + g(k_{min})^{-1}\|f_p^k\|_{-1,p}^2\right].
\end{aligned}$$

The above inequality implies the stability of the CNLF-method provided that $E^{N-\frac{1}{2}} + 2\Delta t C^{N-\frac{1}{2}} \ge 0$. By (2.7) and (2.8),

$$\begin{aligned}
C^{N-\frac{1}{2}} &\ge -\frac{1}{2}\mathcal{C}h^{-2}\left(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2\right) - \frac{1}{2}\mathcal{C}\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right), \\
C^{N-\frac{1}{2}} &\ge -\frac{1}{2}\mathcal{C}h^{-1}\left(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2\right) - \frac{1}{2}\mathcal{C}h^{-1}\left(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2\right).
\end{aligned}$$

Apply these bounds separately to the energy term $E^{N-\frac{1}{2}} + 2\Delta t C^{N-\frac{1}{2}}$.

$$\begin{aligned}
E^{N-\frac{1}{2}} + 2\Delta t C^{N-\frac{1}{2}} &\ge (1 - \Delta t\mathcal{C}h^{-2})(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2) \\
&\qquad + (gS_0 - \Delta t\mathcal{C})(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2), \\
E^{N-\frac{1}{2}} + 2\Delta t C^{N-\frac{1}{2}} &\ge (1 - \Delta t\mathcal{C}h^{-1})(\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2) \\
&\qquad + (gS_0 - \Delta t\mathcal{C}h^{-1})(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2).
\end{aligned}$$

Therefore $E^{N-\frac{1}{2}} + 2\Delta t C^{N-\frac{1}{2}} \ge 0$ provided that

$$\Delta t \le \mathcal{C}^{-1} \max\left\{\min\left\{h^2, gS_0\right\}, \min\left\{h, gS_0 h\right\}\right\}.$$

□

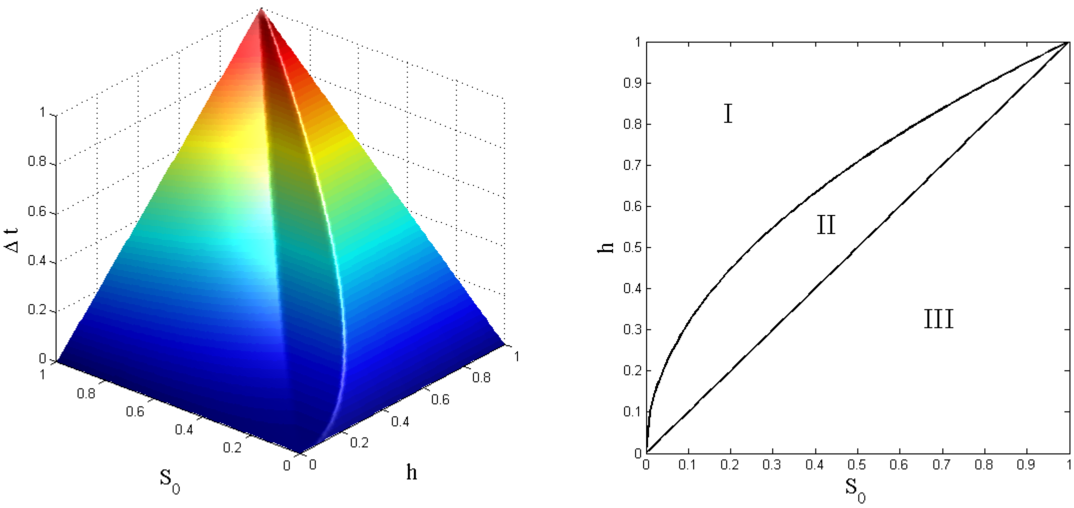

FIG. 3.1. *(left) Surface Plot of the Time Step Condition (right) Cross Section of the Time Step Condition*

A surface plot for the time step condition is given in Figure 3.1, with $g = 1$. For stability, Δt must be on or below the surface. This surface consists of three separate surfaces forming one continuous surface as evident in 3.1 (left). In region I, $\Delta t \leq S_0$, in region II, $\Delta t \leq h^2$, and in region III, $\Delta t \leq S_0 h$. It is also important to notice that this time step condition is independent of k_{min} but sensitive to S_0.

3.2. Error Analysis of CNLF. We analyze the error of the method over long time intervals. Recall that the FEM spaces, X_f^h, X_p^h and Q_f^h satisfy approximation properties of piecewise polynomials of degree $r-1$, r, and $r+1$ as stated previously. Since we assumed that X_f^h and Q_f^h satisfied (LBB^h), there exists some constant C such that if $u \in V := \{v \in X_f : \nabla \cdot v = 0\}$

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega_f)} \leq C \inf_{x_h \in X_f^h} \|u - x_h\|_{H^1(\Omega_f)}, \tag{3.3}$$

(see, for example, Girault and Raviart [18]). Let $N \in \mathbf{N}$ be given. Denote $t^n = n\Delta t$ and $T = N\Delta t$. If $T = \infty$ then $N = \infty$. In order to conduct error analysis for CNLF, we first introduce the following discrete norms.

$$|||v|||_{L^2(0,T;H^s(\Omega_{f,p}))} := \left(\sum_{k=1}^{N} \|v^k\|^2_{H^s(\Omega_{f,p})} \Delta t \right)^{1/2},$$
$$|||v|||_{L^\infty(0,T;H^s(\Omega_{f,p}))} := \max_{0 \leq k \leq N} \|v^k\|_{H^s(\Omega_{f,p})}.$$

Our error analysis of CNLF will use the lemma below.

LEMMA 3.2. *(Consistency Errors) The following inequalities hold:*

$$\Delta t \sum_{k=1}^{N-1} \left\| u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 \leq \frac{(\Delta t)^4}{40} \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))}, \tag{3.4}$$

$$\Delta t \sum_{k=1}^{N-1} \|\phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t} \|_p^2 \leq \frac{(\Delta t)^4}{40} \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))}, \tag{3.5}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(u^k - \frac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 \leq \frac{7(\Delta t)^4}{6} \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))}, \tag{3.6}$$

$$\Delta t \sum_{k=1}^{N-1} \left\| \nabla \left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2} \right) \right\|_p^2 \leq \frac{7(\Delta t)^4}{6} \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}. \tag{3.7}$$

Proof. We will prove (3.4) and (3.6). The proofs for the other inequalities are similar. We prove the first inequality by integrating by parts twice and the Cauchy-Schwarz inequality.

$$\begin{aligned}
&\sum_{k=1}^{N-1} \left\| u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t} \right\|_f^2 \\
&\quad = \frac{1}{4(\Delta t)^2} \int_{\Omega_f} \sum_{k=1}^{N-1} \left(\int_{t^k}^{t^{k+1}} (t - t^{k+1}) u_{tt}\, dt + \int_{t^{k-1}}^{t^k} (t - t^{k-1}) u_{tt}\, dt \right)^2 dx \\
&\quad = \frac{1}{4(\Delta t)^2} \int_{\Omega_f} \sum_{k=1}^{N-1} \left(\int_{t^k}^{t^{k+1}} \frac{(t - t^{k+1})^2}{2} u_{ttt}\, dt + \int_{t^{k-1}}^{t^k} \frac{(t - t^{k-1})^2}{2} u_{ttt}\, dt \right)^2 dx \\
&\quad \leq \frac{1}{4(\Delta t)^2} \int_{\Omega_f} \sum_{k=1}^{N-1} \frac{(\Delta t)^5}{20} \left(\int_{t^{k-1}}^{t^{k+1}} |u_{ttt}|^2\, dt \right) dx \\
&\quad \leq \frac{(\Delta t)^3}{40} \int_{\Omega_f} \int_0^T |u_{ttt}|^2\, dt\, dx \leq \frac{(\Delta t)^3}{40} \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))}.
\end{aligned}$$

This next inequality is proved similarly.

$$\begin{aligned}
&\sum_{k=1}^{N-1} \left\| \nabla \left(u^k - \frac{u^{k+1} + u^{k-1}}{2} \right) \right\|_f^2 = \int_{\Omega_f} \sum_{k=1}^{N-1} \left| \nabla \left(\frac{u^k - u^{k+1}}{2} + \frac{u^k - u^{k-1}}{2} \right) \right|^2 dx \\
&\quad = \frac{1}{4} \int_{\Omega_f} \sum_{k=1}^{N-1} |(\nabla u^k - \nabla u^{k+1}) + (\nabla u^k - \nabla u^{k-1})|^2\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\int_{\Omega_f}\sum_{k=1}^{N-1}\left|\int_{t^{k+1}}^{t^k}\nabla u_t\,dt + \int_{t^{k-1}}^{t^k}\nabla u_t\,dt\right|^2 dx\\
&= \frac{1}{4}\int_{\Omega_f}\sum_{k=1}^{N-1}\left|\int_{t^{k+1}}^{t^k}(t-t^k)'\nabla u_t dt + \int_{t^{k-1}}^{t^k}(t-t^k)'\nabla u_t dt\right|^2 dx\\
&= \frac{1}{4}\int_{\Omega_f}\sum_{k=1}^{N-1}\Bigg|-\Delta t\int_{t^{k-1}}^{t^{k+1}}\nabla u_{tt}\,dt + \int_{t^k}^{t^{k+1}}(t-t^k)\nabla u_{tt}\,dt\\
&\quad + \int_{t^{k-1}}^{t^k}(t^k-t)\nabla u_{tt}\,dt|^2 dx\\
&\le \frac{1}{2}\int_{\Omega_f}\sum_{k=1}^{N-1}\left((\Delta t)^2\left|\int_{t^{k-1}}^{t^{k+1}}\nabla u_{tt}\,dt\right|^2 + \left|\int_{t^k}^{t^{k+1}}(t-t^k)\nabla u_{tt}\,dt\right|^2\right.\\
&\quad \left. + \left|\int_{t^{k-1}}^{t^k}(t^k-t)\nabla u_{tt}\,dt\right|^2\right) dx\\
&\le \frac{1}{2}\int_{\Omega_f}\sum_{k=1}^{N-1}\left((\Delta t)^3\int_{t^{k-1}}^{t^{k+1}}|\nabla u_{tt}|^2\,dt + \frac{(\Delta t)^3}{3}\int_{t^{k-1}}^{t^{k+1}}|\nabla u_{tt}|^2\,dt\right) dx\\
&\le \frac{7(\Delta t)^3}{6}\|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))}.
\end{aligned}
$$

□

We now prove convergence with optimal rates over long time intervals under condition (3.2). Denote $e_f^n = u^n - u_h^n$ and $e_p^n = \phi^n - \phi_h^n$.

THEOREM 3.3. *(Convergence of CNLF) Consider the CNLF method* (3.1). *Suppose that the time step condition* (3.2) *holds and* u, p, ϕ *satisfy the following regularity conditions.*

$$
\begin{aligned}
&u \in L^2(0,T;H^{r+2}(\Omega_f)) \cap L^\infty(0,T;H^{r+1}(\Omega_f)) \cap H^3(0,T;H^1(\Omega_f)),\\
&p \in L^2(0,T;L^2(\Omega_f)),\\
&\phi \in L^2(0,T;H^{r+2}(\Omega_p)) \cap L^\infty(0,T;H^{r+1}(\Omega_p)) \cap H^3(0,T;H^1(\Omega_p)).
\end{aligned}
$$

Then, for any $0 \le t_N \le \infty$, *there is a positive constant* $\widehat{C}$ *independent of the mesh width and time step such that*

$$
\begin{aligned}
&\frac{\alpha}{2}(\|e_f^N\|_f^2 + \|e_f^{N-1}\|_f^2 + \frac{\beta}{2}(\|e_p^N\|_p^2 + \|e_p^{N-1}\|_p^2)\\
&\quad + \Delta t\sum_{k=1}^{N-1}\left(\frac{\nu}{4}\|\nabla(e_f^{k+1}+e_f^{k-1})\|_f^2 + \frac{gk_{min}}{4}\|\nabla(e_p^{k+1}+e_p^{k-1})\|_p^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\leq \widehat{C}\Bigg\{ & h^{2r}\Big\{|||u|||^2_{L^2(0,T;H^{r+1}(\Omega_f))} + |||\phi|||^2_{L^2(0,T;H^{r+1}(\Omega_p))}\Big\} \\
& + h^{2r+2}\Big\{\|u_t\|^2_{L^2(0,T;H^{r+1}(\Omega_f))} + \|\phi_t\|_{L^2(0,T;H^{r+1}(\Omega_p))} + |||u|||^2_{L^\infty(0,T;H^{r+1}(\Omega_f))} \\
& + |||\phi|||^2_{L^\infty(0,T;H^{r+1}(\Omega_p))} + |||p|||^2_{L^2(0,T;H^{r+1}(\Omega_f))}\Big\} \\
& + (\Delta t)^4\Big\{\|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} \\
& + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}\Big\} + \Delta t(\|\nabla e^1_f\|^2_f + \|\nabla e^0_f\|^2_f + \|\nabla e^1_p\|^2_p + \|\nabla e^0_p\|^2_p) \\
& + \|e^1_f\|^2_f + \|e^0_f\|^2_f + \|e^1_p\|^2_p + \|e^0_p\|^2_p \qquad \Bigg\}.
\end{aligned}
$$

Proof. Recall that solution $u^k = u(t^k)$ where $t^k = k\Delta t$, satisfies (2.4). Consider CNLF over the discretely divergence free space $V^h := \{v_h \in X^h_f : (q_h, \nabla \cdot v_h)_f = 0\,\forall q_h \in Q^h_f\}$ instead of X^h_f. Subtract (3.1) from (2.4) evaluated at time t^k. Note that since $v_h \in V^h$, the Stokes pressure term, $\left(\frac{p_h^{k+1}+p_h^{k-1}}{2}, \nabla \cdot v_h\right)$ is equal to zero, and can therefore be omitted from the equation. We have:

$$
\begin{aligned}
\left(u^k_t - \tfrac{u_h^{k+1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(u^k - \tfrac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) & \\
- \left(p^k, \nabla \cdot v_h\right)_f + c_I\left(v_h, \phi^k - \phi^k_h\right) &= 0, \\
gS_0\left(\phi^k_t - \tfrac{\phi_h^{k+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\phi^k - \tfrac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) - c_I\left(u^k - u^k_h, \psi_h\right) &= 0.
\end{aligned}
$$

Since v_h is discretely divergence free, $\left(p^k, \nabla \cdot v_h\right)_f = \left(p^k - \lambda^k_h, \nabla \cdot v_h\right)_f$, for any $\lambda_h \in Q^h_f$.

After rearranging terms, the error equations become

$$
\begin{aligned}
&\left(\frac{u^{k+1}-u_h^{k+1}}{2\Delta t} - \frac{u^{k-1}-u_h^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(\frac{u^{k+1}+u^{k-1}}{2} - \frac{u_h^{k+1}+u_h^{k-1}}{2}, v_h\right) \\
&\qquad + c_I\left(v_h, \phi^k - \phi^k_h\right) = \left(\frac{u^{k+1}-u^{k-1}}{2\Delta t}, v_h\right)_f - (u^k_t, v_h)_f \\
&\qquad - a_f\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}, v_h\right) + \left(p^k - \lambda^k_h, \nabla \cdot v_h\right)_f,
\end{aligned}
$$

$$
\begin{aligned}
&gS_0\left(\frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t} - \frac{\phi_h^{k+1}-\phi_h^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\frac{\phi^{k+1}+\phi^{k-1}}{2} - \frac{\phi_h^{k+1}+\phi_h^{k-1}}{2}, \psi_h\right) \\
&\qquad - c_I\left(u^k - u^k_h, \psi_h\right) = gS_0\left(\frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}, \psi_h\right)_p - gS_0(\phi^k_t, \psi_h)_p \\
&\qquad - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}, \psi_h\right).
\end{aligned}
$$

The consistency errors are:

$$\varepsilon_f^k(v_h) = \left(\frac{u^{k+1}-u^{k-1}}{2\Delta t}, v_h\right)_f - (u_t^k, v_h)_f - a_f\left(u^k - \frac{u^{k+1}+u^{k-1}}{2}, v_h\right),$$
$$\varepsilon_p^k(\psi_h) = gS_0\left(\frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}, \psi_h\right)_p - gS_0(\phi_t^k, \psi_h)_p - a_p\left(\phi^k - \frac{\phi^{k+1}+\phi^{k-1}}{2}, \Psi_h\right).$$

Split the error terms into:

$$e_f^{k+1} = u^{k+1} - u_h^{k+1} = (u^{k+1} - \tilde{u}^{k+1}) + (\tilde{u}^{k+1} - u_h^{k+1}) = \eta_f^{k+1} + \xi_f^{k+1},$$
$$e_p^{k+1} = \phi^{k+1} - \phi_h^{k+1} = (\phi^{k+1} - \tilde{\phi}^{k+1}) + (\tilde{\phi}^{k+1} - \phi_h^{k+1}) = \eta_p^{k+1} + \xi_p^{k+1}.$$

Take $\tilde{u}^{k+1} \in V^h$ and $\tilde{\phi}^{k+1} \in X_p^h$ so that $\xi_f^{k+1} \in V^h$. Rearranging error equations gives

$$\begin{aligned}
&\left(\frac{\xi_f^{k+1}-\xi_f^{k-1}}{2\Delta t}, v_h\right)_f + a_f\left(\frac{\xi_f^{k+1}+\xi_f^{k-1}}{2}, v_h\right) + c_I(v_h, \xi_p^k) \\
&\qquad = -\left(\frac{\eta_f^{k+1}-\eta_f^{k-1}}{2\Delta t}, v_h\right)_f - a_f\left(\frac{\eta_f^{k+1}+\eta_f^{k-1}}{2}, v_h\right) \\
&\qquad\quad - c_I(v_h, \eta_p^k) + \varepsilon_f^k(v_h) + \left(p^k - \lambda_h^k, \nabla\cdot v_h\right)_f,
\end{aligned}$$

$$\begin{aligned}
&gS_0\left(\frac{\xi_p^{k+1}-\xi_p^{k-1}}{2\Delta t}, \psi_h\right)_p + a_p\left(\frac{\xi_p^{k+1}+\xi_p^{k-1}}{2}, \psi_h\right) - c_I(\xi_f^k, \psi_h) \\
&\qquad = -gS_0\left(\frac{\eta_p^{k+1}-\eta_p^{k-1}}{2\Delta t}, \psi_h\right)_p - a_p\left(\frac{\eta_p^{k+1}+\eta_p^{k-1}}{2}, \psi_h\right) \\
&\qquad\quad + c_I(\eta_f^k, \psi_h) + \varepsilon_p^k(\psi_h).
\end{aligned}$$

Choosing $v_h = \xi_f^{k+1} + \xi_f^{k-1} \in V^h$ and $\psi_h = \xi_p^{k+1} + \xi_p^{k-1} \in X_p^h$ and adding both error equations produces

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\|\xi_f^{k+1}\|_f^2 + gS_0\|\xi_p^{k+1}\|_p^2 - \|\xi_f^{k-1}\|_f^2 - gS_0\|\xi_p^{k-1}\|_p^2\right) \\
&\quad + \left[c_I(\xi_f^{k+1}+\xi_f^{k-1}, \xi_p^k) - c_I(\xi_f^k, \xi_p^{k+1}+\xi_p^{k-1})\right] \\
&\quad + \frac{1}{2}\left[a_f(\xi_f^{k+1}+\xi_f^{k-1}, \xi_f^{k+1}+\xi_f^{k-1}) + a_p(\xi_p^{k+1}+\xi_p^{k-1}, \xi_p^{k+1}+\xi_p^{k-1})\right] \\
&= -\frac{1}{2\Delta t}\left[\left(\eta_f^{k+1}-\eta_f^{k-1}, \xi_f^{k+1}+\xi_f^{k-1}\right)_f + gS_0\left(\eta_p^{k+1}-\eta_p^{k-1}, \xi_p^{k+1}+\xi_p^{k-1}\right)_p\right] \\
&\quad - \frac{1}{2}\left[a_f\left(\eta_f^{k+1}+\eta_f^{k-1}, \xi_f^{k+1}+\xi_f^{k-1}\right) + a_p\left(\eta_p^{k+1}+\eta_p^{k-1}, \xi_p^{k+1}+\xi_p^{k-1}\right)\right] \\
&\quad - \left[c_I(\xi_f^{k+1}+\xi_f^{k-1}, \eta_p^k) - c_I(\eta_f^k, \xi_p^{k+1}+\xi_p^{k-1})\right] \\
&\quad + \varepsilon_f^k(\xi_f^{k+1}+\xi_f^{k-1}) + \left(p^k - \lambda_h^k, \nabla\cdot(\xi_f^{k+1}+\xi_f^{k-1})\right)_f + \varepsilon_p^k(\xi_p^{k+1}+\xi_p^{k-1}).
\end{aligned}$$

Split the coupled terms on the left hand side in the following way:

$$
\begin{aligned}
c_I(\xi_f^{k+1}+\xi_f^{k-1},\xi_p^k)-&c_I(\xi_f^k,\xi_p^{k+1}+\xi_p^{k-1})\\
&=\Big(c_I(\xi_f^{k+1},\xi_p^k)-c_I(\xi_f^k,\xi_p^{k+1})\Big)-\Big(c_I(\xi_f^k,\xi_p^{k-1})-c_I(\xi_f^{k-1},\xi_p^k)\Big)\\
&=C_\xi^{k+\frac{1}{2}}-C_\xi^{k-\frac{1}{2}}.
\end{aligned}
$$

Denote the ξ energy terms by

$$
E_\xi^{k+1/2}:=\|\xi_f^{k+1}\|_f^2+gS_0\|\xi_p^{k+1}\|_p^2+\|\xi_f^k\|_f^2+gS_0\|\xi_p^k\|_p^2.
$$

Applying the coercivity of $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$ we have

$$
\begin{aligned}
E_\xi^{k+1/2}&+2\Delta t C_\xi^{k+\frac{1}{2}}-E_\xi^{k-1/2}-2\Delta t C_\xi^{k-\frac{1}{2}}\\
&\qquad+\Delta t\Big(\nu\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2+gk_{min}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1}\|_p^2\Big)\\
\le&-\Big[\Big(\eta_f^{k+1}-\eta_f^{k-1},\xi_f^{k+1}+\xi_f^{k-1}\Big)_f+gS_0\left(\eta_p^{k+1}-\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1}\right)_p\Big]\\
&-\Delta t\Big[a_f\Big(\eta_f^{k+1}+\eta_f^{k-1},\xi_f^{k+1}+\xi_f^{k-1}\Big)+a_p\left(\eta_p^{k+1}+\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1}\right)\Big]\\
&-2\Delta t\Big[c_I(\xi_f^{k+1}+\xi_f^{k-1},\eta_p^k)-c_I(\eta_f^k,\xi_p^{k+1}+\xi_p^{k-1})\Big]\\
&+2\Delta t\Big(\varepsilon_f^k(\xi_f^{k+1}+\xi_f^{k-1})+(p^k-\lambda_h^k,\nabla\cdot(\xi_f^{k+1}+\xi_f^{k-1}))_f+\varepsilon_p^k(\xi_p^{k+1}+\xi_p^{k-1})\Big).
\end{aligned}
$$

Now we bound the right hand side of the inequality from above. We begin by bounding the first term on the right using the standard Cauchy-Schwarz, Poincaré (2.2), and Young inequalities.

$$
\begin{aligned}
(\eta_f^{k+1}-\eta_f^{k-1},&\xi_f^{k+1}+\xi_f^{k-1})_f+gS_0(\eta_p^{k+1}-\eta_p^{k-1},\xi_f^{k+1}+\xi_p^{k-1})_p\\
&\le\frac{3C_{P,f}^2}{\nu\Delta t}\|\eta_f^{k+1}-\eta_f^{k-1}\|_f^2+\frac{5gS_0^2C_{P,p}^2}{2k_{min}\Delta t}\|\eta_p^{k+1}-\eta_p^{k-1}\|_p^2\\
&\qquad+\Delta t\frac{\nu}{12}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2+\Delta t\frac{gk_{min}}{10}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2.
\end{aligned}
$$

Next, we apply the continuity of the bilinear forms $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$ to bound the second term on the right. To simplify, let $M=\left(\nu+\dfrac{\alpha C}{\sqrt{k_{min}}}\right)$ as in (2.3).

$$
\begin{aligned}
a_f(\eta_f^{k+1}+\eta_f^{k-1},&\xi_f^{k+1}+\xi_f^{k-1})+a_p(\eta_p^{k+1}+\eta_p^{k-1},\xi_p^{k+1}+\xi_p^{k-1})\\
&\le M\|\nabla(\eta_f^{k+1}+\eta_f^{k-1})\|_f\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f\\
&\qquad+gk_{max}\|\nabla(\eta_p^{k+1}+\eta_p^{k-1})\|_p\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p\\
&\le\frac{3M^2}{\nu}\|\nabla(\eta_f^{k+1}+\eta_f^{k-1})\|_f^2+\frac{5gk_{max}^2}{2k_{min}}\|\nabla(\eta_p^{k+1}+\eta_p^{k-1})\|_p^2\\
&\quad+\frac{\nu}{12}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2+\frac{gk_{min}}{10}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2
\end{aligned}
$$

We bound the coupled terms on the right hand side using the trace (2.1), Poincaré (2.2), and Young inequalities. Let $C = C_{\Omega_f}^2 C_{\Omega_p}^2 C_{P,f} C_{P,p} g^2$. Then

$$\begin{aligned}
&c_I(\xi_f^{k+1}+\xi_f^{k-1},\eta_p^k) - c_I(\eta_f^k,\xi_p^{k+1}+\xi_p^{k-1}) \\
&\quad\le g\left(\|(\xi_f^{k+1}+\xi_f^{k-1})\cdot\hat{n}_f\|_I\|\eta_p^k\|_I + \|\eta_f^k\cdot\hat{n}_f\|_I\|\xi_p^{k+1}+\xi_p^{k-1}\|_I\right) \\
&\quad\le C_{\Omega_f}C_{\Omega_p} g\Big(\|\xi_f^{k+1}+\xi_f^{k-1}\|_f^{1/2}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^{1/2}\|\eta_p^k\|_p^{1/2}\|\nabla\eta_p^k\|_p^{1/2} \\
&\qquad + \|\xi_p^{k+1}+\xi_p^{k-1}\|_p^{1/2}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^{1/2}\|\eta_f^k\|_f^{1/2}\|\nabla\eta_f^k\|_f^{1/2}\Big) \\
&\quad\le \sqrt{C}\left(\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f\|\nabla\eta_p^k\|_p + \|\nabla\eta_f^k\|_f\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p\right) \\
&\quad\le \frac{6C}{\nu}\|\nabla\eta_f^k\|_f^2 + \frac{5C}{gk_{min}}\|\nabla\eta_p^k\|_p^2 \\
&\qquad + \frac{\nu}{24}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2 + \frac{gk_{min}}{20}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2.
\end{aligned}$$

Finally, we bound the consistency errors, ε_f^k and ε_p^k, and the pressure term, $\left(p^k-\lambda_h^k,\nabla\cdot(\xi_f^{k+1}+\xi_f^{k-1})\right)_f$, as follows.

$$\begin{aligned}
\varepsilon_f^k(\xi_f^{k+1}+\xi_f^{k-1}) &= \left(\frac{u^{k+1}-u^{k-1}}{2\Delta t},\xi_f^{k+1}+\xi_f^{k-1}\right) - (u_t^k,\xi_f^{k+1}+\xi_f^{k-1})_f \\
&\qquad - a_f\left(u^k - \frac{u^{k+1}+u^{k-1}}{2},\xi_f^{k+1}+\xi_f^{k-1}\right) \\
&\le \left\|u_t^k - \frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f \|\xi_f^{k+1}+\xi_f^{k-1}\|_f \\
&\qquad + M\left\|\nabla\left(u^k-\frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f \\
&\le \frac{6C_{P,f}^2}{\nu}\left\|u_t^k-\frac{u^{k+1}-u^{k-1}}{2\Delta t}\right\|_f^2 + \frac{6M^2}{\nu}\left\|\nabla\left(u^k-\frac{u^{k+1}+u^{k-1}}{2}\right)\right\|_f^2 \\
&\qquad + \frac{\nu}{12}\|\nabla(\xi_f^{k+1}+\xi_f^{k-1})\|_f^2,
\end{aligned}$$

$$\begin{aligned}
\varepsilon_p^k(\xi_p^{k+1}+\xi_p^{k-1}) &= gS_0\left(\frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t},\xi_p^{k+1}+\xi_p^{k-1}\right)_p - gS_0(\phi_t^k,\xi_p^{k+1}+\xi_p^{k-1})_p \\
&\quad - a_p\left(\phi^k-\frac{\phi^{k+1}+\phi^{k-1}}{2},\xi_p^{k+1}+\xi_p^{k-1}\right) \\
&\le gS_0\left\|\phi_t^k-\frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p\|\xi_p^{k+1}+\xi_p^{k-1}\|_p \\
&\quad + gk_{max}\left\|\nabla\left(\phi^k-\frac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p \\
&\le \frac{5gS_0^2C_{P,p}^2}{k_{min}}\left\|\phi_t^k-\frac{\phi^{k+1}-\phi^{k-1}}{2\Delta t}\right\|_p^2 + \frac{5gk_{max}}{k_{min}}\left\|\nabla\left(\phi^k-\frac{\phi^{k+1}+\phi^{k-1}}{2}\right)\right\|_p^2 \\
&\quad + \frac{gk_{min}}{10}\|\nabla(\xi_p^{k+1}+\xi_p^{k-1})\|_p^2,
\end{aligned}$$

$$\left(p^k - \lambda_h^k, \nabla \cdot (\xi_f^{k+1} + \xi_f^{k-1})\right)_f \leq \|p^k - \lambda_h^k\|_f \|\nabla \cdot (\xi_f^{k+1} + \xi_f^{k-1})\|_f$$
$$\leq \frac{6d}{\nu}\|p^k - \lambda_h^k\|_f^2 + \frac{\nu}{24}\|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2.$$

Having bounded each term on the right hand side from above, we now subsume the ξ terms on the right into the left hand side of the inequality.

$$\begin{aligned}
&E_\xi^{k+\frac{1}{2}} + 2\Delta t C_\xi^{k+\frac{1}{2}} - E_\xi^{k-\frac{1}{2}} - 2\Delta t C_\xi^{k-\frac{1}{2}} \\
&\qquad + \Delta t\left(\frac{\nu}{2}\|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2 + \frac{gk_{min}}{2}\|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2\right) \\
&\leq (\Delta t)^{-1}\left\{\frac{3C_{P,f}^2}{\nu}\|\eta_f^{k+1} - \eta_f^{k-1}\|_f^2 + \frac{5gS_0^2C_{P,p}^2}{2k_{min}}\|\eta_p^{k+1} - \eta_p^{k-1}\|_p^2\right\} \\
&+ \Delta t\Bigg\{\frac{3M^2}{\nu}\|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f^2 + \frac{5gk_{max}^2}{2k_{min}}\|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_p^2 + \frac{12C}{\nu}\|\nabla\eta_f^k\|_f^2 \\
&+ \frac{10C}{gk_{min}}\|\nabla\eta_p^k\|_p^2 + \frac{12C_{P,f}^2}{\nu}\left\|u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t}\right\|_f^2 \\
&+ \frac{12M^2}{\nu}\left\|\nabla\left(u^k - \frac{u^{k+1} + u^{k-1}}{2}\right)\right\|_f^2 + \frac{12d}{\nu}\|p^k - \lambda_h^k\|_f^2 \\
&+ \frac{5gS_0^2C_{P,p}^2}{k_{min}}\left\|\phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t}\right\|_p^2 + \frac{5gk_{max}^2}{k_{min}}\left\|\nabla\left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2}\right)\right\|_p^2\Bigg\}.
\end{aligned}$$

Sum this inequality from $k = 1, ..., N-1$. Then

$$\begin{aligned}
&E_\xi^{N-\frac{1}{2}} + 2\Delta t C_\xi^{N-\frac{1}{2}} - E_\xi^{\frac{1}{2}} \\
&\qquad + \Delta t\sum_{k=1}^{N-1}\left(\frac{\nu}{2}\|\nabla\left(\xi_f^{k+1} + \xi_f^{k-1}\right)\|_f^2 + \frac{gk_{min}}{2}\|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2\right) \\
&\leq (\Delta t)^{-1}\sum_{k=1}^{N-1}\left[\frac{3C_{P,f}^2}{\nu}\|\eta_f^{k+1} - \eta_f^{k-1}\|_f^2 + \frac{5gS_0^2C_{P,p}^2}{2k_{min}}\|\eta_p^{k+1} - \eta_p^{k-1}\|_p^2\right] \\
&+ \Delta t\sum_{k=1}^{N-1}\Bigg[\frac{3M^2}{\nu}\|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f^2 + \frac{5gk_{max}^2}{2k_{min}}\|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_p^2 \\
&+ \frac{12C}{\nu}\|\nabla\eta_f^k\|_f^2 + \frac{10C}{gk_{min}}\|\nabla\eta_p^k\|_p^2 + \frac{12C_{P,f}^2}{\nu}\left\|u_t^k - \frac{u^{k+1} - u^{k-1}}{2\Delta t}\right\|_f^2 \\
&+ \frac{12M^2}{\nu}\left\|\nabla\left(u^k - \frac{u^{k+1} + u^{k-1}}{2}\right)\right\|_f^2 + \frac{12d}{\nu}\|p^k - \lambda_h\|_f^2 \\
&+ \frac{10gS_0^2C_{P,p}^2}{k_{min}}\left\|\phi_t^k - \frac{\phi^{k+1} - \phi^{k-1}}{2\Delta t}\right\|_p^2 + \frac{10gk_{max}^2}{k_{min}}\left\|\nabla\left(\phi^k - \frac{\phi^{k+1} + \phi^{k-1}}{2}\right)\right\|_p^2\Bigg].
\end{aligned}$$

We want to bound this in terms of norms instead of summations. Using Cauchy-Schwarz and other basic inequalities, we bound the first term on the right hand side as follows.

$$\begin{aligned}\sum_{k=1}^{N-1} \|\eta_f^{k+1} - \eta_f^{k-1}\|_f^2 &= \sum_{k=1}^{N-1} \left\| \int_{t^{k-1}}^{t^{k+1}} \eta_{f,t} dt \right\|_f^2 \\ &\le \sum_{k=1}^{N-1} \int_{\Omega_f} (2\Delta t) \int_{t^{k-1}}^{t^{k+1}} |\eta_{f,t}|^2 dt \quad dx \\ &\le 4\Delta t \|\eta_{f,t}\|^2_{L^2(0,T;L^2(\Omega_f))}.\end{aligned} \tag{3.8}$$

We treat the second term similarly.

$$\sum_{k=1}^{N-1} \|\eta_p^{k+1} - \eta_p^{k-1}\|_f^2 \le 4\Delta t \|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))}. \tag{3.9}$$

We bound the remaining η terms using Cauchy-Schwartz and the discrete norms.

$$\begin{aligned}\sum_{k=1}^{N-1} \|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f^2 &\le 2 \sum_{k=1}^{N-1} \left(\|\nabla \eta_f^{k+1}\|_f^2 + \|\nabla \eta_f^{k-1}\|_f^2 \right) \\ &\le 4 \sum_{k=0}^{N} \|\nabla \eta_f^k\|_f^2 \le 4(\Delta t)^{-1} |||\nabla \eta_f|||^2_{L^2(0,T;L^2(\Omega_f))},\end{aligned} \tag{3.10}$$

$$\sum_{k=1}^{N-1} \|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_f^2 \le 4(\Delta t)^{-1} |||\nabla \eta_p|||^2_{L^2(0,T;L^2(\Omega_p))}, \tag{3.11}$$

$$\sum_{k=1}^{N-1} \|\nabla \eta_f^k\|_f^2 \le (\Delta t)^{-1} |||\nabla \eta_f|||^2_{L^2(0,T;L^2(\Omega_f))}, \tag{3.12}$$

$$\sum_{k=1}^{N-1} \|\nabla \eta_p^k\|_p^2 \le (\Delta t)^{-1} |||\nabla \eta_p|||^2_{L^2(0,T;L^2(\Omega_p))}, \tag{3.13}$$

$$\sum_{k=1}^{N-1} \|p^k - \lambda_h^k\|_f^2 \le (\Delta t)^{-1} |||p - \lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))}. \tag{3.14}$$

Recall from the proof of stability that since (3.2) holds, we have the following lower bound for the energy terms.

$$E_\xi^{N-1/2} + 2\Delta t C_\xi^{N-\frac{1}{2}} \ge \alpha(\|\xi_f^N\|_f^2 + \|\xi_f^{N-1}\|_f^2) + \beta(\|\xi_p^N\|_p^2 + \|\xi_p^{N-1}\|_p^2) \ge 0$$

Here $\alpha = \min\{1 - \Delta t \mathcal{C}^{-1} h^{-1}, 1 - \Delta t \mathcal{C}^{-1} h^{-2}\}$ and $\beta = \min\{g S_0 - \Delta t \mathcal{C}^{-1} h^{-1}, g S_0 - \Delta t \mathcal{C}^{-1}\}$ are both positive because of the time step condition (3.2).

After applying bounds (3.8)-(3.14), along with (3.4)-(3.7), and absorbing all the constants into one constant, $\widehat{C}_1$, the inequality becomes

$$
\begin{aligned}
&\alpha(\|\xi_f^N\|_f^2 + \|\xi_f^{N-1}\|_f^2) + \beta(\|\xi_p^N\|_p^2 + \|\xi_p^{N-1}\|_p^2) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2}\|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2 + \frac{gk_{min}}{2}\|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2 \right) \\
&\quad \leq \widehat{C}_1 \Big\{ \|\eta_{f,t}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + |||\nabla\eta_f|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&\qquad + |||\nabla\eta_p|||^2_{L^2(0,T;L^2(\Omega_p))} + |||p-\lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&\qquad + (\Delta t)^4 \Big(\|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad\quad + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big) \Big\} + E_\xi^{1/2} + 2\Delta t C_\xi^{\frac{1}{2}}.
\end{aligned} \tag{3.15}
$$

Recall that $e_f^N = u^N - u_h^N$ and $e_p^N = \phi^N - \phi_h^N$. Use the triangle inequality on the error equation to split the error terms into terms of η and ξ.

$$
\begin{aligned}
&\frac{\alpha}{2}(\|e^N{}_f\|_f^2 + \|e_f^{N-1}\|_f^2 + \frac{\beta}{2}(\|e_p^N\|_p^2 + \|e_p^{N-1}\|_p^2) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{4}\|\nabla(e_f^{k+1} + e_f^{k-1})\|_f^2 + \frac{gk_{min}}{4}\|\nabla(e_p^{k+1} + e_p^{k-1})\|_p^2 \right) \\
&\quad \leq \alpha(\|\xi_f^N\|_f^2 + \|\xi_f^{N-1}\|_f^2) + \beta(\|\xi_p^N\|_p^2 + \|\xi_p^{N-1}\|_p^2) \\
&\qquad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2}\|\nabla(\xi_f^{k+1} + \xi_f^{k-1})\|_f^2 + \frac{gk_{min}}{2}\|\nabla(\xi_p^{k+1} + \xi_p^{k-1})\|_p^2 \right) \\
&\qquad + \alpha(\|\eta_f^N\|_f^2 + \|\eta_f^{N-1}\|_f^2) + \beta(\|\eta_p^N\|_p^2 + \|\eta_p^{N-1}\|_p^2) \\
&\qquad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{2}\|\nabla(\eta_f^{k+1} + \eta_f^{k-1})\|_f^2 + \frac{gk_{min}}{2}\|\nabla(\eta_p^{k+1} + \eta_p^{k-1})\|_p^2 \right)
\end{aligned} \tag{3.16}
$$

Note that $\|\eta_{f,p}^N\|_{f,p}^2, \|\eta_{f,p}^{N-1}\|_{f,p}^2 \leq |||\eta_{f,p}|||^2_{L^\infty(0,T;L^2(\Omega_{f,p}))}$. Using this, the previous bounds for η terms, applying inequality (3.15), and absorbing constants into a new constant, $\widehat{C}_2$ produces

$$
\begin{aligned}
&\frac{\alpha}{2}(\|e^N{}_f\|_f^2 + \|e_f^{N-1}\|_f^2 + \frac{\beta}{2}(\|e_p^N\|_p^2 + \|e_p^{N-1}\|_p^2) \\
&\quad + \Delta t \sum_{k=1}^{N-1} \left(\frac{\nu}{4}\|\nabla(e_f^{k+1} + e_f^{k-1})\|_f^2 + \frac{gk_{min}}{4}\|\nabla(e_p^{k+1} + e_p^{k-1})\|_p^2 \right) \\
&\quad \leq \widehat{C}_2 \Big\{ \|\eta_{f,t}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + |||\nabla\eta_f|||^2_{L^2(0,T;L^2(\Omega_f))}
\end{aligned}
$$

$$
\begin{aligned}
&+ |||\nabla \eta_p|||^2_{L^2(0,T;L^2(\Omega_p))} + |||p-\lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&+ (\Delta t)^4 \Big(\|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))} + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} \\
&\qquad + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))} \Big) \\
&+ |||\eta_f|||^2_{L^\infty(0,T;L^2(\Omega_f))} + |||\eta_p|||^2_{L^\infty(0,T;L^2(\Omega_p))} \Big\} \\
&+ \|\xi_f^1\|_f^2 + gS_0\|\xi_p^1\|_p^2 + \|\xi_f^0\|_f^2 + gS_0\|\xi_p^0\|_p^2 + 2\Delta t C_\xi^{1/2}.
\end{aligned} \tag{3.17}
$$

The coupled terms on the right hand side can be bounded by:

$$
C_\xi^{1/2} \leq \frac{C}{2}\left(\|\nabla \xi_p^0\|_p^2 + \|\nabla \xi_p^1\|_p^2 + \|\nabla \xi_f^0\|_f^2 + \|\nabla \xi_f^1\|_f^2\right).
$$

Since (3.17) holds for any $\tilde{u} \in V^h$, $\lambda_h \in Q_f^h$, and $\tilde{\phi} \in X_p^h$, we may take the infimum over V^h, Q_f^h, and X_p^h. By (3.3), we may bound the infimum over V^h by the infimum over X_f^h so the following holds for some positive constant $\widehat{C}_4$:

$$
\begin{aligned}
&\frac{\alpha}{2}(\|e_f^N\|_f^2 + \|e_f^{N-1}\|_f^2 + \frac{\beta}{2}(\|e_p^N\|_p^2 + \|e_p^{N-1}\|_p^2) \\
&+ \Delta t \sum_{k=1}^{N-1}\left(\frac{\nu}{4}\|\nabla(e_f^{k+1}+e_f^{k-1})\|_f^2 + \frac{gk_{min}}{4}\|\nabla(e_p^{k+1}+e_p^{k-1})\|_p^2\right) \\
&\leq \widehat{C}_3 \Bigg\{ \inf_{\tilde{u}\in X_f^h} \Big\{ \|\eta_{f,t}\|^2_{L^2(0,T;L^2(\Omega_f))} + |||\eta_f|||^2_{L^2(0,T;H^1(\Omega_f))} + |||\eta_f|||^2_{L^\infty(0,T;L^2(\Omega_f))} \\
&\quad + \|\xi_f^1\|_f^2 + \|\xi_f^0\|_f^2 + \Delta t(\|\nabla\xi_f^1\|_f^2 + \|\nabla \xi_f^0\|_f^2)\Big\} + \inf_{\lambda_h\in Q_f^h} |||p-\lambda_h|||^2_{L^2(0,T;L^2(\Omega_f))} \\
&\quad + \inf_{\tilde{\phi}\in X_p^h} \Big\{ \|\eta_{p,t}\|^2_{L^2(0,T;L^2(\Omega_p))} + |||\eta_p|||_{L^2(0,T;H^1(\Omega_p))} + |||\eta_p|||^2_{L^\infty(0,T;L^2(\Omega_p))} \\
&\quad + gS_0(\|\xi_p^1\|_p^2 + \|\xi_p^0\|_p^2) + \Delta t(\|\nabla \xi_p^1\|_p^2 + \|\nabla\xi_p^0\|_p^2)\Big\} + (\Delta t)^4 \Big\{ \|u_{ttt}\|^2_{L^2(0,T;L^2(\Omega_f))} \\
&\quad + \|\phi_{ttt}\|^2_{L^2(0,T;L^2(\Omega_p))} + \|u_{tt}\|^2_{L^2(0,T;H^1(\Omega_f))} + \|\phi_{tt}\|^2_{L^2(0,T;H^1(\Omega_p))}\Big\}\Bigg\}.
\end{aligned}
$$

After applying the approximation assumptions we get the final result. □

4. Numerical Experiments. Using the exact solutions introduced by Mu and Zhu in [12], we conduct numerical experiments to verify the stability and predicted rates of convergence of CNLF. First, we test the convergence rate of the method. Then, we will test stability in various ways. In both tests, we use the same domain and exact solutions. We utilize FreeFem++ software for all calculations.

$$\Omega_f = (0,1)\times(1,2), \qquad \Omega_p = (0,1)\times(0,1), \qquad I = \{(x,1) : x\in(0,1)\}$$
$$u(x,y,t) = \left((x^2(y-1)^2+y)\cos(t), (\frac{2}{3}x(1-y)^3+2-\pi\sin(\pi x))\cos(t)\right)$$
$$p(x,y,t) = (2-\pi\sin(\pi x))\sin(\frac{\pi}{2}y)\cos(t)$$
$$\phi(x,y,t) = (2-\pi\sin(\pi x))(1-y-\cos(\pi y))\cos(t)$$

4.1. Convergence Experiment. To test the rate of convergence, we set all parameters, α, ν, S_0, κ, g, equal to one. We use Taylor-Hood elements (P2-P1) for the Stokes problem and piecewise quadratics (P2) for the Darcy problem. We set the boundary condition on the problem to be inhomogeneous Dirichlet: $u_h = u$ on $\partial\Omega_f/I$, and similar for the Darcy pressure, ϕ. The initial condition and first two terms are chosen to correspond with the exact solutions. We set the mesh size, h, equal to the time step, Δt. This satisfies the time step condition (3.2) when $S_0 = 1$. The errors for various values of h are given in Table 4.1. We denote $L^\infty(0,1;L^2(\Omega_{f,p}))$ by $L^\infty_{f,p}$.

$h=\Delta t$	$\|\|u-u^h\|\|_{L^\infty_f}$	rate	$\|\|p-p^h\|\|_{L^\infty_f}$	rate	$\|\|\phi-\phi^h\|\|_{L^\infty_p}$	rate
$\frac{1}{10}$	0.000862671		0.156045		0.00654407	
$\frac{1}{20}$	0.000177135	2.28	0.0377064	2.05	0.00146515	2.16
$\frac{1}{40}$	3.54644e-5	2.32	0.0089672	2.07	0.00034904	2.07
$\frac{1}{80}$	6.72106e-6	2.40	0.00215951	2.05	8.70886e-5	2.00

TABLE 4.1
Rates of Convergence

The rates of convergence in the table exhibit second order convergence for u, p, and ϕ. This agrees with the error analysis.

4.2. Stability Experiments. To test the stability of the method, we set the body force and source functions, f_f and f_p, equal to zero, change the boundary conditions to be equal to zero except along the interface, I, and calculate the energy at each time level, $E(n) = \|u_h^n\|_f^2 + \|u_h^{n-1}\|_f^2 + gS_0\|\phi_h^n\|_p^2 + gS_0\|\phi_h^{n-1}\|_p^2$. Let $h = \Delta t = \frac{1}{20}$. We compute the energy over the time interval $(0,10)$ for various values of S_0 and plot the energy versus the time step, n.

As evident in 4.1, the energy decays to zero only for $S_0 = 1.0$, which satisfies our time step condition. The other values of S_0 violate the condition and we see the energy for the system increasing as time progresses.

The CFL-type stability condition, (3.2), is independent of k_{min}. We ran the same stability tests for $S_0 = O(1)$ and k_{min} is small. The energy rapidly decreased towards zero, as expected. However, when the time interval was extended, the energy began to grown again due, we believe, to the accumulation of numerical noise in the "unstable mode" of Leapfrog, see, for example, Durran [19]. Leapfrog is only marginally stable due to an undamped oscillatory mode, referred to here as the "unstable mode". To connect the energy growth to the unstable mode, we computed two energy modes of u and ϕ: $\|u_h^{n+1} - u_h^{n-1}\|_f^2$, $\|\phi_h^{n+1} - \phi_h^{n-1}\|_p^2$, the unstable energy modes, and $\|u_h^{n+1} + u_h^{n-1}\|_f^2$, and $\|\phi_h^{n+1} + \phi_h^{n-1}\|_p^2$, the stable energy modes.

In both pictures of 4.2, the growth in the energy, $E(n)$, corresponds with spurious oscillations in unstable energy modes of both u and ϕ. The stable energy modes decay

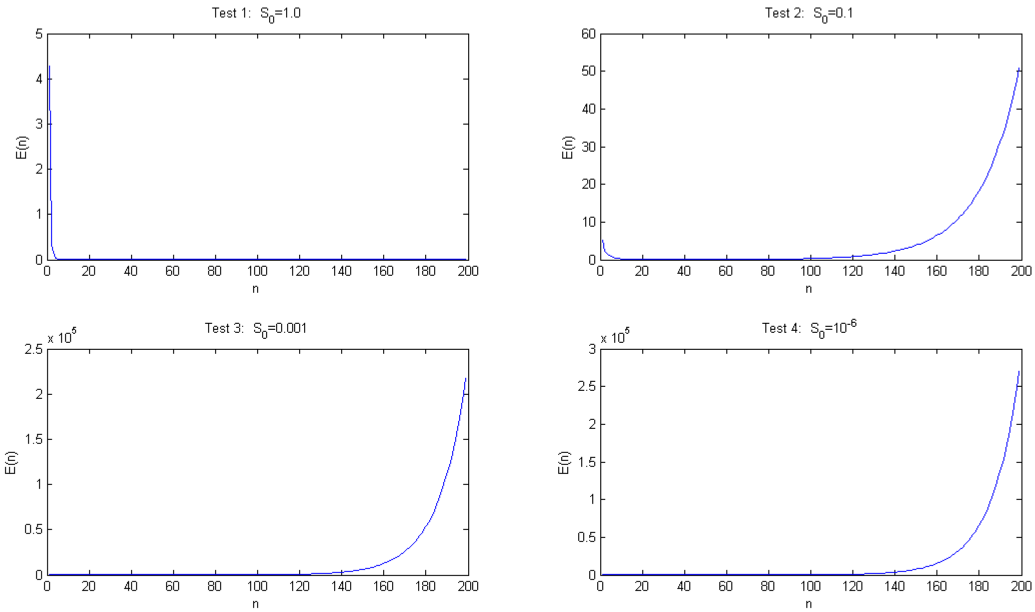

FIG. 4.1. *Test 1: CFL Condition holds, Test 2-4: CFL Condition Violated*

to zero in machine precision. Thus, the rise in energy in this case is exactly in the unstable mode of CNLF.

4.3. CNLF and Time Filtering. One popular way to counteract this effect of the unstable mode in CNLF in geophysical fluid dynamics is to use time filters, see, for example Jablonowski and Williamson [20]. For this initial test we begin with the Robert-Asselin Filter, or RA-filter ([21], [22]). At every time step, after computing $u_h^{k+1}, p_h^{k+1}, \phi_h^{k+1}$, we update the previous k^{th} values and replace them with a filtered value, given below.

$$\overline{w}_h^k = w_h^k + \alpha(\overline{w}_h^{k-1} - 2w_h^k + w_h^{k+1}), \text{ where } w = u, p, \text{ or } \phi, 0 \leq \alpha \leq 1.$$

The RA-filter damps the computational mode in Leapfrog (see e.g. Durran [19]). However, the analytical theory of the RA-filter applied to CNLF remains an interesting open problem. For this initial test we chose $\alpha = 0.20$. For more discussion on the choice of the parameter α see, for example [20] p. 437. After applying the RA-filter to the stability test performed in 4.1, with $S_0 = O(1)$ and $k_{min} = 0.1$, we see the energy, $E(n)$ decay to zero in machine precision, as well as the unstable energy mode, Figure 4.3.

5. Conclusion. Analysis of the Crank Nicolson Leap-Frog method applied to the Stokes-Darcy equations lead us to a CFL-type time step condition, (3.2), sufficient for stability and convergence. However, the sensitivity of this condition to small values of S_0, is restrictive in cases of confined aquifers in which S_0 can be very small. See p.566 in Domenico and Mifflin, [23], for values of specific storage for different materials. Numerical experiments confirmed sensitivity to S_0. Theoretical analysis showed that the CFL-type condition for stability is insensitive to k_{min} being small when we have

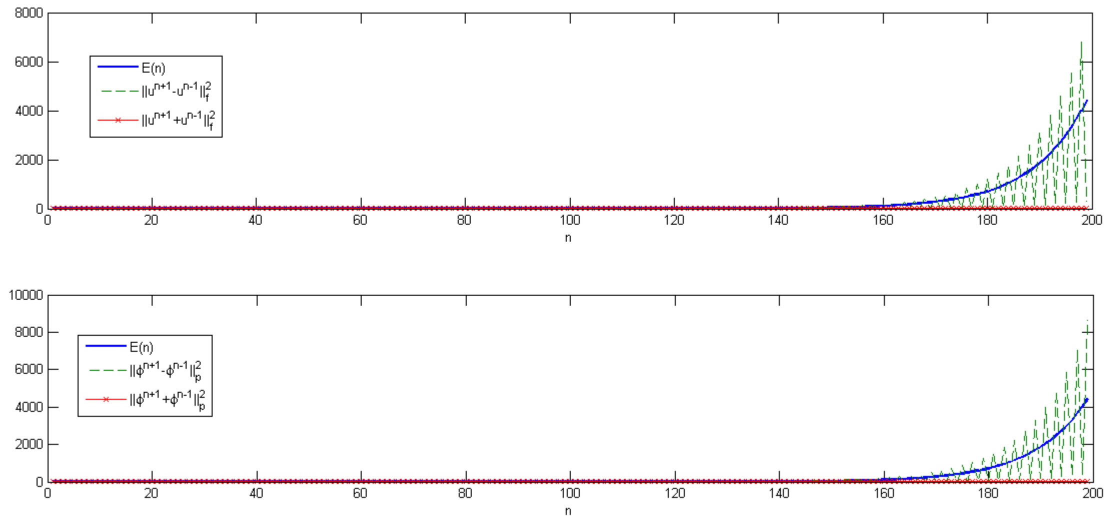

FIG. 4.2. *Energy Plots for* $k_{min} = 0.1$

$S_0 = O(1)$. However, the numerical tests revealed the problem of growth of numerical noise in unstable modes causing an energy blow-up in finite time. Time filters seem to add stability, but their analytical foundations is an open problem. The convergence analysis and numerical experiments show that this method is second order in time and space provided our stability condition is satisfied, approximation assumptions (2.5) hold for at least $r = 2$, and our choice of initial method to compute (u_h^1, p_h^1, ϕ_h^1) is accurate enough.

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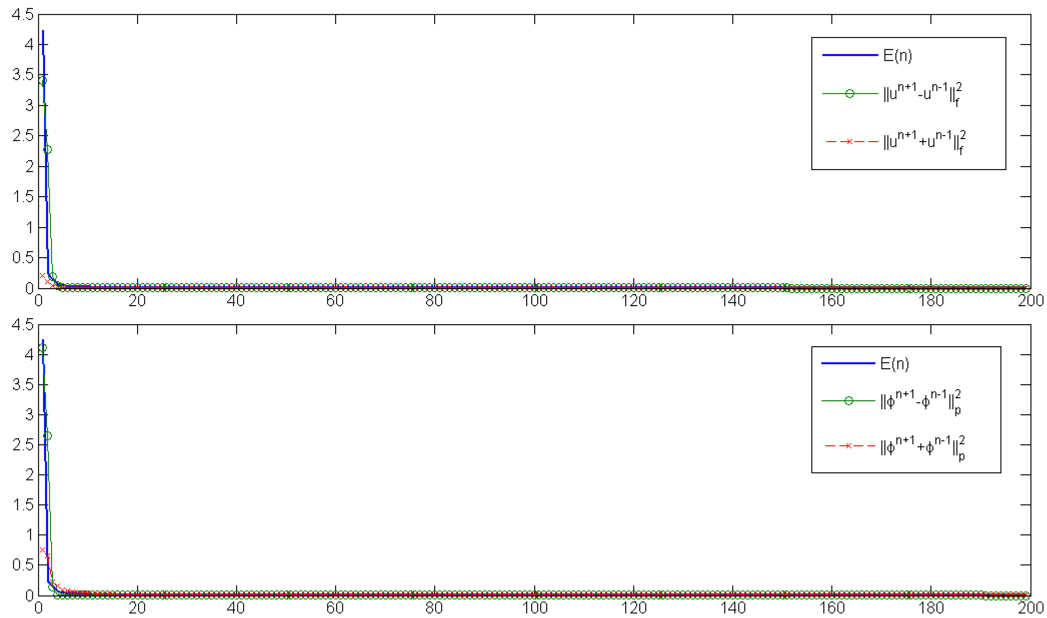

FIG. 4.3. *Energy Plots vs. time (n) for* $k_{min} = 0.1$ *for CNLF + RA Filter*

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