DG APPROXIMATION OF COUPLED NAVIER-STOKES AND DARCY EQUATIONS BY BEAVER-JOSEPH-SAFFMAN INTERFACE CONDITION

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Abstract. In this work, we couple the incompressible steady Navier-Stokes equations with the Darcy equations, by means of the Beaver-Joseph-Saffman's condition on the interface. Under mild regularity conditions on the data, we prove existence of a weak solution as well as some a priori estimates. We establish uniqueness under smallness restrictions on the data, similar to those that guatantee uniqueness of the solution of the Navier-Stokes equations. Then we propose a discontinuous Galerkin scheme for discretizing the equations and do its numerical analysis.

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1. Introduction

Consider a fluid occupying a bounded domain $\Omega \subset \mathbb{R}^2$, decomposed into two *disjoint* subdomains Ω_1 and Ω_2. Let $\boldsymbol{u}$ denote the fluid velocity in Ω_1 and for $i = 1, 2$, let p_i be the fluid pressure in Ω_i. The fluid motion is modelled by the Navier-Stokes equations in Ω_1 and by the Darcy equations in Ω_2. The viscosity, that acts on Ω_1, is denoted by μ, assumed to be a positive constant, and the viscous effects are neglected in Ω_2. The body forces $\boldsymbol{f}_1$ and f_2 respectively act on Ω_1 and Ω_2 . The permeability $\boldsymbol{K}$ defined in Ω_2 is a positive definite symmetric tensor, that is allowed to vary in space. The partial differential equations modelling the fluid are:

$$-\nabla \cdot (2\mu \boldsymbol{D}(\boldsymbol{u}) - p_1 \boldsymbol{I}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{f}_1, \quad \text{in} \quad \Omega_1, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in} \quad \Omega_1, \tag{1.2}$$

$$-\nabla \cdot \boldsymbol{K} \nabla p_2 = f_2, \quad \text{in} \quad \Omega_2. \tag{1.3}$$

Here, $\boldsymbol{I}$ is the 2×2 identity tensor and $\boldsymbol{D}(\boldsymbol{v})$ is the deformation tensor defined by $\boldsymbol{D}(\boldsymbol{v}) = 1/2(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$. Equations (1.1) and (1.2) represent the momentum conservation and the incompressibility condition and equation (1.3) is the continuity equation for the Darcy flow with velocity $\boldsymbol{u} = -\boldsymbol{K} \nabla p_2$. As usual, we write formally:

$$\boldsymbol{u} \cdot \nabla \boldsymbol{v} = \sum_{i=1}^{2} u_i \frac{\partial \boldsymbol{v}}{\partial x_i} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i}.$$

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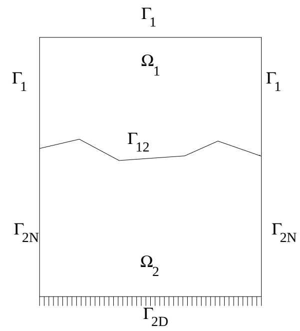

FIGURE 1. Coupled domains with interface Γ_{12}.

We denote by $\partial\Omega$ the boundary of Ω and by $\partial\Omega_i$ the boundary of Ω_i; all three boundaries are assumed to be Lipschitz continuous. We also define the interface $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$, and the boundaries $\Gamma_i = \partial\Omega_i \setminus \Gamma_{12}$ for $i = 1, 2$. Finally, the boundary Γ_2 is decomposed into two disjoint open sets $\Gamma_2 = \Gamma_{2D} \cup \Gamma_{2N}$, as in Figure 1. We denote by $\boldsymbol{n}_{\Omega_i}$ the exterior unit vector normal to $\partial\Omega_i$. System (1.1)–(1.3) is complemented by the boundary conditions below. First we prescribe standard conditions on Γ_i:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{0}, \quad \text{on} \quad \Gamma_1, && (1.4)\\ p_2 &= 0, \quad \text{on} \quad \Gamma_{2D}, && (1.5)\\ \boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{\Omega_2} &= 0, \quad \text{on} \quad \Gamma_{2N}. && (1.6) \end{aligned}$$

Here we assume on one hand that Γ_1 is not reduced to a straight line, and on the other hand that $|\Gamma_{2D}| > 0$. Now, let $\boldsymbol{n}_{12}$ be the unit normal vector to Γ_{12} pointing from Ω_1 to Ω_2 and let $\boldsymbol{\tau}_{12}$ be the unit tangent vector to Γ_{12}. On the interface Γ_{12}, we prescribe the following interface conditions:

$$\boldsymbol{u} \cdot \boldsymbol{n}_{12} = -\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{12}, \tag{1.7}$$

$$\big((-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}\big) \cdot \boldsymbol{n}_{12} + \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}) = p_2, \tag{1.8}$$

$$\boldsymbol{u} \cdot \boldsymbol{\tau}_{12} = -2\mu G\big(\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}_{12}\big) \cdot \boldsymbol{\tau}_{12}. \tag{1.9}$$

Condition (1.7) represents continuity of the fluid velocity's normal component, (1.8) represents the balance of forces acting across the interface and (1.9) is the Beaver-Joseph-Saffman's condition [5,31]. The constant $G > 0$ is given and is usually obtained from experimental data.

A problem of the form (1.1)–(1.9) has several variational formulations. We propose the following formulation in adequate Sobolev spaces to be specified further on:

$$\begin{aligned} &\forall \boldsymbol{v}, \forall q, \quad 2\mu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla p_2, \nabla q)_{\Omega_2} \\ &+(p_2 - \tfrac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + \tfrac{1}{G}(\boldsymbol{u} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u} \cdot \boldsymbol{n}_{12}, q)_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}, \\ &\forall q, \quad -(\nabla \cdot \boldsymbol{u}, q)_{\Omega_1} = 0. \end{aligned}$$

Here we use the inner-product notation for scalar functions f, g, vector functions $\boldsymbol{f}, \boldsymbol{g}$ and matrices $\boldsymbol{F}, \boldsymbol{G}$ on a domain $\mathcal{O} \subset \mathbb{R}^2$ or on a Lipschitz-continuous plane curve γ:

$$(f, g)_{\mathcal{O}} = \int_{\mathcal{O}} fg, \quad (\boldsymbol{f}, \boldsymbol{g})_{\mathcal{O}} = \int_{\mathcal{O}} \boldsymbol{f} \cdot \boldsymbol{g}, \quad (\boldsymbol{F}, \boldsymbol{G})_{\mathcal{O}} = \sum_{ij} \int_{\mathcal{O}} F_{ij} G_{ij},$$

$$(f, g)_{\gamma} = \int_{\gamma} fg, \quad (\boldsymbol{f}, \boldsymbol{g})_{\gamma} = \int_{\gamma} \boldsymbol{f} \cdot \boldsymbol{g}.$$

We approximate this variational formulation with the following discontinuous Galerkin method, symmetric or non-symmetric for the elliptic operators, and with an upwind Lesaint-Raviart discretization of the convection operator, again in appropriate finite-element spaces:

$$\forall \boldsymbol{v}, \forall q, \quad 2\mu \sum_{E\in\mathcal{E}_1^h} (D(\boldsymbol{U}), D(\boldsymbol{v}))_E + \sum_{E\in\mathcal{E}_2^h} (\boldsymbol{K}\nabla P_2, \nabla q)_E + \sum_{E\in\mathcal{E}_1^h} (\boldsymbol{U}\cdot\nabla\boldsymbol{U}, \boldsymbol{v})_E + \frac{1}{2}\sum_{E\in\mathcal{E}_1^h} (\nabla\cdot\boldsymbol{U}, \boldsymbol{U}\cdot\boldsymbol{v})_E$$
$$-2\mu \sum_{e\in\Gamma_1^h\cup\Gamma_1} (\{D(\boldsymbol{U})\boldsymbol{n}_e\}, [\boldsymbol{v}])_e + 2\epsilon_1\mu \sum_{e\in\Gamma_1^h\cup\Gamma_1} (\{D(\boldsymbol{v})\boldsymbol{n}_e\}, [\boldsymbol{U}])_e + \mu \sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\sigma_e}{|e|}([\boldsymbol{U}], [\boldsymbol{v}])_e$$
$$+ \sum_{E\in\mathcal{E}_1^h} (|\{\boldsymbol{U}\}\cdot\boldsymbol{n}_E|(\boldsymbol{U}^{\text{int}} - \boldsymbol{U}^{\text{ext}}), \boldsymbol{v}^{\text{int}})_{\partial E_-\setminus\Gamma_{12}} - \frac{1}{2}\sum_{e\in\Gamma_1^h\cup\Gamma_1} ([\boldsymbol{U}]\cdot\boldsymbol{n}_e, \{\boldsymbol{U}\cdot\boldsymbol{v}\})_e$$
$$-\sum_{E\in\mathcal{E}_1^h} (P_1, \nabla\cdot\boldsymbol{v})_E + \sum_{e\in\Gamma_1^h\cup\Gamma_1} (\{P\}, [\boldsymbol{v}]\cdot\boldsymbol{n}_e)_e$$
$$-\sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} (\{\boldsymbol{K}\nabla P_2\cdot\boldsymbol{n}_e\}, [q])_e + \epsilon_2 \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} (\{\boldsymbol{K}\nabla q\cdot\boldsymbol{n}_e\}, [P_2])_e + \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} \frac{\sigma_e}{|e|}([P_2], [q])_e$$
$$+(P_2 - \frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{U}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{U}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}$$
$$= (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2},$$
$$\forall q, \quad -\sum_{E\in\mathcal{E}_1^h} (q, \nabla\cdot\boldsymbol{U})_E + \sum_{e\in\Gamma_1^h\cup\Gamma_1} (\{q\}, [\boldsymbol{U}]\cdot\boldsymbol{n}_e)_e = 0.$$

Here $\mathcal{E}_i^h$ is a triangulation of $\overline{\Omega}_i$, Γ_i^h denotes the set of edges of $\mathcal{E}_i^h$, interior to Ω_i, ϵ_i are parameters that determine the symmetry or anti-symmetry of the discrete elliptic operators and σ_i are appropriate weights chosen on each edge to enforce stability. These quantities are made precise in Section 3.

The coupled problem (1.1)–(1.9) arises from important applications such as groundwater contamination through lakes and rivers. In the case where the momentum equation (1.1) is replaced by the linear Stokes equation $-\nabla\cdot(2\mu D(\boldsymbol{u}) - p_1\mathbf{I}) = \boldsymbol{f}_1$ and the interface condition (1.8) is replaced by the linear interface condition $\big((-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}\big)\cdot\boldsymbol{n}_{12} = p_2$, the resulting linear coupled problem has been studied in the literature: a weak formulation is analyzed in [14,22] and several numerical discretizations are proposed in [14,15,22,27–29].

In this work, we show that under mild regularity restrictions on the data and the domain, the system of equations (1.1)–(1.9) has at least one weak solution and the solution is unique if the data satisfy a smallness condition analogous to the condition that guarantees uniqueness of the solution to the Navier-Stokes equations. We then analyze the above numerical scheme, prove existence of a solution, its convergence to a weak solution and for adequate data, prove uniqueness and a priori error estimates.

1.1. Notation and preliminaries

Let Ω be a bounded, open connected domain in the plane with a Lipschitz-continuous boundary $\partial\Omega$, cf. [21]. The space $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω, its dual space is denoted by $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\overline{\Omega})$ denotes the space of restrictions to $\overline{\Omega}$ of the functions of $\mathcal{D}(\mathbb{R}^2)$. For a given integer m, we shall use the classical Sobolev space $H^m(\Omega)$ (see [1])

$$H^m(\Omega) = \{v \in L^2(\Omega)\,;\, \partial^k v \in L^2(\Omega)\ \forall |k| \le m\},$$

where $|k| = k_1 + k_2$, with k_1 and k_2 non-negative integers and

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1}\partial x_2^{k_2}}.$$

This space is equipped with the seminorm

$$|v|_{H^m(\Omega)} = \left[\sum_{|k|=m} \int_\Omega |\partial^k v|^2 \right]^{1/2} ,$$

and norm (for which it is a Hilbert space)

$$\|v\|_{H^m(\Omega)} = \left[\sum_{0\le|k|\le m} |v|^2_{H^k(\Omega)} \right]^{1/2} .$$

Let γ be a Lipschitz-continuous plane curve in $\overline{\Omega}$. The trace space of functions of $H^1(\Omega)$ on γ is denoted by $H^{1/2}(\gamma)$, normed by

$$\|v\|_{H^{1/2}(\gamma)} = \left(\int_\gamma \int_\gamma \frac{|v(x)-v(y)|^2}{|x-y|^2} dx\, dy \right)^{1/2} .$$

Its dual space, is $H^{-1/2}(\gamma)$. We shall also use the space $H^{1/2}_{00}(\gamma)$. When γ is not a closed curve, its norm is

$$\|v\|_{H^{1/2}_{00}(\gamma)} = \left(\|v\|^2_{H^{1/2}(\gamma)} + \|\frac{v}{\rho^{1/2}}\|^2_{L^2(\gamma)} \right)^{1/2} ,$$

where ρ is the distance function to the end-points of γ. Thus $H^{1/2}_{00}(\gamma)$ coincides with $H^{1/2}(\gamma)$ when γ is a closed curve. The definitions of these spaces are extended straightforwardly to vectors, with the same notation.

For functions that vanish on $\partial\Omega$, we define

$$H^1_0(\Omega) = \{v \in H^1(\Omega)\,;\, v|_{\partial\Omega} = 0\}.$$

We shall also need the following space of functions with zero mean value:

$$L^2_0(\Omega) = \{q \in L^2(\Omega)\,;\, \int_\Omega q = 0\}.$$

1.2. The boundary conditions at the interface

Let $(\boldsymbol{u}, p_1, p_2)$ be any solution of (1.1)–(1.3) in $H^1(\Omega_1)^2 \times L^2(\Omega_1) \times H^1(\Omega_2)$. First, let us give a meaning to the boundary conditions (1.7)–(1.9). For this, we assume from now on that the interface Γ_{12} is a *curvilinear polygonal line* of class $\mathcal{C}^{1,1}$ (cf. [21]). This implies that the normal vector $\boldsymbol{n}_{12}$ has a finite number of points of discontinuity, say P_i, $0 \le i \le L$, and Γ_{12} can be decomposed into L curvilinear segments S_i of class $\mathcal{C}^{1,1}$, with end-points P_{i-1} and P_i, for $1 \le i \le L$. This is the case when Γ_{12} is a polygonal line. As far as the data are concerned, we assume that

$$\boldsymbol{f}_1 \in L^2(\Omega_1)^2 \ ,\ f_2 \in L^2(\Omega_2) \ ,\ \boldsymbol{K} \in L^\infty(\Omega_2)^{2\times 2}, \tag{1.10}$$

and we suppose that $\boldsymbol{K}$ is uniformly bounded and positive definite in Ω_2: there exists $\lambda_{\max} > 0$ and $\lambda_{\min} > 0$ such that

$$\text{a.e. } \boldsymbol{x} \in \Omega_2 \ ,\ \lambda_{\min}|\boldsymbol{x}|^2 \le \boldsymbol{K}\boldsymbol{x}\cdot\boldsymbol{x} \le \lambda_{\max}|\boldsymbol{x}|^2. \tag{1.11}$$

To begin with, these assumptions on f_2 and $\boldsymbol{K}$ imply that any solution $p_2 \in H^1(\Omega_2)$ of (1.3) is such that $\nabla\cdot(\boldsymbol{K}\nabla p_2)$ belongs to $L^2(\Omega_2)$ and therefore $\boldsymbol{K}\nabla p_2$ belongs to $H(\text{div};\Omega_2)$. It follows from standard properties of this space (see for instance [2]), that the normal trace $(\boldsymbol{K}\nabla p_2)\cdot\boldsymbol{n}_{\Omega_2}$ belongs to $H^{-1/2}(\partial\Omega_2)$ and the following Green's formula holds:

$$\forall q \in H^1(\Omega_2), \quad -(\nabla\cdot\boldsymbol{K}\nabla p_2, q)_{\Omega_2} = (\boldsymbol{K}\nabla p_2, \nabla q)_{\Omega_2} - \langle(\boldsymbol{K}\nabla p_2)\cdot\boldsymbol{n}_{\Omega_2}, q\rangle_{\partial\Omega_2} , \tag{1.12}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. Furthermore, the normal trace $(\boldsymbol{K}\nabla p_2) \cdot \boldsymbol{n}_{12}$ is well defined in the dual space of $H^{1/2}_{00}(\Gamma_{12})$. Hence condition (1.7) is meaningful in a distributional sense, and since $\boldsymbol{u} \cdot \boldsymbol{n}_{12}$ belongs at least to $L^4(\Gamma_{12})$, as $\boldsymbol{u}$ belongs to $H^{1/2}(\Gamma_{12})^2$, (1.7) implies in particular that

$$(\boldsymbol{K}\nabla p_2) \cdot \boldsymbol{n}_{12} \in L^4(\Gamma_{12}).$$

Next, assumption (1.10) on $\boldsymbol{f}_1$ implies that any solution $(\boldsymbol{u}, p_1)$ in $H^1(\Omega_1)^2 \times L^2(\Omega_1)$ of (1.1), (1.2) is such that

$$\nabla \cdot (2\mu \boldsymbol{D}(\boldsymbol{u}) - p_1 \boldsymbol{I}) \in L^{4/3}(\Omega_1)^2,$$

i.e. each line of $2\mu\boldsymbol{D}(\boldsymbol{u}) - p_1\boldsymbol{I}$ belongs to the space

$$H^{4/3}(\mathrm{div};\Omega_1) = \{\boldsymbol{v} \in L^2(\Omega_1)^2\,;\, \nabla \cdot \boldsymbol{v} \in L^{4/3}(\Omega_1)\}.$$

A standard argument proves that $\mathcal{D}(\overline{\Omega}_1)^2$ is dense in $H^{4/3}(\mathrm{div};\Omega_1)$. This allows us to extend Green's formula

$$(\nabla \cdot \boldsymbol{v}, \varphi)_{\Omega_1} = -(\boldsymbol{v}, \nabla\varphi)_{\Omega_1} + \langle \boldsymbol{v} \cdot \boldsymbol{n}_{\Omega_1}, \varphi \rangle_{\partial\Omega_1}, \tag{1.13}$$

to all $\varphi \in H^1(\Omega_1)$ and all $\boldsymbol{v} \in H^{4/3}(\mathrm{div};\Omega_1)$ and to show in particular that the normal trace $\boldsymbol{v} \cdot \boldsymbol{n}_{12}$ is well-defined as an element of the dual space of $H^{1/2}_{00}(\Gamma_{12})$. Thus $(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}$ belongs to the dual space of $H^{1/2}_{00}(\Gamma_{12})^2$, but we cannot apply it by duality to $\boldsymbol{n}_{12}$ because in general, $\boldsymbol{n}_{12}$ does not belong to $H^{1/2}_{00}(\Gamma_{12})^2$. *At this point, we use the assumption that* Γ_{12} *is a curvilinear polygon of class* $\mathcal{C}^{1,1}$. On each segment S_i of Γ_{12}, let v be an arbitrary smooth function that vanishes at the end-points P_{i-1} and P_i. Then $v\,\boldsymbol{n}_{12}$ belongs to $H^{1/2}_{00}(S_i)^2$ and the duality

$$\langle(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}, v\,\boldsymbol{n}_{12}\rangle_{S_i}$$

is well-defined. Thus, we express (1.8) weakly as

$$\forall v \in H^{1/2}_{00}(S_i), \quad \langle(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}, v\,\boldsymbol{n}_{12}\rangle_{S_i} = (p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}), v)_{S_i}, \quad 1 \le i \le L. \tag{1.14}$$

As v is arbitrary, (1.14) implies that (1.8) holds in the sense of distributions on each S_i, hence on Γ_{12}. Furthermore, by virtue of (1.8), the fact that $p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u})$ belongs at least to $L^2(\Gamma_{12})$ shows that

$$((2\mu\boldsymbol{D}(\boldsymbol{u}) - p_1\boldsymbol{I})\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12} \in L^2(\Gamma_{12}). \tag{1.15}$$

Similarly, in each S_i, we express (1.9) as

$$\forall v \in H^{1/2}_{00}(S_i), \quad \langle(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}, v\,\boldsymbol{\tau}_{12}\rangle_{S_i} = \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}, v)_{S_i},$$

and since $\boldsymbol{\tau}_{12} \cdot \boldsymbol{n}_{12} = 0$, this reduces to

$$\forall v \in H^{1/2}_{00}(S_i), \quad \langle -2\mu\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}_{12}, v\,\boldsymbol{\tau}_{12}\rangle_{S_i} = \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}, v)_{S_i}, \quad 1 \le i \le L. \tag{1.16}$$

Therefore (1.9) holds in the sense of distributions on each S_i, hence on Γ_{12} and

$$2\mu(\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12} \in L^4(\Gamma_{12}). \tag{1.17}$$

1.3. Weak formulation

The above considerations suggest to set our problem in the functional spaces $\boldsymbol{X}, M_1$ and M_2:

$$\boldsymbol{X} = \{\boldsymbol{v} \in H^1(\Omega_1)^2\,;\, \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1\}, \tag{1.18}$$

$$M_1 = L^2(\Omega_1), \tag{1.19}$$

$$M_2 = \{q \in H^1(\Omega_2)\,; \quad q = 0 \quad \text{on} \quad \Gamma_{2D}\}. \tag{1.20}$$

We now recall the Poincaré and Korn's inequalities and the trace and Sobolev inequalities: there exist constants $\mathcal{P}_1$, C_0, C_1, C_4 and $\tilde{C}_4$, that only depend on Ω_1, and $\mathcal{P}_2$ that only depends on Ω_2, such that for all $\boldsymbol{v} \in \boldsymbol{X}$,

$$\|\boldsymbol{v}\|_{L^2(\Omega_1)} \leq \mathcal{P}_1 |\boldsymbol{v}|_{H^1(\Omega_1)}\,, \tag{1.21}$$

$$\|\boldsymbol{v}\|_{L^4(\Omega_1)} \leq \tilde{C}_4 |\boldsymbol{v}|_{H^1(\Omega_1)}\,, \tag{1.22}$$

$$|\boldsymbol{v}|_{H^1(\Omega_1)} \leq C_1 \|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega_1)}\,, \tag{1.23}$$

$$\|\boldsymbol{v}\|_{L^2(\Gamma_{12})} \leq C_0 |\boldsymbol{v}|_{H^1(\Omega_1)}, \quad \|\boldsymbol{v}\|_{L^4(\Gamma_{12})} \leq C_4 |\boldsymbol{v}|_{H^1(\Omega_1)}\,, \tag{1.24}$$

and for all $q \in M_2$,

$$\|q\|_{L^2(\Omega_2)} \leq \mathcal{P}_2 |q|_{H^1(\Omega_2)}\,; \tag{1.25}$$

moreover, owing to (1.11),

$$\frac{1}{\sqrt{\lambda_{\max}}} \|\boldsymbol{K}^{1/2} \nabla q\|_{L^2(\Omega_2)} \leq |q|_{H^1(\Omega_2)} \leq \frac{1}{\sqrt{\lambda_{\min}}} \|\boldsymbol{K}^{1/2} \nabla q\|_{L^2(\Omega_2)}\,. \tag{1.26}$$

Inequality (1.23) is known as Korn's first inequality and holds on $\boldsymbol{X}$ because Γ_1 is not reduced to a straight line.

Note that in view of (1.8), there is no undetermined constant in p_1. Thus, let $(\boldsymbol{u}, p_1, p_2) \in \boldsymbol{X} \times M_1 \times M_2$ be a solution of (1.1)–(1.9). Choose $\boldsymbol{v} \in \boldsymbol{X}$, take the scalar product of (1.1) with $\boldsymbol{v}$ over Ω_1 and apply Green's formula:

$$2\mu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v})_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + \langle(-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{\Omega_1}, \boldsymbol{v}\rangle_{\partial\Omega_1} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1}. \tag{1.27}$$

Since $\boldsymbol{v} = \boldsymbol{0}$ on Γ_1, the boundary term in (1.27) reduces to

$$\langle(-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{\Omega_1}, \boldsymbol{v}\rangle_{\partial\Omega_1} = \langle(-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}}. \tag{1.28}$$

Next, take the scalar product of (1.3) with $q \in M_2$ over Ω_2 and apply Green's formula:

$$(\boldsymbol{K}\nabla p_2, \nabla q)_{\Omega_2} - \langle \boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{\Omega_2}, q\rangle_{\partial\Omega_2} = (f_2, q)_{\Omega_2}. \tag{1.29}$$

Then, we have by adding (1.27) and (1.29) and by using (1.6), (1.28) and the orientation of $\boldsymbol{n}_{12}$:

$$\begin{aligned} &2\mu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla p_2, \nabla q)_{\Omega_2} \\ &+\langle(-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}} + \langle \boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{12}, q\rangle_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}. \end{aligned} \tag{1.30}$$

On one hand, we rewrite $\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{n}_{12})\boldsymbol{n}_{12} + (\boldsymbol{v} \cdot \boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}$. The regularities (1.15) and (1.17) imply:

$$\langle(-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}} = \big((-2\mu \boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} + \big(((-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12}, \boldsymbol{v} \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}}.$$

Therefore (1.8) and (1.9) give

$$\langle(-2\mu \boldsymbol{D}(\boldsymbol{u}) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}} = (p_2 - \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}), \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}}. \tag{1.31}$$

On the other hand, (1.7) yields

$$\langle \boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{12}, q\rangle_{\Gamma_{12}} = -(\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}.$$

Collecting these results, we propose the following variational formulation: find $(\boldsymbol{u}, p_1, p_2) \in \boldsymbol{X} \times M_1 \times M_2$ such that:

$$(Q)\begin{cases} \forall(\boldsymbol{v}, q) \in \boldsymbol{X} \times M_2, \quad 2\mu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_{\Omega_1} - (p_1, \nabla\cdot\boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla p_2, \nabla q)_{\Omega_2} \\ +(p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}, \\ \forall q \in M_1, \quad -(\nabla\cdot\boldsymbol{u}, q)_{\Omega_1} = 0. \end{cases}$$

Lemma 1.1. *Let the interface* Γ_{12} *be a* curvilinear polygon *of class* $\mathcal{C}^{1,1}$ *and assume that the data satisfy (1.10) and (1.11). Then any solution* $(\boldsymbol{u}, p_1, p_2) \in \boldsymbol{X} \times M_1 \times M_2$ *of the coupled problem (1.1)–(1.9) is also solution to the variational problem (Q). Conversely, any solution of problem (Q) satisfies (1.1)–(1.9).*

Proof. We have just shown that (1.1)–(1.9) implies Problem (Q). Conversely, let us assume that $(\boldsymbol{u}, p_1, p_2)$ is a solution of Problem (Q). We first choose $\boldsymbol{v} \in \mathcal{D}(\Omega_1)^2$ and $q = 0$; next $\boldsymbol{v} = \boldsymbol{0}$ and $q \in \mathcal{D}(\Omega_2)$. Then in the sense of distributions on Ω_1 and on Ω_2 we obtain:

$$-\nabla\cdot(2\mu\boldsymbol{D}(\boldsymbol{u})) + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p_1 = \boldsymbol{f}_1, \tag{1.32}$$

$$-\nabla\cdot\boldsymbol{K}\nabla p_2 = f_2. \tag{1.33}$$

Next, we multiply (1.32) by $\boldsymbol{v} \in \boldsymbol{X}$, (1.33) by $q \in M_2$, apply Green's formulas (1.12) and (1.13), which are valid here, add the two equations and compare with (Q). This gives:

$$\begin{aligned} \forall(\boldsymbol{v}, q) \in \boldsymbol{X} \times M_2, (p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} \\ = \langle(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{\Omega_1}, \boldsymbol{v}\rangle_{\partial\Omega_1} + \langle -\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{\Omega_2}, q\rangle_{\partial\Omega_2}. \end{aligned} \tag{1.34}$$

By choosing $\boldsymbol{v} = \boldsymbol{0}$ in (1.34), we deduce:

$$\forall q \in M_2, \quad \langle \boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{\Omega_2}, q\rangle_{\partial\Omega_2} = (\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}. \tag{1.35}$$

In particular, the choice $q|_{\Gamma_{12}} = 0$ in (1.35) implies (1.6):

$$\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{\Omega_2} = 0, \quad \text{on} \quad \Gamma_{2N}.$$

Hence, in view of the orientation of $\boldsymbol{n}_{12}$, (1.35) reduces to

$$\forall q \in M_2, \quad \langle \boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{12}, q\rangle_{\Gamma_{12}} = -(\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}},$$

whence (1.7). With this information and the boundary condition of $\boldsymbol{X}$, (1.34) reduces to

$$\forall\boldsymbol{v} \in \boldsymbol{X}, (p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - \langle(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}} = 0. \tag{1.36}$$

Fix a curvilinear segment S_i of Γ_{12} and choose in (1.36) $\boldsymbol{v} = v\,\boldsymbol{n}_{12}$, where v is a smooth function defined in Ω_1 that vanishes in a neighborhood of $\partial\Omega_1 \setminus S_i$. Then $\boldsymbol{v} \in \boldsymbol{X}$ and we recover (1.14) that implies (1.8). A similar argument yields (1.16) that implies (1.9). Whence the equivalence between the two problems. □

Now, we denote by $\boldsymbol{Y}$ the product space

$$\boldsymbol{Y} = \boldsymbol{X} \times M_2\,,$$

equipped with the norm

$$\forall (\boldsymbol{v},q) \in \boldsymbol{Y}, \quad \|(\boldsymbol{v},q)\|_Y = \Big(2\mu \|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla q\|^2_{L^2(\Omega_2)}\Big)^{1/2}, \tag{1.37}$$

and associated scalar product:

$$((\boldsymbol{v},q),(\boldsymbol{w},r))_Y = 2\mu(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\boldsymbol{w}))_{\Omega_1} + (\boldsymbol{K}\nabla q, \nabla r)_{\Omega_2}.$$

The norm (1.37) is equivalent to the product norm of $\boldsymbol{Y}$:

$$\forall (\boldsymbol{v},q) \in \boldsymbol{Y}, \quad \|(\boldsymbol{v},q)\| = \Big(|\boldsymbol{v}|^2_{H^1(\Omega_1)} + |q|^2_{H^1(\Omega_2)}\Big)^{1/2},$$

via (1.23) and (1.26). Therefore, $\boldsymbol{Y}$ is a Hilbert space for the norm (1.37). Then, we define the space of divergence-free functions:

$$\boldsymbol{V}=\{\boldsymbol{v} \in \boldsymbol{X}\,;\, \nabla\cdot\boldsymbol{v} = 0 \text{ in } \Omega_1\},$$

and the associated subspace of $\boldsymbol{Y}$:

$$\boldsymbol{W} = \boldsymbol{V} \times M_2.$$

It is also a Hilbert space for the norm and scalar product of $\boldsymbol{Y}$. By restricting the test functions $\boldsymbol{v}$ to $\boldsymbol{V}$ in Problem (Q), we obtain a second variational formulation:

$$(P)\begin{cases} \text{Find} \quad (\boldsymbol{u},p_2) \in \boldsymbol{W} \quad \text{such that} \\ \forall (\boldsymbol{v},q) \in \boldsymbol{W}, \quad 2\mu(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla p_2,\nabla q)_{\Omega_2} \\ +(p_2 - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} = (\boldsymbol{f}_1,\boldsymbol{v})_{\Omega_1} + (f_2,q)_{\Omega_2}. \end{cases}$$

In order to prove that problems (P) and (Q) are equivalent, we use the following result.

Lemma 1.2. *For each $q \in L^2(\Omega_1)$, there exists $\boldsymbol{v} \in \boldsymbol{X}$, such that*

$$\nabla\cdot\boldsymbol{v} = q \quad \text{in } \Omega_1 \quad , \; |\boldsymbol{v}|_{H^1(\Omega_1)} \le \frac{1}{\kappa}\left(\frac{|\Omega|}{|\Omega_2|}\right)^{1/2} \|q\|_{L^2(\Omega_1)}, \tag{1.38}$$

where κ only depends on Ω.

Proof. The proof is easy and is not new. Let us reproduce it for the readers' sake. The idea is to extend q by a constant function in Ω_2 so that the extended function, say $\tilde{q}$, has mean-value zero in Ω. Clearly, the constant function is:

$$c = -\frac{1}{|\Omega_2|}\int_{\Omega_1} q.$$

As $\tilde{q} \in L^2_0(\Omega)$, by the classical inf-sup condition between $H^1_0(\Omega)^2$ and $L^2_0(\Omega)$ (see for instance [17]), there exists $\tilde{\boldsymbol{v}} \in H^1_0(\Omega)^2$ such that

$$-\nabla\cdot\tilde{\boldsymbol{v}} = \tilde{q} \quad \text{in } \Omega \quad , \quad \|\nabla\,\tilde{\boldsymbol{v}}\|_{L^2(\Omega)} \le \frac{1}{\kappa}\|\tilde{q}\|_{L^2(\Omega)}, \tag{1.39}$$

where the isomorphism constant κ only depends on Ω. Then (1.38) follows from the facts that

$$\|\tilde{q}\|_{L^2(\Omega)} \le \left(\frac{|\Omega|}{|\Omega_2|}\right)^{1/2} \|q\|_{L^2(\Omega_1)},$$

and the restriction of $\boldsymbol{v}$ to Ω_1 belongs to $\boldsymbol{X}$. □

Proposition 1.3. *Problems (P) and (Q) are equivalent: if $(\boldsymbol{u}, p_1, p_2)$ is a solution of problem (Q), then $(\boldsymbol{u}, p_2)$ is a solution of problem (P); conversely, if $(\boldsymbol{u}, p_2)$ is a solution of problem (P), then there exists a unique $p_1 \in M_1$ such that $(\boldsymbol{u}, p_1, p_2)$ is a solution of problem (Q).*

Proof. Clearly, it suffices to prove that if $(\boldsymbol{u}, p_2)$ is a solution of problem (P), then there exists a pressure $p_1 \in M_1$ such that $(\boldsymbol{u}, p_1, p_2)$ is a solution of problem (Q). This is a consequence of Lemma 1.2 and the Babuška–Brezzi's theory (cf. [4], [8] or [17]). Indeed, on one hand, (1.38) is equivalent to the inf-sup condition:

$$\inf_{q \in M_1} \sup_{\boldsymbol{v} \in \boldsymbol{X}} \frac{(\nabla \cdot \boldsymbol{v}, q)_{\Omega_1}}{|\boldsymbol{v}|_{H^1(\Omega_1)} \|q\|_{L^2(\Omega_1)}} \geq \beta \,, \tag{1.40}$$

where

$$\beta = \kappa \left(\frac{|\Omega_2|}{|\Omega|} \right)^{1/2} .$$

In turn, (1.40) implies trivially with the same constant β

$$\inf_{q \in M_1} \sup_{(\boldsymbol{v}, q) \in \boldsymbol{Y}} \frac{(\nabla \cdot \boldsymbol{v}, q)_{\Omega_1}}{\|(\boldsymbol{v}, q)\|_Y \|q\|_{L^2(\Omega_1)}} \geq \frac{\beta}{\sqrt{2\mu}} . \tag{1.41}$$

On the other hand, given $(\boldsymbol{u}, p_2)$ in $\boldsymbol{W}$, the mapping

$$\begin{aligned} (\boldsymbol{v}, q) \in \boldsymbol{Y} \mapsto 2\mu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} &+ (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K} \nabla p_2, \nabla q)_{\Omega_2} \\ &+ (p_2 - \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}), \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u} \cdot \boldsymbol{n}_{12}, q)_{\Gamma_{12}} \\ &- (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} - (f_2, q)_{\Omega_2} \end{aligned}$$

defines an element ℓ of the dual space $\boldsymbol{Y}'$, and since $(\boldsymbol{u}, p_2)$ solves (P), then ℓ vanishes on $\boldsymbol{W}$. Therefore the inf-sup condition (1.41) implies that there exists a unique $p_1 \in M_1$ such that

$$\forall (\boldsymbol{v}, q) \in \boldsymbol{Y} \,, \; \ell(\boldsymbol{v}, q) = (\nabla \cdot \boldsymbol{v}, p_1) \,.$$

This is precisely the statement of problem (Q). □

2. Existence and uniqueness of a weak solution

Since problems (P) and (Q) are equivalent, it suffices to construct a weak solution of problem (P). As the spaces $\boldsymbol{V}$ and M_2 are separable, then the space $\boldsymbol{W}$ is separable and there exist two sequences of smooth functions, $\{\boldsymbol{\Phi}_m\}_{m \geq 1}$ and $\{\varphi_m\}_{m \geq 1}$, with $\boldsymbol{\Phi}_m \in \boldsymbol{V} \cap H^2(\Omega_1)^2$ and $\varphi_m \in M_2 \cap H^2(\Omega_2)$ such that $\{(\boldsymbol{\Phi}_m, \varphi_m)\}_{m \geq 1}$ is a basis of $\boldsymbol{W}$. Now, define the finite-dimensional space:

$$\boldsymbol{W}_m = \mathrm{Vect}\{(\boldsymbol{\Phi}_i, \varphi_i) \,; \, 1 \leq i \leq m\} \,.$$

Then the Galerkin approximation of problem (P) is

$$(P_m) \begin{cases} \text{Find} \quad (\boldsymbol{u}_m, p_m) \in \boldsymbol{W}_m \quad \text{such that} \\ \forall 1 \leq i \leq m, \quad 2\mu(\boldsymbol{D}(\boldsymbol{u}_m), \boldsymbol{D}(\boldsymbol{\Phi}_i))_{\Omega_1} + (\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{\Phi}_i)_{\Omega_1} + (\boldsymbol{K} \nabla p_m, \nabla \varphi_i)_{\Omega_2} \\ +(p_m - \frac{1}{2}\boldsymbol{u}_m \cdot \boldsymbol{u}_m, \boldsymbol{\Phi}_i \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{u}_m \cdot \boldsymbol{n}_{12}, \varphi_i)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}_m \cdot \boldsymbol{\tau}_{12}, \boldsymbol{\Phi}_i \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{\Phi}_i)_{\Omega_1} + (f_2, \varphi_i)_{\Omega_2}. \end{cases}$$

Proposition 2.1. *Assume that the data satisfy (1.10) and (1.11). Then, for each m, problem (P_m) has at least one solution $(\boldsymbol{u}_m, p_m) \in \boldsymbol{W}_m$ and there exists a constant $\mathcal{C}$, independent of m, such that any solution of problem (P_m) satisfies the uniform bound*

$$\|(\boldsymbol{u}_m, p_m)\|_Y \leq \mathcal{C}. \tag{2.1}$$

Proof. Since $\boldsymbol{W}_m$ is a Hilbert space in finite dimension, we use Brouwer's Fixed Point Theorem. To this end, we introduce the mapping: $\mathcal{F}_m : \boldsymbol{W}_m \to \boldsymbol{W}_m$ defined for all $(\boldsymbol{v}, q) \in \boldsymbol{W}_m$ by:

$$\forall (\boldsymbol{w}, r) \in \boldsymbol{W}_m, \quad (\mathcal{F}_m(\boldsymbol{v}, q), (\boldsymbol{w}, r))_Y = 2\mu(\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{w}))_{\Omega_1} + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{w})_{\Omega_1} + (\boldsymbol{K}\nabla q, \nabla r)_{\Omega_2}$$
$$+(q - \frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v}, \boldsymbol{w} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{v} \cdot \boldsymbol{n}_{12}, r)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{v} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{w} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{f}_1, \boldsymbol{w})_{\Omega_1} - (f_2, r)_{\Omega_2}.$$

Clearly, $\mathcal{F}_m$ defines a mapping from $\boldsymbol{W}_m$ into itself. As the dimension of $\boldsymbol{W}_m$ is finite, considering the operations involved, this mapping is continuous. Furthermore, any zero of $\mathcal{F}_m$ is a solution of problem (P_m).

Let us evaluate $(\mathcal{F}_m(\boldsymbol{v}, q), (\boldsymbol{v}, q))_Y$:

$$(\mathcal{F}_m(\boldsymbol{v}, q), (\boldsymbol{v}, q))_Y = 2\mu\|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla q\|^2_{L^2(\Omega_2)} + (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v})_{\Omega_1} - \frac{1}{2}(\boldsymbol{v} \cdot \boldsymbol{v}, \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}$$
$$+\frac{1}{G}\|\boldsymbol{v} \cdot \boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} - (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} - (f_2, q)_{\Omega_2}.$$

However, the two nonlinear terms cancel because, by Green's formula (1.13)

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \quad (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v})_{\Omega_1} = -\frac{1}{2}(\nabla \cdot \boldsymbol{v}, \boldsymbol{v} \cdot \boldsymbol{v})_{\Omega_1} + \frac{1}{2}(\boldsymbol{v} \cdot \boldsymbol{n}_{\Omega_1}, \boldsymbol{v} \cdot \boldsymbol{v})_{\partial\Omega_1} = \frac{1}{2}(\boldsymbol{v} \cdot \boldsymbol{n}_{\Omega_1}, \boldsymbol{v} \cdot \boldsymbol{v})_{\partial\Omega_1}.$$

Thus, by using (1.21), (1.23), (1.25) and (1.26), we have

$$(\mathcal{F}_m(\boldsymbol{v}, q), (\boldsymbol{v}, q))_Y \geq \frac{1}{2}\left(2\mu\|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla q\|^2_{L^2(\Omega_2)} - (\frac{1}{2\mu}\mathcal{P}_1^2 C_1^2\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + \frac{\mathcal{P}_2^2}{\lambda_{\min}}\|f_2\|^2_{L^2(\Omega_2)})\right).$$

Therefore, $(\mathcal{F}_m(\boldsymbol{v}, q), (\boldsymbol{v}, q))_Y \geq 0$ provided that

$$\|(\boldsymbol{v}, q)\|_Y = \left(\frac{1}{2\mu}\mathcal{P}_1^2 C_1^2\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + \frac{\mathcal{P}_2^2}{\lambda_{\min}}\|f_2\|^2_{L^2(\Omega_2)}\right)^{1/2},$$

i.e. for all $(\boldsymbol{v}, q) \in \boldsymbol{W}_m$ (equipped with the norm $\|(\boldsymbol{v}, q)\|_{\boldsymbol{Y}}$) on the surface of the sphere centered at the origin, with radius

$$\mathcal{C} = \left(\frac{1}{2\mu}\mathcal{P}_1^2 C_1^2\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + \frac{\mathcal{P}_2^2}{\lambda_{\min}}\|f_2\|^2_{L^2(\Omega_2)}\right)^{1/2}. \tag{2.2}$$

Then a classical variant of Brouwer's Fixed Point Theorem implies that $\mathcal{F}_m$ has at least one zero in this ball. This yields existence of one solution $(\boldsymbol{u}_m, p_m)$ of (P_m) and the bound (2.1) with the constant $\mathcal{C}$ defined by (2.2).

Finally, the same argument shows that any solution of (P_m) satisfies this bound. □

Theorem 2.2. *Assume that the data satisfy (1.10) and (1.11). Then problem (Q) has at least one solution $(\boldsymbol{u}, p_1, p_2)$ in $\boldsymbol{V} \times M_1 \times M_2$ and every solution $(\boldsymbol{u}, p_2)$ of problem (P) satisfies the a priori estimate:*

$$\|(\boldsymbol{u}, p_2)\|_Y \leq \mathcal{C}, \tag{2.3}$$

with the constant $\mathcal{C}$ of (2.2).

Proof. For each $m \geq 1$, let $(\boldsymbol{u}_m, p_m)$ be a solution of (P_m) in $\boldsymbol{W}_m$. It stems from the uniform bound (2.1) that there exists a pair of functions $(\boldsymbol{u}, p_2)$ in $\boldsymbol{W}$ and a subsequence of m, still denoted by m, such that

$$\lim_{m\to\infty} \boldsymbol{u}_m = \boldsymbol{u} \quad \text{weakly in} \quad \boldsymbol{V},$$

$$\lim_{m\to\infty} p_m = p_2 \quad \text{weakly in} \quad H^1(\Omega_2).$$

Then the Sobolev imbeddings imply that the above convergences are strong in $L^t(\Omega)$ for any $t < \infty$. In particular, by extracting another subsequence,

$$\lim_{m\to\infty} \boldsymbol{u}_m = \boldsymbol{u} \quad \text{strongly in} \quad L^4(\Omega_1)^2.$$

Furthermore, since for any bounded Lipschitz-continuous domain $\mathcal{O}$, the trace operator is continuous from $H^1(\mathcal{O})$ into $H^{1/2}(\partial\mathcal{O})$ for the weak topology, we have that

$$\lim_{m\to\infty} \boldsymbol{u}_m|_{\partial\Omega_1} = \boldsymbol{u}|_{\partial\Omega_1} \quad \text{weakly in} \quad H^{1/2}(\partial\Omega_1)^2,$$

$$\lim_{m\to\infty} p_m|_{\partial\Omega_2} = p_2|_{\partial\Omega_2} \quad \text{weakly in} \quad H^{1/2}(\partial\Omega_2).$$

Hence the Sobolev imbeddings imply that (by extracting another subsequence)

$$\lim_{m\to\infty} \boldsymbol{u}_m|_{\partial\Omega_1} = \boldsymbol{u}|_{\partial\Omega_1} \quad \text{strongly in} \quad L^4(\partial\Omega_1)^2.$$

Therefore, by a standard argument, we pass to the limit in the equations of (P_m) as m tends to infinity and we derive that the limit pair $(\boldsymbol{u}, p_2)$ satisfies problem (P).

The a priori estimate (2.3) is obtained as in the proof of Proposition 2.1. Finally, existence of p_1 follows from Proposition 1.3. □

The next corollary is straightforward.

Corollary 2.3. *Under the assumptions of Theorem 2.2, every solution $(\boldsymbol{u}, p_2)$ to problem (P) satisfies the a priori estimate:*

$$\mu\|\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(\Omega_1)} + \frac{1}{2}\|\boldsymbol{K}^{1/2}\nabla p_2\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} \leq \frac{1}{2}\mathcal{C}^2\,, \tag{2.4}$$

with $\mathcal{C}$ defined by (2.2).

Theorem 2.4. *There is a constant C^* that only depends on the constants in the inequalities (1.21)–(1.24) such that if the condition holds*

$$32\mu^4 > C^*\left(\mathcal{P}_1^2 C_1^2\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + 2\mu\frac{\mathcal{P}_2^2}{\lambda_{\min}}\|f_2\|^2_{L^2(\Omega_2)}\right)\,, \tag{2.5}$$

then problem (Q) has a unique weak solution.

Proof. Assume that $(\boldsymbol{u}^1, p_2^1)$ and $(\boldsymbol{u}^2, p_2^2)$ are two solutions of problem (P). Their difference, say $(\boldsymbol{w}, z_2)$ satisfies:

$$\begin{aligned}&\forall(\boldsymbol{v}, q) \in \boldsymbol{W}, \quad 2\mu(\boldsymbol{D}(\boldsymbol{w}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{K}\nabla z_2, \nabla q)_{\Omega_2} + (\boldsymbol{w}\cdot\nabla\boldsymbol{u}^1, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{u}^2\cdot\nabla\boldsymbol{w}, \boldsymbol{v})_{\Omega_1}\\ &+\frac{1}{G}(\boldsymbol{w}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} + (z_2 - \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{u}^1), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{w}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{u}^2\cdot\boldsymbol{w}, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} = 0\,.\end{aligned}$$

By choosing $(\boldsymbol{v}, q) = (\boldsymbol{w}, z_2) \in \boldsymbol{W}$ and applying Green's formula (1.13) and the boundary condition on the functions of $\boldsymbol{X}$, this equation becomes

$$\begin{aligned}&2\mu\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla z_2\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{w}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}\\ &\quad + (\boldsymbol{w}\cdot\nabla\boldsymbol{u}^1, \boldsymbol{w})_{\Omega_1} + \frac{1}{2}[(\boldsymbol{w}\cdot\boldsymbol{w}, \boldsymbol{u}^2\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{w}\cdot(\boldsymbol{u}^1 + \boldsymbol{u}^2), \boldsymbol{w}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}] = 0\,.\end{aligned} \tag{2.6}$$

Applying formulas (1.22)–(1.24), the term in brackets in (2.6) is bounded above by

$$\|\boldsymbol{w}\|^2_{L^4(\Gamma_{12})}\left(\|\boldsymbol{u}^1\|_{L^2(\Gamma_{12})}+2\|\boldsymbol{u}^2\|_{L^2(\Gamma_{12})}\right)$$
$$\le C_4^2 C_0 C_1^3\frac{1}{\sqrt{\mu}}\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)}\left(\sqrt{\mu}\|\boldsymbol{D}(\boldsymbol{u}^1)\|_{L^2(\Omega_1)}+2\sqrt{\mu}\|\boldsymbol{D}(\boldsymbol{u}^2)\|_{L^2(\Omega_1)}\right).$$

Similarly, applying (1.22) and (1.23), the first non-linear term is bounded above by

$$\|\boldsymbol{w}\|^2_{L^4(\Omega_1)}|\boldsymbol{u}^1|_{H^1(\Omega_1)}\le C_1^3\tilde{C}_4^2\frac{1}{\sqrt{\mu}}\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)}\left(\sqrt{\mu}\|\boldsymbol{D}(\boldsymbol{u}^1)\|_{L^2(\Omega_1)}\right).$$

Hence, using the a priori estimate (2.4), the second line in (2.6) is bounded above by

$$\frac{C_1^3}{2\sqrt{\mu}}\left(\frac{3}{2}C_4^2C_0+\tilde{C}_4^2\right)\mathcal{C}\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)}.$$

Thus if

$$4\mu^{3/2}>C_1^3\left(\frac{3}{2}C_4^2C_0+\tilde{C}_4^2\right)\mathcal{C},$$

then $(\boldsymbol{w},z_2)=(\boldsymbol{0},0)$. Of course, since problems (P) and (Q) are equivalent, this also implies that the pressures in Ω_1 coincide in $L^2(\Omega_1)$. □

3. Numerical scheme

¿From now on, we assume that all boundaries are polygonal lines. Let us introduce the notation related to the spatial discretization and then present the numerical method. We consider a *regular* family of triangulations of $\overline{\Omega}$, denoted by $\mathcal{E}^h$, composed of triangles with maximum diameter h. We assume that all vertices of Γ_{12} and $\partial\Omega$ are vertices of $\mathcal{E}^h$ and we assume that all segments of Γ_{12} are composed of segments of $\mathcal{E}^h$. Therefore the restriction of $\mathcal{E}^h$ to Ω_i is also a regular family of triangulations of $\overline{\Omega}_i$; we denote it by $\mathcal{E}_i^h$ and observe that the two meshes $\mathcal{E}_i^h$ coincide at the interface Γ_{12}. This restriction simplifies the discussion, but it can be relaxed. By regular, we mean that there exists a constant $\gamma>0$, independent of h, such that (cf. Ciarlet [9]):

$$\forall E\in\mathcal{E}^h,\quad \frac{h_E}{\rho_E}=\gamma_E\le\gamma, \tag{3.1}$$

where h_E denotes the diameter of E (bounded above by h) and ρ_E denotes the diameter of the ball inscribed in E.

For $i=1,2$, let Γ_i^h denote the set of edges of $\mathcal{E}_i^h$ interior to Ω_i. To each edge e of $\mathcal{E}^h$ we associate once and for all a unit normal vector $\boldsymbol{n}_e$. For the edges in Γ_i^h, this can be done by ordering the triangles of $\mathcal{E}_i^h$ and orienting the normal in the direction of increasing numbers. For $e\in\Gamma_{12}$, we set $\boldsymbol{n}_e=\boldsymbol{n}_{12}$, i.e. $\boldsymbol{n}_e$ is the exterior normal to Ω_1. For a boundary edge $e\in\Gamma_i$, $\boldsymbol{n}_e$ coincides with the outward normal vector $\boldsymbol{n}_{\Omega_i}$. If $\boldsymbol{n}_e$ points from the element E^1 to the element E^2, the jump $[]_e$ and average $\{\}_e$ of a function φ are given by:

$$[\varphi]_e=\varphi|_{E^1}-\varphi|_{E^2},\quad \{\varphi\}_e=\frac{1}{2}\varphi|_{E^1}+\frac{1}{2}\varphi|_{E^2}.$$

By convention, for an edge on Γ_i, the jump and average are defined to be equal to the trace of the function on that edge. Also, by convention, we suppress the index e when there is no ambiguity.

Our scheme is based on discontinuous finite element spaces of possibly different orders for the Navier-Stokes velocity and the Darcy pressure. Let $k_1 \geq 1$ and $k_2 \geq 1$ be two integers; we set

$$\begin{aligned}
\boldsymbol{X}^h &= \{\boldsymbol{v} \in L^2(\Omega_1)^2\,;\, \forall E \in \mathcal{E}_1^h, \quad \boldsymbol{v}|_E \in I\!P_{k_1}(E)^2\},\\
M_1^h &= \{q \in L^2(\Omega_1)\,;\, \forall E \in \mathcal{E}_1^h, \quad q|_E \in I\!P_{k_1-1}(E)\},\\
M_2^h &= \{q \in L^2(\Omega_2)\,;\, \forall E \in \mathcal{E}_1^h, \quad q|_E \in I\!P_{k_2}(E)\}.
\end{aligned}$$

We associate with the discrete spaces $\boldsymbol{X}^h$ and M_2^h the mesh-dependent norms:

$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \quad \|\boldsymbol{v}\|_X = \left(2\sum_{E\in\mathcal{E}_1^h} \|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(E)} + \sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\sigma_e}{|e|}\|[\boldsymbol{v}]\|^2_{L^2(e)}\right)^{1/2}, \tag{3.2}$$

$$\forall q \in M_2^h, \quad \|q\|_{M_2} = \left(\sum_{E\in\mathcal{E}_2^h} \|\boldsymbol{K}^{1/2}\nabla q\|^2_{L^2(E)} + \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} \frac{\sigma_e}{|e|}\|[q]\|^2_{L^2(e)}\right)^{1/2}, \tag{3.3}$$

and we take the L^2 norm on M_1^h:

$$\forall q \in M_1^h, \quad \|q\|_{M_1} = \|q\|_{L^2(\Omega_1)}\,. \tag{3.4}$$

The penalty parameters $\sigma_e > 0$ in (3.2), (3.3) may vary from edge to edge and satisfy:

$$\begin{aligned}
\forall e \in \Gamma_1^h \cup \Gamma_1, \quad 1 \leq \sigma^1_{\text{min}} \leq \sigma_e \leq \sigma^1_{\text{max}}\,,\\
\forall e \in \Gamma_2^h \cup \Gamma_{2D}, \quad 1 \leq \sigma^2_{\text{min}} \leq \sigma_e \leq \sigma^2_{\text{max}}\,,
\end{aligned} \tag{3.5}$$

where σ^i_{min} and σ^i_{max}, $i = 1, 2$ are independent of h. For convenience, we also introduce the broken norm for $i = 1, 2$:

$$|\!|\!|\varphi|\!|\!|_{L^2(\Omega_i)} = \left(\sum_{E\in\mathcal{E}_i^h} \|\varphi\|^2_{L^2(E)}\right)^{1/2}, \tag{3.6}$$

and the jump bilinear forms:

$$J^1(\varphi,\psi) = \sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\sigma_e}{|e|}\int_e [\varphi][\psi]\;,\; J^2(\varphi,\psi) = \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} \frac{\sigma_e}{|e|}\int_e [\varphi][\psi]\,. \tag{3.7}$$

Next, we define the usual NIPG, SIPG and IIPG bilinear forms for Navier-Stokes and Darcy problems. The parameters $\epsilon_1, \epsilon_2 \in \{+1, 0, -1\}$ allow to switch from symmetric to non-symmetric bilinear forms, with the option that if $\epsilon_1 = \epsilon_2 = 0$, we obtain the Incomplete Interior Penalty Galerkin method introduced for elliptic problems in [13]. The forms a_S, b_S and a_D (the index S for Stokes and D for Darcy) correspond to the DG discretization of the viscous term, divergence term and Darcy term respectively, namely

$$\forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{X}^h, \quad a_S(\boldsymbol{u},\boldsymbol{v}) = 2\mu \sum_{E\in\mathcal{E}_1^h} (\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_E - 2\mu \sum_{e\in\Gamma_1^h\cup\Gamma_1} (\{\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}_e\},[\boldsymbol{v}])_e$$
$$+2\epsilon_1\mu \sum_{e\in\Gamma_1^h\cup\Gamma_1} (\{\boldsymbol{D}(\boldsymbol{v})\boldsymbol{n}_e\},[\boldsymbol{u}])_e + \mu \sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\sigma_e}{|e|}([\boldsymbol{u}],[\boldsymbol{v}])_e,$$
$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \forall p \in M_1^h, \quad b_S(\boldsymbol{v},p) = -\sum_{E\in\mathcal{E}_1^h} (p,\nabla\cdot\boldsymbol{v})_E + \sum_{e\in\Gamma_1^h\cup\Gamma_1} (\{p\},[\boldsymbol{v}]\cdot\boldsymbol{n}_e)_e,$$
$$\forall p,q \in M_2^h, \quad a_D(p,q) = \sum_{E\in\mathcal{E}_2^h} (\boldsymbol{K}\nabla p,\nabla q)_E - \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} (\{\boldsymbol{K}\nabla p\cdot\boldsymbol{n}_e\},[q])_e$$
$$+\epsilon_2 \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} (\{\boldsymbol{K}\nabla q\cdot\boldsymbol{n}_e\},[p])_e + \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} \frac{\sigma_e}{|e|}([p],[q])_e.$$

We then discretize the trilinear form of the convection term by using the upwinding of Lesaint-Raviart [23] adapted to DG. It has been thoroughly studied in [18, 19]. For this, we introduce the following notation. For an element $E \in \mathcal{E}^h$, we denote by $\boldsymbol{n}_E$ the outward normal to E, and we denote by $\boldsymbol{v}^{\text{int}}$ (resp. $\boldsymbol{v}^{\text{ext}}$) the trace of the function $\boldsymbol{v}$ on a side of E coming from the interior of E (resp. the exterior of E). When the side of E belongs to $\partial\Omega$, then by convention we set $\boldsymbol{v}^{\text{int}} = \boldsymbol{v}$ and $\boldsymbol{v}^{\text{ext}} = \boldsymbol{0}$. Then we define:

$$\forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{X}^h, \quad d_h(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = \sum_{E\in\mathcal{E}_1^h} (\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w})_E + \frac{1}{2}\sum_{E\in\mathcal{E}_1^h} (\nabla\cdot\boldsymbol{u},\boldsymbol{v}\cdot\boldsymbol{w})_E - \frac{1}{2}\sum_{e\in\Gamma_1^h\cup\Gamma_1} ([\boldsymbol{u}]\cdot\boldsymbol{n}_e,\{\boldsymbol{v}\cdot\boldsymbol{w}\})_e, \tag{3.8}$$

$$\forall \boldsymbol{z},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{X}^h, \quad \ell_h(\boldsymbol{z},\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = \sum_{E\in\mathcal{E}_1^h} (\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{v}^{\text{int}}-\boldsymbol{v}^{\text{ext}}),\boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{z})\setminus\Gamma_{12}}, \tag{3.9}$$

$$\forall \boldsymbol{z},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{X}^h, \quad c_{NS}(\boldsymbol{z},\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = d_h(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) + \ell_h(\boldsymbol{z},\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}), \tag{3.10}$$

where

$$\partial E_-(\boldsymbol{z}) = \{\boldsymbol{x}\in\partial E\,;\,\{\boldsymbol{z}(\boldsymbol{x})\}\cdot\boldsymbol{n}_E < 0\}.$$

Note that on one hand, the first argument of c_{NS} only appears in the definition of the integration region $\partial E_-(\boldsymbol{z})$, and on the other hand $-\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E$ can be replaced by $|\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E|$ when $\boldsymbol{z} = \boldsymbol{u}$.

With these forms and spaces, we propose the following numerical scheme: Find $(\boldsymbol{U},P_1,P_2) \in \boldsymbol{X}^h \times M_1^h \times M_2^h$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \forall q \in M_2^h, \quad a_S(\boldsymbol{U},\boldsymbol{v}) + b_S(\boldsymbol{v},P_1) + a_D(P_2,q) + c_{NS}(\boldsymbol{U},\boldsymbol{U};\boldsymbol{U},\boldsymbol{v})$$
$$+(P_2 - \frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U},\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{U}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{U}\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}$$
$$= (\boldsymbol{f}_1,\boldsymbol{v})_{\Omega_1} + (f_2,q)_{\Omega_2}, \tag{3.11}$$
$$\forall q \in M_1^h, \quad b_S(\boldsymbol{U},q) = 0. \tag{3.12}$$

Note that, in this scheme, the integrals on Γ_{12} present no ambiguity because the traces of functions come either from Ω_1 or from Ω_2.

We end this section with the consistency of (3.11), (3.12). To guarantee this consistency, we assume that each solution $(\boldsymbol{u},p_1,p_2)$ of problem (Q) is sufficiently smooth for the dualities $\langle(-2\mu D(\boldsymbol{u})+p_1\boldsymbol{I})\boldsymbol{n}_e,\boldsymbol{v}\rangle_e$, respectively $\langle\boldsymbol{K}\nabla p_2\boldsymbol{n}_e,q\rangle_e$, to be well-defined on each segment e of $\mathcal{E}_1^h$, respectively $\mathcal{E}_2^h$, for any polynomial functions $\boldsymbol{v}$ and q. By virtue of Green's formula (1.13), respectively (1.12), such dualities are well-defined on the boundary ∂E of any triangle E of $\mathcal{E}_1^h$, respectively $\mathcal{E}_2^h$, but this is not necessarily true on individual segments of $\mathcal{E}_1^h$, respectively, $\mathcal{E}_2^h$, because this property requires a little more regularity. Since this is a delicate question, we assume that this regularity holds.

Proposition 3.1. *Let $(\boldsymbol{u}, p_1, p_2)$ be a solution to (1.1)–(1.9), sufficiently smooth as above. Then, $(\boldsymbol{u}, p_1, p_2)$ satisfies equations (3.11) and (3.12).*

Proof. Multiply (1.1) by a test function $\boldsymbol{v} \in \boldsymbol{X}^h$, apply Green's formula (1.13) over each element E and sum over all elements in $\mathcal{E}_1^h$:

$$2\mu \sum_{E\in\mathcal{E}_1^h} (\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_E - \sum_{E\in\mathcal{E}_1^h} (p_1, \nabla\cdot\boldsymbol{v})_E + \sum_{E\in\mathcal{E}_1^h} \langle(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_E, \boldsymbol{v}\rangle_{\partial E}$$
$$+ \sum_{E\in\mathcal{E}_1^h} (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_E = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1}.$$

The above regularity assumption on $\boldsymbol{u}$ and p_1 and the boundary condition (1.4) imply that

$$2\mu \sum_{E\in\mathcal{E}_1^h} (\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_E - \sum_{E\in\mathcal{E}_1^h} (p_1, \nabla\cdot\boldsymbol{v})_E + \sum_{E\in\mathcal{E}_1^h} (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_E + \sum_{e\in\Gamma_1^h\cup\Gamma_1} \langle\{(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_e\}, [\boldsymbol{v}]\rangle_e$$
$$+ \sum_{e\in\Gamma_{12}} \langle(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_e + 2\epsilon_1\mu \sum_{e\in\Gamma_1^h\cup\Gamma_1} (\{D(\boldsymbol{v})\boldsymbol{n}_e\}, [\boldsymbol{u}])_e = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1}. \quad (3.13)$$

Similarly, multiply (1.3) by a test function $q_2 \in M_2^h$, apply Green's formula (1.12) over each element E, sum over all elements in $\mathcal{E}_2^h$ and use the boundary conditions (1.5), (1.6) and the above regularity assumption on p_2:

$$\sum_{E\in\mathcal{E}_2^h} (\boldsymbol{K}\nabla p_2, \nabla q)_E - \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} \langle\{\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_e\}, [q]\rangle_e$$
$$+\epsilon_2 \sum_{e\in\Gamma_2^h\cup\Gamma_{2D}} (\{\boldsymbol{K}\nabla q\cdot\boldsymbol{n}_e\}, [p_2])_e + \sum_{e\in\Gamma_{12}} \langle\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{12}, q\rangle_{\Gamma_{12}} = (f_2, q)_{\Omega_2}. \quad (3.14)$$

Then add (3.13) and (3.14); this gives

$$a_S(\boldsymbol{u}, \boldsymbol{v}) + b_S(\boldsymbol{v}, p_1) + a_D(p_2, q) + c_{NS}(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})$$
$$+ \sum_{e\in\Gamma_{12}} \langle(-2\mu\boldsymbol{D}(\boldsymbol{u}) + p_1\boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_e + \sum_{e\in\Gamma_{12}} \langle\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{12}, q\rangle_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}.$$

In this equation, the terms on the interface Γ_{12} are the same as in equation (1.30). Those terms are handled by the same argument as in the proof of Lemma 1.1, and we obtain (3.11). The second equation (3.12) is simply obtained by multiplying (1.2) by a test function $q \in M_1^h$, by integrating over one element E, summing over all elements in $\mathcal{E}_1^h$ and by using (1.4) and the regularity of $\boldsymbol{u}$. ☐

4. Properties of the discrete spaces and forms

For analyzing problem (3.11), (3.12), we require on one hand that the discrete spaces satisfy the analogues of (1.21)–(1.25), i.e. the analogues of Poincaré's, Korn's, Sobolev and trace inequalities in suitable norms, and on the other hand that the discrete forms a_S, a_D, c_{NS} satisfy adequate continuity and coercivity properties and the form b_S satisfies an inf-sup condition. All of these, except possibly the trace theorems (1.24) have been established previously.

4.1. **Poincaré, Korn, Sobolev and trace inequalities**

Let us start with the trace inequalities satisfied by the components of the functions of $\boldsymbol{X}^h$ on the interface Γ_{12}. In fact, they are valid for any function z_h in the space :

$$\Theta_h = \{z_h\,;\, \forall E \in \mathcal{E}_1^h, z_h|_E \in I\!P_k(E)\}\,,$$

where $k \geq 1$ is an integer that may vary from one element to the next. The proof is based on a comparison with the Crouzeix-Raviart elements of degree one, an idea used by Brenner in [6] and by Girault, Rivière & Wheeler in [19]. Indeed, since the Crouzeix-Raviart elements of degree one (cf. Crouzeix & Raviart [12]), i.e. the functions in the space

$$CR^h = \{\theta_h\,;\, \forall E \in \mathcal{E}_1^h, \theta_h|_E \in I\!P_1(E)\,,\, \forall e \in \Gamma_h \cup \Gamma_1, \int_e [\theta_h] = 0\}\,,$$

satisfy the trace inequality (cf. Girault & Wheeler [20] or Bernardi & Girault [3]) for any real number $r \geq 2$:

$$\forall \theta_h \in CR^h\,,\ \|\theta_h\|_{L^r(\Gamma_{12})} \leq C(r) |\!|\!| \nabla\, \theta_h |\!|\!|_{L^2(\Omega_1)}\,, \tag{4.1}$$

with a constant $C(r)$, independent of h, we establish the trace inequality (4.11) by transforming z_h into a Crouzeix-Raviart element. This is achieved in two steps: first, we reduce the degree of z_h by interpolating it with the Lagrange interpolation operator I_h of degree one *defined independently in each E*. Next we transform $I_h(z_h)$ into an element of CR^h by removing the mean-value of its jump across each edge of $\Gamma_h \cup \Gamma_1$, as is done in [18]. More precisely, let $\boldsymbol{a}_i$, $i = 1,2,3$ denote the vertices of E, then in E, $I_h(z_h)$ is the polynomial of $I\!P_1$ defined by $I_h(z_h)(\boldsymbol{a}_i) = z_h(\boldsymbol{a}_i)$. Note that, in general, $I_h(z_h)$ is not continuous across interelement boundaries. Finally, we define $CR(z_h) \in CR^h$ by:

$$CR(z_h) = I_h(z_h) - \sum_{e \in \Gamma_h \cup \Gamma_1} \frac{1}{|e|}\left(\int_e [I_h(z_h)]_e\right)\lambda_e\,, \tag{4.2}$$

where for any e in $\Gamma_h \cup \Gamma_1$, λ_e is the basis function defined as follows. Let $\boldsymbol{b}_e$ be the midpoint of e and let E be the triangle of $\mathcal{E}_1^h$ adjacent to e such that $\boldsymbol{n}_e$ is exterior to E. Then $\lambda_e|_{E'} = 0$ for all $E' \neq E$, $\lambda_e|_E \in I\!P_1$, $\lambda_e(\boldsymbol{b}_e) = 1$ and $\lambda_e(\boldsymbol{b}_{e'}) = 0$ if $e' \neq e$. It is easy to check that $CR(z_h)$ belongs to CR^h. The trace theorem follows from the following set of lemmas. The first lemma estimates the broken and jump norms of $I_h(z_h)$. In the rest of the section, we denote by $\hat{E}$ the reference element, i.e. the triangle with vertices $(0,0), (1,0)$ and $(0,1)$. For each element $E \in \mathcal{E}_h$, there is an affine mapping F_E from $\hat{E}$ onto E. As usual, we denote $\hat{z} = z \circ F_E$ and $\hat{\nabla}\hat{z}$ the gradient of $\hat{z}$ on $\hat{E}$.

Lemma 4.1. *If $\mathcal{E}_1^h$ satisfies (3.1), then there exists constants C_1 and C_2, independent of h, such that, for all $z_h \in \Theta_h$:*

$$|\!|\!| \nabla\, I_h(z_h) |\!|\!|_{L^2(\Omega_1)} \leq C_1 |\!|\!| \nabla\, z_h |\!|\!|_{L^2(\Omega_1)}\,,$$
$$\left(\sum_{e \in \Gamma_1^h \cup \Gamma_1} \int_e \frac{1}{|e|}[I_h(z_h)]_e^2\right)^{1/2} \leq C_2 |\!|\!| \nabla\, z_h |\!|\!|_{L^2(\Omega_1)} + \left(\sum_{e \in \Gamma_1^h \cup \Gamma_1} \int_e \frac{1}{|e|}[z_h]_e^2\right)^{1/2}. \tag{4.3}$$

Proof. Let E be an element of $\mathcal{E}_1^h$. We write

$$\|\nabla\, I_h(z_h)\|_{L^2(E)} \leq \|\nabla(I_h(z_h) - z_h)\|_{L^2(E)} + \|\nabla\, z_h\|_{L^2(E)}\,.$$

For the first term, passing to the reference element $\hat{E}$, using the fact that I_h preserves the constant functions and is invariant by affine transformation, the equivalence of norms on the space $\mathbb{P}_k$ and (3.1), we derive:

$$\|\nabla(I_h(z_h) - z_h)\|_{L^2(E)} \le c_1|E|^{1/2}\frac{1}{\rho_E}\|\hat{\nabla}(\hat{I}(\hat{z}) - \hat{z})\|_{L^2(\hat{E})} \le c_2|E|^{1/2}\frac{1}{\rho_E}\|\hat{\nabla}\,\hat{z}\|_{L^2(\hat{E})} \le c_3\gamma\|\nabla\, z_h\|_{L^2(E)}\,.$$

This proves the first part of (4.3).

For the second part, we write:

$$\left(\sum_{e\in\Gamma_1^h\cup\Gamma_1}\int_e \frac{1}{|e|}[I_h(z_h)]^2\right)^{1/2} \le \left(\sum_{e\in\Gamma_1^h\cup\Gamma_1}\int_e \frac{1}{|e|}[I_h(z_h) - z_h]^2\right)^{1/2} + \left(\sum_{e\in\Gamma_1^h\cup\Gamma_1}\int_e \frac{1}{|e|}[z_h]^2\right)^{1/2}. \tag{4.4}$$

Let e be an interior segment adjacent to two elements E_1 and E_2; the case of $e \in \Gamma_1$ is similar and simpler. Then

$$\int_e \frac{1}{|e|}[I_h(z_h) - z_h]^2 \le 2\int_e \frac{1}{|e|}\left((I_h(z_h) - z_h)|_{E_1}^2 + (I_h(z_h) - z_h)|_{E_2}^2\right)\,. \tag{4.5}$$

If $e \in \Gamma_1$, then there is only one element E in this expression. Switching to the reference segment $\hat{e} = [0,1]$, using the finite dimension of the space, the fact that I_h preserves the constant functions and the regularity of $\mathcal{E}_1^h$ given by (3.1), we obtain

$$\int_e \frac{1}{|e|}(I_h(z_h) - z_h)|_E^2 = \int_{\hat{e}}(\hat{I}(\hat{z}) - \hat{z})^2 \le c_1\|\hat{I}(\hat{z}) - \hat{z}\|_{L^2(\hat{E})}^2 \le c_2\|\hat{\nabla}\,\hat{z}\|_{L^2(\hat{E})}^2 \le c_2\gamma^2\|\nabla\, z_h\|_{L^2(E)}^2\,. \tag{4.6}$$

Then the second part of (4.3) follows by summing (4.6) over all segments $e \in \Gamma_1^h \cup \Gamma_1$ and using (4.5) and (4.4). □

Lemma 4.2. *If $\mathcal{E}_1^h$ satisfies (3.1), then for each real number $r \ge 2$, there exists a constant $C(r)$, independent of h, such that*

$$\forall z_h \in \Theta_h,\quad \|z_h - I_h(z_h)\|_{L^r(\Gamma_{12})} \le C(r)h^{1/r}\left(\sum_{e\in\Delta_{12}}\|\nabla\, z_h\|_{L^2(E)}^2\right)^{1/2}, \tag{4.7}$$

where Δ_{12} denotes the union of all elements $E \in \mathcal{E}_1^h$ that are adjacent to Γ_{12}.

Proof. Let e be any segment on Γ_{12}. Switching to the reference segment $\hat{e} = [0,1]$, we obtain as above:

$$\begin{aligned}\|z_h - I_h(z_h)\|_{L^r(e)} &= |e|^{1/r}\|\hat{z} - \hat{I}(\hat{z})\|_{L^r(\hat{e})} \le c_1|e|^{1/r}\|\hat{\nabla}\,\hat{z}\|_{L^2(\hat{e})}\\ &\le c_2|e|^{1/r}\|\hat{\nabla}\,\hat{z}\|_{L^2(\hat{E})} \le c_3|e|^{1/r}\gamma\|\nabla\, z_h\|_{L^2(E)}\,,\end{aligned} \tag{4.8}$$

where $\hat{E}$ is the reference element with side $\hat{e}$ and E is the element of $\mathcal{E}_1^h$ that is adjacent to e. Then (4.7) follows by raising (4.8) to the r-th power, summing over all edges e of Γ_{12} and applying Jensen's inequality that is valid since $r \ge 2$. □

Lemma 4.3. *If $\mathcal{E}_1^h$ satisfies (3.1), then for each real number $r \ge 2$, there exists a constant $C(r)$, independent of h, such that*

$$\forall z_h \in \Theta_h,\quad \|I_h(z_h) - CR(z_h)\|_{L^r(\Gamma_{12})} \le C(r)h^{1/r}\left(\|\!|\nabla\, z_h\|\!|_{L^2(\Omega_1)} + \left(\sum_{e\in\Gamma_1^h\cup\Gamma_1}\int_e \frac{1}{|e|}[z_h]_e^2\right)^{1/2}\right). \tag{4.9}$$

Proof. We have by definition

$$\|I_h(z_h) - CR(z_h)\|_{L^r(\Gamma_{12})} = \left\| \sum_{e' \in \Gamma_1^h \cup \Gamma_1} \left(\int_{e'} \frac{1}{|e'|}[I_h(z_h)]_{e'} \right) \lambda_{e'} \right\|_{L^r(\Gamma_{12})} .$$

For any $e \in \Gamma_{12}$, there are at most two terms in the above sum for which $\lambda_{e'}|_e \neq 0$; they correspond to the two edges, say e_i and e_j, of the triangle E that is adjacent to e. Therefore

$$\|I_h(z_h) - CR(z_h)\|_{L^r(\Gamma_{12})} \leq \left(\sum_{e \in \Gamma_{12}} \int_e \left| \frac{1}{|e_i|}(\int_{e_i} [I_h(z_h)]_{e_i})\lambda_{e_i} + \frac{1}{|e_j|}(\int_{e_j} [I_h(z_h)]_{e_j})\lambda_{e_j} \right|^r \right)^{1/r} .$$

Applying

$$\forall \alpha \geq 0, \beta \geq 0, (\alpha+\beta)^r \leq 2^{r/r'}(\alpha^r + \beta^r) \ , \ \frac{1}{r} + \frac{1}{r'} = 1 \, ,$$

that follows from Cauchy-Schwarz inequality and is valid for any real number $r > 1$, switching to the reference segment $[0,1]$ and applying Cauchy-Schwarz inequality, we have on one segment $e \in \Gamma_{12}$:

$$\begin{aligned} &\int_e \left| \frac{1}{|e_i|}(\int_{e_i} [I_h(z_h)]_{e_i})\lambda_{e_i} + \frac{1}{|e_j|}(\int_{e_j} [I_h(z_h)]_{e_j})\lambda_{e_j} \right|^r \\ &\quad \leq 2^{r/r'}|e|\Big(\big(\int_{e_i} \frac{1}{|e_i|}[I_h(z_h)]^2_{e_i}\big)^{r/2} \|\hat{\lambda}_i\|^r_{L^r(\hat{e})} + \big(\int_{e_j} \frac{1}{|e_j|}[I_h(z_h)]^2_{e_j}\big)^{r/2} \|\hat{\lambda}_j\|^r_{L^r(\hat{e})}\Big) . \end{aligned} \tag{4.10}$$

When summing (4.10) over all segments $e \in \Gamma_{12}$, each segment e_i adjacent to e is counted at most once. Therefore, considering that $\|\hat{\lambda}_i\|^r_{L^r(\hat{e})}$ and $\|\hat{\lambda}_j\|^r_{L^r(\hat{e})}$ take only two constant values independent of h, we obtain (4.9) by applying Jensen's inequality and (4.3). □

It remains to estimate $\|CR(z_h)\|_{L^r(\Gamma_{12})}$. As mentioned above, we know that (4.1) holds. Its proof is based on a particular regularization of θ_h by the Scott-Zhang interpolant (cf. [33]). We refer to [6] and to Crouzeix [10] for a slightly different regularization. Then collecting the above results, we deduce the following trace theorem.

Theorem 4.4. *If $\mathcal{E}_1^h$ satisfies (3.1), then for each real number $r \geq 2$, there exists a constant $C^t(r)$, independent of h, such that*

$$\forall z_h \in \Theta_h, \quad \|z_h\|_{L^r(\Gamma_{12})} \leq C^t(r) \left(\|\!|\nabla z_h|\!\|^2_{L^2(\Omega_1)} + \sum_{e \in \Gamma_1^h \cup \Gamma_1} \int_e \frac{1}{|e|}[z_h]^2_e \right)^{1/2} . \tag{4.11}$$

As far as Poincaré's inequalities are concerned, we have the following proposition established, in a more general setting, by Brenner in [6].

Proposition 4.5. *Let the triangulation $\mathcal{E}^h$ satisfy (3.1). There exist constants $C_5^{(i)}$, independent of h, but depending on $\sigma^i_{\min}$, $i = 1, 2$, such that*

$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \quad \|\boldsymbol{v}\|_{L^2(\Omega_1)} \leq C_5^{(1)} \left(\|\!|\nabla \boldsymbol{v}|\!\|^2_{L^2(\Omega_1)} + J^1(\boldsymbol{v},\boldsymbol{v}) \right)^{1/2} , \tag{4.12}$$

$$\forall q \in M_2^h, \quad \|q\|_{L^2(\Omega_2)} \leq C_5^{(2)} \left(\|\!|\nabla q|\!\|^2_{L^2(\Omega_2)} + J^2(q,q) \right)^{1/2} . \tag{4.13}$$

For Sobolev's inequality, we have the next proposition. The proof is given in [19], when the jump norm runs over all edges of the triangulation. The above approach gives a slightly different proof, valid in more general situations.

Proposition 4.6. *Let the triangulation $\mathcal{E}_1^h$ satisfy (3.1). For each real number $r \geq 2$, there exists a constant $C(r)$, independent of h, but depending on $\sigma^1_{\min}$, such that*

$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \quad \|\boldsymbol{v}\|_{L^r(\Omega_1)} \leq C(r) \left(\|\!|\nabla \boldsymbol{v}|\!\|^2_{L^2(\Omega_1)} + J^1(\boldsymbol{v},\boldsymbol{v}) \right)^{1/2} . \tag{4.14}$$

Of course, (4.14) reduces to (4.12) when $r = 2$.

The following Korn's inequality follows immediately from a result in [7], again established in a more general setting.

Proposition 4.7. *Let the triangulation $\mathcal{E}_1^h$ satisfy (3.1). There exists a constant C_6, independent of h, but depending on $\sigma^1_{\min}$, such that*

$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \quad \|\!|\nabla \boldsymbol{v}|\!\|_{L^2(\Omega_1)} \leq C_6 \|\boldsymbol{v}\|_X . \tag{4.15}$$

Then, as is done by Rivière in [27], by combining the above results, we obtain:

$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \quad \|\boldsymbol{v}\|_{L^2(\Omega_1)} \leq C_7 \|\boldsymbol{v}\|_X, \tag{4.16}$$

$$\forall q \in M_2^h, \quad \|q\|_{L^2(\Omega_2)} \leq C_8 \|q\|_{M_2}. \tag{4.17}$$

We can easily check that $C_7 = C(2)(1 + C_6^2)^{1/2}$ and $C_8 = C_5^{(2)}(1 + \lambda_{\min}^{-1})^{1/2}$.

4.2. Continuity and coercivity of forms a_S, a_D and c_{NS}

The continuity and coercivity of a_S and a_D stated in the next lemma are well-known (cf. for instance [18]).

Lemma 4.8. *Let the triangulation $\mathcal{E}_1^h$ satisfy (3.1). There exist constants C_9 and C_9', independent of h and μ, such that*

$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \quad C_9 \mu \|\boldsymbol{v}\|_X^2 \leq a_S(\boldsymbol{v},\boldsymbol{v}), \tag{4.18}$$

$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \forall \boldsymbol{w} \in \boldsymbol{X}^h, \quad a_S(\boldsymbol{v},\boldsymbol{w}) \leq C_9' \mu \|\boldsymbol{v}\|_X \|\boldsymbol{w}\|_X. \tag{4.19}$$

Similarly, let the triangulation $\mathcal{E}_2^h$ satisfy (3.1). There exist constants C_{10} and C_{10}', independent of h, such that:

$$\forall q \in M_2^h, \quad C_{10} \|q\|_{M_2}^2 \leq a_D(q,q), \tag{4.20}$$

$$\forall q \in M_2^h, \forall r \in M_2^h, \quad a_D(q,r) \leq C_{10}' \|q\|_{M_2} \|r\|_{M_2}. \tag{4.21}$$

When $\epsilon_1 = -1$, respectively $\epsilon_2 = -1$, (4.18), respectively (4.20), holds provided $\sigma_1^{\min}$, respectively $\sigma_2^{\min}$, is large enough.

The form c_{NS} is linear with respect to its arguments except the first one. The following lemma shows that it is bounded independently of this first argument. We skip the proof that can easily be obtained following the proof of Lemma 6.4 in [18].

Lemma 4.9. *Let the triangulation $\mathcal{E}_1^h$ satisfy (3.1). There exists a constant C and for any real number $r > 2$, a constant C_r, independent of h, but dependent on $\sigma_1^{\min}$, such that for all $\boldsymbol{u} \in \boldsymbol{V}^h$, $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{X}^h$*

$$\begin{aligned} c_{NS}(\boldsymbol{z},\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) &\leq C(\|\!|\nabla \boldsymbol{v}|\!\|^2_{L^2(\Omega_1)} + J^1(\boldsymbol{v},\boldsymbol{v}))^{1/2} \|\boldsymbol{u}\|_{L^4(\Omega_1)} \|\boldsymbol{w}\|_{L^4(\Omega_1)} \\ &+ C_r h^{2/r} (\|\!|\nabla \boldsymbol{u}|\!\|^2_{L^2(\Omega_1)} + J^1(\boldsymbol{u},\boldsymbol{u}))^{1/2} (\|\!|\nabla \boldsymbol{v}|\!\|^2_{L^2(\Omega_1)} + J^1(\boldsymbol{v},\boldsymbol{v}))^{1/2} (\|\!|\nabla \boldsymbol{w}|\!\|^2_{L^2(\Omega_1)} + J^1(\boldsymbol{w},\boldsymbol{w}))^{1/2}. \end{aligned} \tag{4.22}$$

Using (3.5), (4.14) and (4.15), Lemma 4.9 yields that there is a constant N independent of h and μ such that:

$$\sup_{\substack{\boldsymbol{u} \in \boldsymbol{V}^h \\ \boldsymbol{z}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}^h}} \frac{c_{NS}(\boldsymbol{z}, \boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w})}{\|\boldsymbol{u}\|_X \|\boldsymbol{v}\|_X \|\boldsymbol{w}\|_X} \leq N. \tag{4.23}$$

We prove next that in a certain sense, the form c_{NS} is Lipschitz-continuous with respect to its first argument. We first need a preliminary result given below.

Lemma 4.10. *If $\mathcal{E}_1^h$ satisfies (3.1), there exists a constant C, independent of h, but dependent on $\sigma_1^{\min}$, such that for all edges $e \in \Gamma_1^h \cup \Gamma_1$ and for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}^h$, we have*

$$\begin{aligned} \Big| \sum_{E \in \mathcal{E}_1^h} (\{\boldsymbol{u}\} \cdot \boldsymbol{n}_E (\boldsymbol{u}^{\text{ext}} - \boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{u},\boldsymbol{v}) \cap e} - \sum_{E \in \mathcal{E}_1^h} (\{\boldsymbol{u}\} \cdot \boldsymbol{n}_E (\boldsymbol{u}^{\text{ext}} - \boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{v},\boldsymbol{u}) \cap e} \Big| \\ \leq C \|\boldsymbol{u} - \boldsymbol{v}\|_{L^2(\Delta_e)} \frac{\sigma_e}{|e|} \|[\boldsymbol{u}]\|_{L^2(e)} \|[\boldsymbol{w}]\|_{L^2(e)}, \end{aligned} \tag{4.24}$$

where

$$\partial E_-^*(\boldsymbol{u}, \boldsymbol{v}) = \{\boldsymbol{x} \in \partial E\,;\, \{\boldsymbol{u}(\boldsymbol{x})\} \cdot \boldsymbol{n}_E < 0 \text{ and } \{\boldsymbol{v}(\boldsymbol{x})\} \cdot \boldsymbol{n}_E \neq 0\},$$

and Δ_e is the union of elements of $\mathcal{E}_1^h$ adjacent to e.

Proof. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}^h$ and define the set

$$\partial E_-(\boldsymbol{u}, -\boldsymbol{v}) = \{\boldsymbol{x} \in \partial E\,;\, \{\boldsymbol{u}(\boldsymbol{x})\} \cdot \boldsymbol{n}_E < 0 \text{ and } \{\boldsymbol{v}(\boldsymbol{x})\} \cdot \boldsymbol{n}_E > 0\}.$$

Consider first an edge e in Γ_1^h. The proof is based on the identity (see formula (5.32), Chapter IV, [17]):

$$\begin{aligned} A &:= \sum_{E \in \mathcal{E}_1^h} (\{\boldsymbol{u}\} \cdot \boldsymbol{n}_E (\boldsymbol{u}^{\text{ext}} - \boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{u},\boldsymbol{v}) \cap e} - \sum_{E \in \mathcal{E}_1^h} (\{\boldsymbol{u}\} \cdot \boldsymbol{n}_E (\boldsymbol{u}^{\text{ext}} - \boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{v},\boldsymbol{u}) \cap e} \\ &= - \sum_{E \in \mathcal{E}_1^h} (\{\boldsymbol{u}\} \cdot \boldsymbol{n}_E (\boldsymbol{u}^{\text{ext}} - \boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{ext}} - \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{u},-\boldsymbol{v}) \cap e}\,, \end{aligned}$$

and on the remark that on $\partial E_-(\boldsymbol{u}, -\boldsymbol{v})$, we have

$$|\{\boldsymbol{u}\} \cdot \boldsymbol{n}_E| < |\{\boldsymbol{u} - \boldsymbol{v}\} \cdot \boldsymbol{n}_E|\,. \tag{4.25}$$

Therefore

$$|A| \leq \|\{\boldsymbol{u} - \boldsymbol{v}\} \cdot \boldsymbol{n}_E\|_{L^\infty(e)} \|[\boldsymbol{u}]\|_{L^2(e)} \|[\boldsymbol{w}]\|_{L^2(e)}. \tag{4.26}$$

As $\boldsymbol{u}$ and $\boldsymbol{v}$ belong to a finite-dimensional space in each element E, we easily deduce from (3.1) that

$$\frac{|e|}{\sigma_e} \|\{\boldsymbol{u} - \boldsymbol{v}\}\|_{L^\infty(e)} \leq C \|\boldsymbol{u} - \boldsymbol{v}\|_{L^2(\Delta_e)}\,, \tag{4.27}$$

where Δ_e is the union of all elements of $\mathcal{E}_1^h$ adjacent to e, and C is a constant that depends on $\sigma_1^{\min}$, but not on h. Then (4.24) follows easily from (4.26) and (4.27).

Next, we prove the result for an edge e in Γ_1. In this case, we easily obtain that

$$A = - \sum_{E \in \mathcal{E}_1^h} (\boldsymbol{u} \cdot \boldsymbol{n}_E \boldsymbol{u}, \boldsymbol{w})_{\partial E_-(\boldsymbol{u},-\boldsymbol{v}) \cap e} + \sum_{E \in \mathcal{E}_1^h} (\boldsymbol{u} \cdot \boldsymbol{n}_E \boldsymbol{u}, \boldsymbol{w})_{\partial E_-(\boldsymbol{v},-\boldsymbol{u}) \cap e}.$$

The proof is concluded as above by noting that (4.25) holds also on $\partial E_-(\boldsymbol{v}, -\boldsymbol{u})$. ∎

Proposition 4.11. *If $\mathcal{E}_1^h$ satisfies (3.1), there exists a constant C^ℓ, independent of h, but dependent on $\sigma_1^{\min}$, such that for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}^h$, we have*

$$\Big|\ell_h(\boldsymbol{u},\boldsymbol{u};\boldsymbol{u},\boldsymbol{w}) - \ell_h(\boldsymbol{v},\boldsymbol{v};\boldsymbol{v},\boldsymbol{w})\Big| \le C^\ell \|\boldsymbol{u}-\boldsymbol{v}\|_X \|\boldsymbol{w}\|_X (\|\boldsymbol{u}\|_X + \|\boldsymbol{v}\|_X). \tag{4.28}$$

Proof. We first note that for any $\boldsymbol{u} \in \boldsymbol{X}^h$, on any fixed edge e, we have either $\{\boldsymbol{u}\}\cdot\boldsymbol{n}_e \equiv 0$ or $\{\boldsymbol{u}\}\cdot\boldsymbol{n}_e \neq 0$ except possibly on a finite number of points, in which case $\{\boldsymbol{u}\}\cdot\boldsymbol{n}_e \neq 0$ a.e.. Therefore, $\Gamma_1^h \cup \Gamma_1$ can be partitioned into

$$\Gamma_1^h \cup \Gamma_1 = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3,$$

with

$$\begin{aligned}
\mathcal{F}_1 &= \{e\,;\, \{\boldsymbol{u}\}\cdot\boldsymbol{n}_e = 0 \text{ on } e \quad \text{and } \{\boldsymbol{v}\}\cdot\boldsymbol{n}_e \neq 0 \text{ on } e \quad \text{a.e.}\},\\
\mathcal{F}_2 &= \{e\,;\, \{\boldsymbol{v}\}\cdot\boldsymbol{n}_e = 0 \text{ on } e \quad \text{and } \{\boldsymbol{u}\}\cdot\boldsymbol{n}_e \neq 0 \text{ on } e \quad \text{a.e.}\},\\
\mathcal{F}_3 &= \Gamma_1^h \cup \Gamma_1 \setminus (\mathcal{F}_1 \cup \mathcal{F}_2).
\end{aligned}$$

We then have

$$\begin{aligned}
\ell_h(\boldsymbol{u},\boldsymbol{u};\boldsymbol{u},\boldsymbol{w}) - \ell_h(\boldsymbol{v},\boldsymbol{v};\boldsymbol{v},\boldsymbol{w}) &= \sum_{i=1}^{3}\sum_{e\in\mathcal{F}_i}\sum_{E\in\mathcal{E}_1^h} (\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}} - \boldsymbol{u}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{u})\cap e}\\
&\quad - \sum_{i=1}^{3}\sum_{e\in\mathcal{F}_i}\sum_{E\in\mathcal{E}_1^h} (\{\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{v}^{\text{int}} - \boldsymbol{v}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}.
\end{aligned}$$

We now consider each subset $\mathcal{F}_i$ separately:

$$\begin{aligned}
&\sum_{e\in\mathcal{F}_1}\sum_{E\in\mathcal{E}_1^h} \big((\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}} - \boldsymbol{u}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{u})\cap e} - (\{\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{v}^{\text{int}} - \boldsymbol{v}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}\big)\\
&\qquad = \sum_{e\in\mathcal{F}_1}\sum_{E\in\mathcal{E}_1^h} (\{\boldsymbol{u}-\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{v}^{\text{int}} - \boldsymbol{v}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}\\
&\qquad \le C(J^1(\boldsymbol{v},\boldsymbol{v}))^{1/2}\|\boldsymbol{u}-\boldsymbol{v}\|_{L^4(\Omega_1)}\|\boldsymbol{w}\|_{L^4(\Omega_1)},
\end{aligned}$$

similarly

$$\begin{aligned}
&\sum_{e\in\mathcal{F}_2}\sum_{E\in\mathcal{E}_1^h} \big((\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}} - \boldsymbol{u}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{u})\cap e} - (\{\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{v}^{\text{int}} - \boldsymbol{v}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}\big)\\
&\qquad \le C(J^1(\boldsymbol{u},\boldsymbol{u}))^{1/2}\|\boldsymbol{u}-\boldsymbol{v}\|_{L^4(\Omega_1)}\|\boldsymbol{w}\|_{L^4(\Omega_1)};
\end{aligned}$$

finally

$$\begin{aligned}
&\sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_1^h} \big((\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}} - \boldsymbol{u}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{u})\cap e} - (\{\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{v}^{\text{int}} - \boldsymbol{v}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}\big)\\
&= \sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_1^h} \big((\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}} - \boldsymbol{u}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{u})\cap e} - (\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}} - \boldsymbol{u}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}\big)\\
&\qquad + \sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_1^h} (\{\boldsymbol{u}-\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}} - \boldsymbol{u}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}\\
&\qquad + \sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_1^h} (\{\boldsymbol{v}\}\cdot\boldsymbol{n}_E((\boldsymbol{u}^{\text{int}} - \boldsymbol{v}^{\text{int}}) - (\boldsymbol{u}^{\text{ext}} - \boldsymbol{v}^{\text{int}})), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}.
\end{aligned} \tag{4.29}$$

The first line in the right-hand side of (4.29) is equivalently rewritten as

$$\sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_1^h}\Big((\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\rm int}-\boldsymbol{u}^{\rm ext}),\boldsymbol{w}^{\rm int})_{\partial E_-^*(\boldsymbol{u},\boldsymbol{v})\cap e}-(\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\rm int}-\boldsymbol{u}^{\rm ext}),\boldsymbol{w}^{\rm int})_{\partial E_-^*(\boldsymbol{v},\boldsymbol{u})\cap e}\Big)$$

and in view of Lemma 4.10, is bounded by:

$$C\|\boldsymbol{u}-\boldsymbol{v}\|_{L^2(\Omega_1)}\sum_{e\in\mathcal{F}_3}\frac{\sigma_e}{|e|}\|[\boldsymbol{u}]\|_{L^2(e)}\|[\boldsymbol{w}]\|_{L^2(e)}\le C\|\boldsymbol{u}-\boldsymbol{v}\|_{L^2(\Omega_1)}J_1(\boldsymbol{u},\boldsymbol{u})^{1/2}J_1(\boldsymbol{w},\boldsymbol{w})^{1/2}.$$

The second and third lines in the right-hand side of (4.29) are easily bounded respectively by

$$C\|\boldsymbol{u}-\boldsymbol{v}\|_{L^4(\Omega_1)}J_1(\boldsymbol{u},\boldsymbol{u})^{1/2}\|\boldsymbol{w}\|_{L^4(\Omega_1)}\quad\text{and}\quad C\|\boldsymbol{v}\|_{L^4(\Omega_1)}J_1(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v})^{1/2}\|\boldsymbol{w}\|_{L^4(\Omega_1)}.$$

Then (4.28) follows from the above bounds, (4.14) and (4.15). □

The following lemma states the positivity of form c_{NS}.

Lemma 4.12. *The nonlinear form c_{NS} satisfies the following property:*

$$\forall\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{X}^h,\quad c_{NS}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v},\boldsymbol{v})=\frac{1}{2}\sum_{E\in\mathcal{E}_1^h}\||\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E|^{1/2}(\boldsymbol{v}^{\rm int}-\boldsymbol{v}^{\rm ext})\|^2_{L^2(\partial E_-(\boldsymbol{u})\setminus\partial\Omega_1)}$$
$$+\||\boldsymbol{u}\cdot\boldsymbol{n}_{\Omega_1}|^{1/2}\boldsymbol{v}\|^2_{L^2(\Gamma_{1-}(\boldsymbol{u}))}+\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{n}_{12},\boldsymbol{v}\cdot\boldsymbol{v})_{\Gamma_{12}},\tag{4.30}$$

where

$$\Gamma_{1-}(\boldsymbol{u})=\{\boldsymbol{x}\in\Gamma_1\,;\,\{\boldsymbol{u}(\boldsymbol{x})\}\cdot\boldsymbol{n}_{\Omega_1}<0\}.$$

Proof. Integrating by parts the first term in the definition of c_{NS} in (3.10), we obtain:

$$\forall\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{X}^h,\quad c_{NS}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})=-\sum_{E\in\mathcal{E}_1^h}(\boldsymbol{u}\cdot\nabla\boldsymbol{w},\boldsymbol{v})_E-\frac{1}{2}\sum_{E\in\mathcal{E}_1^h}(\nabla\cdot\boldsymbol{u},\boldsymbol{v}\cdot\boldsymbol{w})_E$$
$$+\sum_{E\in\mathcal{E}_1^h}(|\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E|\boldsymbol{v}^{\rm ext},\boldsymbol{w}^{\rm ext}-\boldsymbol{w}^{\rm int})_{\partial E_-(\boldsymbol{u})\setminus\partial\Omega_1}+\frac{1}{2}\sum_{e\in\Gamma_1^h}([\boldsymbol{u}]\cdot\boldsymbol{n}_e,\{\boldsymbol{v}\cdot\boldsymbol{w}\})_e$$
$$+\frac{1}{2}\sum_{e\in\Gamma_1}(|\boldsymbol{u}\cdot\boldsymbol{n}_e|,\boldsymbol{v}\cdot\boldsymbol{w})_e+\sum_{e\in\Gamma_{12}}(\boldsymbol{u}\cdot\boldsymbol{n}_{12},\boldsymbol{v}\cdot\boldsymbol{w})_e.$$

Therefore, choosing $\boldsymbol{v}=\boldsymbol{w}$ in (3.10) and in the above equation, and adding the resulting equations yields:

$$2c_{NS}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v},\boldsymbol{v})=\sum_{E\in\mathcal{E}_1^h}\||\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E|^{1/2}(\boldsymbol{v}^{\rm int}-\boldsymbol{v}^{\rm ext})\|^2_{L^2(\partial E_-(\boldsymbol{u})\setminus\partial\Omega_1)}+2(|\boldsymbol{u}\cdot\boldsymbol{n}_{\Omega_1}|,\boldsymbol{v}\cdot\boldsymbol{v})_{\Gamma_{1-}(\boldsymbol{u})}+(\boldsymbol{u}\cdot\boldsymbol{n}_{12},\boldsymbol{v}\cdot\boldsymbol{v})_{\Gamma_{12}},$$

whence (4.30). □

4.3. An inf-sup condition

In this section, we prove an inf-sup condition for the form b_S.

Proposition 4.13. *Let the triangulation $\mathcal{E}_1^h$ satisfy (3.1). There is a constant $\beta^* > 0$ independent of h such that*

$$\inf_{q\in M_1^h} \sup_{\boldsymbol{v}\in \boldsymbol{X}^h} \frac{b_S(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_X \|q\|_{M_1}} \geq \beta^*. \tag{4.31}$$

Proof. Clearly, it is equivalent to showing that, for any given $q \in M_1^h$, there is a function $\boldsymbol{v} \in \boldsymbol{X}^h$ and positive constants γ_1, γ_2 such that

$$b_S(\boldsymbol{v},q) \geq \gamma_1 \|q\|_{M_1}^2 \quad , \quad \|\boldsymbol{v}\|_X \leq \gamma_2 \|q\|_{M_1}.$$

Fix $q \in M_1^h$ and let $\tilde{q} \in L_0^2(\Omega)$ and $\tilde{\boldsymbol{v}} \in H_0^1(\Omega)^2$ be defined as in the proof of Lemma 1.2. Let $\boldsymbol{v} = R_h(\tilde{\boldsymbol{v}}|_{\Omega_1})$ where $R_h \in \mathcal{L}(H^1(\Omega_1)^2; \boldsymbol{X}^h)$ is the Raviart-Thomas operator [17,26] satisfying the following properties

$$\forall \boldsymbol{v} \in H^1(\Omega_1)^2, \quad \forall E \in \mathcal{E}_1^h, \quad \forall q \in I\!P_{k_1-1}(E), \quad (q, \nabla\cdot(R_h(\boldsymbol{v}) - \boldsymbol{v}))_E = 0, \tag{4.32}$$

$$\forall \boldsymbol{v} \in H^1(\Omega_1)^2, \quad \forall e \in \Gamma_h^1 \cup \Gamma_1, \quad \forall q \in I\!P_{k_1-1}(e), \quad (q, (R_h(\boldsymbol{v}) - \boldsymbol{v})\cdot \boldsymbol{n}_e)_e = 0, \tag{4.33}$$

$$\forall \boldsymbol{v} \in H^1(\Omega_1)^2, \quad \forall E \in \mathcal{E}_1^h, \quad |\boldsymbol{v} - R_h(\boldsymbol{v})|_{H^1(E)} + h_E^{-1}\|\boldsymbol{v} - R_h(\boldsymbol{v})\|_{L^2(E)} \leq C|\boldsymbol{v}|_{H^1(E)}. \tag{4.34}$$

¿From (1.39), properties (4.32) and (4.33), we can easily obtain:

$$b_S(\boldsymbol{v},q) = \|q\|_{L^2(\Omega_1)}^2.$$

Next, we bound $\|\boldsymbol{v}\|_X$. We have from (4.34) and (1.39)

$$\begin{aligned}\sum_{E\in\mathcal{E}_1^h} \|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(E)}^2 &\leq \sum_{E\in\mathcal{E}_1^h} \|\nabla \boldsymbol{v}\|_{L^2(E)}^2 \leq 2\sum_{E\in\mathcal{E}_1^h} \|\nabla \tilde{\boldsymbol{v}}\|_{L^2(E)}^2 + 2\sum_{E\in\mathcal{E}_1^h} \|\nabla(\boldsymbol{v} - \tilde{\boldsymbol{v}})\|_{L^2(E)}^2 \\ &\leq C\|\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega_1)}^2 \leq \frac{C}{\kappa^2}\|\tilde{q}\|_{L^2(\Omega)}^2 \leq \frac{C}{\kappa^2}\frac{|\Omega|}{|\Omega_2|}\|q\|_{L^2(\Omega_1)}^2.\end{aligned}$$

The jump term is bounded using property (4.34) and a trace theorem:

$$J^1(\boldsymbol{v},\boldsymbol{v}) = J^1(\boldsymbol{v}-\tilde{\boldsymbol{v}}, \boldsymbol{v}-\tilde{\boldsymbol{v}}) \leq C\|\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega_1)}^2 \leq \frac{C}{\kappa^2}\frac{|\Omega|}{|\Omega_2|}\|q\|_{L^2(\Omega_1)}^2.$$

Thus, there is a constant C that only depends on Ω and Ω_2 such that

$$\|\boldsymbol{v}\|_X \leq C\|q\|_{L^2(\Omega_1)}.$$

This concludes the proof. □

5. Existence and uniqueness of numerical solution

As in the weak formulation, we introduce an equivalent numerical scheme by restricting the space of velocities to discrete divergence-free velocities:

$$\boldsymbol{V}^h = \{\boldsymbol{v} \in \boldsymbol{X}^h : \forall q \in M_1^h, \quad b_S(\boldsymbol{v},q) = 0\}.$$

Scheme (3.11), (3.12) then becomes: find $(\boldsymbol{U}, P_2) \in \boldsymbol{V}^h \times M_2^h$ such that

$$\begin{aligned}&\forall \boldsymbol{v} \in \boldsymbol{X}^h, \forall q \in M_2^h, \quad a_S(\boldsymbol{U},\boldsymbol{v}) + a_D(P_2,q) + c_{NS}(\boldsymbol{U},\boldsymbol{U};\boldsymbol{U},\boldsymbol{v}) \\ &+(P_2 - \frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{U}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{U}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} \\ &= (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q)_{\Omega_2}.\end{aligned} \tag{5.1}$$

Clearly, if $(\boldsymbol{U}, P_1, P_2)$ is a solution to (3.11), (3.12), then $(\boldsymbol{U}, P_2)$ is also a solution of (5.1). Conversely, if $(\boldsymbol{U}, P_2)$ solves (5.1), it is easy to prove that by virtue of Proposition 4.13, there exists a unique $P_1 \in M_1^h$ such that $(\boldsymbol{U}, P_1, P_2)$ solves (3.11), (3.12). Therefore, it suffices to show that (5.1) has a solution $(\boldsymbol{U}, P_2)$.

Theorem 5.1. *Let the triangulation $\mathcal{E}_1^h$ satisfy (3.1). Problem (5.1) has at least one solution $(\boldsymbol{U}, P_2)$ and all solutions $(\boldsymbol{U}, P_2)$ satisfy the boound:*

$$(\mu \|\boldsymbol{U}\|_X^2 + \|P_2\|_{M_2}^2)^{1/2} \le \tilde{C}, \quad \textit{with} \quad \tilde{C} = \left(\frac{1}{\min(C_9, C_{10})} \left(\frac{C_7^2}{C_9 \mu} \|\boldsymbol{f}_1\|_{L^2(\Omega_1)}^2 + \frac{C_8^2}{C_{10}} \|f_2\|_{L^2(\Omega_2)}^2 \right) \right)^{1/2}. \tag{5.2}$$

Proof. The proof is similar to that of Proposition 2.1: let $\boldsymbol{W}^h = \boldsymbol{V}^h \times M_2^h$ and let the mapping $\mathcal{F}^h : \boldsymbol{W}^h \mapsto \boldsymbol{W}^h$ be defined for all $(\boldsymbol{v}, q) \in \boldsymbol{W}^h$ as follows:

$$\begin{aligned}\forall (\boldsymbol{w}, r) \in \boldsymbol{W}^h, \quad (\mathcal{F}^h(\boldsymbol{v}, q), (\boldsymbol{w}, r))_{W^h} &= a_S(\boldsymbol{v}, \boldsymbol{w}) + c_{NS}(\boldsymbol{v}, \boldsymbol{v}; \boldsymbol{v}, \boldsymbol{w}) + a_D(q, r) - (\boldsymbol{f}_1, \boldsymbol{w})_{\Omega_1} - (f_2, r)_{\Omega_2} \\ &+ (q - \frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v}, \boldsymbol{w} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{v} \cdot \boldsymbol{n}_{12}, r)_{\Gamma_{12}} + \frac{1}{G} (\boldsymbol{v} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{w} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}}.\end{aligned}$$

Here $(\cdot, \cdot)_{W^h}$ is the inner-product on $\boldsymbol{W}^h$ defined by:

$$((\boldsymbol{v}, q), (\boldsymbol{w}, r))_{W^h} = 2\mu \sum_{E \in \mathcal{E}_1^h} (\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{w}))_E + \mu J^1(\boldsymbol{v}, \boldsymbol{w}) + \sum_{E \in \mathcal{E}_2^h} (\boldsymbol{K} \nabla q, \nabla r)_E + J^2(q, r),$$

associated with the norm

$$\|(\boldsymbol{v}, q)\|_{W^h} = (\mu \|\boldsymbol{v}\|_X^2 + \|q\|_{M_2}^2)^{1/2}.$$

Let us derive a lower bound for $(\mathcal{F}^h(\boldsymbol{v}, q), (\boldsymbol{v}, q))_{W^h}$:

$$\begin{aligned}(\mathcal{F}^h(\boldsymbol{v}, q), (\boldsymbol{v}, q))_{W^h} &= a_S(\boldsymbol{v}, \boldsymbol{v}) + c_{NS}(\boldsymbol{v}, \boldsymbol{v}; \boldsymbol{v}, \boldsymbol{v}) + a_D(q, q) - (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} - (f_2, q)_{\Omega_2} \\ &- \frac{1}{2} (\boldsymbol{v} \cdot \boldsymbol{v}, \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G} \|\boldsymbol{v} \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2.\end{aligned}$$

By Lemmas 4.8 and 4.12, we have with the constants of (4.18) and (4.20)

$$(\mathcal{F}^h(\boldsymbol{v}, q), (\boldsymbol{v}, q))_{W^h} \ge C_9 \mu \|\boldsymbol{v}\|_X^2 + C_{10} \|q\|_{M_2}^2 + \frac{1}{G} \|\boldsymbol{v} \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 - (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} - (f_2, q)_{\Omega_2}.$$

By Cauchy-Schwarz inequality and (4.16), (4.17) we have

$$(\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} \le C_7 \|\boldsymbol{f}_1\|_{L^2(\Omega_1)} \|\boldsymbol{v}\|_X \ , \ (f_2, q)_{\Omega_2} \le C_8 \|f_2\|_{L^2(\Omega_2)} \|q\|_{M_2}.$$

This implies that

$$\begin{aligned}(\mathcal{F}^h(\boldsymbol{v}, q), (\boldsymbol{v}, q))_{W^h} &\ge \frac{1}{2} \big(C_9 \mu \|\boldsymbol{v}\|_X^2 + C_{10} \|q\|_{M_2}^2 - \frac{C_7^2}{C_9 \mu} \|\boldsymbol{f}_1\|_{L^2(\Omega_1)}^2 - \frac{C_8^2}{C_{10}} \|f_2\|_{L^2(\Omega_2)}^2 \big) \\ &\ge \frac{1}{2} \left(\min(C_9, C_{10}) \|(\boldsymbol{v}, q)\|_{W^h}^2 - \frac{C_7^2}{C_9 \mu} \|\boldsymbol{f}_1\|_{L^2(\Omega_1)}^2 - \frac{C_8^2}{C_{10}} \|f_2\|_{L^2(\Omega_2)}^2 \right).\end{aligned}$$

Thus, $(\mathcal{F}^h(\boldsymbol{v}, q), (\boldsymbol{v}, q))_{W^h} \ge 0$ provided that

$$\|(\boldsymbol{v}, q)\|_{W^h} = \left(\frac{1}{\min(C_9, C_{10})} \left(\frac{C_7^2}{C_9 \mu} \|\boldsymbol{f}_1\|_{L^2(\Omega_1)}^2 + \frac{C_8^2}{C_{10}} \|f_2\|_{L^2(\Omega_2)}^2 \right) \right)^{1/2}.$$

As in the proof of Proposition 2.1, this yields existence of at least one solution $(\boldsymbol{U}, P)$ of (5.1), that satisfies (5.2). Finally, the same calculation gives that all solutions of (5.1) satisfy (5.2). □

Theorem 5.2. *Under the condition*

$$\mu^2 > \frac{2}{C_9}\frac{1}{\min(C_9, C_{10})}(N + C^\ell + C^t(2)C^t(4)^2(C_6^2+1)^{3/2})\left(C_7^2\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + \mu C_8^2\|f_2\|^2_{L^2(\Omega)}\right), \tag{5.3}$$

the discrete problem (5.1) has a unique solution.

Proof. Assume that $(\boldsymbol{U}^1, P_2^1)$ and $(\boldsymbol{U}^2, P_2^2)$ are two solutions of (5.1) and let $\boldsymbol{W} = \boldsymbol{U}^1 - \boldsymbol{U}^2$ and $Z_2 = P_2^1 - P_2^2$. We have

$$\forall \boldsymbol{v} \in \boldsymbol{X}^h, \quad \forall q \in M_2^h, \quad a_S(\boldsymbol{W}, \boldsymbol{v}) + a_D(Z_2, q) + c_{NS}(\boldsymbol{U}^1, \boldsymbol{U}^1; \boldsymbol{U}^1, \boldsymbol{v}) - c_{NS}(\boldsymbol{U}^2, \boldsymbol{U}^2; \boldsymbol{U}^2, \boldsymbol{v})$$
$$+\frac{1}{G}(\boldsymbol{W}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} + (Z_2 - \frac{1}{2}\boldsymbol{W}\cdot\boldsymbol{U}^1, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\frac{1}{2}\boldsymbol{U}^2\cdot\boldsymbol{W}, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{W}\cdot\boldsymbol{n}_{12}, q_2)_{\Gamma_{12}} = 0.$$

By choosing $\boldsymbol{v} = \boldsymbol{W}$ and $q = Z_2$ and by using Lemma 4.8, we obtain:

$$C_9\mu\|\boldsymbol{W}\|_X^2 + C_{10}\|Z_2\|_{M_2}^2 + \frac{1}{G}\|\boldsymbol{W}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} + c_{NS}(\boldsymbol{U}^1, \boldsymbol{U}^1; \boldsymbol{U}^1, \boldsymbol{W}) - c_{NS}(\boldsymbol{U}^2, \boldsymbol{U}^2; \boldsymbol{U}^2, \boldsymbol{W}) + A \le 0. \tag{5.4}$$

where A is defined and bounded below:

$$A = -\frac{1}{2}(\boldsymbol{W}\cdot\boldsymbol{U}^1, \boldsymbol{W}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{U}^2\cdot\boldsymbol{W}, \boldsymbol{W}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \le \|\boldsymbol{W}\|_{L^4(\Gamma_{12})}\|\boldsymbol{W}\|_{L^2(\Gamma_{12})}(\|\boldsymbol{U}^1\|_{L^4(\Gamma_{12})} + \|\boldsymbol{U}^2\|_{L^4(\Gamma_{12})}).$$

Using (4.11) and the fact that $\sigma^1_{\min} \ge 1$, we can write

$$\|\boldsymbol{W}\|_{L^4(\Gamma_{12})}\|\boldsymbol{W}\|_{L^2(\Gamma_{12})} \le C^t(2)C^t(4)\left(\|\nabla \boldsymbol{W}\|^2_{L^2(\Omega_1)} + J^1(\boldsymbol{W}, \boldsymbol{W})\right).$$

Thus using again (4.11) for $\boldsymbol{U}^1$ and $\boldsymbol{U}^2$, applying (4.15) and finally (5.2), we derive

$$|A| \le C^t(2)C^t(4)^2(C_6^2+1)^{3/2}\frac{1}{\sqrt{\mu}}\|\boldsymbol{W}\|_X^2(\sqrt{\mu}\|\boldsymbol{U}^1\|_X + \sqrt{\mu}\|\boldsymbol{U}^2\|_X)$$
$$\le C^t(2)C^t(4)^2(C_6^2+1)^{3/2}\frac{2\tilde{\mathcal{C}}}{\sqrt{\mu}}\|\boldsymbol{W}\|^2_{X_1}.$$

The two terms involving c_{NS} in (5.4) can be rewritten as:

$$c_{NS}(\boldsymbol{U}^1, \boldsymbol{U}^1; \boldsymbol{U}^1, \boldsymbol{W}) - c_{NS}(\boldsymbol{U}^2, \boldsymbol{U}^2; \boldsymbol{U}^2, \boldsymbol{W}) = d_h(\boldsymbol{U}^1; \boldsymbol{W}, \boldsymbol{W}) + d_h(\boldsymbol{W}; \boldsymbol{U}^2, \boldsymbol{W})$$
$$+ \ell_h(\boldsymbol{U}^1, \boldsymbol{U}^1; \boldsymbol{U}^1, \boldsymbol{W}) - \ell_h(\boldsymbol{U}^2, \boldsymbol{U}^2; \boldsymbol{U}^2, \boldsymbol{W}).$$

Using (4.23) and (5.2), the first two terms are bounded as:

$$\left|d_h(\boldsymbol{U}^1; \boldsymbol{W}, \boldsymbol{W}) + d_h(\boldsymbol{W}; \boldsymbol{U}^2, \boldsymbol{W})\right| \le N\|\boldsymbol{W}\|_X^2(\|\boldsymbol{U}^1\|_X + \|\boldsymbol{U}^2\|_X) \le \frac{2}{\sqrt{\mu}}N\tilde{\mathcal{C}}\|\boldsymbol{W}\|_X^2.$$

¿From Lemma 4.11 and (5.2), we have

$$\left|\ell_h(\boldsymbol{U}^1, \boldsymbol{U}^1; \boldsymbol{U}^1, \boldsymbol{W}) - \ell_h(\boldsymbol{U}^2, \boldsymbol{U}^2; \boldsymbol{U}^2, \boldsymbol{W})\right| \le C^\ell\|\boldsymbol{W}\|_X^2(\|\boldsymbol{U}^1\|_X + \|\boldsymbol{U}^2\|_X) \le \frac{2}{\sqrt{\mu}}C^\ell\tilde{\mathcal{C}}\|\boldsymbol{W}\|_X^2.$$

Combining the above bounds, inequality (5.4) becomes

$$\left(C_9\mu - \frac{2\tilde{\mathcal{C}}}{\sqrt{\mu}}(N + C^\ell + C^t(2)C^t(4)^2(C_6^2+1)^{3/2})\right)\|\boldsymbol{W}\|_X^2 + C_{10}\|Z_2\|_{M_2}^2 + \frac{1}{G}\|\boldsymbol{W}\cdot\boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 \le 0.$$

Thus, we have $\boldsymbol{W} = \mathbf{0}$ and $Z_2 = 0$ if

$$\mu^{3/2} > \frac{2\tilde{\mathcal{C}}}{C_9}(N + C^\ell + C^t(2)C^t(4)^2(C_6^2+1)^{3/2}).$$

This is the case if (5.3) holds. □

6. Error estimates

¿From the inf-sup condition (4.31), we derive an approximation operator $\tilde{R}_h \in \mathcal{L}(H^1(\Omega_1)^2, \boldsymbol{X}^h)$ satisfying:

$$\forall \boldsymbol{v} \in \boldsymbol{X}, \quad \forall q \in M_1^h, \quad b_S(\tilde{R}_h(\boldsymbol{v}) - \boldsymbol{v}, q) = 0, \tag{6.5}$$

$$\forall s \in [1, k_1+1], \quad \forall \boldsymbol{v} \in \boldsymbol{X} \cap H^s(\Omega_1)^2, \quad \|\tilde{R}_h(\boldsymbol{v}) - \boldsymbol{v}\|_X \le C_a h^{s-1}|\boldsymbol{v}|_{H^s(\Omega_1)}, \tag{6.6}$$

with C_a independent of h. Let us recall briefly its construction; the ideas can be found in [32]. It suffices to correct a standard approximation operator, say $\Pi_h \in \mathcal{L}(H^1(\Omega_1)^2, \boldsymbol{X}^h \cap \boldsymbol{X})$ such as the Scott-Zhang interpolant [33] and set

$$\forall \boldsymbol{v} \in \boldsymbol{X}, \; \tilde{R}_h(\boldsymbol{v}) = \Pi_h(\boldsymbol{v}) + \boldsymbol{c}_h(\boldsymbol{v}),$$

where $\boldsymbol{c}_h(\boldsymbol{v}) \in \boldsymbol{X}^h$ satisfies

$$\forall q \in M_1^h, \; b_S(\boldsymbol{c}_h(\boldsymbol{v}), q) = b_S(\boldsymbol{v} - \Pi_h(\boldsymbol{v}), q) = \int_{\Omega_1} \nabla\cdot(\boldsymbol{v} - \Pi_h(\boldsymbol{v}))q. \tag{6.7}$$

By virtue of the Babuška-Brezzi's theory (cf. [4], [8] or [17]), (6.7) has a solution $\boldsymbol{c}_h(\boldsymbol{v})$, unique in $(\boldsymbol{V}^h)^\perp$ (the orthogonal being taken with respect to $\|\cdot\|_X$), and

$$\|\boldsymbol{c}_h(\boldsymbol{v})\|_X \le \frac{1}{\beta^\star}\|\nabla\cdot(\boldsymbol{v} - \Pi_h(\boldsymbol{v}))\|_{L^2(\Omega_1)}.$$

Then (6.6) follows from the approximation properties of Π_h and (6.5) follows from (6.7).

Remark 6.1. For $k = 1, 2, 3$, [18] presents another operator P_h that is based on the non-conforming elements of [12], [16] and [11]. This operator has the advantage of being quasi-local and hence satisfies optimal approximation properties in Banach spaces such as L^r and $W^{1,r}$ for any number $r \ge 2$, without restricting the mesh or the domain. However, this is not fundamental here and therefore we do not use it in the present work.

Theorem 6.2. *Assume that the solution $(\boldsymbol{u}, p_1, p_2)$ of problem (P) belongs to $H^{k_1+1}(\Omega_1)^2 \times H^{k_1}(\Omega_1) \times H^{k_2+1}(\Omega_2)$. In addition, assume that*

$$C_9\mu^{3/2} \ge 2C^t(2)C^t(4)^2(C_6^2+1)^{3/2}\tilde{\mathcal{C}} + \mathcal{C}C_1\left(\sqrt{2}NC_a + \frac{C_4}{\sqrt{2}}C^t(2)C^t(4)(C_6^2+1)\right). \tag{6.8}$$

Then, there exists a constant C independent of h and μ such that

$$\mu\|\boldsymbol{u}-\boldsymbol{U}\|_X^2 + \|p_2 - P_2\|_{M_2}^2 \le Ch^{2k_1}|\boldsymbol{u}|_{H^{k_1+1}(\Omega_1)}^2\left(\frac{\mathcal{C}^2+\tilde{\mathcal{C}}^2}{\mu^2} + \mu + 1\right) + Ch^{2k_2}\left(1+\frac{1}{\mu}\right)\|p_2\|_{H^{k_2+1}(\Omega_2)}^2 + Ch^{2k_1}\frac{1}{\mu}\|p_1\|_{H^{k_1}(\Omega_1)}^2.$$

Proof. Set $\tilde{\boldsymbol{u}} = \tilde{R}_h(\boldsymbol{u})$ and take for $\tilde{p}_1$ and $\tilde{p}_2$ any optimal interpolant of p_1 and p_2 respectively. Let us denote the numerical and approximation errors by

$$\boldsymbol{\chi} = \boldsymbol{U} - \tilde{\boldsymbol{u}}, \quad \xi_1 = P_1 - \tilde{p}_1, \quad \xi_2 = P_2 - \tilde{p}_2,$$
$$\boldsymbol{\zeta} = \boldsymbol{u} - \tilde{\boldsymbol{u}}, \quad \eta_1 = p_1 - \tilde{p}_1, \quad \eta_2 = p_2 - \tilde{p}_2.$$

The error equations are:

$$\begin{aligned}
&\forall \boldsymbol{v} \in \boldsymbol{X}^h, \forall q_2 \in M_2^h, \quad a_S(\boldsymbol{\chi}, \boldsymbol{v}) + a_D(\xi_2, q_2) + b_S(\boldsymbol{v}, \xi_1) + c_{NS}(\boldsymbol{U}, \boldsymbol{U}; \boldsymbol{U}, \boldsymbol{v}) - c_{NS}(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) \\
&+(\xi_2, \boldsymbol{v}\cdot \boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{\chi}\cdot \boldsymbol{n}_{12}, q_2)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{\chi}\cdot \boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{U} - \boldsymbol{u}\cdot\boldsymbol{u}, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \\
&= a_S(\boldsymbol{\zeta}, \boldsymbol{v}) + a_D(\eta_2, q_2) + b_S(\boldsymbol{v}, \eta_1) \\
&+(\eta_2, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{\zeta}\cdot\boldsymbol{n}_{12}, q_2)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{\zeta}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}, \\
&\forall q_1 \in M_1^h, \quad b_S(\boldsymbol{\chi}, q_1) = b_S(\boldsymbol{\zeta}, q_1) = 0,
\end{aligned}$$

where we use (6.5) in the last equality. In the above equations, take $\boldsymbol{v} = \boldsymbol{\chi}, q_1 = \xi_1$, $q_2 = \xi_2$ and use Lemma 4.8:

$$\begin{aligned}
&C_9\mu\|\boldsymbol{\chi}\|_X^2 + C_{10}\|\xi_2\|_{M_2}^2 + \frac{1}{G}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} + c_{NS}(\boldsymbol{U}, \boldsymbol{U}; \boldsymbol{U}, \boldsymbol{\chi}) - c_{NS}(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{u}, \boldsymbol{\chi}) \\
&- \frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{U} - \boldsymbol{u}\cdot\boldsymbol{u}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} = a_S(\boldsymbol{\zeta}, \boldsymbol{\chi}) + a_D(\eta_2, \xi_2) + b_S(\boldsymbol{\chi}, \eta_1) \\
&+ (\eta_2, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{\zeta}\cdot\boldsymbol{n}_{12}, \xi_2)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{\zeta}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}.
\end{aligned} \tag{6.9}$$

As $\boldsymbol{u}$ has no jumps and vanishes on Γ_1, we can write:

$$\begin{aligned}
&c_{NS}(\boldsymbol{U}, \boldsymbol{U}; \boldsymbol{U}, \boldsymbol{\chi}) - c_{NS}(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{u}, \boldsymbol{\chi}) = c_{NS}(\boldsymbol{U}, \boldsymbol{U}; \boldsymbol{U}, \boldsymbol{\chi}) - c_{NS}(\boldsymbol{U}, \boldsymbol{u}; \boldsymbol{u}, \boldsymbol{\chi}) \\
&= c_{NS}(\boldsymbol{U}, \boldsymbol{U}; \boldsymbol{\chi}, \boldsymbol{\chi}) + c_{NS}(\boldsymbol{U}, \boldsymbol{\chi}; \tilde{\boldsymbol{u}}, \boldsymbol{\chi}) - c_{NS}(\boldsymbol{U}, \boldsymbol{\zeta}; \tilde{\boldsymbol{u}}, \boldsymbol{\chi}) - c_{NS}(\boldsymbol{U}, \boldsymbol{u}; \boldsymbol{\zeta}, \boldsymbol{\chi}).
\end{aligned}$$

Likewise,

$$\begin{aligned}
\frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{U} - \boldsymbol{u}\cdot\boldsymbol{u}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} &= \frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{\chi}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\boldsymbol{\chi}\cdot\boldsymbol{u}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \\
&-\frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{\zeta}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{\zeta}\cdot\boldsymbol{u}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}.
\end{aligned}$$

Then substituting into (6.9) and applying Lemma 4.12, we obtain the following error equation

$$\begin{aligned}
&C_9\mu\|\boldsymbol{\chi}\|_X^2 + C_{10}\|\xi_2\|_{M_2}^2 + \frac{1}{G}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} \\
&+ \frac{1}{2}\sum_{E\in\mathcal{E}_1^h} \||\{\boldsymbol{U}\}\cdot\boldsymbol{n}_E|^{1/2}[\boldsymbol{\chi}]\|^2_{L^2(\partial E_-(\boldsymbol{U})\setminus\partial\Omega_1)} + \||\boldsymbol{U}\cdot\boldsymbol{n}_{\Omega_1}|^{1/2}\boldsymbol{\chi}\|^2_{L^2(\Gamma_{1-}(\boldsymbol{U}))} \\
&+ \frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{n}_{12}, \boldsymbol{\chi}\cdot\boldsymbol{\chi})_{\Gamma_{12}} + c_{NS}(\boldsymbol{U}, \boldsymbol{\chi}; \tilde{\boldsymbol{u}}, \boldsymbol{\chi}) - \frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{\chi}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{\chi}\cdot\boldsymbol{u}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \\
&= c_{NS}(\boldsymbol{U}, \boldsymbol{\zeta}; \tilde{\boldsymbol{u}}, \boldsymbol{\chi}) + c_{NS}(\boldsymbol{U}, \boldsymbol{u}; \boldsymbol{\zeta}, \boldsymbol{\chi}) + a_S(\boldsymbol{\zeta}, \boldsymbol{\chi}) + a_D(\eta_2, \xi_2) + b_S(\boldsymbol{\chi}, \eta_1) \\
&- \frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{\zeta}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{\zeta}\cdot\boldsymbol{u}, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \\
&+ (\eta_2, \boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{\zeta}\cdot\boldsymbol{n}_{12}, \xi_2)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{\zeta}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}.
\end{aligned} \tag{6.10}$$

We now derive upper bounds for the terms on the third line of (6.10). Using (4.11), (4.15) and (5.2), we have

$$\frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{n}_{12},\boldsymbol{\chi}\cdot\boldsymbol{\chi})_{\Gamma_{12}} \le \frac{C^t(2)C^t(4)^2}{2}(C_6^2+1)^{3/2}\|\boldsymbol{U}\|_X\|\boldsymbol{\chi}\|_X^2$$
$$\le \frac{C^t(2)C^t(4)^2}{2}(C_6^2+1)^{3/2}\frac{\tilde{\mathcal{C}}}{\sqrt{\mu}}\|\boldsymbol{\chi}\|_X^2. \tag{6.11}$$

Using (4.23), the fact that $\boldsymbol{\chi}\in\boldsymbol{V}^h$ by (6.5), we have

$$c_{NS}(\boldsymbol{U},\boldsymbol{\chi};\tilde{\boldsymbol{u}},\boldsymbol{\chi}) \le N\|\tilde{\boldsymbol{u}}\|_X\|\boldsymbol{\chi}\|_X^2.$$

But, from property (6.6), (1.23) and (2.4), we obtain

$$\|\tilde{\boldsymbol{u}}\|_X \le C_a C_1\frac{\mathcal{C}}{\sqrt{2\mu}}\|\boldsymbol{u}\|_X.$$

Thus,

$$c_{NS}(\boldsymbol{U},\boldsymbol{\chi};\tilde{\boldsymbol{u}},\boldsymbol{\chi}) \le NC_aC_1\frac{\mathcal{C}}{\sqrt{2\mu}}\|\boldsymbol{\chi}\|_X^2. \tag{6.12}$$

Similarly, using (4.11), (4.15) and (5.2), we have

$$\frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{\chi},\boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \le C^t(2)C^t(4)^2(C_6^2+1)^{3/2}\frac{\tilde{\mathcal{C}}}{2\sqrt{\mu}}\|\boldsymbol{\chi}\|_{X_1}^2. \tag{6.13}$$

Finally, using (4.11), (4.15), (1.23), (1.24) and (2.4), we have

$$\frac{1}{2}(\boldsymbol{\chi}\cdot\boldsymbol{u},\boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \le C^t(2)C^t(4)C_4C_1(C_6^2+1)\frac{\mathcal{C}}{2\sqrt{2\mu}}\|\boldsymbol{\chi}\|_X^2. \tag{6.14}$$

Hence if (6.8) holds, the left-hand side of equation (6.10) is bounded below by

$$\frac{C_9\mu}{2}\|\boldsymbol{\chi}\|_X^2 + C_{10}\|\xi_2\|_{M_2}^2 + \frac{1}{G}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2.$$

Now, it remains to bound the terms in the right-hand side of (6.10). The terms on the fourth line of (6.10) are analyzed as in [18].

$$c_{NS}(\boldsymbol{U},\boldsymbol{\zeta};\tilde{\boldsymbol{u}},\boldsymbol{\chi}) + c_{NS}(\boldsymbol{U},\boldsymbol{u};\boldsymbol{\zeta},\boldsymbol{\chi}) + a_S(\boldsymbol{\zeta},\boldsymbol{\chi}) + a_D(\eta_2,\xi_2) + b_S(\boldsymbol{\chi},\eta_1)$$
$$\le \frac{C_9\mu}{8}\|\boldsymbol{\chi}\|_X^2 + \frac{C_{10}}{4}\|\xi_2\|_{M_2}^2 + Ch^{2k_1}|\boldsymbol{u}|_{H^{k_1+1}(\Omega_1)}^2(\frac{\mathcal{C}^2}{\mu^2}+\mu) + Ch^{2k_2}\|p_2\|_{H^{k_2+1}(\Omega_2)}^2 + Ch^{2k_1}\frac{1}{\mu}\|p_1\|_{H^{k_1}(\Omega_1)}^2. \tag{6.15}$$

Next, by inserting into $\boldsymbol{\zeta}$ the Scott-Zhang interpolant Π_h (or the standard Lagrange interpolant) and applying Theorem 4.4 to $\Pi_h(\boldsymbol{u})-\tilde{R}_h(\boldsymbol{u})$, we can write:

$$\|\boldsymbol{\zeta}\|_{L^2(\Gamma_{12})} \le C\left(\|\nabla(\Pi_h(\boldsymbol{u})-\boldsymbol{u})\|_{L^2(\Omega_1)}^2 + \|\nabla\boldsymbol{\zeta}\|_{L^2(\Omega_1)}^2 + \sum_{e\in\Gamma_1^h\cup\Gamma_1}\int_e\frac{1}{|e|}[\boldsymbol{\zeta}]_e^2\right)^{1/2}$$
$$\le Ch^{k_1}|\boldsymbol{u}|_{H^{k_1+1}(\Omega_1)}.$$

Then the integrals on the interface Γ_{12} are handled as follows:

$$\frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{\zeta},\boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \le C\|\boldsymbol{U}\|_X\|\boldsymbol{\chi}\|_X\|\boldsymbol{\zeta}\|_{L^2(\Gamma_{12})} \le \frac{C_9\mu}{8}\|\boldsymbol{\chi}\|_X^2 + C\frac{\tilde{\mathcal{C}}^2}{\mu^2}h^{2k_1}|\boldsymbol{u}|^2_{H^{k_1+1}(\Omega_1)}.$$

Similarly,

$$\frac{1}{2}(\boldsymbol{\zeta}\cdot\boldsymbol{u},\boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \le \frac{C_9\mu}{8}\|\boldsymbol{\chi}\|_X^2 + C\frac{\mathcal{C}^2}{\mu^2}h^{2k_1}|\boldsymbol{u}|^2_{H^{k_1+1}(\Omega_1)}.$$

We also have

$$(\eta_2,\boldsymbol{\chi}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \le \frac{C_9\mu}{8}\|\boldsymbol{\chi}\|_X^2 + \frac{C}{\mu}h^{2k_2}|p_2|^2_{H^{k_2+1}(\Omega_2)},$$

and

$$(\boldsymbol{\zeta}\cdot\boldsymbol{n}_{12},\xi_2)_{\Gamma_{12}} \le \frac{C_{10}}{4}\|\xi_2\|^2_{M_2} + Ch^{2k_1}|\boldsymbol{u}|^2_{H^{k_1+1}(\Omega_2)}.$$

Finally,

$$\frac{1}{G}(\boldsymbol{\zeta}\cdot\boldsymbol{\tau}_{12},\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} \le \frac{1}{2G}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} + Ch^{2k_1}|\boldsymbol{u}|^2_{H^{k_1+1}(\Omega_2)}.$$

The final result is obtained by combining all bounds and using triangle inequality. □

Theorem 6.3. *Under the assumptions of Theorem 6.2, we have*

$$\|p_1 - P_1\|_{M_1} \le C\Big((\mu + \frac{1+\mathcal{C}+\tilde{\mathcal{C}}}{\sqrt{\mu}})h^{k_1}|\boldsymbol{u}|_{H^{k_1+1}(\Omega_1)} + h^{k_2}|p_2|_{H^{k_2+1}(\Omega_2)} + h^{k_1}|p_1|_{H^{k_1}(\Omega_1)}$$
$$+\mu\|\boldsymbol{U}-\boldsymbol{u}\|_X + \frac{\mathcal{C}+\tilde{\mathcal{C}}}{\sqrt{\mu}}\|\boldsymbol{U}-\tilde{\boldsymbol{u}}\|_X + \frac{1}{G}\|(\boldsymbol{U}-\boldsymbol{u})\cdot\boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})} + \|P_2-\tilde{p}_2\|_{M_2}\Big)$$

with a constant C independent of h and μ.

Proof. ¿From the inf-sup condition (4.31), we have

$$\|P_1 - \tilde{p}_1\|_{M_1} \le \frac{1}{\beta^*}\sup_{\boldsymbol{v}\in\boldsymbol{X}^h}\frac{b_S(\boldsymbol{v},P_1-\tilde{p}_1)}{\|\boldsymbol{v}\|_X}.$$

The error equation can be written as follows:

$$\begin{aligned}\forall \boldsymbol{v}\in\boldsymbol{X}_h,\quad b_S(\boldsymbol{v},P_1-\tilde{p}_1) &= -a_S(\boldsymbol{U}-\boldsymbol{u},\boldsymbol{v}) + c_{NS}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) - c_{NS}(\boldsymbol{U},\boldsymbol{U};\boldsymbol{U},\boldsymbol{v})\\ &+ b_S(\boldsymbol{v},p_1-\tilde{p}_1) - \frac{1}{G}((\boldsymbol{U}-\boldsymbol{u})\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} + (p_2-P_2,\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\\ &+ \frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{U},\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u},\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + ((\boldsymbol{U}-\boldsymbol{u})\cdot\boldsymbol{n}_{12},q_2)_{\Gamma_{12}}.\end{aligned}$$

It suffices to bound the terms in the right-hand side using similar techniques as in the proof of Theorem 6.2. The details are skipped for the sake of brevity. □

An immediate consequence is the following result.

Corollary 6.4. *Under the assumptions of Theorem 6.2, there is a constant C independent of h such that*

$$\|p_1 - P_1\|_{M_1} \le C(h^{k_1}|\boldsymbol{u}|_{H^{k_1+1}(\Omega_1)} + h^{k_2}|p_2|_{H^{k_2+1}(\Omega_2)} + h^{k_1}|p_1|_{H^{k_1}(\Omega_1)}).$$

7. Conclusions

In this work, we presented a mathematical model for the coupling of surface flow and subsurface flow. Because of the nonlinear convection term in the Navier-Stokes equations, inertial forces were added in the balance of forces across the interface between surface and subsurface. We derived a complete analysis of the weak problem, and proposed a numerical scheme based on discontinuous finite element methods. Optimal error estimates were obtained. As future work, we will validate our model using numerical simulations and experimental data.

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