SUPERCONVERGENCE OF FINITE ELEMENT DISCRETIZATION OF TIME RELAXATION MODELS OF ADVECTION

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Abstract. The nodal accuracy of finite element discretizations of advection equations including a time relaxation term is studied. Worst case error estimates have been proven for this combination by energy methods. By considering the Cauchy problem with uniform meshes, precise Fourier analysis of the error is possible. This analysis shows (1)the worst case upper bounds are sharp, (2)time relaxation stabilization does not degrade superconvergence of the usual FEM, and (3)higher order time relaxation is preferable to maintain small numerical errors.

Key words. superconvergence, time relaxation, deconvolution

1. Introduction. We consider an approach to eliminating oscillations (forcing them to decay rapidly in time) induced by unresolved scales in conservation laws and convection dominated problems. To reduce the problem to its simplest form (which permits a more exact analysis) consider the advection equation: find $u = u(x,t)$ defined for $x \in \mathbb{R}, t \geq 0$ and satisfying

$$\begin{aligned} u_t = u_x, -\infty < x < \infty, 0 < t \leq T, \\ u(x,0) = f(x), -\infty < x < \infty. \end{aligned} \tag{1.1}$$

Let over-bar denote a local averaging over radius $O(\delta)$ (defined precisely in Section 1.2). Thus, given an approximate solution u^h its average is denoted $\overline{u^h}$ and the fluctuation is $(u^h)' := u^h - \overline{u^h}$. Let $S^{h,\mu}(\mathbb{R})$ denote a finite element space of smoothest splines defined on a uniform mesh (Section 2). The zeroth order example of the approximations we consider is: given a parameter $\alpha > 0$, find $u^h : [0,T] \to S^{h,\mu}(\mathbb{R})$ satisfying

$$\begin{aligned} (u^h_t - u^h_x, v^h) - \alpha((u^h)', v^h) = 0, \forall v^h \in S^{h,\mu}(\mathbb{R}), \\ u^h(x,0) = I^h(f)(x), \end{aligned} \tag{1.2}$$

where I^h is the usual spline interpolation operator. This is the usual Galerkin approximation plus a time relaxation/stabilization term intended to drive small fluctuations to zero exponentially fast, see section 1.1.

The variational multi-scale framework (see Hughes, Mazzei and Jansen [HMJ00]) gives some insight into this mechanism. Briefly, let $r(w) := w_t - w_x$ denote the residual of w in (1.1) and decompose u^h as $u^h = \overline{u^h} + (u^h)'$. Setting alternately $v^h = \overline{v^h}$ and $v^h = (v^h)'$ in (1.2) gives the equivalent coupled system

$$\begin{aligned} (\overline{u^h_t} - \overline{u^h_x}, \overline{v^h}) = (r((u^h)'), \overline{v^h}) = 0, \forall \overline{v^h} \in \overline{S^{h,\mu}(\mathbb{R})}, \\ ((u^h)'_t - (u^h)'_x, (v^h)') - \alpha((u^h)', v^h) = (r(\overline{u^h}), (v^h)'), \forall (v^h)' \in (S^{h,\mu}(\mathbb{R}))'. \end{aligned}$$

The second equation suggests that the relaxation term will tend to derive fluctuations to zero while the first suggests that its effects on the means will be limited to the projected residual term on the RHS.

There are various other realizations of the same idea. For example, when the map $u^h \to (u^h)'$ is not positive semi-definite, (as can arise, e.g., [LS06] and Pruett [P06]),

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the relaxation term should be instead $-\alpha((u^h)', (v^h)')$. The most important variant, analyzed herein and introduced by Stolz, Adams and Kleiser in their computations of turbulent compressible flows [AS02], [SA99], [SAK01a], [SAK01b], [SAK02] (see also [Gue04]), is a higher order time relaxation operator. Briefly, (see Section 3.2) given a deconvolution operator D_N, i.e., a bounded linear operator on $L^2(\mathbb{R})$ with the property

$$\phi = D_N \overline{\phi} + O(\delta^{2N+2}) \text{ for smooth } \phi, \tag{1.3}$$

the higher order[1], generalized fluctuation is

$$(u^h)^* := u^h - D_N \overline{u^h}.$$

The higher order time relaxation discretization is then: given $\alpha > 0$, find $u^h : [0,T] \to S^{h,\mu}(\mathbb{R})$ satisfying

$$(u^h_t - u^h_x, v^h) - \alpha((u^h)^*, v^h) = 0, \forall v^h \in S^{h,\mu}(\mathbb{R}), \tag{1.4}$$
$$u^h(x,0) = I^h(f)(x),$$

Note that since $\phi = \overline{\phi} + O(\delta^2)$ (1.2) is the $N = 0$ case of (1.4).

If $\alpha = 0$, (1.4) reduces to the usual FEM which superconverges at the nodes with rate $O(h^{2\mu})$. We show that the added stabilization term in (1.4) preserves this property. In Theorem 3.10 we prove

$$\left\| u(t) - u^h(t) \right\|_{l_{2,h}} \leq C\{h^{2\mu}||f||_{H^{2\mu+1}(\mathbb{R})} + \alpha\delta^{2N+2}[||f||_{H^{2N+2}(\mathbb{R})} + h^{2\mu-2}||f||_{H^{2N+2\mu}(\mathbb{R})}]\}.$$

1.1. The genesis of the time relaxation term. The time relaxation term combines a numerical regularization with a physical model. Because of that it is particularly interesting computationally: it induces a deviation from an exact discretization of (1.1) intended to move the computed solution's behavior closer to the behavior of the physics (1.1) is often intended to model. In theoretical work on the derivation of conservation laws, regularizations of Chapman-Enskog expansions in Rosenau [R89], Schochet and E. Tadmor [ST92] produced conservation laws with a time relaxation term. This added time relaxation operator is a lower order perturbation and thus (since the equation does not change order or type) questions of well-posedness and boundary conditions are transparent.; in combination with a large eddy simulation model, it has produced positive results for the Navier-Stokes equations at high Reynolds numbers and a mathematical foundation for its inclusion in models for turbulent flow has been derived, [LN05], [LN05b], and [ELN06] . It can also be used quite independently of any turbulence model (and has been so used in compressible flow calculations). As a stand alone regularization, it has been successful for the Euler equations for shock-entropy wave interaction and other tests, [AS02], [SAK01a] , [SAK01b], [SAK02], including aerodynamic noise prediction and control, Guenanff [Gue04]. It was observed to ensure sufficient numerical entropy dissipation for numerical solution of conservation laws, Adams and Stolz [AS02], p.393.

1.2. Averaging by discrete differential filters. We study herein averaging by a *discrete differential filter*[2] (Germano [Ger86] and Manica and Kaya-Merdan

[1] As N increases to moderate values $\phi \to D_N\overline{\phi}$ becomes quite close to sharp spectral cutoff, see the figures in [LN05b].

[2] The "best" filter depends on the neds of application at hand. Scale space analysis suggests the gaussian filter as the generic case. The above differential filter arises as the first subdiagonal Padé approximation to it in wavenumber space, [GL00]. It is also very convenient for both mathematical analysis and FEM implementation.

[MM06]). Germano's proposal of differential filtering for fluid velocities plays a key role in a number of models of turbulence including the alpha-model, [FHT01], the zeroth order model, [LL03], [LL06a], [LL06b], and deconvolution models, [AS02],[SAK01a] , [SAK01b], [SAK02] and [LMNR06].

Let δ be the user selected averaging radius (typically $\delta = O(h)$ in computations). Given $\phi \in L^2(\mathbb{R})$ its discrete average $\overline{\phi} \in S^{h,\mu}(\mathbb{R})$ is the unique solution of

$$\delta^2(\overline{\phi}_x, v_x^h) + (\overline{\phi}, v^h) = (\phi, v^h), \forall v^h \in S^{h,\mu}(\mathbb{R}). \tag{1.5}$$

Associated with (1.5) define the discrete Laplacian operator $\triangle^h : L^2(\mathbb{R}) \to S^{h,\mu}(\mathbb{R})$ and projection operator $\Pi^h : L^2(\mathbb{R}) \to S^{h,\mu}(\mathbb{R})$ by

$$(\phi_x, v_x^h) = (-\triangle^h \phi, v^h), \forall v^h \in S^{h,\mu}(\mathbb{R}) \text{ , and}$$
$$(\phi, v^h) = (\Pi^h \phi, v^h), \forall v^h \in S^{h,\mu}(\mathbb{R}).$$

With these definitions, the discrete filter (1.3) can be written

$$\overline{\phi} = (-\delta^2 \triangle^h + \Pi^h)^{-1}(\Pi^h \phi) \text{ , or} \tag{1.6}$$
$$(-\delta^2 \triangle^h + \Pi^h)\overline{\phi} = (\Pi^h \phi). \tag{1.7}$$

2. Notation and preliminaries. The fundamental connection between the Galerkin method with splines and an associated difference scheme at the nodes was made by V. Thomée in the early 1970's in [T72], [T73], see also [TW74], papers of mathematical power and beauty. We shall use the techniques introduced in these papers and shall thus follow the notation in them closely. Define, following, e.g., Schoenberg [Sc73], Thomée [T73], the B-spline of order $\mu \geq 2$. Let χ denote the characteristic function of $[-1, 1]$. Define ϕ and ϕ_l^h by $\phi = \chi^{*\mu}$ and $\phi_l^h(x) = \phi(h^{-1}x - l)$. We take $S^{h,\mu}(\mathbb{R})$ to be the space of splines of at most power growth:

$$S^{h,\mu}(\mathbb{R}) = \{\sum_l c_l \phi_l^h(x) : c_l = O(|l|^q) \text{ as } |l| \to \infty \text{ for some } q\}.$$

The splines in $S^{h,\mu}(\mathbb{R})$ are $C^{\mu-2}$ functions which reduce to polynomials of degree $\mu - 1$ on each interval $[jh, (j+1)h]$ for μ even and on $[(j - \frac{1}{2})h, (j + \frac{1}{2})h]$ for μ odd. The usual spline interpolation operator is denoted I_h, that is, $I_h(v)$ is that element of $S^{h,\mu}(\mathbb{R})$ satisfying $I_h(v)(lh) = v(lh), l \in \mathbb{Z}$. For $\mu \geq 2$ and integer σ, $0 \leq \sigma \leq 2\mu - 2$, define the trigonometric polynomials (scaled to be independent of h)

$$g_{\mu,\sigma}(\theta) = h^{\sigma-1}(-i)^{\sigma-2\nu} \sum_{l=-\infty}^{\infty} (\frac{\partial^{\sigma-\nu}}{\partial x^{\sigma-\nu}} \phi_0^h, \frac{\partial^\nu}{\partial x^\nu} \phi_l^h) e^{-il\theta} \text{ , where } \nu = [\sigma].$$

For σ even, $g_{\mu,\sigma}(\theta)$ is a real, positive trigonometric polynomial, [T73].

Norms associated with doubly infinite sequences will be useful. For $h > 0$, the $l_{2,h}$ norm of a function $v(x)$ and sequence $c = (c_j)_{j=-\infty}^{\infty}$ are defined by

$$||v||_{l_{2,h}}^2 = h \sum_{j \in \mathbb{Z}} |v(jh)|^2 \text{ and } ||c||_{l_2}^2 = \sum_{j \in \mathbb{Z}} |c_j|^2.$$

For $c \in l_2$ the discrete Fourier transform of c is $\widetilde{c}(\theta) = (\mathcal{F}c)(\theta) = \sum_{j \in \mathbb{Z}} c_j e^{ij\theta}$.

3. Superconvergence at the nodes. First consider (1.2), i.e., the case $N = 0$ in (1.4). Expand $u^h(x,t) = \sum_{j\in\mathbb{Z}} c_j(t)\phi_j^h(x)$. Taking the discrete Fourier transform (denoted by an over-tilda) of (1.2) gives

$$hg_{\mu,0}(h\theta)\frac{d}{dt}\widetilde{c}(\theta,t) - ig_{\mu,1}(h\theta)\widetilde{c}(\theta,t) + \alpha[hg_{\mu,0}(h\theta)\widetilde{c}(\theta,t) - \widetilde{(\overline{u^h},\phi^h)}(\theta,t)] = 0. \quad (3.1)$$

The difference between various methods of filtering/averaging and between the time relaxation discretization and the usual Galerkin method lies in the last term on the LHS of the above. The analysis of this last term will be more compact and clear by identifying two trigonometric polynomials that occur frequently. Accordingly, define, suppressing dependence on all parameters except θ,

$$\omega(\theta) := \frac{g_{\mu,0}(\theta)}{(\frac{\delta}{h})^2 g_{\mu,2}(\theta) + g_{\mu,0}(\theta)},$$
$$d(\theta) := 1 - \omega(\theta) = \frac{(\frac{\delta}{h})^2 g_{\mu,2}(\theta)}{(\frac{\delta}{h})^2 g_{\mu,2}(\theta) + g_{\mu,0}(\theta)}$$

3.1. Fourier analysis of discrete differential filters. Consider first the discrete differential filter. If we write

$$u^h(x,t) = \sum_{j\in\mathbb{Z}} c_j(t)\phi_j^h(x)\ ,\ \overline{u}^h(x) = \sum_{j\in\mathbb{Z}} \overline{c}_j(t)\phi_j^h(x)$$
$$\text{and } (u^h)'(x) = \sum_{j\in\mathbb{Z}} c_j'(t)\phi_j^h(x)$$

then taking the discrete Fourier transform of the filter step (1.5) relates the coefficients of the filtered and unfiltered quantities by

$$\delta^2 h^{-1} g_{\mu,2}(h\theta)\widetilde{\overline{c}}(\theta,t) + hg_{\mu,0}(h\theta)\widetilde{\overline{c}}(\theta,t) = hg_{\mu,0}(h\theta)\widetilde{c}(\theta,t).$$

Thus, we can express the discrete Fourier transform of averaged quantities in terms of non-averaged by

$$\widetilde{\overline{c}}(\theta,t) = \omega(h\theta)\widetilde{c}(\theta,t), \quad (3.2)$$
$$\text{and } \widetilde{c'}(\theta,t) = d(h\theta)\widetilde{c}(\theta,t) \quad (3.3)$$

3.2. Fourier analysis of discrete deconvolution operators. We consider higher order time relaxation operators as well as the zeroth order case above. To do so we must perform a careful Fourier analysis of the van Cittert approximate deconvolution algorithm with the discrete differential filter. We shall go through the first two steps of the deconvolution algorithm careful before analyzing the general case.

Given

$$u^h(x) = \sum_{j\in\mathbb{Z}} c_j\phi_j^h(x) \text{ and } \overline{u^h}(x) = \sum_{j\in\mathbb{Z}} \overline{c}_j(t)\phi_j^h(x),$$

the van Cittert approximate deconvolution algorithm, [BB98], is given as follows.

Algorithm 3.1 (van Cittert deconvolution algorithm). ***Given*** $\overline{u^h}(x)$. ***Set*** $u^0(x) = \overline{u^h}(x)$. ***Then compute*** $\overline{u^0}$ ***and set***

$$u^1 = u^0 + \overline{u} - \overline{u^0},$$

Next compute $\overline{u^1}$ ***and set***

$$u^2 = u^1 + \overline{u} - \overline{u^0},$$

For $n = 0, 1, 2, \cdots, N-1$ ***, given*** u^n***, compute*** $\overline{u^n}$ ***and set***

$$u^{n+1} = u^n + \overline{u} - \overline{u^n},$$

The map $D_N : L^2(\mathbb{R}) \to S^{h,\mu}(\mathbb{R}) \subset L^2(\mathbb{R})$ by $D_N : \overline{u} \to u^N$, i.e.,

$$D_N \overline{u} = u^N$$

is a bounded linear map, [BIL06], [DE06]. It will be convenient to denote by $H_N : L^2(\mathbb{R}) \to S^{h,\mu}(\mathbb{R}) \subset L^2(\mathbb{R})$ the bounded linear map $H_N : u \to D_N \overline{u}$, i.e.,

$$H_N u := D_N \overline{u}.$$

It is necessary to find the symbols of D_N and H_N . To be sure of the correct path, we first take the discrete Fourier transforms of the first few steps in the van Cittert algorithm.

The algorithm takes data

$$u^h(x) = \sum_{j \in \mathbb{Z}} c_j \phi_j^h(x) \Leftrightarrow \widetilde{c}(\theta) = \sum_{j \in \mathbb{Z}} c_j e^{ij\theta},$$

and filters it by

$$\overline{u^h}(x) = \sum_{j \in \mathbb{Z}} \overline{c}_j(t) \phi_j^h(x) \Leftrightarrow \widetilde{\overline{c}}(\theta, t) = \omega(h\theta) \widetilde{c}(\theta, t).$$

Algorithm 3.1 begins by setting $u^0(x) = \overline{u^h}(x)$ where if $u^0(x) = \sum_{j \in \mathbb{Z}} c_j^0 \phi_j^h(x)$, then $\widetilde{c}^0(\theta, t) = \omega(h\theta) \widetilde{c}(\theta, t)$. Next Algorithm 3.1 computes $d^0 \in S^{h,\mu}(\mathbb{R})$ by

$$d^0 = \overline{u^0},$$

where if $d^0(x) = \sum_{j \in \mathbb{Z}} d_j^0 \phi_j^h(x)$, and $\widetilde{d^0}(\theta) = \sum_{j \in \mathbb{Z}} d_j^0 e^{ij\theta}$ then we have

$$\widetilde{d^0}(\theta) = \omega(h\theta) \widetilde{c^0}(\theta) = \omega(h\theta)^2 \widetilde{c}(\theta).$$

Next, it updates by $u^1 = u^0 + \overline{u} - d^0$. If $u^1(x) = \sum_{j \in \mathbb{Z}} c_j^1 \phi_j^h(x)$, and $\widetilde{c^1}(\theta) = \sum_{j \in \mathbb{Z}} c_j^1 e^{ij\theta}$, we have

$$\begin{aligned} &\widetilde{c^1}(\theta) = \widetilde{c^0}(\theta) + \widetilde{\overline{c}}(\theta) - \widetilde{d}(\theta), \text{ which implies} \\ &\widetilde{c^1}(\theta) = [2\omega(h\theta) - \omega(h\theta)^2] \widetilde{c}(\theta). \end{aligned}$$

Proceed similarly: since $d^1 = \overline{u^1}$, there follows

$$\widetilde{d^1}(\theta) = \omega(h\theta)[2\omega(h\theta) - \omega(h\theta)^2] \widetilde{c}(\theta).$$

Next since $u^2 = u^1 + \overline{u} - d^1$, we compute

$$\widetilde{c^2}(\theta) = [3\omega(h\theta) - 3\omega(h\theta)^2 + \omega(h\theta)^3] \widetilde{c}(\theta).$$

Eliminating the intermediate steps, u^N can be written

$$u^N = \sum_{n=0}^{N} (I - (-\triangle^h + \Pi^h)^{-1})^n \overline{u}.$$

Thus, for general $N = 0, 1, \cdots$, the operator $H_N : L^2(\mathbb{R}) \to S^{h,\mu}(\mathbb{R})$ can be expressed as

$$H_N : u \to \sum_{n=0}^{N} (I - (-\triangle^h + \Pi^h)^{-1})^n \overline{u}$$

.

Proposition 3.2. *Let the Nth van Cittert deconvolution approximation be written*

$$u^N(x) = \sum_{j\in\mathbb{Z}} c_j^N \phi_j^h(x) \text{ , and } \widetilde{c^N}(\theta) = \sum_{j\in\mathbb{Z}} c_j^N e^{ij\theta}.$$

Then

$$\begin{aligned} \widetilde{c^N}(\theta) &= \sum_{n=0}^{N} (1 - \omega(h\theta))^n \omega(h\theta) \widetilde{c}(\theta) \\ &\Longleftrightarrow \\ \widetilde{H_N}(\theta) &= \sum_{n=0}^{N} (1 - \omega(h\theta))^n \omega(h\theta) \end{aligned}$$

Further,

$$\widetilde{H_n}(\theta) = [1 - (1 - \omega(h\theta))^{N+1}]$$

Proof. The first two formulas follow by a simple induction argument and the last by summing the geometric series. □

Consider the generalized fluctuation

$$u^* := u - H_N u.$$

Proposition 3.3. *We have*

$$\widetilde{u^*}(\theta) = d(h\theta)^{N+1} \widetilde{c}(\theta)$$

Proof. We have, by Proposition 3.2,

$$\widetilde{u^*}(\theta) = \{1 - [1 - (1 - \omega(h\theta))^{N+1}]\} \widetilde{c}(\theta) = d(h\theta)^{N+1} \widetilde{c}(\theta).$$

□

3.3. Fourier analysis of the time relaxation discretization. Consider first the case $N = 0$. The last term on the RHS of (3.1) is thus given by

$$\widetilde{((u^h)', \phi^h)}(\theta, t) = hg_{\mu,0}(h\theta)\frac{(\frac{\delta}{h})^2 g_{\mu,2}(h\theta)}{\delta^2 hg_{\mu,2}(\theta) + hg_{\mu,0}(\theta)}\widetilde{c}(\theta, t)$$

Equation (3.1) becomes

$$\frac{d}{dt}\widetilde{c}(\theta, t) = a_0(h\theta)\widetilde{c}(\theta, t) \text{ , where}$$
$$a_0(\theta) \ : = \frac{i}{h}\frac{g_{\mu,1}(\theta)}{g_{\mu,0}(\theta)} - \alpha d(\theta).$$

Solving gives

$$\widetilde{c}(\theta, t) = \exp\left\{\frac{t}{h}\left[i\frac{g_{\mu,1}(h\theta)}{g_{\mu,0}(h\theta)} - \alpha hd(h\theta)\right]\right\}\widetilde{c}(\theta, 0) \ , \tag{3.4}$$

or, equivalently (3.5)

$$\widetilde{c}(\theta, t + \triangle t) = \exp\left\{\frac{\triangle t}{h}\left[i\frac{g_{\mu,1}(h\theta)}{g_{\mu,0}(h\theta)} - (\alpha h)d(h\theta)\right]\right\}\widetilde{c}(\theta, t). \tag{3.6}$$

Let $\lambda = \frac{\triangle t}{h}$, then the symbol for the nodal finite difference scheme induced by the zeroth order ($N = 0$), time relaxation-finite element discretization (1.2) of (1.1) is

$$a_0(\theta) := \exp\left\{\lambda\left[i\frac{g_{\mu,1}(\theta)}{g_{\mu,0}(\theta)} - \alpha hd(\theta)\right]\right\} \tag{3.7}$$

The symbol of the solution operator for the true equation (1.1) is

$$\exp\{\lambda i\theta\} \ ,$$

so the nodal accuracy of the FEM-time relaxation discretization (1.2) is precisely the order of contact of the last two symbols at $\theta = 0$, the usual definition of accuracy in Fourier space for finite difference schemes.

The evaluation of the accuracy of (1.2) depends on the asymptotics (established by V. Thomée [T73], Lemma 3.1) of $g_{\mu,0}(\theta)$, $g_{\mu,1}(\theta)$ and $g_{\mu,2}(\theta)$. We recall next this key result.

Lemma 3.4. *For integer* $\sigma, 0 \leq \sigma \leq 2\mu$

$$g_{\mu,\sigma}(\theta) = \theta^\sigma \widehat{\chi}(\theta)^{2\mu} + R_{\mu,\sigma}(\theta),$$

where, as $|\theta| \to 0$,

$$R_{\mu,\sigma}(\theta) = O(\theta^{2\mu}) \text{ , for even } \sigma,$$
$$R_{\mu,\sigma}(\theta) = O(\theta^{2\mu+1}) \text{ , for odd } \sigma.$$

Thus,

$$g_{\mu,1}(\theta)/g_{\mu,0}(\theta) = \theta + O(\theta^{2\mu+1}),$$
$$g_{\mu,2}(\theta)/g_{\mu,0}(\theta) = \theta^2 + O(\theta^{2\mu}).$$

Proof. See Thomée [T73], Lemma 3.1. □

In the case of the Nth order time relaxation discretization, the analysis is similar. We have then

$$\widetilde{c}(\theta,t) = \exp\left\{\frac{t}{h}\left[i\frac{g_{\mu,1}(h\theta)}{g_{\mu,0}(h\theta)} - \alpha h d(h\theta)^{N+1}\right]\right\}\widetilde{c}(\theta,0)\ , \tag{3.8}$$

or, equivalently

$$\widetilde{c}(\theta,t+\triangle t) = \exp\left\{\lambda\left[i\frac{g_{\mu,1}(\theta)}{g_{\mu,0}(\theta)} - \alpha h d(h\theta)^{N+1}\right]\right\}\widetilde{c}(\theta,t). \tag{3.9}$$

Thus, the symbol for the nodal finite difference scheme induced by the zeroth order, time relaxation-finite element discretization (1.2) of (1.1) is

$$a_N(\theta) := \exp\left\{\lambda\left[i\frac{g_{\mu,1}(\theta)}{g_{\mu,0}(\theta)} - \alpha h d(\theta)^{N+1}\right]\right\} \tag{3.10}$$

LEMMA 3.5. *Let $\delta = O(h^{\varepsilon})$ for some $\varepsilon > 0$. Then,*

$$d(h\theta) = \frac{\delta^2\theta^2}{1+\delta^2\theta^2} + \delta^2 h^{2\mu-2} O(\theta^{2\mu}),$$

and

$$d(h\theta)^{N+1} = \left(\frac{\delta^2\theta^2}{1+\delta^2\theta^2}\right)^{N+1} + \delta^{2N+2}h^{2\mu-2}O(\theta^{2N+2\mu}).$$

Further,

$$d(\theta) = \frac{\left(\frac{\delta}{h}\right)^2\theta^2}{1+\left(\frac{\delta}{h}\right)^2\theta^2} + \left(\frac{\delta}{h}\right)^2 O(\theta^{2\mu}),\ \textit{and}$$

$$d(\theta)^{N+1} = \left(\frac{\left(\frac{\delta}{h}\right)^2\theta^2}{1+\left(\frac{\delta}{h}\right)^2\theta^2}\right)^{N+1} + \left(\frac{\delta}{h}\right)^{2N+2} O(\theta^{2N+2\mu}).$$

Proof. Divide the numerator and denominator of $d(\theta)$ by $g_{\mu,0}(\theta)$. Using $g_{\mu,2}(\theta)/g_{\mu,0}(\theta) = \theta^2 + O(\theta^{2\mu})$ (Lemma 3.4) gives

$$d(\theta) = (\frac{\delta}{h})^2\frac{\theta^2+O(\theta^{2\mu})}{1+(\frac{\delta}{h})^2\theta^2+(\frac{\delta}{h})^2O(\theta^{2\mu})},$$

so that, as $|\theta| \to 0$

$$d(h\theta) = (\frac{\delta}{h})^2\frac{h^2\theta^2+h^{2\mu}O(\theta^{2\mu})}{1+\delta^2\theta^2+\delta^2h^{2\mu-2}O(\theta^{2\mu})}$$

from which the first result follows.

For the second result, apply the binomial theorem:

$$\begin{aligned} d(h\theta)^{N+1} = {} & \left(\delta^2\frac{\theta^2}{1+\delta^2\theta^2}\right)^{N+1} + \\ & +\delta^{2N+2}h^{2\mu-2}O(\theta^{2N+2\mu}) + \delta^{2N+2}h^{4\mu-4}O(\theta^{2N-2+4\mu}) + \\ & + \cdots . \end{aligned}$$

Provided $\mu > 1$ the third and subsequent terms on the above right hand side are higher order and the second result follows.

The final two results follow by rescaling the first two by $\theta \leftarrow \theta/h$. □

LEMMA 3.6. *Let $\delta < 1$, $\alpha < 1$, $h < 1$, $N \geq 0$. The symbol $a_N(\theta)$, given by (3.10), satisfies*

$$a_N(\theta) = \exp\{i\lambda\theta + \psi_N(\theta, h, \delta, \alpha)\}$$

where

$$\psi_N(\theta) = -\alpha h \left(\frac{\delta^2\theta^2}{1+\delta^2\theta^2}\right)^{N+1} + \alpha h \left(\frac{\delta}{h}\right)^{2N+2} O(\theta^{2N+2\mu}) + O(\theta^{2\mu+1}).$$

Proof. For $N \geq 0$, obviously,

$$\psi_N(\theta) := -i\lambda\theta + \lambda \left[i\frac{g_{\mu,1}(\theta)}{g_{\mu,0}(\theta)} - (\alpha h)d(\theta)^{N+1}\right].$$

Using Lemma 3.4, we find

$$\lambda i \frac{g_{\mu,1}(\theta)}{g_{\mu,0}(\theta)} = i\lambda\theta + O(\theta^{2\mu+1}), \tag{3.11}$$

Using Lemma 3.4 gives for $N = 0$

$$\alpha h d(\theta) = \alpha h \left(\frac{\left(\frac{\delta}{h}\right)^2 \theta^2}{1+\left(\frac{\delta}{h}\right)^2\theta^2} + \left(\frac{\delta}{h}\right)^2 O(\theta^{2\mu}) \right).$$

while for any $N \geq 0$

$$\alpha h d(\theta)^{N+1} = \alpha h \left(\left(\frac{\left(\frac{\delta}{h}\right)^2 \theta^2}{1+\left(\frac{\delta}{h}\right)^2\theta^2}\right)^{N+1} + \left(\frac{\delta}{h}\right)^{2N+2} O(\theta^{2N+2\mu}) \right). \tag{3.12}$$

Equations (3.11) and (3.12) yield the claimed result. □

Further properties can be inferred from the symbol. For example, the scheme is stable if $|a(\theta)| \leq 1$ and dissipative if $|a(\theta)| < 1$ for $\theta \neq 0, 2\pi$.

LEMMA 3.7. *Let $N = 0$. The scheme (1.2) is dissipative of order 2: there is a constant C such that*

$$\operatorname{Re}\psi_0(\theta) \leq C\alpha\delta|\theta|^2, |\theta| \leq \pi.$$

Proof. We have $\operatorname{Re}\psi_0(\theta) = -\alpha h d(\theta)$. Since $g_{\mu,0}(\theta) > 0, g_{\mu,2}(\theta) > 0$ for $0 < |\theta| \leq \pi$ dissipativity follows. The upper bound follows from Lemma 3.4. □

3.4. Nodal error of the associated difference scheme. Define

$$P_N(\theta) := \frac{1}{h}\left[i\frac{g_{\mu,1}(h\theta)}{g_{\mu,0}(h\theta)} - \alpha h d(h\theta)^{N+1}\right],$$

so that at the nodes

$$u^h(lh, t) = \mathcal{F}^{-1}\left\{\exp(tP_N(\theta)\widehat{f}(\theta))\right\}(lh), l \in \mathbb{Z}.$$

Lemma 3.8. *For $|h\theta| \le \pi, \mu > 1, N \ge 0$*

$$|P_N(\theta) - i\theta| \le Ch^{2\mu}|\theta|^{2\mu+1} + C\alpha\delta^{2N+2}\left\{\left(\frac{|\theta|^2}{1+\delta^2|\theta|^2}\right)^{N+1} + h^{2\mu-2}|\theta|^{2N+2\mu}\right\}$$
$$\le Ch^{2\mu}|\theta|^{2\mu+1} + C\alpha\delta^{2N+2}\left\{|\theta|^{2N+2} + h^{2\mu-2}|\theta|^{2N+2\mu}\right\}.$$

Proof. Starting with

$$P_N(\theta) := \frac{1}{h}\left[i\frac{g_{\mu,1}(h\theta)}{g_{\mu,0}(h\theta)} - \alpha h d(h\theta)^{N+1}\right],$$

change variables by $\widetilde{\theta} = h\theta$. Then,

$$hP_N(h^{-1}\widetilde{\theta}) := i\frac{g_{\mu,1}(\widetilde{\theta})}{g_{\mu,0}(\widetilde{\theta})} - \alpha h d(\widetilde{\theta})^{N+1}$$

We have shown in Lemma 3.6 that for $|\widetilde{\theta}| \le \pi$

$$hP_N(h^{-1}\widetilde{\theta}) = i\frac{g_{\mu,1}(\widetilde{\theta})}{g_{\mu,0}(\widetilde{\theta})} - (\alpha h)d(\widetilde{\theta})^{N+1} =$$
$$= i\widetilde{\theta} + O(\widetilde{\theta}^{2\mu+1}) - (\alpha h)\left(\left(\frac{\left(\frac{\delta}{h}\right)^2\widetilde{\theta}^2}{1+\left(\frac{\delta}{h}\right)^2\widetilde{\theta}^2}\right)^{N+1} + \left(\frac{\delta}{h}\right)^{2N+2}O(\widetilde{\theta}^{2N+2\mu})\right).$$

Thus,

$$|hP_N(h^{-1}\widetilde{\theta}) - i\widetilde{\theta}| \le C\alpha h\left(\left(\frac{\left(\frac{\delta}{h}\right)^2|\widetilde{\theta}|^2}{1+\left(\frac{\delta}{h}\right)^2|\widetilde{\theta}|^2}\right)^{N+1} + \left(\frac{\delta}{h}\right)^{2N+2}|\widetilde{\theta}|^{2N+2\mu}\right) + C|\widetilde{\theta}|^{2\mu+1},$$

from which the result follows by letting $\widetilde{\theta} = h\theta$. □

Lemma 3.9. *We have for $0 \le t \le T$*

$$e^{tP_N(\theta)} - e^{ti\theta} = t(P_N(\theta) - i\theta)\int_0^1 \exp(stP_N(\theta) + (1-s)ti\theta)ds,$$

and thus for $|h\theta| \le \pi$

$$|e^{tP_N(\theta)} - e^{ti\theta}| \le C|P_N(\theta) - i\theta|.$$

Proof. A calculation. □

With the above calculations the error estimate now follows easily using techniques usual for the Fourier analysis of finite difference schemes.

Theorem 3.10. *The error in (1.1) at the nodes satisfies*

$$\left\|u(t) - u^h(t)\right\|_{l_{2,h}} \le C\{h^{2\mu}||f||_{H^{2\mu+1}(\mathbb{R})} + \alpha\delta^{2N+2}[||f||_{H^{2N+2}(\mathbb{R})} + h^{2\mu-2}||f||_{H^{2N+2\mu}(\mathbb{R})}].]$$

Proof. Indeed, let the error be denoted by

$$e(t) := u(t) - u^h(t).$$

Then, for any $s > 0$

$$\|e(t)\|_{l_{2,h}}^2 \leq C \int_{-\frac{\pi}{h}}^{+\frac{\pi}{h}} |e^{tP_N(\theta)} - e^{ti\theta}|^2 |\widehat{f}(\theta)|^2 d\theta + h^{2s} \int_{-\infty}^{+\infty} |\theta|^{2s} |\widehat{f}(\theta)|^2 d\theta$$

$$\leq C \int_{-\frac{\pi}{h}}^{+\frac{\pi}{h}} [h^{2\mu}|\theta|^{2\mu+1} + \alpha\delta^{2N+2}\left\{|\theta|^{2N+2} + h^{2\mu-2}|\theta|^{2N+2\mu}\right\}]^2 |\widehat{f}(\theta)|^2 d\theta + h^{2s}||f||_{H^s(\mathbb{R})}^2$$

picking $s = 2\mu$

$$\leq C\{h^{4\mu}||f||^2_{H^{2\mu+1}(\mathbb{R})} + \alpha^2\delta^{4N+4}[||f||^2_{H^{2N+2}(\mathbb{R})} + h^{4\mu-4}||f||^2_{H^{2N+2\mu}(\mathbb{R})}].$$

□

4. Conclusions. The dominant terms in Theorem 3.10 are $h^{2\mu}$ and $\alpha\delta^{2N+2}$ If the parameter $\alpha = O(1)$ then this suggests that if, as usual $\delta = O(h)$, not degrading the approximation's basic error requires at least $N \geq \mu - 1$ deconvolution steps. For example, with cubic splines, $\mu = 4$ so $N \geq 3$ steps. If $\alpha \to \infty$ as $\delta \to 0$ (as in theory in [ST92], [LN05] and experiments in [SA99]), then even more deconvolution steps are necessary. If $N = 0$ then the added stabilization term is the simple and perhaps most commonly seen fluctuation about the local mean but the nodal error is degraded. When coupled with the simplest discretization of piecewise linears, $\mu = 2$, the error at the nodes is $O(h^4 + \delta^2)$-significantly larger than desired. We conclude from this analysis that *using higher order deconvolution operators and higher order generalized fluctuations is recommended* to preserve nodal accuracy.

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