Problem 1. Let \( \{f_n\} \) be a sequence of \( C^\infty \) functions on a compact interval \( I \) such that for each \( k \geq 0 \) there exists \( M_k \) such that
\[
|f^{(k)}_n(x)| \leq M_k \quad \text{for all } n \text{ and } x \in I.
\]
Prove that there exists a subsequence converging uniformly, together with the derivatives of all orders, to a \( C^\infty \) function.

**Hint:** A function \( f \) is \( C^\infty \) means that \( f \in C^k \) for all \( k \). You may consider using a diagonalization argument.

Problem 2. Compute the surface integral
\[
I = \iint_{\Sigma} \frac{x dydz + y dzdx + z dxdy}{(x^2 + y^2 + z^2)^{3/2}}
\]
for each of the following cases:

1. \( \Sigma = \{ (x, y, z) : x^2 + y^2 + z^2 = t^2 \} \);
2. \( \Sigma = \partial V \) where \( V \) is a bounded smooth closed region that does not include the origin;
3. \( \Sigma = \partial V \) where \( V \) is a bounded smooth closed region that contains the origin.

Problem 3. Assume \( f : \mathbb{R}^n \to \mathbb{R} \) is a continuous function. Given that for all \( x, y \in \mathbb{R}^n \) we have
\[
f\left(\frac{x + y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y),
\]
show that actually for any \( \lambda \in [0, 1] \), and any \( x, y \in \mathbb{R}^n \) we have
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).
\]

*You don’t need to use any convexity properties, but if you do: you must prove all of them. You can not assume that \( f \) is differentiable.*

**Hint:** The result is obviously true for \( \lambda = 1/2 \). You may try proving it for \( \lambda = 1/4 \) and \( 3/4 \), and from there to try to find a pattern.

Problem 4.

(a) Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a continuous map. Assume that \( U \subset \mathbb{R}^n \) is connected. Show that \( f(U) \subset \mathbb{R}^m \) is connected.

(b) Show that there is no \( f : \mathbb{R}^2 \to \mathbb{R} \) such that

- \( f \) is continuous
- \( f \) is injective
- for any open set \( U \subset \mathbb{R}^2 \) we have \( f(U) \) is open
Problem 5. Let $M_{n \times n}$ denote the vector space of $n \times n$ real matrices. Prove that there are neighborhoods $U$ and $V$ of the identity matrix $I_n$ such that for every $A \in U$ there is a unique $X \in V$ such that $X^4 = A$, where here $X^4$ is a matrix power.

Hint: Implicit or inverse function theorem.

Problem 6. Let $F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapping satisfying $F(0) = 0$. Suppose that

$$
\sum_{i,j=1}^{n} \left| \frac{\partial F_i}{\partial x_j}(0) \right|^2 = c < 1.
$$

Prove that there is a ball $B$ in $\mathbb{R}^n$ centered at 0 such that

$$
f(B) \subset B.
$$

Hint: Note that here $F$ may NOT be $C^1$, and hence no implicit/inverse function theorem. Try using differentiability.