Analysis of hp Discontinuous Galerkin Methods for Incompressible Two-Phase Flow

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Abstract

In this paper, we prove the convergence of a class of discontinuous Galerkin methods for solving the fully coupled incompressible two-phase flow problem in the non-degenerate case. Estimates in both the mesh size and the polynomial degrees are obtained. Numerical convergence rates confirm the theoretical results.

Key words: porous media, multiphase flow, NIPG, SIPG, IIPG, h and p-version, Newton-Raphson

1 Introduction

This paper is devoted to the numerical analysis of high order primal discontinuous Galerkin methods for solving the incompressible two-phase flow problem arising in porous media [6,26,19]. The unknowns of the proposed fully coupled scheme are the global pressure of Chavent and Jaffré [8] and the non-wetting

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phase saturation, which are approximated by discontinuous piecewise polynomials of different degree. We show stability and hp convergence of the method under the condition that the diffusion coefficient for the saturation equation is bounded below away from zero.

Many discretization techniques have been applied and analyzed for incompressible two-phase flow in both non-degenerate and degenerate cases. These numerical methods are of low order and include finite differences [26,3,12], finite volume methods [23,21,20], mixed finite elements coupled with Galerkin method or finite volume method [8,13,10,14,25,9]. There are however few works on the application of discontinuous Galerkin methods to incompressible two-phase flow.

Since the late nineties, discontinuous Galerkin (DG) methods have been applied to a wide range of applications ranging from solid mechanics to fluid mechanics. One of the attractive features of these locally mass conservative methods is the ability to easily increase the order of approximation on each mesh element. Many of the proposed DG methods are sequential approaches where at each time step a discrete pressure equation and a discrete saturation equation are solved successively [27,28,5,24,18]. The underlying variational problem is based on the non-symmetric interior penalty Galerkin method (NIPG) [29], the symmetric interior penalty Galerkin method (SIPG) [33,2] or the incomplete interior penalty Galerkin method (IIPG) [11,31]. All three methods are very similar to each other and involve the use of stabilizing penalty terms.

More recently, in [17,16], we introduce a fully coupled DG method for the wetting phase pressure and non-wetting phase saturation. We numerically show convergence of the scheme with respect to the mesh size and the polynomial degree. We also obtain accurate solutions for the quarter-five spot benchmark problem. The advantages of the coupled approach over the sequential one, are that no slope limiting techniques are used. The overshoot and undershoot phenomena occurring near the saturation front are small, stable and decrease with h and p refinement.

The outline of the paper is as follows. Section 2 contains the coupled system of partial differential equations and assumptions on the data. The DG scheme is formulated in Section 3. A priori bounds are derived in Section 4, and are followed by the hp error estimates in Section 5. Finally, Section 6 gives some numerical examples. Concluding remarks end this paper.

2 Model Problem and Notation

Let Ω be a polygonal porous medium in $\mathbb{R}^2$. The incompressible flow of the wetting phase (such as water) and non-wetting phase (such as oil) in Ω over a time interval $[0, T]$, is described by Darcy's law and the continuity equation for each phase. If we denote the wetting and non-wetting phase by the subscript $\alpha = w$ and $\alpha = n$ respectively, we can write the continuity equation satisfied by each phase saturation s_α as:

$$\frac{\partial(\phi s_\alpha)}{\partial t} + \nabla \cdot u_\alpha = q_\alpha, \quad \alpha = w, n, \tag{1}$$

where the phase velocity u_α follows Darcy's law:

$$u_\alpha = -\lambda_\alpha K \nabla p_\alpha, \quad \alpha = w, n. \tag{2}$$

Here, the phase pressure is denoted by p_α. The other coefficients are the porosity ϕ, the permeability K, the phase mobility λ_α and the source function q_α.

In addition, we assume the following closure relations to hold:

$$s_w + s_n = 1, \tag{3}$$

$$p_c = p_n - p_w, \tag{4}$$

where p_c is the capillary pressure. Several models for the mobilities and capillary pressure are available in the literature [22]: two popular ones are the Brooks-Corey and the Van Genuchten models.

Based on the work of Chavent and Jaffré [8], we reformulate the model problem by using the global pressure defined by:

$$\forall (x,t) \in \Omega \times [0,T], \quad p(x,t) = p_n(x,t) + p_c(1 - s_{nr}) - \int_{1-s_{nr}}^{s_w(x,t)} \frac{\lambda_w(\xi)}{\lambda_t(\xi)} p_c'(\xi) d\xi, \tag{5}$$

where s_{nr} (resp. s_{wr}) is the residual saturation of the non-wetting phase (resp. wetting phase). The total mobility λ_t is defined as the sum of the phase mobilities ($\lambda_t = \lambda_w + \lambda_n$). Mathematically, the global pressure is well-defined for all values of s_n in $[1 - s_{wr}, s_{nr}]$. An equivalent formulation of (1) can then be obtained for the primary variables (p, s_n):

$$-\nabla \cdot (\lambda_t K \nabla p) = q_w + q_n, \tag{6}$$

$$\phi \frac{\partial s_n}{\partial t} + \nabla \cdot (\lambda_w K \nabla p - \frac{\lambda_w \lambda_n}{\lambda_t} K \nabla p_c) = -q_w. \tag{7}$$

In the analysis that follows, we make the following assumptions on the coefficients in (6), (7).

- Assumption H1. The function $\gamma = \frac{\lambda_w \lambda_n}{\lambda_t} p_c'$ is Lipschitz continuous with Lipschitz constant C_γ. It is also bounded above and below: $0 < \underline{\gamma} \leq \gamma \leq \overline{\gamma}$.
- Assumption H2. The mobilities λ_t, λ_w are Lipschitz continuous with Lipschitz constant C_λ.

- Assumption H3. The mobilities $\lambda_w, \lambda_n, \lambda_t$ are bounded functions of saturation:

$$0 < \underline{\lambda}_t \leq \lambda_t \leq \overline{\lambda}_t, \quad 0 \leq \lambda_w \leq \overline{\lambda}_w, \quad 0 \leq \lambda_n \leq \overline{\lambda}_n.$$

- Assumption H4. The tensor K is symmetric positive definite and uniformly bounded above and below. There are constants $\overline{k} > 0, \underline{k} > 0$ such that:

$$\forall x, \quad \underline{k} x^T x \leq x^T K x \leq \overline{k} x^T x.$$

- Assumption H5. The porosity is bounded above and below.

$$\underline{\phi} \leq \phi \leq \overline{\phi}.$$

We close the system (2), (3), (4), (5), (6), (7) by the following initial and boundary conditions. We decompose the boundary $\partial\Omega$ into disjoint sets Γ_D and Γ_N and we denote by n the unit outward normal to $\partial\Omega$.

$$\forall x \in \Omega, \quad s_n(x,0) = s_n^0(x), \tag{8}$$

$$\forall x \in \Gamma_D, \quad s_n(x,t) = s_n^{dir}, \quad p(x,t) = p^{dir}, \tag{9}$$

$$\forall x \in \Gamma_N, \quad u_w \cdot n = u_n \cdot n = 0. \tag{10}$$

It is understood that Γ_D contains both inflow and outflow boundaries whereas Γ_N corresponds to the no-flow boundary.

We propose a discontinuous finite element discretization of (6), (7). For this, we introduce a non-degenerate quasi-uniform subdivision of Ω, made of either triangles or quadrilaterals. The quasiuniformity assumption is only needed for the p-version, i.e. for deriving error estimates in terms of the polynomial degree. The set of interior edges is denoted by Γ_h. To each edge e in Γ_h, we associate a unit normal vector n_e. For a boundary edge, n_e coincides with the outward normal. The discrete space of discontinuous piecewise polynomials of degree $r \geq 1$ is denoted by $\mathcal{D}_r(\mathcal{E}_h)$:

$$\mathcal{D}_r(\mathcal{E}_h) = \{v \in L^2(\Omega) : \forall E \in \mathcal{E}_h : v|_E \in \mathbb{P}_r(E)\}.$$

For any function $v \in \mathcal{D}_r(\mathcal{E}_h)$, we denote the jump and average over a given edge e by $[v]$ and $\{v\}$ respectively. We assume that n_e is outward to E_e^1:

$$\forall e = \partial E_e^1 \cap \partial E_e^2, \quad [v]|_e = v|_{E_e^1} - v|_{E_e^2}, \quad \{v\}|_e = 0.5 v|_{E_e^1} + 0.5 v|_{E_e^2},$$
$$\forall e = \partial E_e^1 \cap \partial \Omega, \quad [v]|_e = v|_{E_e^1}, \quad \{v\}|_e = v|_{E_e^1}.$$

We also denote by $\tilde{C}$ the constant that only depends on the maximum number of neighbors that one mesh element can have so that the following inequality holds. Let A be any quantity depending on E_e^1 or E_e^2:

$$\forall i = 1, 2, \quad (\sum_{e \in \Gamma_h} A(E_e^i))^{1/2} \leq \frac{\sqrt{\tilde{C}}}{2} (\sum_{E \in \mathcal{E}_h} A(E))^{1/2}. \tag{11}$$
$$(\sum_{e \in \Gamma_D} A(E_e^1))^{1/2} \leq \sqrt{\tilde{C}} (\sum_{E \in \mathcal{E}_h} A(E))^{1/2}. \tag{12}$$

Let $H^k(\mathcal{O})$ be the usual Sobolev space on $\mathcal{O} \subset \mathbb{R}^d, d \geq 1$ with norm $\|\cdot\|_{k,\mathcal{O}}$. We now recall well-known facts that will be used in the error analysis.

Lemma 1 *There is a constant C_2 independent of h and r such that*

$$\forall v \in \mathcal{D}_r(\mathcal{E}_h), \quad \|v\|_{0,\Omega} \leq C_2 (\sum_{E \in \mathcal{E}_h} \|\nabla v\|_{0,E}^2 + \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{1}{|e|} \|[v]\|_{0,e}^2)^{1/2}, \tag{13}$$

where $|e|$ denotes the measure of e.

Lemma 2 *Let γ_0 and γ_1 denote the usual trace operators. There is a constant*

C_t independent of h such that if E is a triangle or quadrilateral:

$$\forall v \in H^k(E), k \geq 1, \forall e \subset \partial E, \|\gamma_0 v\|_{0,e} \leq C_t h^{-1/2}(\|v\|_{0,E} + h\|\nabla v\|_{0,E}), \quad (14)$$

$$\forall v \in H^k(E), k \geq 2, \forall e \subset \partial E, \|\gamma_1 v\|_{0,e} \leq C_t h^{-1/2}(\|\nabla v\|_{0,E} + h\left\|\nabla^2 v\right\|_{0,E}). \quad (15)$$

Lemma 3 *Let E be a mesh element. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(k) = (k+1)(k+2)$ if E is a triangle, and by $f(k) = k^2$ if E is a quadrilateral. There is a constant C_t independent of h and k such that:*

$$\forall v \in \mathbb{P}_k(E), \forall e \subset \partial E, \|\gamma_0 v\|_{0,e} \leq C_t \sqrt{\frac{f(k)}{h}} \|v\|_{0,E}. \quad (16)$$

In the case of the triangle, if θ_E denotes the smallest angle, an exact expression for C_t is given by:

$$C_t = \sqrt{2 \cot \theta_E \frac{h}{|e|}}.$$

The proofs of these results can be found in the literature: see Lemma 2.1 in [2] or (1.3) in [7] for Lemma 1, see Theorem 3.10 in [1] for Lemma 2, see Theorem 3 in [32] and the proof of Theorem 9 in [15] for the case of triangle for Lemma 3 and Lemma 2.1 in [30] for the case of quadrilateral for Lemma 3.

3 Numerical Scheme

Let $\Delta t > 0$ be a time step such that $T = N\Delta t$ with $N \in \mathbb{N}$. Let $t^i = i\Delta t$. For each $1 \leq i \leq N$, the approximations $(P^i, S_n^i) \in \mathcal{D}_{r_p}(\mathcal{E}_h) \times \mathcal{D}_{r_s}(\mathcal{E}_h)$ of the functions $(p(\cdot, t^i), s_n(\cdot, t^i))$ satisfy the following set of equations:

Initial condition

$$\forall v \in \mathcal{D}_{r_s}(\mathcal{E}_h), \quad \int_\Omega S_n^0 v = \int_\Omega s_n^0 v. \tag{17}$$

Pressure equation

$$\begin{aligned}
&\forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad \forall i \geq 0, \\
&\sum_{E\in\mathcal{E}_h} \int_E \lambda_t(S_n^{i+1}) K \nabla P^{i+1} \cdot \nabla z + \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \int_e [P^{i+1}][z] \\
&- \sum_{e\in\Gamma_h} \int_e \{\lambda_t(S_n^{i+1}) K \nabla P^{i+1} \cdot n_e\}[z] - \sum_{e\in\Gamma_D} \int_e (\lambda_t(s_n^{dir}) K \nabla P^{i+1} \cdot n_e) z \\
&+\varepsilon \sum_{e\in\Gamma_h} \int_e \{\lambda_t(S_n^{i+1}) K \nabla z \cdot n_e\}[P^{i+1}] + \varepsilon \sum_{e\in\Gamma_D} \int_e (\lambda_t(s_n^{dir}) K \nabla z \cdot n_e) P^{i+1} \\
&= \varepsilon \sum_{e\in\Gamma_D} \int_e (\lambda_t(s_n^{dir}) K \nabla z \cdot n_e) p^{dir} + \sigma_p \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \int_e p^{dir} z + \int_\Omega (q_w + q_n) z.
\end{aligned} \tag{18}$$

Saturation equation

$$\begin{aligned}
&\forall v \in \mathcal{D}_{r_s}(\mathcal{E}_h), \quad \forall i \geq 0, \\
&\int_\Omega \frac{\phi}{\Delta t}(S_n^{i+1} - S_n^i) v - \sum_{E\in\mathcal{E}_h} \int_E \lambda_w(S_n^{i+1}) K \nabla P^{i+1} \cdot \nabla v + \sum_{E\in\mathcal{E}_h} \int_E \gamma(S_n^{i+1}) K \nabla S_n^{i+1} \cdot \nabla v \\
&+ \sum_{e\in\Gamma_h} \int_e \{\lambda_w(S_n^{i+1}) K \nabla P^{i+1} \cdot n_e\}[v] + \sum_{e\in\Gamma_D} \int_e (\lambda_w(s_n^{dir}) K \nabla P^{i+1} \cdot n_e) v \\
&- \sum_{e\in\Gamma_h} \int_e \{\gamma(S_n^{i+1}) K \nabla S_n^{i+1} \cdot n_e\}[v] - \sum_{e\in\Gamma_D} \int_e (\gamma(s_n^{dir}) K \nabla S_n^{i+1} \cdot n_e) v \\
&-\varepsilon \sum_{e\in\Gamma_h} \int_e \{\lambda_w(S_n^{i+1}) K \nabla v \cdot n_e\}[P^{i+1}] - \varepsilon \sum_{e\in\Gamma_D} \int_e (\lambda_w(s_n^{dir}) K \nabla v \cdot n_e) P^{i+1} \\
&+\varepsilon \sum_{e\in\Gamma_h} \int_e \{\gamma(S_n^{i+1}) K \nabla v \cdot n_e\}[S_n^{i+1}] + \varepsilon \sum_{e\in\Gamma_D} \int_e \gamma(s_n^{dir}) K \nabla v \cdot n_e) S_n^{i+1} \\
&+\sigma_s \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|} \int_e [S_n^{i+1}][v] = \sigma_s \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|} \int_e s_n^{dir} v \\
&-\varepsilon \sum_{e\in\Gamma_D} \int_e (\lambda_w(s_n^{dir}) K \nabla v \cdot n_e) p^{dir} + \varepsilon \sum_{e\in\Gamma_D} \int_e (\gamma(s_n^{dir}) K \nabla v \cdot n_e) s_n^{dir} - \int_\Omega q_w v.
\end{aligned} \tag{19}$$

The scheme (17), (18), (19) yields a coupled system of nonlinear equations that can be solved using the Newton-Raphson technique. These equations contain two types of parameters: the coefficient ϵ that takes the values $\{-1,0,+1\}$ corresponding to the generalization of the SIPG, IIPG and NIPG methods and the penalty coefficients $\sigma_p > 0, \sigma_s > 0$. In this work, we show that σ_p and σ_s need to be chosen large enough for stability and convergence of the scheme.

4 A priori Estimates

In this section, we prove existence of the numerical solution by using the Leray-Schauder theorem [34]. For this, we first prove a priori estimates for the discrete global pressure and non-wetting phase saturation.

Proposition 4 *Assume that the penalty parameter satisfies*

$$\sigma_p > (1-\varepsilon)^2 \frac{6(\overline{\lambda_t})^2 \overline{k} \tilde{C} C_t^2}{\underline{\lambda_t}} + \frac{\underline{\lambda_t} \underline{k}}{3}. \tag{20}$$

Then, there is a constant C independent of h, r_p, r_s and Δt such that

$$\begin{aligned} \forall 1 \le m \le N, \quad & \sum_{i=0}^{m-1} \Big(\sum_{E \in \mathcal{E}_h} \|K^{1/2} \nabla P^{i+1}\|_{0,E}^2 + \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{f(r_p)}{|e|} \left\| [P^{i+1}] \right\|_{0,e}^2 \Big) \\ & \le C \sum_{i=0}^{m-1} \|q_w + q_n\|_{0,\Omega}^2 + C \sum_{i=0}^{m-1} \sum_{e \in \Gamma_D} \frac{f(r_p)}{|e|} \left\| p^{dir} \right\|_{0,e}^2 . \end{aligned} \tag{21}$$

PROOF. Let us put $z = P^{i+1}$ in pressure equation (18) so we obtain:

$$\sum_{E\in\mathcal{E}_h}\int_E \lambda_t(S_n^{i+1})K\nabla P^{i+1}\cdot\nabla P^{i+1} + \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_p)}{|e|}\int_e [P^{i+1}][P^{i+1}]$$
$$= (1-\epsilon)\sum_{e\in\Gamma_h}\int_e \{\lambda_t(S_n^{i+1})K\nabla P^{i+1}\cdot n_e\}[P^{i+1}] + (1-\epsilon)\sum_{e\in\Gamma_D}\int_e (\lambda_t(s_n^{dir})K\nabla P^{i+1}\cdot n_e)P^{i+1}$$
$$+\varepsilon\sum_{e\in\Gamma_D}\int_e (\lambda_t(s_n^{dir})K\nabla P^{i+1}\cdot n_e)p^{dir} + \sigma_p\sum_{e\in\Gamma_D}\frac{f(r_p)}{|e|}\int_e p^{dir}P^{i+1} + \int_\Omega (q_w+q_n)P^{i+1}$$
$$= B_1 + \cdots + B_5.$$

We now bound each term B_i in the right-hand side of the equation above. In what follows, the numbers ϵ_i are positive real numbers to be defined later. Using Assumption H3, H4 and Cauchy-Schwarz inequality, we have

$$|B_1| \le (1-\varepsilon)\overline{\lambda_t}(\overline{k})^{\frac{1}{2}}\sum_{e\in\Gamma_h}\left\|\{K^{\frac{1}{2}}\nabla P^{i+1}\}\right\|_{0,e}\left\|[P^{i+1}]\right\|_{0,e}.$$

We now fix an interior edge e and denote by E_e^1 and E_e^2 the two elements sharing the edge e. Using the trace inequality (16) and (11) , we have:

$$\sum_{e\in\Gamma_h}\left\|\{K^{\frac{1}{2}}\nabla P^{i+1}\}\right\|_{0,e}\left\|[P^{i+1}]\right\|_{0,e} \le \sum_{e\in\Gamma_h}\frac{1}{2}(\|K^{1/2}\nabla P^{i+1}|_{E_e^1}\|_{0,e} + \|K^{1/2}\nabla P^{i+1}|_{E_e^2}\|_{0,e})\|[P^{i+1}]\|_{0,e}$$
$$\le \frac{1}{2}C_t\sqrt{\frac{f(r_p)}{h}}\sum_{e\in\Gamma_h}(\|K^{1/2}\nabla P^{i+1}\|_{0,E_e^1} + \|K^{1/2}\nabla P^{i+1}\|_{0,E_e^2})\|[P^{i+1}]\|_{0,e}$$
$$\le (\sum_{e\in\Gamma_h}\frac{C_t^2 f(r_p)}{4h}\|[P^{i+1}]\|_{0,e}^2)^{1/2}\Big((\sum_{e\in\Gamma_h}\|K^{1/2}\nabla P^{i+1}\|_{0,E_e^1}^2)^{1/2} + (\sum_{e\in\Gamma_h}\|K^{1/2}\nabla P^{i+1}\|_{0,E_e^2}^2)^{1/2}\Big)$$
$$\le (\sum_{e\in\Gamma_h}\frac{\tilde{C}C_t^2 f(r_p)}{4h}\|[P^{i+1}]\|_{0,e}^2)^{1/2}(\sum_{E\in\mathcal{E}_h}\|K^{1/2}\nabla P^{i+1}\|_{0,E}^2)^{1/2}.$$

Therefore, we have the following bound for B_1:

$$|B_1| \le \frac{\varepsilon_1}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla P^{i+1}\right\|_{0,E}^2 + (1-\varepsilon)^2\frac{(\overline{\lambda_t})^2\overline{k}\tilde{C}C_t^2}{8\varepsilon_1}\sum_{e\in\Gamma_h}\frac{f(r_p)}{h}\left\|[P^{i+1}]\right\|_{0,e}^2$$
$$\le \frac{\varepsilon_1}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla P^{i+1}\right\|_{0,E}^2 + (1-\varepsilon)^2\frac{(\overline{\lambda_t})^2\overline{k}\tilde{C}C_t^2}{8\varepsilon_1}\sum_{e\in\Gamma_h}\frac{f(r_p)}{|e|}\left\|[P^{i+1}]\right\|_{0,e}^2. \quad (22)$$

Similarly, we have for B_2:

$$|B_2| \leq \frac{\varepsilon_1}{2} \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla P^{i+1} \right\|_{0,E}^2 + (1-\varepsilon)^2 \frac{(\overline{\lambda_t})^2 \overline{k} \tilde{C} C_t^2}{2\varepsilon_1} \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \left\| P^{i+1} \right\|_{0,e}^2 . \quad (23)$$

Similarly, we have for B_3:

$$|B_3| \leq \frac{\varepsilon_2}{2} \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla P^{i+1} \right\|_{0,E}^2 + \frac{(\overline{\lambda_t})^2 \overline{k} \tilde{C} C_t^2}{2\varepsilon_2} \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \left\| p^{dir} \right\|_{0,e}^2 . \quad (24)$$

The term B_4 is simply bounded by Cauchy-Schwarz and Young's inequalities.

$$|B_4| \leq \varepsilon_4 \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \left\| P^{i+1} \right\|_{0,e}^2 + \frac{{\sigma_p}^2}{4\varepsilon_4} \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \left\| p^{dir} \right\|_{0,e}^2 . \quad (25)$$

Finally, the last term B_5 is bounded using Cauchy-Schwarz inequality and (13).

$$\begin{aligned}
|B_5| &\leq \|q_w + q_n\|_{0,\Omega} \|P^{i+1}\|_{0,\Omega} \\
&\leq C_2 \|q_w + q_n\|_{0,\Omega} \Big(\sum_{E\in\mathcal{E}_h} \|\nabla P^{i+1}\|_{0,E}^2 + \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{1}{|e|} \|[P^{i+1}]\|_{0,e}^2 \Big)^{1/2} \\
&\leq \epsilon_5 \underline{k} \Big(\sum_{E\in\mathcal{E}_h} \|\nabla P^{i+1}\|_{0,E}^2 + \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{1}{|e|} \|[P^{i+1}]\|_{0,e}^2 \Big) + \frac{C_2^2}{4\epsilon_5 \underline{k}} \|q_w + q_n\|_{0,\Omega}^2 \\
&\leq \epsilon_5 \Big(\sum_{E\in\mathcal{E}_h} \|K^{1/2} \nabla P^{i+1}\|_{0,E}^2 + \underline{k} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{1}{|e|} \|[P^{i+1}]\|_{0,e}^2 \Big) + \frac{C_2^2}{4\epsilon_5 \underline{k}} \|q_w + q_n\|_{0,\Omega}^2 \\
&\leq \epsilon_5 \Big(\sum_{E\in\mathcal{E}_h} \|K^{1/2} \nabla P^{i+1}\|_{0,E}^2 + \underline{k} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \|[P^{i+1}]\|_{0,e}^2 \Big) + \frac{C_2^2}{4\epsilon_5 \underline{k}} \|q_w + q_n\|_{0,\Omega}^2 .
\end{aligned} \quad (26)$$

Combining the bounds (22)-(26) we obtain:

$$\begin{aligned}
&\Big(\underline{\lambda}_t - \epsilon_1 - \frac{\epsilon_2}{2} - \epsilon_5 \Big) \sum_{E\in\mathcal{E}_h} \|K^{1/2} \nabla P^{i+1}\|_{0,E}^2 \\
&+ \Big(\sigma_p - (1-\varepsilon)^2 \frac{(\overline{\lambda_t})^2 \overline{k} \tilde{C} C_t^2}{2\varepsilon_1} - \epsilon_4 - \underline{k} \epsilon_5 \Big) \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \left\| [P^{i+1}] \right\|_{0,e}^2 \\
&\leq \frac{C_2^2}{4\epsilon_5 \underline{k}} \|q_w + q_n\|_{0,\Omega}^2 + \Big(\frac{(\overline{\lambda_t})^2 \overline{k} \tilde{C} C_t^2}{2\varepsilon_2} + \frac{{\sigma_p}^2}{4\varepsilon_4} \Big) \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \left\| p^{dir} \right\|_{0,e}^2 .
\end{aligned} \quad (27)$$

Thus, if we choose

$$\epsilon_1 = \frac{\epsilon_2}{2} = \epsilon_5 = \frac{\underline{\lambda}_t}{6}$$

and

$$\epsilon_4 = \frac{\sigma_p}{2}$$

we have:

$$\begin{aligned}
&\frac{\underline{\lambda}_t}{2} \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla P^{i+1}\|^2_{0,E} \\
&+\Big(\frac{\sigma_p}{2} - (1-\varepsilon)^2 \frac{3(\overline{\lambda_t})^2\overline{k}\tilde{C}C_t^2}{\underline{\lambda}_t} - \frac{\underline{\lambda}_t\underline{k}}{6}\Big) \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \left\|[P^{i+1}]\right\|^2_{0,e} \\
&\leq \frac{3C_2^2}{2\underline{\lambda}_t\underline{k}}\|q_w+q_n\|^2_{0,\Omega} + \Big(\frac{3(\overline{\lambda_t})^2\overline{k}\tilde{C}C_t^2}{2\underline{\lambda}_t} + \frac{\sigma_p}{2}\Big) \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \left\|p^{dir}\right\|^2_{0,e}.
\end{aligned} \tag{28}$$

The final result is obtained by summing over i.

Proposition 5 *Assume that (20) holds and that*

$$\sigma_s > (1-\epsilon)^2 \frac{12(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}} + \frac{\underline{k}\underline{\gamma}}{6}. \tag{29}$$

There is a constant C independent of h, r_p, r_s and Δt such that:

$$\begin{aligned}
\forall 1 \leq m \leq N, \quad & \sum_{i=0}^{m-1} \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla S_n^{i+1}\|^2_{0,E} + \sum_{i=0}^{m-1} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|}\|[S_n^{i+1}]\|^2_{0,e} + C\frac{\underline{\phi}}{\Delta t}\|S_n^m\|^2_{0,\Omega} \\
&\leq C\frac{\underline{\phi}}{\Delta t}\|s_n^0\|^2_{0,\Omega} + C\Big(1 + \frac{f(r_p)}{f(r_s)} + \frac{f(r_s)}{f(r_p)}\Big) \sum_{i=0}^{m-1} \|q_w+q_n\|^2_{0,\Omega} + C\sum_{i=0}^{m-1}\|q_w\|^2_{0,\Omega} \\
&+C\Big(1 + \frac{f(r_p)}{f(r_s)} + \frac{f(r_s)}{f(r_p)}\Big) \sum_{i=0}^{m-1} \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \left\|p^{dir}\right\|^2_{0,e} + C\sum_{i=0}^{m-1} \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|} \left\|s_n^{dir}\right\|^2_{0,e}.
\end{aligned} \tag{30}$$

PROOF. Choosing $v = S_n^{i+1}$ in (19) gives:

$$
\begin{aligned}
&\sum_{E\in\mathcal{E}_h}\int_E \gamma(S_n^{i+1})K\nabla S_n^{i+1}\cdot\nabla S_n^{i+1} + \sigma_s \sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_s)}{|e|}\int_e [S_n^{i+1}]^2 \\
&+\int_\Omega \frac{\phi}{\Delta t}(S_n^{i+1}-S_n^i)S_n^{i+1} = \sum_{E\in\mathcal{E}_h}\int_E \lambda_w(S_n^{i+1})K\nabla P^{i+1}\cdot\nabla S_n^{i+1} \\
&-\sum_{e\in\Gamma_h}\int_e \{\lambda_w(S_n^{i+1})K\nabla P^{i+1}\cdot n_e\}[S_n^{i+1}] - \sum_{e\in\Gamma_D}\int_e (\lambda_w(s_n^{dir})K\nabla P^{i+1}\cdot n_e)S_n^{i+1} \\
&+(1-\epsilon)\sum_{e\in\Gamma_h}\int_e \{\gamma(S_n^{i+1})K\nabla S_n^{i+1}\cdot n_e\}[S_n^{i+1}] + (1-\epsilon)\sum_{e\in\Gamma_D}\int_e (\gamma(s_n^{dir})K\nabla S_n^{i+1}\cdot n_e)S_n^{i+1} \\
&+\varepsilon \sum_{e\in\Gamma_h}\int_e \{\lambda_w(S_n^{i+1})K\nabla S_n^{i+1}\cdot n_e\}[P^{i+1}] + \varepsilon\sum_{e\in\Gamma_D}\int_e (\lambda_w(s_n^{dir})K\nabla S_n^{i+1}\cdot n_e)P^{i+1} \\
&+\sigma_s\sum_{e\in\Gamma_D}\frac{f(r_s)}{|e|}\int_e s_n^{dir}S_n^{i+1} - \varepsilon\sum_{e\in\Gamma_D}\int_e (\lambda_w(s_n^{dir})K\nabla S_n^{i+1}\cdot n_e)p^{dir} \\
&+\varepsilon\sum_{e\in\Gamma_D}\int_e (\gamma(s_n^{dir})K\nabla S_n^{i+1}\cdot n_e)s_n^{dir} - \int_\Omega q_w S_n^{i+1} \\
&= D_1 + \cdots + D_{11}.
\end{aligned}
$$

We now bound each term D_i.

$$|D_1| \le \varepsilon_2 \sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla S_n^{i+1}\right\|_{0,E}^2 + \frac{\overline{\lambda_w}^2}{4\varepsilon_2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla P^{i+1}\right\|_{0,E}^2. \tag{31}$$

The term D_2 is bounded like B_1:

$$|D_2| \le \frac{\varepsilon_3}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla P^{i+1}\right\|_{0,E}^2 + \frac{(\overline{\lambda}_w)^2\overline{k}\tilde{C}C_t^2}{8\varepsilon_3}\sum_{e\in\Gamma_h}\frac{f(r_p)}{|e|}\left\|[S_n^{i+1}]\right\|_{0,e}^2. \tag{32}$$

The term D_3 is bounded like B_2:

$$|D_3| \le \frac{\varepsilon_3}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla P^{i+1}\right\|_{0,E}^2 + \frac{(\overline{\lambda}_w)^2\overline{k}\tilde{C}C_t^2}{2\varepsilon_3}\sum_{e\in\Gamma_D}\frac{f(r_p)}{|e|}\left\|S_n^{i+1}\right\|_{0,e}^2. \tag{33}$$

The term D_4 is bounded like B_1:

$$|D_4| \le \frac{\varepsilon_5}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla S_n^{i+1}\right\|_{0,E}^2 + (1-\epsilon)^2\frac{(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{8\varepsilon_5}\sum_{e\in\Gamma_h}\frac{f(r_s)}{|e|}\left\|[S_n^{i+1}]\right\|_{0,e}^2. \tag{34}$$

The term D_5 is bounded like B_2:

$$|D_5| \le \frac{\varepsilon_5}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla S_n^{i+1}\right\|_{0,E}^2 + (1-\epsilon)^2\frac{(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{2\varepsilon_5}\sum_{e\in\Gamma_D}\frac{f(r_s)}{|e|}\left\|S_n^{i+1}\right\|_{0,e}^2. \tag{35}$$

The term D_6 is bounded like B_1:

$$|D_6| \leq \frac{\varepsilon_7}{2} \sum_{E \in \mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla S_n^{i+1} \right\|_{0,E}^2 + \frac{(\overline{\lambda}_w)^2 \overline{k} \tilde{C} C_t^2}{8\varepsilon_7} \sum_{e \in \Gamma_h} \frac{f(r_s)}{|e|} \left\| [P^{i+1}] \right\|_{0,e}^2 . \quad (36)$$

The term D_7 is bounded like B_2:

$$|D_7| \leq \frac{\varepsilon_7}{2} \sum_{E \in \mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla S_n^{i+1} \right\|_{0,E}^2 + \frac{(\overline{\lambda}_w)^2 \overline{k} \tilde{C} C_t^2}{2\varepsilon_7} \sum_{e \in \Gamma_D} \frac{f(r_s)}{|e|} \left\| P^{i+1} \right\|_{0,e}^2 . \quad (37)$$

The term D_8 is bounded like B_4:

$$|D_8| \leq \varepsilon_9 \sum_{e \in \Gamma_D} \frac{f(r_s)}{|e|} \left\| S_n^{i+1} \right\|_{0,e}^2 + \frac{{\sigma_s}^2}{4\varepsilon_9} \sum_{e \in \Gamma_D} \frac{f(r_s)}{|e|} \left\| s_n^{dir} \right\|_{0,e}^2 . \quad (38)$$

The term D_9 is bounded like D_7:

$$|D_9| \leq \frac{\varepsilon_{10}}{2} \sum_{E \in \mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla S_n^{i+1} \right\|_{0,E}^2 + \frac{(\overline{\lambda}_w)^2 \overline{k} \tilde{C} C_t^2}{2\varepsilon_{10}} \sum_{e \in \Gamma_D} \frac{f(r_s)}{|e|} \left\| p^{dir} \right\|_{0,e}^2 . \quad (39)$$

The term D_{10} is bounded like D_7:

$$|D_{10}| \leq \frac{\varepsilon_{11}}{2} \sum_{E \in \mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla S_n^{i+1} \right\|_{0,E}^2 + \frac{(\overline{\gamma})^2 \overline{k} \tilde{C} C_t^2}{2\varepsilon_{11}} \sum_{e \in \Gamma_D} \frac{f(r_s)}{|e|} \left\| s_n^{dir} \right\|_{0,e}^2 . \quad (40)$$

The term D_{11} is bounded like B_5:

$$|D_{11}| \leq \epsilon_{12} (\sum_{E \in \mathcal{E}_h} \| K^{1/2} \nabla S_n^{i+1} \|_{0,E}^2 + \underline{k} \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{f(r_s)}{|e|} \| [S_n^{i+1}] \|_{0,e}^2) + \frac{C_2^2}{4\epsilon_{12} \underline{k}} \| q_w \|_{0,\Omega}^2 . \quad (41)$$

Combining the bounds (31)-(41), we have:

$$\left(\underline{\gamma} - \epsilon_2 - \epsilon_5 - \epsilon_7 - \frac{\epsilon_{10}}{2} - \frac{\epsilon_{11}}{2} - \epsilon_{12}\right) \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla S_n^{i+1}\|_{0,E}^2$$

$$+\left(\sigma_s - \frac{(\overline{\lambda}_w)^2\overline{k}\tilde{C}C_t^2 f(r_p)}{2\varepsilon_3 f(r_s)} - (1-\epsilon)^2\frac{(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{2\varepsilon_5} - \epsilon_9 - \underline{k}\epsilon_{12}\right) \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|}\|[S_n^{i+1}]\|_{0,e}^2$$

$$+\int_\Omega \frac{\phi}{\Delta t}(S_n^{i+1} - S_n^i)S_n^{i+1} \leq$$

$$\left(\frac{\overline{\lambda_w}^2}{4\varepsilon_2} + \epsilon_3\right) \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla P^{i+1}\right\|_{0,E}^2 + \frac{(\overline{\lambda}_w)^2\overline{k}\tilde{C}C_t^2 f(r_s)}{2\varepsilon_7 f(r_p)} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|}\left\|[P^{i+1}]\right\|_{0,e}^2$$

$$+\frac{(\overline{\lambda}_w)^2\overline{k}\tilde{C}C_t^2 f(r_s)}{2\varepsilon_{10} f(r_p)} \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|}\left\|p^{dir}\right\|_{0,e}^2 + \left(\frac{{\sigma_s}^2}{4\varepsilon_9} + \frac{(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{2\varepsilon_{11}}\right) \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|}\left\|s_n^{dir}\right\|_{0,e}^2$$

$$+\frac{C_2^2}{4\epsilon_{12}\underline{k}}\|q_w\|_{0,\Omega}^2.$$

Thus, taking

$$\epsilon_2 = \epsilon_5 = \epsilon_7 = \frac{\epsilon_{10}}{2} = \frac{\epsilon_{11}}{2} = \epsilon_{12} = \frac{\underline{\gamma}}{12},$$

and

$$\frac{(\overline{\lambda}_w)^2\overline{k}\tilde{C}C_t^2 f(r_p)}{2\varepsilon_3 f(r_s)} = \epsilon_9 = \frac{\sigma_s}{4},$$

we obtain:

$$\frac{\underline{\gamma}}{2}\sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla S_n^{i+1}\|_{0,E}^2$$

$$+\left(\frac{\sigma_s}{2} - (1-\epsilon)^2\frac{6(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}} - \frac{\underline{k}\underline{\gamma}}{12}\right) \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|}\|[S_n^{i+1}]\|_{0,e}^2 + \frac{\phi}{\Delta t}\int_\Omega (S_n^{i+1} - S_n^i)S_n^{i+1}$$

$$\leq \left(\frac{3\overline{\lambda_w}^2}{\underline{\gamma}} + \frac{2\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2 f(r_p)}{\sigma_s f(r_s)}\right) \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla P^{i+1}\right\|_{0,E}^2 + \frac{6(\overline{\lambda}_w)^2\overline{k}\tilde{C}C_t^2 f(r_s)}{\underline{\gamma} f(r_p)} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|}\left\|[P^{i+1}]\right\|_{0,e}^2$$

$$+\frac{3(\overline{\lambda}_w)^2\overline{k}\tilde{C}C_t^2 f(r_s)}{\underline{\gamma} f(r_p)} \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|}\left\|p^{dir}\right\|_{0,e}^2 + \left(\sigma_s + \frac{3(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}}\right) \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|}\left\|s_n^{dir}\right\|_{0,e}^2$$

$$+\frac{3C_2^2}{\underline{\gamma}\underline{k}}\|q_w\|_{0,\Omega}^2.$$

Therefore, if

$$\sigma_s > (1-\epsilon)^2\frac{12(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}} + \frac{\underline{k}\underline{\gamma}}{6},$$

then there is a constant C independent of h, r_s, r_p and Δt such that

$$\sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla S_n^{i+1}\|_{0,E}^2 + \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|}\|[S_n^{i+1}]\|_{0,e}^2 + C\frac{\phi}{\Delta t}\int_\Omega (S_n^{i+1} - S_n^i)S_n^{i+1}$$
$$\leq C(1+\frac{f(r_p)}{f(r_s)}) \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla P^{i+1}\right\|_{0,E}^2 + C\frac{f(r_s)}{f(r_p)} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|}\left\|[P^{i+1}]\right\|_{0,e}^2$$
$$+C\frac{f(r_s)}{f(r_p)}\sum_{e\in\Gamma_D}\frac{f(r_p)}{|e|}\left\|p^{dir}\right\|_{0,e}^2 + C\sum_{e\in\Gamma_D}\frac{f(r_s)}{|e|}\left\|s_n^{dir}\right\|_{0,e}^2 + C\|q_w\|_{0,\Omega}^2.$$

We now sum over i and use the fact that $\|S_n^0\|_{0,\Omega} \leq \|s_n^0\|_{0,\Omega}$ (obtained from (17)).

$$\sum_{i=0}^{m-1}\sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla S_n^{i+1}\|_{0,E}^2 + \sum_{i=0}^{m-1}\sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|}\|[S_n^{i+1}]\|_{0,e}^2 + C\frac{\phi}{\Delta t}\|S_n^m\|_{0,\Omega}^2$$
$$\leq C\frac{\phi}{\Delta t}\|s_n^0\|_{0,\Omega}^2 + C(1+\frac{f(r_p)}{f(r_s)})\sum_{i=0}^{m-1}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla P^{i+1}\right\|_{0,E}^2 + C\frac{f(r_s)}{f(r_p)}\sum_{i=0}^{m-1}\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_p)}{|e|}\left\|[P^{i+1}]\right\|_{0,e}^2$$
$$+C\frac{f(r_s)}{f(r_p)}\sum_{i=0}^{m-1}\sum_{e\in\Gamma_D}\frac{f(r_p)}{|e|}\left\|p^{dir}\right\|_{0,e}^2 + C\sum_{i=0}^{m-1}\sum_{e\in\Gamma_D}\frac{f(r_s)}{|e|}\left\|s_n^{dir}\right\|_{0,e}^2 + C\sum_{i=0}^{m-1}\|q_w\|_{0,\Omega}^2. \tag{42}$$

From (21) and (42), we have:

$$\sum_{i=0}^{m-1}\sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla S_n^{i+1}\|_{0,E}^2 + \sum_{i=0}^{m-1}\sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|}\|[S_n^{i+1}]\|_{0,e}^2 + C\frac{\phi}{\Delta t}\|S_n^m\|_{0,\Omega}^2$$
$$\leq C\frac{\phi}{\Delta t}\|s_n^0\|_{0,\Omega}^2 + C(1+\frac{f(r_p)}{f(r_s)}+\frac{f(r_s)}{f(r_p)})\sum_{i=0}^{m-1}\|q_w+q_n\|_{0,\Omega}^2 + C\sum_{i=0}^{m-1}\|q_w\|_{0,\Omega}^2$$
$$+C\left(1+\frac{f(r_p)}{f(r_s)}+\frac{f(r_s)}{f(r_p)}\right)\sum_{i=0}^{m-1}\sum_{e\in\Gamma_D}\frac{f(r_p)}{|e|}\left\|p^{dir}\right\|_{0,e}^2 + C\sum_{i=0}^{m-1}\sum_{e\in\Gamma_D}\frac{f(r_s)}{|e|}\left\|s_n^{dir}\right\|_{0,e}^2.$$

Theorem 6 *There exists a solution to (17), (18), (19).*

PROOF. The existence of S_n^0 is trivial. Let $P = (P^i)_{1\leq i\leq N}$ and $S_n = (S_n^i)_{1\leq i\leq N}$ be the sequences of approximations satisfying (18) and (19). Let $X = \mathcal{D}_{r_p}(\mathcal{E}_h)^N \times \mathcal{D}_{r_s}(\mathcal{E}_h)^N$ and let $G : X \to X$ such that $G(P, S_n) = (\hat{P}, \hat{S}_n)$ where $(\hat{P}, \hat{S}_n)$ is the solution of the following system of linear equations:

$$\forall v \in \mathcal{D}_{r_s}(\mathcal{E}_h), \quad \int_\Omega \hat{S}_n^0 v = \int_\Omega s_n^0 v. \tag{43}$$

$$
\begin{aligned}
&\forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad \forall i \geq 0, \\
&\sum_{E\in\mathcal{E}_h} \int_E \lambda_t(S_n^{i+1}) K \nabla \hat{P}^{i+1} \cdot \nabla z + \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \int_e [\hat{P}^{i+1}][z] \\
&- \sum_{e\in\Gamma_h} \int_e \{\lambda_t(S_n^{i+1}) K \nabla \hat{P}^{i+1} \cdot n_e\}[z] - \sum_{e\in\Gamma_D} \int_e (\lambda_t(s_n^{dir}) K \nabla \hat{P}^{i+1} \cdot n_e) z \\
&+\varepsilon \sum_{e\in\Gamma_h} \int_e \{\lambda_t(S_n^{i+1}) K \nabla z \cdot n_e\}[\hat{P}^{i+1}] + \varepsilon \sum_{e\in\Gamma_D} \int_e (\lambda_t(s_n^{dir}) K \nabla z \cdot n_e) \hat{P}^{i+1} \\
&= \varepsilon \sum_{e\in\Gamma_D} \int_e (\lambda_t(s_n^{dir}) K \nabla z \cdot n_e) p^{dir} + \sigma_p \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \int_e p^{dir} z + \int_\Omega (q_w + q_n) z.
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
&\forall v \in \mathcal{D}_{r_s}(\mathcal{E}_h), \quad \forall i \geq 0, \\
&\int_\Omega \frac{\phi}{\Delta t}(\hat{S}_n^{i+1} - \hat{S}_n^i) v - \sum_{E\in\mathcal{E}_h} \int_E \lambda_w(S_n^{i+1}) K \nabla \hat{P}^{i+1} \cdot \nabla v + \sum_{E\in\mathcal{E}_h} \int_E \gamma(S_n^{i+1}) K \nabla \hat{S}_n^{i+1} \cdot \nabla v \\
&+ \sum_{e\in\Gamma_h} \int_e \{\lambda_w(S_n^{i+1}) K \nabla \hat{P}^{i+1} \cdot n_e\}[v] + \sum_{e\in\Gamma_D} \int_e (\lambda_w(s_n^{dir}) K \nabla \hat{P}^{i+1} \cdot n_e) v \\
&- \sum_{e\in\Gamma_h} \int_e \{\gamma(S_n^{i+1}) K \nabla \hat{S}_n^{i+1} \cdot n_e\}[v] - \sum_{e\in\Gamma_D} \int_e (\gamma(s_n^{dir}) K \nabla \hat{S}_n^{i+1} \cdot n_e) v \\
&-\varepsilon \sum_{e\in\Gamma_h} \int_e \{\lambda_w(S_n^{i+1}) K \nabla v \cdot n_e\}[\hat{P}^{i+1}] - \varepsilon \sum_{e\in\Gamma_D} \int_e (\lambda_w(s_n^{dir}) K \nabla v \cdot n_e) \hat{P}^{i+1} \\
&+\varepsilon \sum_{e\in\Gamma_h} \int_e \{\gamma(S_n^{i+1}) K \nabla v \cdot n_e\}[\hat{S}_n^{i+1}] + \varepsilon \sum_{e\in\Gamma_D} \int_e \gamma(s_n^{dir}) K \nabla v \cdot n_e) \hat{S}_n^{i+1} \\
&+\sigma_s \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|} \int_e [\hat{S}_n^{i+1}][v] = \sigma_s \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|} \int_e s_n^{dir} v \\
&-\varepsilon \sum_{e\in\Gamma_D} \int_e (\lambda_w(s_n^{dir}) K \nabla v \cdot n_e) p^{dir} + \varepsilon \sum_{e\in\Gamma_D} \int_e (\gamma(s_n^{dir}) K \nabla v \cdot n_e) s_n^{dir} - \int_\Omega q_w v.
\end{aligned}
\tag{45}
$$

The operator G is well-defined only if there exists a unique solution to (43), (44), (45). But this system of equations is linear and can be solved sequentially at each time step. Indeed, (44) corresponds to a DG discretization of an elliptic equation satisfied by $\hat{P}$ and (45) corresponds to a DG discretization of a parabolic equation satisfied by $\hat{S}_n$. Furthermore, it is easy to see that the operator G is continuous as this follows from the continuity of the functions

$\lambda_t, \lambda_w, \lambda_n$ and γ. Finally, the operator G is a compact operator. Indeed, one can show that it transforms bounded sets into bounded sets (relatively compact sets in finite-dimensional spaces) by deriving a priori estimates similar to (21), (30) for $(\hat{P}, \hat{S}_n)$.

Now by construction, for any $\alpha \in [0,1]$, the problem $(P, S_n) = \alpha G(P, S_n)$ has exactly the same solution as the scheme (18)-(19) with $\alpha p^{dir}, \alpha s_n^{dir}, \alpha s_n^0, \alpha q_w$ and αq_n. Since we have $\left\|\alpha p^{dir}\right\|_{0,e} \leq \left\|p^{dir}\right\|_{0,e}$, $\left\|\alpha s_n^{dir}\right\|_{0,e} \leq \left\|s_n^{dir}\right\|_{0,e}$, $\|\alpha s_n^0\|_{0,\Omega} \leq \|s_n^0\|_{0,\Omega}$, $\|\alpha q_w\|_{0,\Omega} \leq \|q_w\|_{0,\Omega}$, and $\|\alpha q_n\|_{0,\Omega} \leq \|q_n\|_{0,\Omega}$, the a priori estimates (21) and (30) are uniformly satisfied for any $\alpha \in [0,1]$ and any solution of $(P, S_n) = \alpha G(P, S_n)$. Therefore, from Leray-Schauder's theorem, there exists a fixed point for G; so there exists at least one solution to (18)-(19).

5 Error Analysis

We now derive a priori error estimates for (18), (19). For $1 \leq i \leq N$, let us denote the numerical errors by

$$\xi^i = S_n^i - \tilde{s}_n^i, \chi^i = \tilde{s}_n^i - s_n^i, \tau^i = P^i - \tilde{p}^i, \theta^i = \tilde{p}^i - p^i, \tag{46}$$

where $\tilde{s}_n \in \mathcal{D}_{r_s}(\mathcal{E}_h)$ and $\tilde{p} \in \mathcal{D}_{r_p}(\mathcal{E}_h)$ are approximations of the exact solutions s_n and p. Here, we use the notation $s_n^i = s_n(t^i), \tilde{s}_n^i = \tilde{s}_n(t^i)$ and similarly for p^i and $\tilde{p}^i$. We assume that

$$\forall t \in [0,T], \quad \tilde{p}(t) \in W^{1,\infty}(\Omega), \quad \tilde{s}(t) \in W^{1,\infty}(\Omega), \tag{47}$$

and that the following optimal bounds hold for any $E \in \mathcal{E}_h$ and $t > 0$ (see [4]): there is a constant C independent of h, r_s, r_p and Δt such that

$$\forall 0 \leq q \leq \kappa_s, \quad \|s_n(t) - \tilde{s}_n(t)\|_{q,E} \leq C \frac{h^{\min(r_s+1,\kappa_s)}}{r_s^{\kappa_s - q}} \|s_n(t)\|_{\kappa_s,E}, \tag{48}$$

$$\forall 0 \leq q \leq \kappa_p, \quad \|p(t) - \tilde{p}(t)\|_{q,E} \leq C \frac{h^{\min(r_p+1,\kappa_p)}}{r_p^{\kappa_p - q}} \|p(t)\|_{\kappa_p,E}. \tag{49}$$

We first prove two lemmas that contain bounds of the discrete errors τ^i and ξ^i.

Lemma 7 *If*

$$\sigma_p > 8(1-\varepsilon)^2 \frac{(\overline{\lambda_t})^2 \overline{k} C_t^2 \tilde{C}}{\underline{\lambda}_t},$$

then, there is a constant M independent of h, r_s, r_p and Δt such that:

$$\begin{aligned}
\forall i \geq 0, \quad & \sum_{E \in \mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla \tau^{i+1} \right\|_{0,E}^2 + \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{f(r_p)}{|e|} \left\| [\tau^{i+1}] \right\|_{0,e}^2 \\
& \leq \left(\frac{72 \overline{\lambda_t}^2 \overline{k} C_t^4 \tilde{C}^2}{\underline{\lambda}_t} + 3\sigma_p C_t^2 \tilde{C} \right) \frac{f(r_p)}{M h^2} \|\theta^{i+1}\|_{0,\Omega}^2 \\
& + \frac{1}{M} \left(\frac{2(\overline{\lambda_t})^2 \overline{k}}{\underline{\lambda}_t} + \frac{6 \overline{\lambda_t}^2 \overline{k}^2 C_t^2 \tilde{C}}{\sigma_p f(r_p)} + \frac{72 \overline{\lambda_t}^2 \overline{k} C_t^4 \tilde{C}^2 f(r_p)}{\underline{\lambda}_t} + 3\sigma_p^2 C_t^2 \tilde{C} f(r_p) \right) \sum_{E \in \mathcal{E}_h} \left\| \nabla \theta^{i+1} \right\|_{0,E}^2 \\
& + \frac{6 (\overline{\lambda_t})^2 (\overline{k})^2 C_t^2 \tilde{C}}{M \sigma_p} \frac{h^2}{f(r_p)} \sum_{E \in \mathcal{E}_h} \left\| \nabla^2 \theta^{i+1} \right\|_{0,E}^2 \\
& + \frac{1}{M} \left(\frac{4 C_\lambda^2 \|\nabla \tilde{p}^{i+1}\|_\infty^2 \overline{k}}{\underline{\lambda}_t} + \frac{3 C_\lambda^2 C_t^2 \tilde{C} \|\nabla \tilde{p}^{i+1}\|_\infty^2 (\overline{k})^2 f(r_s)}{4 \sigma_p f(r_p)} \right) \|\xi^{i+1}\|_{0,\Omega}^2 \\
& + \frac{1}{M} \left(\frac{4 C_\lambda^2 \|\nabla \tilde{p}^{i+1}\|_\infty^2 \overline{k}}{\underline{\lambda}_t} + \frac{3 \tilde{C} C_\lambda^2 C_t^2 \overline{k}^2 \|\nabla \tilde{p}^{i+1}\|_\infty^2}{4 \sigma_p f(r_p)} \right) \|\chi^{i+1}\|_{0,\Omega}^2 \\
& + \frac{3 \tilde{C} C_\lambda^2 C_t^2 \overline{k}^2 \|\nabla \tilde{p}^{i+1}\|_\infty^2 h^2}{4 M \sigma_p f(r_p)} \sum_{E \in \mathcal{E}_h} \|\nabla \chi^{i+1}\|_{0,E}^2.
\end{aligned}$$

An expression for M is

$$M = \min\left(\frac{\underline{\lambda}_t}{2}, \frac{\sigma_p}{2} - 4(1-\varepsilon)^2 \frac{(\overline{\lambda_t})^2 \overline{k} C_t^2 \tilde{C}}{\underline{\lambda}_t} \right).$$

PROOF. Using the consistency of the scheme and choosing the test function

$z = \tau^{i+1}$, we obtain one error equation for the global pressure:

$$\sum_{E\in\mathcal{E}_h}\int_E \lambda_t(S_n^{i+1})K\nabla\tau^{i+1}\cdot\nabla\tau^{i+1} + \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_p)}{|e|}\int_e[\tau^{i+1}]^2 =$$

$$(1-\epsilon)\sum_{e\in\Gamma_h}\int_e\{\lambda_t(S_n^{i+1})K\nabla\tau^{i+1}\cdot n_e\}[\tau^{i+1}] + (1-\epsilon)\sum_{e\in\Gamma_D}\int_e(\lambda_t(s_n^{dir})K\nabla\tau^{i+1}\cdot n_e)\tau^{i+1}$$

$$-\sum_{E\in\mathcal{E}_h}\int_E \lambda_t(s_n^{i+1})K\nabla\theta^{i+1}\cdot\nabla\tau^{i+1}$$

$$+\sum_{e\in\Gamma_h}\int_e\{\lambda_t(s_n^{i+1})K\nabla\theta^{i+1}\cdot n_e\}[\tau^{i+1}] + \sum_{e\in\Gamma_D}\int_e(\lambda_t(s_n^{dir})K\nabla\theta^{i+1}\cdot n_e)\tau^{i+1}$$

$$-\varepsilon\sum_{e\in\Gamma_h}\int_e\{\lambda_t(s_n^{i+1})K\nabla\tau^{i+1}\cdot n_e\}[\theta^{i+1}] - \varepsilon\sum_{e\in\Gamma_D}\int_e(\lambda_t(s_n^{dir})K\nabla\tau^{i+1}\cdot n_e)\theta^{i+1}$$

$$-\sigma_p\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_p)}{|e|}\int_e[\theta^{i+1}][\tau^{i+1}]$$

$$-\sum_{E\in\mathcal{E}_h}\int_E(\lambda_t(S_n^{i+1})-\lambda_t(s_n^{i+1}))K\nabla\tilde{p}^{i+1}\cdot\nabla\tau^{i+1} \tag{50}$$

$$+\sum_{e\in\Gamma_h}\int_e\{(\lambda_t(S_n^{i+1})-\lambda_t(s_n^{i+1}))K\nabla\tilde{p}^{i+1}\cdot n_e\}[\tau^{i+1}]$$

$$-\varepsilon\sum_{e\in\Gamma_h}\int_e\{(\lambda_t(S_n^{i+1})-\lambda_t(s_n^{i+1}))K\nabla\tau^{i+1}\cdot n_e\}[\tilde{p}^{i+1}]$$

$$= T_1 + \cdots + T_{11}.$$

Next, we bound each term in the right-hand side of (50) using techniques standard to DG methods. In what follows, the quantities ϵ_i are positive real numbers to be defined later. Using Assumption H3, H4 and Cauchy-Schwarz inequality, we have

$$|T_1| \le (1-\varepsilon)\overline{\lambda_t}(\overline{k})^{\frac{1}{2}}\sum_{e\in\Gamma_h}\left\|\{K^{\frac{1}{2}}\nabla\tau^{i+1}\}\right\|_{0,e}\left\|[\tau^{i+1}]\right\|_{0,e}.$$

We now fix an interior edge e and denote by E_e^1 and E_e^2 the two elements sharing the edge e. Using the trace inequality (16) and (11), we have:

$$\sum_{e\in\Gamma_h}\left\|\{K^{\frac{1}{2}}\nabla\tau^{i+1}\}\right\|_{0,e}\left\|[\tau^{i+1}]\right\|_{0,e}\leq\sum_{e\in\Gamma_h}\frac{1}{2}(\|K^{1/2}\nabla\tau^{i+1}|_{E_e^1}\|_{0,e}+\|K^{1/2}\nabla\tau^{i+1}|_{E_e^2}\|_{0,e})\|[\tau^{i+1}]\|_{0,e}$$
$$\leq\frac{1}{2}C_t\sqrt{\frac{f(r_p)}{h}}\sum_{e\in\Gamma_h}(\|K^{1/2}\nabla\tau^{i+1}\|_{0,E_e^1}+\|K^{1/2}\nabla\tau^{i+1}\|_{0,E_e^2})\|[\tau^{i+1}]\|_{0,e}$$
$$\leq(\sum_{e\in\Gamma_h}\frac{C_t^2f(r_p)}{4h}\|[\tau^{i+1}]\|_{0,e}^2)^{1/2}\Big((\sum_{e\in\Gamma_h}\|K^{1/2}\nabla\tau^{i+1}\|_{0,E_e^1}^2)^{1/2}+(\sum_{e\in\Gamma_h}\|K^{1/2}\nabla\tau^{i+1}\|_{0,E_e^2}^2)^{1/2}\Big)$$
$$\leq(\sum_{e\in\Gamma_h}\frac{\tilde{C}C_t^2f(r_p)}{4h}\|[\tau^{i+1}]\|_{0,e}^2)^{1/2}(\sum_{E\in\mathcal{E}_h}\|K^{1/2}\nabla\tau^{i+1}\|_{0,E}^2)^{1/2}.$$

Therefore, we have the following bound for T_1:

$$|T_1|\leq\frac{\varepsilon_1}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2+(1-\varepsilon)^2\frac{(\overline{\lambda_t})^2\overline{k}\tilde{C}C_t^2}{8\varepsilon_1}\sum_{e\in\Gamma_h}\frac{f(r_p)}{h}\left\|[\tau^{i+1}]\right\|_{0,e}^2$$
$$\leq\frac{\varepsilon_1}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2+(1-\varepsilon)^2\frac{(\overline{\lambda_t})^2\overline{k}\tilde{C}C_t^2}{8\varepsilon_1}\sum_{e\in\Gamma_h}\frac{f(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|_{0,e}^2. \quad (51)$$

Similarly, we have for T_2:

$$|T_2|\leq\frac{\varepsilon_1}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2+(1-\varepsilon)^2\frac{(\overline{\lambda_t})^2\overline{k}\tilde{C}C_t^2}{2\varepsilon_1}\sum_{e\in\Gamma_D}\frac{f(r_p)}{|e|}\left\|\tau^{i+1}\right\|_{0,e}^2. \quad (52)$$

The term T_3 is bounded using Assumption H3, H4, Cauchy-Schwarz and Young's inequality.

$$|T_3|\leq\overline{\lambda_t}(\overline{k})^{\frac{1}{2}}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}\left\|\nabla\theta^{i+1}\right\|_{0,E}$$
$$\leq\varepsilon_3\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2+\frac{(\overline{\lambda_t})^2\overline{k}}{4\varepsilon_3}\sum_{E\in\mathcal{E}_h}\left\|\nabla\theta^{i+1}\right\|_{0,E}^2. \quad (53)$$

The terms T_4 and T_5 are bounded in a similar way as the terms T_1 and T_2, except that the trace inequality (15) is used instead of (16).

$$
\begin{aligned}
|T_4| \le \overline{\lambda}_t\overline{k} \sum_{e\in\Gamma_h} \left\|\{\nabla\theta^{i+1}\}\right\|_{0,e} \left\|[\tau^{i+1}]\right\|_{0,e} \\
\le (\sum_{e\in\Gamma_h} \frac{C_t^2\tilde{C}\overline{\lambda_t}^2\overline{k}^2}{4h} \|[\tau^{i+1}]\|_{0,e}^2)^{1/2}\Big((\sum_{E\in\mathcal{E}_h} \|\nabla\theta^{i+1}\|_{0,E}^2)^{1/2} + (\sum_{E\in\mathcal{E}_h} h^2\|\nabla^2\theta^{i+1}\|_{0,E}^2)^{1/2}\Big) \\
\le \varepsilon_4 \sum_{e\in\Gamma_h} \frac{f(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2 + \frac{C_t^2\tilde{C}\overline{\lambda_t}^2\overline{k}^2}{8\varepsilon_4 f(r_p)}(\sum_{E\in\mathcal{E}_h} (\|\nabla\theta^{i+1}\|_{0,E}^2 + h^2\|\nabla^2\theta^{i+1}\|_{0,E}^2)).
\end{aligned}
\tag{54}
$$

Similarly, we have for T_5:

$$
|T_5| \le \varepsilon_4 \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \left\|\tau^{i+1}\right\|_{0,e}^2 + \frac{C_t^2\tilde{C}\overline{\lambda_t}^2\overline{k}^2}{2\varepsilon_4 f(r_p)}(\sum_{E\in\mathcal{E}_h} (\|\nabla\theta^{i+1}\|_{0,E}^2 + h^2\|\nabla^2\theta^{i+1}\|_{0,E}^2)).
\tag{55}
$$

The terms T_6 and T_7 are handled in the same way as the terms T_1 and T_2, with the exception that the trace inequality (14) is used to handle the approximation error term.

$$
\begin{aligned}
|T_6| \le \overline{\lambda}_t(\overline{k})^{\frac{1}{2}} \sum_{e\in\Gamma_h} \left\|\{K^{\frac{1}{2}}\nabla\tau^{i+1}\}\right\|_{0,e} \left\|[\theta^{i+1}]\right\|_{0,e} \\
\le \varepsilon_6 \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla\tau^{i+1}\|_{0,E}^2 + \frac{\overline{\lambda_t}^2\overline{k}C_t^4\tilde{C}^2 f(r_p)}{2\varepsilon_6}(\sum_{E\in\mathcal{E}_h} (h^{-2}\|\theta^{i+1}\|_{0,E}^2 + \|\nabla\theta^{i+1}\|_{0,E}^2)).
\end{aligned}
\tag{56}
$$

Similarly, for T_7, we have:

$$
|T_7| \le \varepsilon_6 \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla\tau^{i+1}\|_{0,E}^2 + \frac{\overline{\lambda_t}^2\overline{k}C_t^4\tilde{C}^2 f(r_p)}{4\varepsilon_6}(\sum_{E\in\mathcal{E}_h} (h^{-2}\|\theta^{i+1}\|_{0,E}^2 + \|\nabla\theta^{i+1}\|_{0,E}^2)).
\tag{57}
$$

Using the trace inequality (15), we have for the term T_8:

$$
|T_8| \le \varepsilon_5 \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2 + \frac{\sigma_p^2 C_t^2\tilde{C} f(r_p)}{2\varepsilon_5}(\sum_{E\in\mathcal{E}_h} (h^{-2}\|\theta^{i+1}\|_{0,E}^2 + \|\nabla\theta^{i+1}\|_{0,E}^2)).
\tag{58}
$$

Using Assumption H2, H4, Cauchy-Schwarz inequality and (47), we have:

$$\begin{aligned}|T_9| &\leq C_\lambda \left\|\nabla \tilde{p}^{i+1}\right\|_\infty (\overline{k})^{\frac{1}{2}} \sum_{E\in\mathcal{E}_h} \|S_n^{i+1} - s_n^{i+1}\|_{0,E} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E} \\ &\leq C_\lambda \left\|\nabla \tilde{p}^{i+1}\right\|_\infty (\overline{k})^{\frac{1}{2}} \sum_{E\in\mathcal{E}_h} (\|\xi^{i+1}\|_{0,E} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_0 + \|\chi^{i+1}\|_{0,E} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}) \\ &\leq 2\varepsilon_9 \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2 + \frac{C_\lambda^2 \|\nabla\tilde{p}^{i+1}\|_\infty^2 \overline{k}}{4\varepsilon_9}\|\xi^{i+1}\|_{0,\Omega}^2 + \frac{C_\lambda^2 \|\nabla\tilde{p}^{i+1}\|_\infty^2 \overline{k}}{4\varepsilon_9}\|\chi^{i+1}\|_{0,\Omega}^2.\end{aligned} \tag{59}$$

The term T_{10} is a summation term over interior edges. We assume that the edge e is shared by the elements E_e^1 and E_e^2. Thus, we have using Assumption H2, H4, Cauchy-Schwarz inequality and (47).

$$|T_{10}| \leq \left\|\nabla\tilde{p}^{i+1}\right\|_\infty \overline{k}\frac{C_\lambda}{2} \sum_{e\in\Gamma_h} (\left\|\xi^{i+1}|_{E_e^1}\right\|_{0,e} + \left\|\xi^{i+1}|_{E_e^2}\right\|_{0,e} + \left\|\chi^{i+1}|_{E_e^1}\right\|_{0,e} + \left\|\chi^{i+1}|_{E_e^2}\right\|_{0,e}) \left\|[\tau^{i+1}]\right\|_{0,e}.$$

Using the trace inequalities (14), (16), we have:

$$\begin{aligned}|T_{10}| &\leq \frac{C_\lambda}{2}\overline{k}\|\nabla\tilde{p}^{i+1}\|_\infty C_t\sqrt{\frac{f(r_s)}{h}} \sum_{e\in\Gamma_h} (\|\xi^{i+1}\|_{0,E_e^1} + \|\xi^{i+1}\|_{0,E_e^2})\|[\tau^{i+1}]\|_{0,e} \\ &+\frac{C_\lambda}{2}\overline{k}\|\nabla\tilde{p}^{i+1}\|_\infty C_t h^{-1/2} \sum_{e\in\Gamma_h} (\|\chi^{i+1}\|_{0,E_e^1} + \|\chi^{i+1}\|_{0,E_e^2} + h\|\nabla\chi^{i+1}\|_{0,E_e^1} + h\|\nabla\chi^{i+1}\|_{0,E_e^2})\|[\tau^{i+1}]\|_{0,e} \\ &\leq \epsilon_{10} \sum_{e\in\Gamma_h} \frac{f(r_s)}{|e|}\|[\tau^{i+1}]\|_{0,e}^2 + \frac{\tilde{C}C_\lambda^2 C_t^2 \overline{k}^2 \|\nabla\tilde{p}^{i+1}\|_\infty^2}{8\epsilon_{10}}\|\xi^{i+1}\|_{0,\Omega}^2 \\ &\quad + \frac{\tilde{C}C_\lambda^2 C_t^2 \overline{k}^2 \|\nabla\tilde{p}^{i+1}\|_\infty^2}{8\epsilon_{10} f(r_s)} \sum_{E\in\mathcal{E}_h} (\|\chi^{i+1}\|_{0,E}^2 + h^2\|\nabla\chi^{i+1}\|_{0,E}^2).\end{aligned} \tag{60}$$

The term T_{11} vanishes if the approximation $\tilde{p}$ is continuous. Otherwise, we can bound it exactly like the term T_6.

$$\begin{aligned}T_{11} &\leq \sum_{e\in\Gamma_h} \left|\int_e \{(\lambda_t(S_n^{i+1}) - \lambda_t(s_n^{i+1}))K\nabla\tau^{i+1}\cdot n_e\}[\theta^{i+1}]\right| \\ &\leq \varepsilon_6 \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla\tau^{i+1}\|_{0,E}^2 + \frac{2\overline{\lambda_t}^2 \overline{k} C_t^4 \tilde{C}^2 f(r_p)}{\varepsilon_6}(\sum_{E\in\mathcal{E}_h} (h^{-2}\|\theta^{i+1}\|_{0,E}^2 + \|\nabla\theta^{i+1}\|_{0,E}^2)).\end{aligned} \tag{61}$$

Combining all the bounds (51)-(61) obtained above we have the following estimate for the pressure equation :

$$
\begin{aligned}
&(\underline{\lambda_t} - \varepsilon_1 - \varepsilon_3 - 3\varepsilon_6 - 2\varepsilon_9) \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla \tau^{i+1} \right\|_{0,E}^2 \\
&+ \Big(\sigma_p - (1-\varepsilon)^2 \frac{(\overline{\lambda_t})^2 \overline{k} C_t^2 \tilde{C}}{8\varepsilon_1} - \varepsilon_4 - \varepsilon_5 - \varepsilon_{10} \frac{f(r_s)}{f(r_p)} \Big) \sum_{e\in\Gamma_h} \frac{f(r_p)}{|e|} \left\| [\tau^{i+1}] \right\|_{0,e}^2 \\
&+ \Big(\sigma_p - (1-\varepsilon)^2 \frac{(\overline{\lambda_t})^2 \overline{k} C_t^2 \tilde{C}}{2\varepsilon_1} - \varepsilon_4 - \varepsilon_5 \Big) \sum_{e\in\Gamma_D} \frac{f(r_p)}{|e|} \left\| \tau^{i+1} \right\|_{0,e}^2 \\
&\leq \Big(\frac{3\overline{\lambda_t}^2 \overline{k} C_t^4 \tilde{C}^2}{\varepsilon_6} + \frac{\sigma_p^2 C_t^2 \tilde{C}}{2\varepsilon_5} \Big) \frac{f(r_p)}{h^2} \| \theta^{i+1} \|_{0,\Omega}^2 \\
&+ \Big(\frac{(\overline{\lambda_t})^2 \overline{k}}{4\varepsilon_3} + \frac{\overline{\lambda_t}^2 \overline{k}^2 C_t^2 \tilde{C}}{\varepsilon_4 f(r_p)} + \frac{3\overline{\lambda_t}^2 \overline{k} C_t^4 \tilde{C}^2 f(r_p)}{\varepsilon_6} + \frac{\sigma_p^2 C_t^2 \tilde{C} f(r_p)}{2\epsilon_5} \Big) \sum_{E\in\mathcal{E}_h} \left\| \nabla \theta^{i+1} \right\|_{0,E}^2 \\
&+ \frac{(\overline{\lambda_t})^2 (\overline{k})^2 C_t^2 \tilde{C}}{\varepsilon_4} \frac{h^2}{f(r_p)} \sum_{E\in\mathcal{E}_h} \left\| \nabla^2 \theta^{i+1} \right\|_{0,E}^2 \\
&+ \Big(\frac{C_\lambda^2 \left\| \nabla \widetilde{p}^{i+1} \right\|_\infty^2 \overline{k}}{4\varepsilon_9} + \frac{C_\lambda^2 C_t^2 \tilde{C} \left\| \nabla \widetilde{p}^{i+1} \right\|_\infty^2 (\overline{k})^2}{8\varepsilon_{10}} \Big) \| \xi^{i+1} \|_{0,\Omega}^2 \\
&+ \Big(\frac{C_\lambda^2 \left\| \nabla \widetilde{p}^{i+1} \right\|_\infty^2 \overline{k}}{4\varepsilon_9} + \frac{\tilde{C} C_\lambda^2 C_t^2 \overline{k}^2 \| \nabla \widetilde{p}^{i+1} \|_\infty^2}{8\epsilon_{10} f(r_s)} \Big) \| \chi^{i+1} \|_{0,\Omega}^2 + \frac{\tilde{C} C_\lambda^2 C_t^2 \overline{k}^2 \| \nabla \widetilde{p}^{i+1} \|_\infty^2 h^2}{8\epsilon_{10} f(r_s)} \sum_{E\in\mathcal{E}_h} \| \nabla \chi^{i+1} \|_{0,E}^2 .
\end{aligned}
\tag{62}
$$

Thus, choosing

$$
\epsilon_1 = \epsilon_3 = 3\epsilon_6 = 2\epsilon_9 = \frac{\underline{\lambda_t}}{8},
$$

and

$$
\epsilon_4 = \epsilon_5 = \epsilon_{10} \frac{f(r_s)}{f(r_p)} = \frac{\sigma_p}{6},
$$

we obtain:

$$\frac{\underline{\lambda_t}}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac12}\nabla\tau^{i+1}\right\|^2_{0,E}$$

$$+\Big(\frac{\sigma_p}{2}-4(1-\varepsilon)^2\frac{(\overline{\lambda_t})^2\overline{k}C_t^2\tilde{C}}{\underline{\lambda_t}}\Big)\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|^2_{0,e}$$

$$\leq\Big(\frac{72\overline{\lambda_t}^2\overline{k}C_t^4\tilde{C}^2}{\underline{\lambda_t}}+3\sigma_pC_t^2\tilde{C}\Big)\frac{f(r_p)}{h^2}\|\theta^{i+1}\|^2_{0,\Omega}$$

$$+\Big(\frac{2(\overline{\lambda_t})^2\overline{k}}{\underline{\lambda_t}}+\frac{6\overline{\lambda_t}^2\overline{k}^2C_t^2\tilde{C}}{\sigma_pf(r_p)}+\frac{72\overline{\lambda_t}^2\overline{k}C_t^4\tilde{C}^2f(r_p)}{\underline{\lambda_t}}+3\sigma_p^2C_t^2\tilde{C}f(r_p)\Big)\sum_{E\in\mathcal{E}_h}\left\|\nabla\theta^{i+1}\right\|^2_{0,E}$$

$$+\frac{6(\overline{\lambda_t})^2(\overline{k})^2C_t^2\tilde{C}}{\sigma_p}\frac{h^2}{f(r_p)}\sum_{E\in\mathcal{E}_h}\left\|\nabla^2\theta^{i+1}\right\|^2_{0,E}$$

$$+\Big(\frac{4C_\lambda^2\left\|\nabla\tilde{p}^{i+1}\right\|^2_\infty\overline{k}}{\underline{\lambda_t}}+\frac{3C_\lambda^2C_t^2\tilde{C}\left\|\nabla\tilde{p}^{i+1}\right\|^2_\infty(\overline{k})^2f(r_s)}{4\sigma_pf(r_p)}\Big)\|\xi^{i+1}\|^2_{0,\Omega}$$

$$+\Big(\frac{4C_\lambda^2\left\|\nabla\tilde{p}^{i+1}\right\|^2_\infty\overline{k}}{\underline{\lambda_t}}+\frac{3\tilde{C}C_\lambda^2C_t^2\overline{k}^2\|\nabla\tilde{p}^{i+1}\|^2_\infty}{4\sigma_pf(r_p)}\Big)\|\chi^{i+1}\|^2_{0,\Omega}+\frac{3\tilde{C}C_\lambda^2C_t^2\overline{k}^2\|\nabla\tilde{p}^{i+1}\|^2_\infty h^2}{4\sigma_pf(r_p)}\sum_{E\in\mathcal{E}_h}\|\nabla\chi^{i+1}\|^2_{0,E}.$$

Lemma 8 *Let $\epsilon_2^s,\epsilon_{10}^s,\epsilon_{13}^s,\epsilon_{15}^s,\epsilon_{23}^s$ be positive real numbers. For $i\geq 0$, define*

$$\rho^{i+1}=\frac{1}{\Delta t}\Big(\frac{\tilde{s}_n^{i+1}-\tilde{s}_n^i}{\Delta t}-\frac{\partial\tilde{s}_n^{i+1}}{\partial t}\Big).$$

Then, the following bound holds:

$$\forall i\geq 0,\quad \frac{\phi}{2\Delta t}(\|\xi^{i+1}\|^2_{0,\Omega}-\|\xi^i\|^2_{0,\Omega})+\frac{\gamma}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac12}\nabla\xi^{i+1}\right\|^2_{0,E}$$

$$+\Big(\sigma_s-\frac{\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2}{8\epsilon_2^s}\frac{f(r_p)}{f(r_s)}-(1-\epsilon)^2\frac{2\overline{\gamma}^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}}-\epsilon_{10}^s-\epsilon_{13}^s-\epsilon_{15}^s-2\epsilon_{23}^s\Big)\sum_{e\in\Gamma_h}\frac{f(r_s)}{|e|}\left\|[\xi^{i+1}]\right\|^2_{0,e}$$

$$+\Big(\sigma_s-\frac{\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2}{2\epsilon_2^s}\frac{f(r_p)}{f(r_s)}-(1-\epsilon)^2\frac{8\overline{\gamma}^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}}-\epsilon_{10}^s-\epsilon_{13}^s-\epsilon_{15}^s\Big)\sum_{e\in\Gamma_D}\frac{f(r_s)}{|e|}\left\|\xi^{i+1}\right\|^2_{0,e}$$

$$\leq\Big(\frac{4(\overline{\lambda_w})^2}{\underline{\gamma}}+\epsilon_2^s\Big)\sum_{E\in\mathcal{E}_h}\left\|K^{\frac12}\nabla\tau^{i+1}\right\|^2_{0,E}$$

$$+\frac{8(\overline{\lambda_w})^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}}\frac{f(r_s)}{f(r_p)}\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|^2_{0,e}$$

$$+\Big(1+\Big(\frac{\tilde{C}C_t^2\overline{k}^2}{8\epsilon_{23}^s}+\frac{4\overline{k}}{\underline{\gamma}}(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|^2_\infty+C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|^2_\infty)\Big)\Big)\|\xi^{i+1}\|^2_{0,\Omega}$$

$$+\frac{\overline{\phi}^2\Delta t^2}{2}\|\rho^{i+1}\|^2_{0,\Omega}+\frac{\overline{\phi}^2}{2}\|\chi_t^{i+1}\|^2_{0,\Omega}$$

$$+\Big(\frac{\sigma_s^2 C_t^2 \tilde{C} f(r_s)}{2\epsilon_{10}^s h^2}+\frac{192\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}h^2}+(\frac{16\overline{k}}{\underline{\gamma}}+\frac{\tilde{C}C_t^2\overline{k}^2}{8\epsilon_{23}^s f(r_s)})(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2+C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|_\infty^2)\Big)\|\chi^{i+1}\|_{0,\Omega}^2$$

$$+\frac{160\overline{\lambda}_w^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}h^2}\|\theta^{i+1}\|_{0,\Omega}^2$$

$$+\Big(\frac{\sigma_s^2 C_t^2 \tilde{C} f(r_s)}{2\epsilon_{10}^s}+\frac{4\overline{\gamma}^2\overline{k}}{\underline{\gamma}}+\frac{C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2}{\epsilon_{15}^s f(r_s)}+\frac{192\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}}$$

$$+\frac{\tilde{C}C_t^2\overline{k}^2 h^2}{8\epsilon_{23}^s f(r_s)}(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2+C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|_\infty^2)\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\chi^{i+1}\|_{0,E}^2$$

$$+\Big(\frac{4\overline{\lambda}_w^2\overline{k}}{\underline{\gamma}}+\frac{C_t^2\tilde{C}\overline{\lambda}_w^2\overline{k}^2}{\epsilon_{13}^s f(r_s)}+\frac{160\overline{\lambda_w}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}}\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\theta^{i+1}\|_{0,E}^2$$

$$+\frac{C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2 h^2}{\epsilon_{15}^s f(r_s)}\sum_{E\in\mathcal{E}_h}\|\nabla^2\chi^{i+1}\|_{0,E}^2+\frac{C_t^2\tilde{C}\overline{\lambda}_w^2\overline{k}^2 h^2}{\epsilon_{13}^s f(r_s)}\sum_{E\in\mathcal{E}_h}\|\nabla^2\theta^{i+1}\|_{0,E}^2.$$

PROOF. Using the consistency of the scheme, choosing the test function $v=\xi^{i+1}$, and defining $\rho^{i+1}=\frac{1}{\Delta t}(\frac{\tilde{s}_n^{i+1}-\tilde{s}_n^i}{\Delta t}-\frac{\partial\tilde{s}_n^{i+1}}{\partial t})$ we obtain one error equation for the non-wetting phase saturation:

$$\int_\Omega\frac{\phi}{\Delta t}(\xi^{i+1}-\xi^i)\xi^{i+1}+\sum_{E\in\mathcal{E}_h}\int_E\gamma(S_n^{i+1})K\nabla\xi^{i+1}\cdot\nabla\xi^{i+1}$$

$$+\sigma_s\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_s)}{|e|}\int_e[\xi^{i+1}]^2=\sum_{E\in\mathcal{E}_h}\int_E\lambda_w(S_n^{i+1})K\nabla\tau^{i+1}\cdot\nabla\xi^{i+1}$$

$$-\sum_{e\in\Gamma_h}\int_e\{\lambda_w(S_n^{i+1})K\nabla\tau^{i+1}\cdot n_e\}[\xi^{i+1}]-\sum_{e\in\Gamma_D}\int_e(\lambda_w(s_n^{dir})K\nabla\tau^{i+1}\cdot n_e)\xi^{i+1}$$

$$+(1-\epsilon)\sum_{e\in\Gamma_h}\int_e\{\gamma(S_n^{i+1})K\nabla\xi^{i+1}\cdot n_e\}[\xi^{i+1}]+(1-\epsilon)\sum_{e\in\Gamma_D}\int_e(\gamma(s_n^{dir})K\nabla\xi^{i+1}\cdot n_e)\xi^{i+1}$$

$$+\varepsilon\sum_{e\in\Gamma_h}\int_e\{\lambda_w(S_n^{i+1})K\nabla\xi^{i+1}\cdot n_e\}[\tau^{i+1}]+\varepsilon\sum_{e\in\Gamma_D}\int_e(\lambda_w(s_n^{dir})K\nabla\xi^{i+1}\cdot n_e)\tau^{i+1}$$

$$-\int_\Omega\Delta t\phi\rho^{i+1}\xi^{i+1}-\int_\Omega\phi\chi_t^{i+1}\xi^{i+1}-\sigma_s\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_s)}{|e|}\int_e[\chi^{i+1}][\xi^{i+1}]$$

$$+\sum_{E\in\mathcal{E}_h}\int_E\lambda_w(s_n^{i+1})K\nabla\theta^{i+1}\cdot\nabla\xi^{i+1}-\sum_{E\in\mathcal{E}_h}\int_E\gamma(s_n^{i+1})K\nabla\chi^{i+1}\cdot\nabla\xi^{i+1}$$

$$-\sum_{e\in\Gamma_h}\int_e\{\lambda_w(s_n^{i+1})K\nabla\theta^{i+1}\cdot n_e\}[\xi^{i+1}]-\sum_{e\in\Gamma_D}\int_e(\lambda_w(s_n^{dir})K\nabla\theta^{i+1}\cdot n_e)\xi^{i+1}$$

$$+\sum_{e\in\Gamma_h}\int_e\{\gamma(s_n^{i+1})K\nabla\chi^{i+1}\cdot n_e\}[\xi^{i+1}]+\sum_{e\in\Gamma_D}\int_e(\gamma(s_n^{dir})K\nabla\chi^{i+1}\cdot n_e)\xi^{i+1}$$

$$+\varepsilon\sum_{e\in\Gamma_h}\int_e\{\lambda_w(s_n^{i+1})K\nabla\xi^{i+1}\cdot n_e\}[\theta^{i+1}]+\varepsilon\sum_{e\in\cup\Gamma_D}\int_e(\lambda_w(s_n^{dir})K\nabla\xi^{i+1}\cdot n_e)\theta^{i+1}$$

$$-\varepsilon\sum_{e\in\Gamma_h}\int_e\{\gamma(s_n^{i+1})K\nabla\xi^{i+1}\cdot n_e\}[\chi^{i+1}]-\varepsilon\sum_{e\in\Gamma_D}\int_e(\gamma(s_n^{dir})K\nabla\xi^{i+1}\cdot n_e)\chi^{i+1}$$

$$+\sum_{E\in\mathcal{E}_h}\int_E(\lambda_w(S_n^{i+1})-\lambda_w(s_n^{i+1}))K\nabla\tilde{p}^{i+1}\cdot\nabla\xi^{i+1}-\sum_{E\in\mathcal{E}_h}\int_E(\gamma(S_n^{i+1})-\gamma(s_n^{i+1}))K\nabla\tilde{s}_n^{i+1}\cdot\nabla\xi^{i+1}$$

$$-\sum_{e\in\Gamma_h}\int_e\{(\lambda_w(S_n^{i+1})-\lambda_w(s_n^{i+1}))K\nabla\tilde{p}^{i+1}\cdot n_e\}[\xi^{i+1}]+\sum_{e\in\Gamma_h}\int_e\{(\gamma(S_n^{i+1})-\gamma(s_n^{i+1}))K\nabla\tilde{s}_n^{i+1}\cdot n_e\}[\xi^{i+1}]$$

$$+\varepsilon\sum_{e\in\Gamma_h}\int_e\{(\lambda_w(S_n^{i+1})-\lambda_w(s_n^{i+1}))K\nabla\xi^{i+1}\cdot n_e\}[\tilde{p}^{i+1}]-\varepsilon\sum_{e\in\Gamma_h}\int_e\{(\gamma(S_n^{i+1})-\gamma(s_n^{i+1}))K\nabla\xi^{i+1}\cdot n_e\}[\tilde{s}_n^{i+1}]$$

$$=A_1+\cdots+A_{26}.$$

We now bound each term in the right-hand side of the equation above. The term A_1 is simply bounded using Assumption H3 and Cauchy-Schwarz inequality

$$|A_1|\leq\varepsilon_1^s\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|_{0,E}^2+\frac{(\overline{\lambda_w})^2}{4\varepsilon_1^s}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2. \tag{63}$$

The term A_2 is bounded in a similar way as for the term T_1:

$$|A_2|\leq\frac{\varepsilon_2^s}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2+\frac{(\overline{\lambda_w})^2\overline{k}\tilde{C}C_t^2}{8\varepsilon_2^s}\sum_{e\in\Gamma_h}\frac{f(r_p)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}^2. \tag{64}$$

The term A_3 is bounded in a similar way as for the term T_2:

$$|A_3|\leq\frac{\varepsilon_2^s}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2+\frac{(\overline{\lambda_w})^2\overline{k}\tilde{C}C_t^2}{2\varepsilon_2^s}\sum_{e\in\Gamma_D}\frac{f(r_p)}{|e|}\left\|\xi^{i+1}\right\|_{0,e}^2. \tag{65}$$

The term A_4 is bounded in a similar way as for the term T_1:

$$|A_4|\leq\frac{\varepsilon_4^s}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|_{0,E}^2+(1-\epsilon)^2\frac{(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{8\varepsilon_4^s}\sum_{e\in\Gamma_h}\frac{f(r_s)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}^2. \tag{66}$$

The term A_5 is bounded in a similar way as for the term T_2:

$$|A_5|\leq\frac{\varepsilon_4^s}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|_{0,E}^2+(1-\epsilon)^2\frac{(\overline{\gamma})^2\overline{k}\tilde{C}C_t^2}{2\varepsilon_4^s}\sum_{e\in\Gamma_D}\frac{f(r_s)}{|e|}\left\|\xi^{i+1}\right\|_{0,e}^2. \tag{67}$$

The term A_6 is bounded in a similar way as for the term T_1:

$$|A_6| \le \frac{\varepsilon_6^s}{2} \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla \xi^{i+1} \right\|_{0,E}^2 + \frac{(\overline{\lambda_w})^2 \overline{k} \tilde{C} C_t^2}{8\varepsilon_6^s} \sum_{e\in\Gamma_h} \frac{f(r_s)}{|e|} \left\| [\tau^{i+1}] \right\|_{0,e}^2. \quad (68)$$

The term A_7 is bounded in a similar way as for the term T_2:

$$|A_7| \le \frac{\varepsilon_6^s}{2} \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla \xi^{i+1} \right\|_{0,E}^2 + \frac{(\overline{\lambda_w})^2 \overline{k} \tilde{C} C_t^2}{2\varepsilon_6^s} \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|} \left\| \tau^{i+1} \right\|_{0,e}^2. \quad (69)$$

The terms A_8 and A_9 are simply bounded using Cauchy-Schwarz's inequality.

$$|A_8| \le \epsilon_8^s \|\xi^{i+1}\|_{0,\Omega}^2 + \frac{\overline{\phi}^2 \Delta t^2}{4\epsilon_s^8} \|\rho^{i+1}\|_{0,\Omega}^2. \quad (70)$$

$$|A_9| \le \epsilon_9^s \|\xi^{i+1}\|_{0,\Omega}^2 + \frac{\overline{\phi}^2}{4\varepsilon_9^s} \|\chi_t^{i+1}\|_{0,\Omega}^2. \quad (71)$$

The term A_{10} is bounded exactly like T_8.

$$|A_{10}| \le \varepsilon_{10}^s \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|} \left\| [\xi^{i+1}] \right\|_{0,e}^2 + \frac{\sigma_s^2 C_t^2 \tilde{C} f(r_s)}{2\varepsilon_{10}^s} \Big(\sum_{E\in\mathcal{E}_h} (h^{-2} \|\chi^{i+1}\|_{0,E}^2 + \|\nabla \chi^{i+1}\|_{0,E}^2) \Big). \quad (72)$$

The term A_{11} is bounded exactly like T_3.

$$|A_{11}| \le \varepsilon_{11}^s \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla \xi^{i+1} \right\|_{0,E}^2 + \frac{(\overline{\lambda_w})^2 \overline{k}}{4\varepsilon_{11}^s} \sum_{E\in\mathcal{E}_h} \left\| \nabla \theta^{i+1} \right\|_{0,E}^2. \quad (73)$$

The term A_{12} is bounded exactly like T_3.

$$|A_{12}| \le \varepsilon_{12}^s \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla \xi^{i+1} \right\|_{0,E}^2 + \frac{(\overline{\gamma})^2 \overline{k}}{4\varepsilon_{12}^s} \sum_{E\in\mathcal{E}_h} \left\| \nabla \chi^{i+1} \right\|_{0,E}^2. \quad (74)$$

The term A_{13} is bounded exactly like T_4.

$$|A_{13}| \le \varepsilon_{13}^s \sum_{e\in\Gamma_h} \frac{f(r_s)}{|e|} \left\| [\xi^{i+1}] \right\|_{0,e}^2 + \frac{C_t^2 \tilde{C} \overline{\lambda_w}^2 \overline{k}^2}{8\varepsilon_{13}^s f(r_s)} \Big(\sum_{E\in\mathcal{E}_h} (\|\nabla \theta^{i+1}\|_{0,E}^2 + h^2 \|\nabla^2 \theta^{i+1}\|_{0,E}^2) \Big). \quad (75)$$

The term A_{14} is bounded exactly like T_5.

$$|A_{14}| \le \varepsilon_{13}^s \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|} \left\| \xi^{i+1} \right\|_{0,e}^2 + \frac{C_t^2 \tilde{C} \overline{\lambda_w}^2 \overline{k}^2}{2\varepsilon_{13}^s f(r_s)} \Big(\sum_{E\in\mathcal{E}_h} (\|\nabla \theta^{i+1}\|_{0,E}^2 + h^2 \|\nabla^2 \theta^{i+1}\|_{0,E}^2) \Big). \quad (76)$$

The term A_{15} is bounded exactly like T_4.

$$|A_{15}| \leq \varepsilon^s_{15} \sum_{e\in\Gamma_h} \frac{f(r_s)}{|e|} \left\|[\xi^{i+1}]\right\|^2_{0,e} + \frac{C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2}{8\varepsilon^s_{15} f(r_s)} (\sum_{E\in\mathcal{E}_h} (\|\nabla\chi^{i+1}\|^2_{0,E} + h^2\|\nabla^2\chi^{i+1}\|^2_{0,E})). \tag{77}$$

The term A_{16} is bounded exactly like T_5.

$$|A_{16}| \leq \varepsilon^s_{15} \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|} \left\|\xi^{i+1}\right\|^2_{0,e} + \frac{C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2}{2\varepsilon^s_{15} f(r_s)} (\sum_{E\in\mathcal{E}_h} (\|\nabla\chi^{i+1}\|^2_{0,E} + h^2\|\nabla^2\chi^{i+1}\|^2_{0,E})). \tag{78}$$

The term A_{17} is bounded exactly like T_6.

$$|A_{17}| \leq \varepsilon^s_{17} \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla\xi^{i+1}\|^2_{0,E} + \frac{\overline{\lambda_w}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{2\varepsilon^s_{17}} (\sum_{E\in\mathcal{E}_h} (h^{-2}\|\theta^{i+1}\|^2_{0,E} + \|\nabla\theta^{i+1}\|^2_{0,E})). \tag{79}$$

The term A_{18} is bounded exactly like T_7.

$$|A_{18}| \leq \varepsilon^s_{17} \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla\xi^{i+1}\|^2_{0,E} + \frac{\overline{\lambda_w}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{4\varepsilon^s_{17}} (\sum_{E\in\mathcal{E}_h} (h^{-2}\|\theta^{i+1}\|^2_{0,E} + \|\nabla\theta^{i+1}\|^2_{0,E})). \tag{80}$$

The term A_{19} is bounded exactly like T_6.

$$|A_{19}| \leq \varepsilon^s_{19} \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla\xi^{i+1}\|^2_{0,E} + \frac{\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{2\varepsilon^s_{19}} (\sum_{E\in\mathcal{E}_h} (h^{-2}\|\chi^{i+1}\|^2_{0,E} + \|\nabla\chi^{i+1}\|^2_{0,E})). \tag{81}$$

The term A_{20} is bounded exactly like T_7.

$$|A_{20}| \leq \varepsilon^s_{19} \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla\xi^{i+1}\|^2_{0,E} + \frac{\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{4\varepsilon^s_{19}} (\sum_{E\in\mathcal{E}_h} (h^{-2}\|\chi^{i+1}\|^2_{0,E} + \|\nabla\chi^{i+1}\|^2_{0,E})). \tag{82}$$

The term A_{21} is bounded exactly like T_9.

$$|A_{21}| \leq 2\varepsilon^s_{21} \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|^2_{0,E} + \frac{C_\lambda^2 \left\|\nabla\widetilde{p}^{i+1}\right\|^2_\infty \overline{k}}{4\varepsilon^s_{21}} \|\xi^{i+1}\|^2_{0,\Omega} + \frac{C_\lambda^2 \left\|\nabla\widetilde{p}^{i+1}\right\|^2_\infty \overline{k}}{4\varepsilon^s_{21}} \|\chi^{i+1}\|^2_{0,\Omega}. \tag{83}$$

The term A_{22} is bounded exactly like T_9.

$$|A_{22}| \leq 2\varepsilon_{21}^s \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|_{0,E}^2 + \frac{C_\gamma^2 \left\|\nabla\tilde{s}_n^{i+1}\right\|_\infty^2 \overline{k}}{4\varepsilon_{21}^s}\|\xi^{i+1}\|_{0,\Omega}^2 + \frac{C_\gamma^2 \left\|\nabla\tilde{s}_n^{i+1}\right\|_\infty^2 \overline{k}}{4\varepsilon_{21}^s}\|\chi^{i+1}\|_{0,\Omega}^2. \tag{84}$$

The term A_{23} is bounded exactly like T_{10}.

$$\begin{aligned}|A_{23}| \leq \epsilon_{23}^s \sum_{e\in\Gamma_h} \frac{f(r_s)}{|e|}\|[\xi^{i+1}]\|_{0,e}^2 + \frac{\tilde{C}C_\lambda^2 C_t^2 \overline{k}^2 \|\nabla\tilde{p}^{i+1}\|_\infty^2}{8\epsilon_{23}^s}\|\xi^{i+1}\|_{0,\Omega}^2 \\ + \frac{\tilde{C}C_\lambda^2 C_t^2 \overline{k}^2 \|\nabla\tilde{p}^{i+1}\|_\infty^2}{8\epsilon_{23}^s f(r_s)} \sum_{E\in\mathcal{E}_h}(\|\chi^{i+1}\|_{0,E}^2 + h^2\|\nabla\chi^{i+1}\|_{0,E}^2). \end{aligned}\tag{85}$$

The term A_{24} is bounded exactly like T_{10}.

$$\begin{aligned}|A_{24}| \leq \epsilon_{23}^s \sum_{e\in\Gamma_h} \frac{f(r_s)}{|e|}\|[\xi^{i+1}]\|_{0,e}^2 + \frac{\tilde{C}C_\gamma^2 C_t^2 \overline{k}^2 \|\nabla\tilde{s}_n^{i+1}\|_\infty^2}{8\epsilon_{23}^s}\|\xi^{i+1}\|_{0,\Omega}^2 \\ + \frac{\tilde{C}C_\gamma^2 C_t^2 \overline{k}^2 \|\nabla\tilde{s}_n^{i+1}\|_\infty^2}{8\epsilon_{23}^s f(r_s)} \sum_{E\in\mathcal{E}_h}(\|\chi^{i+1}\|_{0,E}^2 + h^2\|\nabla\chi^{i+1}\|_{0,E}^2). \end{aligned}\tag{86}$$

The term A_{25} is bounded exactly like T_{11}.

$$|A_{25}| \leq \varepsilon_{19}^s \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla\xi^{i+1}\|_{0,E}^2 + \frac{2\overline{\lambda_w}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\varepsilon_{19}^s}\left(\sum_{E\in\mathcal{E}_h}(h^{-2}\|\theta^{i+1}\|_{0,E}^2 + \|\nabla\theta^{i+1}\|_{0,E}^2)\right). \tag{87}$$

The term A_{26} is bounded exactly like T_{11}.

$$|A_{26}| \leq \varepsilon_{19}^s \sum_{E\in\mathcal{E}_h} \|K^{1/2}\nabla\xi^{i+1}\|_{0,E}^2 + \frac{2\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\varepsilon_{19}^s}\left(\sum_{E\in\mathcal{E}_h}(h^{-2}\|\chi^{i+1}\|_{0,E}^2 + \|\nabla\chi^{i+1}\|_{0,E}^2)\right). \tag{88}$$

Combining the bounds (63)-(88), we obtain:

$$\begin{aligned}&\frac{\phi}{2\Delta t}(\|\xi^{i+1}\|_{0,\Omega}^2 - \|\xi^i\|_{0,\Omega}^2) \\ &+(\underline{\gamma} - \epsilon_1^s - \epsilon_4^s - \epsilon_6^s - \epsilon_{11}^s - \epsilon_{12}^s - 2\epsilon_{17}^s - 4\epsilon_{19}^s - 4\epsilon_{21}^s) \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|_{0,E}^2 \\ &+\left(\sigma_s - \frac{\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2}{8\epsilon_2^s}\frac{f(r_p)}{f(r_s)} - (1-\epsilon^2)\frac{\overline{\gamma}^2\overline{k}\tilde{C}C_t^2}{8\epsilon_4^s} - \epsilon_{10}^s - \epsilon_{13}^s - \epsilon_{15}^s - 2\epsilon_{23}^s\right) \sum_{e\in\Gamma_h} \frac{f(r_s)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}^2 \\ &+\left(\sigma_s - \frac{\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2}{2\epsilon_2^s}\frac{f(r_p)}{f(r_s)} - (1-\epsilon^2)\frac{\overline{\gamma}^2\overline{k}\tilde{C}C_t^2}{2\epsilon_4^s} - \epsilon_{10}^s - \epsilon_{13}^s - \epsilon_{15}^s\right) \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|}\left\|\xi^{i+1}\right\|_{0,e}^2\end{aligned}$$

$$
\begin{aligned}
&\le (\frac{(\overline{\lambda_w})^2}{4\varepsilon_1^s} + \epsilon_2^s) \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla \tau^{i+1} \right\|_{0,E}^2 \\
&+ \frac{(\overline{\lambda_w})^2 \overline{k} \tilde{C} C_t^2}{2\varepsilon_6^s} \frac{f(r_s)}{f(r_p)} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \left\| [\tau^{i+1}] \right\|_{0,e}^2 \\
&+(\varepsilon_8^s + \epsilon_9^s + (\frac{\tilde{C} C_t^2 \overline{k}^2}{8\epsilon_{23}^s} + \frac{\overline{k}}{4\epsilon_{21}^s})(C_\lambda^2 \|\nabla \tilde{p}^{i+1}\|_\infty^2 + C_\gamma^2 \|\nabla \tilde{s}_n^{i+1}\|_\infty^2)) \|\xi^{i+1}\|_{0,\Omega}^2 \\
&+ \frac{\overline{\phi}^2 \Delta t^2}{4\epsilon_s^8} \|\rho^{i+1}\|_{0,\Omega}^2 + \frac{\overline{\phi}^2}{4\varepsilon_9^s} \|\chi_t^{i+1}\|_{0,\Omega}^2 \\
&+\Big(\frac{\sigma_s^2 C_t^2 \tilde{C} f(r_s)}{2\epsilon_{10}^s h^2} + \frac{3\overline{\gamma}^2 \overline{k} C_t^4 \tilde{C}^2 f(r_s)}{\epsilon_{19}^s h^2} + (\frac{\overline{k}}{4\epsilon_{21}^s} + \frac{\tilde{C} C_t^2 \overline{k}^2}{8\epsilon_{23}^s f(r_s)})(C_\lambda^2 \|\nabla \tilde{p}^{i+1}\|_\infty^2 + C_\gamma^2 \|\nabla \tilde{s}_n^{i+1}\|_\infty^2) \Big) \|\chi^{i+1}\|_{0,\Omega}^2 \\
&+ \frac{\overline{\lambda}_w^2 \overline{k} C_t^4 \tilde{C}^2 f(r_s)}{h^2} (\frac{1}{\epsilon_{17}^s} + \frac{2}{\epsilon_{19}^s}) \|\theta^{i+1}\|_{0,\Omega}^2 \\
&+\Big(\frac{\sigma_s^2 C_t^2 \tilde{C} f(r_s)}{2\epsilon_{10}^s} + \frac{\overline{\gamma}^2 \overline{k}}{4\epsilon_{12}^s} + \frac{C_t^2 \tilde{C} \overline{\gamma}^2 \overline{k}^2}{\epsilon_{15}^s f(r_s)} + \frac{3\overline{\gamma}^2 \overline{k} C_t^4 \tilde{C}^2 f(r_s)}{\epsilon_{19}^s} \\
&+ \frac{\tilde{C} C_t^2 \overline{k}^2 h^2}{8\epsilon_{23}^s f(r_s)} (C_\lambda^2 \|\nabla \tilde{p}^{i+1}\|_\infty^2 + C_\gamma^2 \|\nabla \tilde{s}_n^{i+1}\|_\infty^2) \Big) \sum_{E\in\mathcal{E}_h} \|\nabla \chi^{i+1}\|_{0,E}^2 \\
&+\Big(\frac{\overline{\lambda}_w^2 \overline{k}}{4\epsilon_{11}^s} + \frac{C_t^2 \tilde{C} \overline{\lambda}_w^2 \overline{k}^2}{\epsilon_{13}^s f(r_s)} + \overline{\lambda_w}^2 \overline{k} C_t^4 \tilde{C}^2 f(r_s) (\frac{1}{\epsilon_{17}^s} + \frac{2}{\epsilon_{19}^s}) \Big) \sum_{E\in\mathcal{E}_h} \|\nabla \theta^{i+1}\|_{0,E}^2 \\
&+ \frac{C_t^2 \tilde{C} \overline{\gamma}^2 \overline{k}^2 h^2}{\epsilon_{15}^s f(r_s)} \sum_{E\in\mathcal{E}_h} \|\nabla^2 \chi^{i+1}\|_{0,E}^2 + \frac{C_t^2 \tilde{C} \overline{\lambda}_w^2 \overline{k}^2 h^2}{\epsilon_{13}^s f(r_s)} \sum_{E\in\mathcal{E}_h} \|\nabla^2 \theta^{i+1}\|_{0,E}^2.
\end{aligned}
$$

Thus, choosing

$$
\epsilon_1^s = \epsilon_4^s = \epsilon_6^s = \epsilon_{11}^s = \epsilon_{12}^s = 2\epsilon_{17}^s = 4\epsilon_{19}^s = 4\epsilon_{21}^s = \frac{\underline{\gamma}}{16}
$$

we have

$$
\begin{aligned}
&\frac{\underline{\phi}}{2\Delta t}(\|\xi^{i+1}\|_{0,\Omega}^2 - \|\xi^i\|_{0,\Omega}^2) + \frac{\underline{\gamma}}{2} \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla \xi^{i+1} \right\|_{0,E}^2 \\
&+(\sigma_s - \frac{\overline{\lambda}_w^2 \overline{k} \tilde{C} C_t^2}{8\epsilon_2^s} \frac{f(r_p)}{f(r_s)} - (1-\epsilon)^2 \frac{2\overline{\gamma}^2 \overline{k} \tilde{C} C_t^2}{\underline{\gamma}} - \epsilon_{10}^s - \epsilon_{13}^s - \epsilon_{15}^s - 2\epsilon_{23}^s) \sum_{e\in\Gamma_h} \frac{f(r_s)}{|e|} \left\| [\xi^{i+1}] \right\|_{0,e}^2 \\
&+(\sigma_s - \frac{\overline{\lambda}_w^2 \overline{k} \tilde{C} C_t^2}{2\epsilon_2^s} \frac{f(r_p)}{f(r_s)} - (1-\epsilon)^2 \frac{8\overline{\gamma}^2 \overline{k} \tilde{C} C_t^2}{\underline{\gamma}} - \epsilon_{10}^s - \epsilon_{13}^s - \epsilon_{15}^s) \sum_{e\in\Gamma_D} \frac{f(r_s)}{|e|} \left\| \xi^{i+1} \right\|_{0,e}^2 \\
&\le (\frac{4(\overline{\lambda_w})^2}{\underline{\gamma}} + \epsilon_2^s) \sum_{E\in\mathcal{E}_h} \left\| K^{\frac{1}{2}} \nabla \tau^{i+1} \right\|_{0,E}^2 \\
&+ \frac{8(\overline{\lambda_w})^2 \overline{k} \tilde{C} C_t^2}{\underline{\gamma}} \frac{f(r_s)}{f(r_p)} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \left\| [\tau^{i+1}] \right\|_{0,e}^2
\end{aligned}
$$

$$+(\varepsilon_8^s + \epsilon_9^s + (\frac{\tilde{C}C_t^2\overline{k}^2}{8\epsilon_{23}^s} + \frac{4\overline{k}}{\underline{\gamma}})(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2 + C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|_\infty^2))\|\xi^{i+1}\|_{0,\Omega}^2$$
$$+\frac{\overline{\phi}^2\Delta t^2}{4\epsilon_8^s}\|\rho^{i+1}\|_{0,\Omega}^2 + \frac{\overline{\phi}^2}{4\varepsilon_9^s}\|\chi_t^{i+1}\|_{0,\Omega}^2$$
$$+\Big(\frac{\sigma_s^2C_t^2\tilde{C}f(r_s)}{2\epsilon_{10}^s h^2}+\frac{192\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2f(r_s)}{\underline{\gamma}h^2}+(\frac{16\overline{k}}{\underline{\gamma}}+\frac{\tilde{C}C_t^2\overline{k}^2}{8\epsilon_{23}^s f(r_s)})(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2+C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|_\infty^2)\Big)\|\chi^{i+1}\|_{0,\Omega}^2$$
$$+\frac{160\overline{\lambda}_w^2\overline{k}C_t^4\tilde{C}^2f(r_s)}{\underline{\gamma}h^2}\|\theta^{i+1}\|_{0,\Omega}^2$$
$$+\Big(\frac{\sigma_s^2C_t^2\tilde{C}f(r_s)}{2\epsilon_{10}^s} + \frac{4\overline{\gamma}^2\overline{k}}{\underline{\gamma}} + \frac{C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2}{\epsilon_{15}^s f(r_s)} + \frac{192\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2f(r_s)}{\underline{\gamma}}$$
$$+\frac{\tilde{C}C_t^2\overline{k}^2h^2}{8\epsilon_{23}^s f(r_s)}(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2 + C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|_\infty^2)\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\chi^{i+1}\|_{0,E}^2$$
$$+\Big(\frac{4\overline{\lambda}_w^2\overline{k}}{\underline{\gamma}} + \frac{C_t^2\tilde{C}\overline{\lambda}_w^2\overline{k}^2}{\epsilon_{13}^s f(r_s)} + \frac{160\overline{\lambda_w}^2\overline{k}C_t^4\tilde{C}^2f(r_s)}{\underline{\gamma}}\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\theta^{i+1}\|_{0,E}^2$$
$$+\frac{C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2h^2}{\epsilon_{15}^s f(r_s)}\sum_{E\in\mathcal{E}_h}\|\nabla^2\chi^{i+1}\|_{0,E}^2 + \frac{C_t^2\tilde{C}\overline{\lambda}_w^2\overline{k}^2h^2}{\epsilon_{13}^s f(r_s)}\sum_{E\in\mathcal{E}_h}\|\nabla^2\theta^{i+1}\|_{0,E}^2.$$

The final result is obtained by taking $\epsilon_s^8 = \epsilon_s^9 = 0.5$.

Theorem 9 *Assume that $s_n^0 \in H^{r_s}(\Omega)$, and for $1 \le i \le N$, $s_n(t^i) \in H^{r_s+1}(\Omega)$, $p(t^i) \in H^{r_p+1}(\Omega)$, $(s_n)_t(t^i) \in H^{r_s}(\Omega)$ and $(s_n)_{tt} \in L^2([0,T]; H^1(\Omega))$. In addition, assume that*

$$\sigma_s > (1-\epsilon)^2\frac{4\overline{\gamma}^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}}.$$

Then, under the assumptions of Lemma 7 and Lemma 8, and if Δt is small enough, there is a constant C independent of h, r_p, r_s and Δt, but dependent of the quantity $\max((r_s/r_p)^2, 1 + (r_p/r_s)^2)$ such that for any $m \ge 1$:

$$\|\xi^m\|_{0,\Omega}^2 + \Delta t\sum_{i=0}^{m-1}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|_{0,E}^2 + \Delta t\sum_{i=0}^{m-1}\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_s)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}^2 \le \mathcal{C},$$

$$\Delta t\sum_{i=0}^{m-1}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2 + \Delta t\sum_{i=0}^{m-1}\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|_{0,e}^2 \le (1+\frac{r_s^2}{r_p^2})\mathcal{C},$$

with

$$\mathcal{C} = C\Delta t^2\int_0^T \|(s_n)_{tt}\|_{0,\Omega}^2 + C\frac{h^{2r_s}}{r_s^{2r_s}}\|s_n^0\|_{r_s,\Omega}^2 + C\frac{h^{2r_s}}{r_s^{2r_s}}\Delta t\sum_{i=1}^{N}\|(s_n)_t(t^i)\|_{r_s,\Omega}^2$$

$$+C\frac{h^{2r_s}}{r_s^{2r_s-2}}(1+\frac{r_p^2}{r_s^2})\Delta t\sum_{i=1}^{N}\|s_n(t^i)\|^2_{r_s+1,\Omega}+C\frac{h^{2r_p}}{r_p^{2r_p-2}}(1+\frac{r_s^2}{r_p^2}+\frac{r_p^2}{r_s^2}))\Delta t\sum_{i=1}^{N}\|p(t^i)\|^2_{r_p+1,\Omega}.$$

PROOF. We give a detailed proof in the case of the NIPG method, namely with the choice $\epsilon = 1$. The cases corresponding to SIPG and IIPG are handled in the same way; there are additional terms in the derivation and the penalty parameter σ_s must be bounded below:

$$\sigma_s > (1-\epsilon)^2 \frac{4\overline{\gamma}^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}}.$$

The final error estimates are the same with a different constant C. We define the constant

$$L = \max\Big(\frac{4\overline{\lambda}_w^2}{\underline{\gamma}} + \epsilon_2^s, \frac{8\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}}\frac{f(r_s)}{f(r_p)}\Big).$$

From now on, let us assume that $\epsilon = 1$. Using both lemmas, we have:

$$\frac{\phi}{2\Delta t}(\|\xi^{i+1}\|^2_{0,\Omega} - \|\xi^i\|^2_{0,\Omega}) + \frac{\underline{\gamma}}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|^2_{0,E}$$

$$+(\sigma_s - \frac{\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2}{8\epsilon_2^s}\frac{f(r_p)}{f(r_s)} - \epsilon_{10}^s - \epsilon_{13}^s - \epsilon_{15}^s - 2\epsilon_{23}^s)\sum_{e\in\Gamma_h}\frac{f(r_s)}{|e|}\left\|[\xi^{i+1}]\right\|^2_{0,e}$$

$$+(\sigma_s - \frac{\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2}{2\epsilon_2^s}\frac{f(r_p)}{f(r_s)} - \epsilon_{10}^s - \epsilon_{13}^s - \epsilon_{15}^s)\sum_{e\in\Gamma_D}\frac{f(r_s)}{|e|}\left\|\xi^{i+1}\right\|^2_{0,e}$$

$$\leq \frac{L}{M}\Big(\frac{4C_\lambda^2\|\nabla\tilde{p}^{i+1}\|^2_\infty\overline{k}}{\underline{\lambda}_t} + \frac{3C_\lambda^2C_t^2\tilde{C}\|\nabla\tilde{p}^{i+1}\|^2_\infty(\overline{k})^2 f(r_s)}{4\sigma_p f(r_p)}\Big)\|\xi^{i+1}\|^2_{0,\Omega}$$

$$+(1+(\frac{\tilde{C}C_t^2\overline{k}^2}{8\epsilon_{23}^s} + \frac{4\overline{k}}{\underline{\gamma}})(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|^2_\infty + C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|^2_\infty))\|\xi^{i+1}\|^2_{0,\Omega}$$

$$+\frac{\overline{\phi}^2\Delta t^2}{2}\|\rho^{i+1}\|^2_{0,\Omega} + \frac{\overline{\phi}^2}{2}\|\chi_t^{i+1}\|^2_{0,\Omega}$$

$$+\Big(\frac{\sigma_s^2C_t^2\tilde{C}f(r_s)}{2\epsilon_{10}^s h^2} + \frac{192\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}h^2} + (\frac{16\overline{k}}{\underline{\gamma}} + \frac{\tilde{C}C_t^2\overline{k}^2}{8\epsilon_{23}^s f(r_s)})(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|^2_\infty + C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|^2_\infty)\Big)\|\chi^{i+1}\|^2_{0,\Omega}$$

$$+\frac{L}{M}\Big(\frac{4C_\lambda^2\|\nabla\tilde{p}^{i+1}\|^2_\infty\overline{k}}{\underline{\lambda}_t} + \frac{3\tilde{C}C_\lambda^2C_t^2\overline{k}^2\|\nabla\tilde{p}^{i+1}\|^2_\infty}{4\sigma_p f(r_p)}\Big)\|\chi^{i+1}\|^2_{0,\Omega}$$

$$+\Big(\frac{160\overline{\lambda}_w^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}h^2} + (\frac{L}{M})\Big(\frac{72\overline{\lambda}_t^2\overline{k}C_t^4\tilde{C}^2}{\underline{\lambda}_t} + 3\sigma_p C_t^2\tilde{C}\Big)\frac{f(r_p)}{h^2}\Big)\|\theta^{i+1}\|^2_{0,\Omega}$$

$$+\Big(\frac{\sigma_s^2 C_t^2\tilde{C} f(r_s)}{2\epsilon_{10}^s}+\frac{4\overline{\gamma}^2\overline{k}}{\underline{\gamma}}+\frac{C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2}{\epsilon_{15}^s f(r_s)}+\frac{192\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}}$$

$$+\frac{\tilde{C}C_t^2\overline{k}^2 h^2}{8\epsilon_{23}^s f(r_s)}(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2+C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|_\infty^2)+\frac{L}{M}\frac{3h^2\tilde{C}C_\lambda^2C_t^2\overline{k}^2\|\nabla\tilde{p}^{i+1}\|_\infty^2}{4\sigma_p f(r_p)}\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\chi^{i+1}\|_{0,E}^2$$

$$+\Big(\frac{4\overline{\lambda}_w^2\overline{k}}{\underline{\gamma}}+\frac{C_t^2\tilde{C}\overline{\lambda}_w^2\overline{k}^2}{\epsilon_{13}^s f(r_s)}+\frac{160\overline{\lambda_w}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}}\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\theta^{i+1}\|_{0,E}^2$$

$$+\frac{L}{M}\Big(\frac{2(\overline{\lambda}_t)^2\overline{k}}{\underline{\lambda}_t}+\frac{6\overline{\lambda_t}^2\overline{k}^2C_t^2\tilde{C}}{\sigma_p f(r_p)}+\frac{72\overline{\lambda_t}^2\overline{k}C_t^4\tilde{C}^2 f(r_p)}{\underline{\lambda}_t}+3\sigma_p C_t^2\tilde{C}f(r_p)\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\theta^{i+1}\|_{0,E}^2$$

$$+\frac{C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2h^2}{\epsilon_{15}^s f(r_s)}\sum_{E\in\mathcal{E}_h}\|\nabla^2\chi^{i+1}\|_{0,E}^2$$

$$+\Big(\frac{C_t^2\tilde{C}\overline{\lambda}_w^2\overline{k}^2h^2}{\epsilon_{13}^s f(r_s)}+\frac{L}{M}\frac{6(\overline{\lambda}_t)^2(\overline{k})^2C_t^2\tilde{C}}{\sigma_p}\frac{h^2}{f(r_p)}\Big)\sum_{E\in\mathcal{E}_h}\|\nabla^2\theta^{i+1}\|_{0,E}^2,$$

with the constant $M=0.5\min(\underline{\lambda}_t,\sigma_p)$. Next, we choose

$$\epsilon_{10}^s=\epsilon_{13}^s=\epsilon_{15}^s=2\epsilon_{23}^s=\frac{\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2 f(r_p)}{2\epsilon_2^s f(r_s)}=\frac{\sigma_s}{10},$$

In that case, the constant L becomes

$$L=\max\Big(\frac{4\overline{\lambda}_w^2}{\underline{\gamma}}+\frac{5\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2 f(r_p)}{f(r_s)\sigma_s},\frac{8\overline{\lambda}_w^2\overline{k}\tilde{C}C_t^2}{\underline{\gamma}}\frac{f(r_s)}{f(r_p)}\Big),$$

and we obtain:

$$\frac{\underline{\phi}}{2\Delta t}(\|\xi^{i+1}\|_{0,\Omega}^2-\|\xi^i\|_{0,\Omega}^2)+\frac{\underline{\gamma}}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac12}\nabla\xi^{i+1}\right\|_{0,E}^2+\frac{\sigma_s}{2}\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_s)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}^2$$

$$\le\frac{L}{M}\Big(\frac{4C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2\overline{k}}{\underline{\lambda}_t}+\frac{3C_\lambda^2C_t^2\tilde{C}\|\nabla\tilde{p}^{i+1}\|_\infty^2(\overline{k})^2 f(r_s)}{4\sigma_p f(r_p)}\Big)\|\xi^{i+1}\|_{0,\Omega}^2$$

$$+(1+(\frac{2\tilde{C}C_t^2\overline{k}^2}{5\sigma_s}+\frac{4\overline{k}}{\underline{\gamma}})(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2+C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|_\infty^2))\|\xi^{i+1}\|_{0,\Omega}^2$$

$$+\frac14\overline{\phi}^2\Delta t^2\|\rho^{i+1}\|_{0,\Omega}^2+\frac14\overline{\phi}^2\|\chi_t^{i+1}\|_{0,\Omega}^2$$

$$+\Big(\frac{5\sigma_s^2C_t^2\tilde{C}f(r_s)}{\sigma_s h^2}+\frac{192\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}h^2}+(\frac{16\overline{k}}{\underline{\gamma}}+\frac{2\tilde{C}C_t^2\overline{k}^2}{5\sigma_s f(r_s)})(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2+C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|_\infty^2)\Big)\|\chi^{i+1}\|_{0,\Omega}^2$$

$$+\frac{L}{M}\Big(\frac{4C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2\overline{k}}{\underline{\lambda}_t}+\frac{3\tilde{C}C_\lambda^2C_t^2\overline{k}^2\|\nabla\tilde{p}^{i+1}\|_\infty^2}{4\sigma_p f(r_p)}\Big)\|\chi^{i+1}\|_{0,\Omega}^2$$

$$+\Big(\frac{160\overline{\lambda}_w^2\overline{k}C_t^4\tilde{C}^2 f(r_s)}{\underline{\gamma}h^2}+\frac{L}{M}\Big(\frac{72\overline{\lambda_t}^2\overline{k}C_t^4\tilde{C}^2}{\underline{\lambda}_t}+3\sigma_p C_t^2\tilde{C}\Big)\frac{f(r_p)}{h^2}\Big)\|\theta^{i+1}\|_{0,\Omega}^2$$

$$+\Big(5\sigma_s C_t^2\tilde{C}f(r_s)+\frac{4\overline{\gamma}^2\overline{k}}{\underline{\gamma}}+\frac{10C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2}{\sigma_s f(r_s)}+\frac{192\overline{\gamma}^2\overline{k}C_t^4\tilde{C}^2f(r_s)}{\underline{\gamma}}$$

$$+\frac{5\tilde{C}C_t^2\overline{k}^2h^2}{2\sigma_s f(r_s)}(C_\lambda^2\|\nabla\tilde{p}^{i+1}\|_\infty^2+C_\gamma^2\|\nabla\tilde{s}_n^{i+1}\|_\infty^2)$$

$$+\frac{L}{M}\frac{3h^2\tilde{C}C_\lambda^2C_t^2\overline{k}^2\|\nabla\tilde{p}^{i+1}\|_\infty^2}{4\sigma_p f(r_p)}\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\chi^{i+1}\|_{0,E}^2$$

$$+\Big(\frac{4\overline{\lambda}_w^2\overline{k}}{\underline{\gamma}}+\frac{10C_t^2\tilde{C}\overline{\lambda}_w^2\overline{k}^2}{\sigma_s f(r_s)}+\frac{160\overline{\lambda_w}^2\overline{k}C_t^4\tilde{C}^2f(r_s)}{\underline{\gamma}}\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\theta^{i+1}\|_{0,E}^2$$

$$+\frac{L}{M}\Big(\frac{2(\overline{\lambda_t})^2\overline{k}}{\underline{\lambda}_t}+\frac{6\overline{\lambda_t}^2\overline{k}^2C_t^2\tilde{C}}{\sigma_p f(r_p)}+\frac{72\overline{\lambda_t}^2\overline{k}C_t^4\tilde{C}^2f(r_p)}{\underline{\lambda}_t}+3\sigma_p C_t^2\tilde{C}f(r_p)\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\theta^{i+1}\|_{0,E}^2$$

$$+\frac{10C_t^2\tilde{C}\overline{\gamma}^2\overline{k}^2h^2}{\sigma_s f(r_s)}\sum_{E\in\mathcal{E}_h}\|\nabla^2\chi^{i+1}\|_{0,E}^2$$

$$+\Big(\frac{10C_t^2\tilde{C}\overline{\lambda}_w^2\overline{k}^2h^2}{\sigma_s f(r_s)}+\frac{L}{M}\frac{6(\overline{\lambda_t})^2(\overline{k})^2C_t^2\tilde{C}}{\sigma_p}\frac{h^2}{f(r_p)}\Big)\sum_{E\in\mathcal{E}_h}\|\nabla^2\theta^{i+1}\|_{0,E}^2.$$

Therefore, there is a constant C independent of h, r_p, r_s and Δt such that

$$\frac{\phi}{2\Delta t}(\|\xi^{i+1}\|_{0,\Omega}^2-\|\xi^i\|_{0,\Omega}^2)+\frac{\underline{\gamma}}{2}\sum_{E\in\mathcal{E}_h}\left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|_{0,E}^2+\frac{\sigma_s}{2}\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{f(r_s)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}^2$$

$$\le C\Big(1+(1+\frac{f(r_s)}{f(r_p)})\max(\frac{f(r_s)}{f(r_p)},1+\frac{f(r_p)}{f(r_s)})\Big)\|\xi^{i+1}\|_{0,\Omega}^2+C\Delta t^2\|\rho^{i+1}\|_{0,\Omega}^2+C\|\chi_t^{i+1}\|_{0,\Omega}^2$$

$$+C\Big(1+\frac{1}{f(r_s)}+\frac{f(r_s)}{h^2}+(1+\frac{1}{f(r_p)})\max(\frac{f(r_s)}{f(r_p)},1+\frac{f(r_p)}{f(r_s)})\Big)\|\chi^{i+1}\|_{0,\Omega}^2$$

$$+C\Big(\frac{f(r_s)}{h^2}+\frac{f(r_p)}{h^2}\max(\frac{f(r_s)}{f(r_p)},1+\frac{f(r_p)}{f(r_s)})\Big)\|\theta^{i+1}\|_{0,\Omega}^2$$

$$+C\Big(f(r_s)+1+\frac{1+h^2}{f(r_s)}+\frac{h^2}{f(r_p)}\max(\frac{f(r_s)}{f(r_p)},1+\frac{f(r_p)}{f(r_s)})\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\chi^{i+1}\|_{0,E}^2$$

$$+C\Big((1+f(r_p)+\frac{1}{f(r_p)})\max(\frac{f(r_s)}{f(r_p)},1+\frac{f(r_p)}{f(r_s)})\Big)\sum_{E\in\mathcal{E}_h}\|\nabla\theta^{i+1}\|_{0,E}^2$$

$$+C\frac{h^2}{f(r_s)}\sum_{E\in\mathcal{E}_h}\|\nabla^2\chi^{i+1}\|_{0,E}^2+Ch^2\Big(\frac{1}{f(r_s)}+\frac{1}{f(r_p)}\Big)\sum_{E\in\mathcal{E}_h}\|\nabla^2\theta^{i+1}\|_{0,E}^2.$$

Multiplying by $2\Delta t$, summing over $i=0$ to $i=m-1$, using the fact that for any $r\ge 1$, $1\le r^2\le f(r)\le 6r^2$, and using Gronwall's inequality, we obtain that there exists a constant C that is independent of h and Δt but depends on the ratios $(r_s/r_p)^2$ and $(r_p/r_s)^2$ such that for Δt small enough:

$$\underline{\phi}\|\xi^m\|_{0,\Omega}^2 - \underline{\phi}\|\xi^0\|_{0,\Omega}^2 + \underline{\gamma}\Delta t \sum_{i=0}^{m-1} \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|_{0,E}^2 + \sigma_s \Delta t \sum_{i=0}^{m-1} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|} \left\|[\xi^{i+1}]\right\|_{0,e}^2$$

$$\leq C\Delta t^3 \sum_{i=0}^{m-1} \|\rho^{i+1}\|_{0,\Omega}^2 + C\Delta t \sum_{i=0}^{m-1} \|\chi_t^{i+1}\|_{0,\Omega}^2$$
$$+C\Big(\frac{r_s^2}{h^2} + \max(\frac{r_s^2}{r_p^2}, 1 + \frac{r_p^2}{r_s^2})\Big)\|\chi^{i+1}\|_{0,\Omega}^2$$
$$+C\frac{r_s^2}{h^2}\Big(1 + \max(1, \frac{r_p^2}{r_s^2} + \frac{r_p^4}{r_s^4}))\Big)\|\theta^{i+1}\|_{0,\Omega}^2$$
$$+C\Big(r_s^2 + \max(\frac{r_s^2}{r_p^2}, 1 + \frac{r_p^2}{r_s^2})\Big) \sum_{E\in\mathcal{E}_h} \|\nabla\chi^{i+1}\|_{0,E}^2$$
$$+C\Big(r_p^2 \max(\frac{r_s^2}{r_p^2}, 1 + \frac{r_p^2}{r_s^2})\Big) \sum_{E\in\mathcal{E}_h} \|\nabla\theta^{i+1}\|_{0,E}^2$$
$$+C\frac{h^2}{r_s^2} \sum_{E\in\mathcal{E}_h} \|\nabla^2\chi^{i+1}\|_{0,E}^2 + Ch^2\Big(\frac{1}{r_s^2} + \frac{1}{r_p^2}\Big) \sum_{E\in\mathcal{E}_h} \|\nabla^2\theta^{i+1}\|_{0,E}^2.$$

We next bound the error $\|\rho^{i+1}\|_{0,\Omega}$ using a Taylor expansion with integral remainder:

$$\tilde{s}_n^i = \tilde{s}_n^{i+1} - \Delta t \frac{\partial \tilde{s}_n^{i+1}}{\partial t} + \frac{1}{2}\int_{t^i}^{t^{i+1}} (t - t^i)\frac{\partial^2 \tilde{s}_n}{\partial t^2} dt,$$

which easily yields:

$$\|\rho^{i+1}\|_{0,\Omega}^2 \leq \frac{1}{6\Delta t}\int_{t^i}^{t^{i+1}} \|\frac{\partial^2 \tilde{s}_n}{\partial t^2}\|_{0,\Omega}^2 dt. \tag{89}$$

Using the approximation properties (48), (49), and the bound (89), we obtain:

$$\|\xi^m\|_{0,\Omega}^2 + \Delta t \sum_{i=0}^{m-1} \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla\xi^{i+1}\right\|_{0,E}^2 + \Delta t \sum_{i=0}^{m-1} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|} \left\|[\xi^{i+1}]\right\|_{0,e}^2$$

$$\leq C\Delta t^2 \int_0^T \|(s_n)_{tt}\|_{0,\Omega}^2 + C\frac{h^{2r_s}}{r_s^{2r_s}}\|s_n^0\|_{r_s,\Omega}^2 + C\frac{h^{2r_s}}{r_s^{2r_s}}\Delta t \sum_{i=0}^{N-1} \|\frac{\partial s_n^{i+1}}{\partial t}\|_{r_s,\Omega}^2$$
$$+C\frac{h^{2r_s}}{r_s^{2r_s-2}}(1 + \frac{r_p^2}{r_s^2})\Delta t \sum_{i=0}^{N-1} \|s_n^{i+1}\|_{r_s+1,\Omega}^2$$
$$+C\frac{h^{2r_p}}{r_p^{2r_p-2}}(1 + \frac{r_s^2}{r_p^2} + \frac{r_p^2}{r_s^2})\Delta t \sum_{i=0}^{N-1} \|p^{i+1}\|_{r_p+1,\Omega}^2.$$

To obtain the pressure error estimate, we combine Lemma 7 with the equation above. Hence, we obtain:

$$\Delta t \sum_{i=0}^{m-1} \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla \tau^{i+1}\right\|_{0,E}^2 + \Delta t \sum_{i=0}^{m-1} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2$$
$$\leq C\Delta t^2 (1+\frac{r_s^2}{r_p^2}) \int_0^T \|(s_n)_{tt}\|_{0,\Omega}^2 + C\frac{h^{2r_s}}{r_s^{2r_s}}(1+\frac{r_s^2}{r_p^2})\|s_n^0\|_{r_s,\Omega}^2 + C\frac{h^{2r_s}}{r_s^{2r_s}}(1+\frac{r_s^2}{r_p^2})\Delta t \sum_{i=0}^{N-1} \|\frac{\partial s_n^{i+1}}{\partial t}\|_{r_s,\Omega}^2$$
$$+C\frac{h^{2r_s}}{r_s^{2r_s-2}}(1+\frac{r_p^2}{r_s^2})(1+\frac{r_s^2}{r_p^2})\Delta t \sum_{i=0}^{N-1} \|s_n^{i+1}\|_{r_s+1,\Omega}^2 + C\frac{h^{2r_p}}{r_p^{2r_p-2}}(1+\frac{r_s^2}{r_p^2})(1+\frac{r_s^2}{r_p^2}+\frac{r_p^2}{r_s^2})\Delta t \sum_{i=0}^{N-1} \|p^{i+1}\|_{r_p+1,\Omega}^2.$$

A straightforward consequence is the following result.

Corollary 10 *Assume that the ratio $\frac{r_p}{r_s}$ is bounded below and above:*

$$0 < \underline{a} \leq \frac{r_p}{r_s} \leq \overline{a}.$$

Then, there is a constant C independent of h, r_p, r_s and Δt such that

$$\|S_n^m - s_n^m\|_{0,\Omega}^2 + \Delta t \sum_{i=0}^{m-1} \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla (S_n^{i+1} - s_n^{i+1})\right\|_{0,E}^2 + \Delta t \sum_{i=0}^{m-1} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_s)}{|e|} \left\|[S_n^{i+1} - s_n^{i+1}]\right\|_{0,e}^2$$

$$+\Delta t \sum_{i=0}^{m-1} \sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla \tau^{i+1}\right\|_{0,E}^2 + \Delta t \sum_{i=0}^{m-1} \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{f(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2$$

$$\leq C\Delta t^2 + C(\frac{h^{2r_s}}{r_s^{2r_s-2}} + \frac{h^{2r_p}}{r_p^{2r_p-2}}).$$

6 Numerical Results

We consider the simulation of two-phase flow in $\Omega = (0,1)^2$ with the following data.

$$\begin{cases} K(x,y) = 0.5I, \quad \forall (x,y) \in (0,0.5)\times(0,1), \\ K(x,y) = I, \quad \forall (x,y) \in (0.5,1)\times(0,1), \\ \phi(x,y) = 1, \quad \forall (x,y) \in (0,1)^2, \\ \lambda_w(s_n) = (1-s_n)^{\frac{11}{3}}, \\ \lambda_n(s_n) = s_n^2(1-(1-s_n)^{\frac{5}{3}}), \\ p_c(s_n) = (1-s_n)^{-\frac{1}{3}}. \end{cases}$$

The right-hand sides for pressure and saturation equations are taken such that the exact solution is, for $t \geq 0$:

$$\begin{cases} p(x,y,t) = 100(2x-1)^2 e^{0.5x+y-t}, \quad \forall (x,y) \in (0,0.5)\times(0,1), \\ p(x,y,t) = 100(x-0.5)^2 e^{0.5x+y-t}, \quad \forall (x,y) \in (0.5,1)\times(0,1), \\ s_n(x,y,t) = 0.3(2x-1)^2 e^{-1.5x+y-t}, \quad \forall (x,y) \in (0,0.5)\times(0,1), \\ s_n(x,y,t) = 0.3(x-0.5)^2 e^{-1.5x+y-t}, \quad \forall (x,y) \in (0.5,1)\times(0,1). \end{cases}$$

We first present the convergence with respect to a uniform mesh refinement. The initial mesh contains four elements and it is successively refined. The parameters in (18), (19) are chosen as $\epsilon = 1$ and $\sigma_p = \sigma_s = 10$. Table 6 gives the numerical errors in the H_0^1 norm for the non-wetting phase saturation and the global pressure at a given time for polynomial approximations of degree $r_s = r_p = 1$. Table 6 gives the numerical errors for polynomial approximations of degree $r_s = r_p = 2$. We note that optimal convergence rates are obtained.

Second, we investigate the hp convergence of the scheme for all choices of $\epsilon \in$

Table 1

Numerical errors in the H_0^1 norm for (p, s_n) using piecewise linear approximations.

h	H_0^1 error for s_n	saturation rate	H_0^1 error for p	pressure rate
0.5	$1.374814648 \times 10^{-01}$		$1.266150758 \times 10^{+02}$	
0.25	$7.025355749 \times 10^{-02}$	0.968	$1.266150758 \times 10^{+02}$	0.910
0.125	$3.532346005 \times 10^{-02}$	0.992	$3.415982958 \times 10^{+01}$	0.980
0.0625	$1.768656302 \times 10^{-02}$	0.998	$1.707837409 \times 10^{+01}$	1.000
0.0312	$8.846393467 \times 10^{-03}$	0.999	8.528880124	1.002

Table 2

Numerical errors in the H_0^1 norm for (p, s_n) using piecewise quadratic approximations.

h	H_0^1 error for s_n	saturation rate	H_0^1 error for p	pressure rate
0.5	$2.069429403 \times 10^{-02}$		$2.741576805 \times 10^{+01}$	
0.25	$5.332099909 \times 10^{-03}$	1.956	7.253306154	1.918
0.125	$1.343341569 \times 10^{-03}$	1.989	1.829505401	1.987
0.0625	$3.364868585 \times 10^{-04}$	1.997	$4.536003736 \times 10^{-01}$	2.012
0.0312	$8.416253683 \times 10^{-05}$	1.999	$1.125823947 \times 10^{-01}$	2.010

$\{-1, 0, +1\}$ and for the choice $\sigma_p = \sigma_s = 10$. In Fig. 1, we plot the number of degrees of freedom versus the logarithm of the numerical error in the H_0^1 norm for both p (left figure) and s_n (right figure). We consider four different meshes that are obtained by uniformly refining a coarse mesh: they correspond to the curves with diamonds ($h = 0.5$), triangles ($h = 0.25$), squares ($h = 0.125$) and

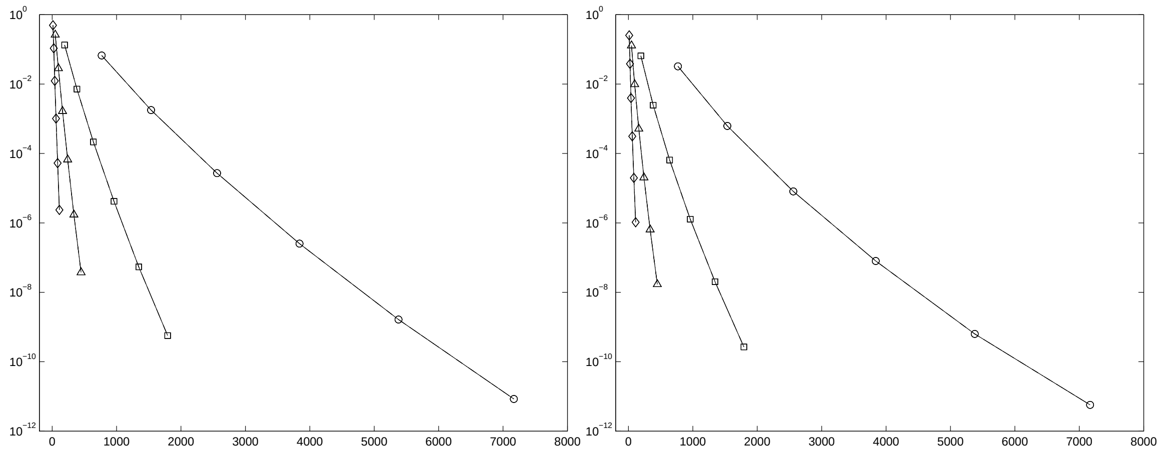

Fig. 1. hp convergence rates for the global pressure (left) and non-wetting phase saturation (right). Each curve corresponds to a fixed mesh and variable polynomial degree from 1 to 6.

circles ($h = 0.0625$). For a fixed mesh, we vary the polynomial degrees from 1 to 6 for both global pressure and non-wetting phase saturation. We observe exponential convergence. There is no noticeable difference between the cases $\epsilon \in \{-1, 0, +1\}$ as the resulting plots coincide. However, our numerical tests show that the SIPG method ($\epsilon = -1$) is very sensitive to the choice of the penalty parameter, which is not the case for the NIPG and IIPG methods. For instance, convergence is obtained for $\sigma_p = \sigma_s = 0.5$ for NIPG and IIPG, but not for SIPG. This can be explained by our theoretical error estimates which give a larger lower bound for the penalty parameters in the case of SIPG. As in [15], one can derive an exact computable lower bound that would yield a stable SIPG method.

7 Conclusions

We have proved convergence of a fully coupled discontinuous Galerkin method for two-phase flow using the global pressure variable. Our estimates are explicit in the mesh size and the polynomial degree. We show that the non-symmetric version of the scheme converges for any positive penalty parameter whereas the symmetric and incomplete versions require the penalty parameter to be sufficiently large. Numerical computations confirm the convergence of the scheme. In a future work, we plan to compare this scheme for benchmark problems to existing discontinuous finite element methods for two-phase flow and to other schemes, such as finite volume methods.

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