Convergence Analysis of the Finite Element Method for a Fundamental Model in Turbulence

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Abstract

This report considers the question of computing accurate approximations to the motion of large structures in turbulent flows. We consider a fundamental closure model used in Large Eddy Simulation, given by $\overline{\mathbf{u}\,\mathbf{u}} - \overline{\mathbf{u}}\,\overline{\mathbf{u}} \approx \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} - \overline{\mathbf{u}}\,\overline{\mathbf{u}}$. We study convergence of approximations to the model that results from this closure and give a bound on the numerical error. Stability and accuracy of the discretization depend on how filtering is performed.

1 Introduction

Large Eddy Simulation (LES) has emerged as one of the most promising approaches in simulation of turbulent flows. Its goal is to compute the large structures of a flow by modeling the effects of small scale structures on the large structures. It uses a space filtering/averaging operation applied to the Navier-Stokes equations, which sifts out the small scales, i.e. those which are of size smaller than the filter width, denoted by $\delta > 0$.

One of the most basic models in turbulence is the Zeroth Order Model. It has been studied analytically and has good properties, such as uniqueness of strong solutions [17, 18]. Our motivation was to use the finite element method to derive a good discretization for this model.

Interestingly, this seemingly straightforward idea is far more intricate than it appears. It demands the study of a correct interpretation of averaging on bounded domains. This is the first step in devising stable discretizations for the model, and has proven to be specially influenced by how filtering is ultimately defined in the computational framework. We find that solving the filter equation in a mesh finer than the one used to solve the model itself does not offer any clear advantages, may even be unstable and of course, increases the computational effort. We also present convergence studies for the discretizations we suggest.

Filtering on bounded domains has long been a matter of discussion in LES. On one hand is the question of commutativity of averaging and differentiation. In general, they only commute in special situations, such as infinite or periodic domains. Therefore, in the presence of boundaries, an extra term is introduced in the equations. On the other hand, when appropriate boundary conditions must be specified, it is not entirely clear how to do that. Put simply, the problem is that the unknowns in the model are all averaged quantities;

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so the averaged velocity, for instance, must be specified at the boundary. However, most of the time, only the velocity itself is known, not its average. See [5] and [8] for a thorough discussion on boundary conditions and LES models.

In our derivation of the model, with the filter chosen, the commutation error is within the modeling error of $O(\delta^2)$, so it can be safely ignored. The imposition of correct boundary conditions also has a lot to do with the filter selection. We have chosen to use a differential filter, because that seems to provide a reasonable extension of filtering by convolution to bounded domains. It also gives us somewhat more freedom: loosely speaking, since in the finite element method all equations are treated weakly, we can impose boundary conditions only on the component of the averaged value that lies in the space we are working with.

We investigate stability and convergence of a semidiscretization of the Zeroth Order Model. In order to assess its accuracy, the numerical error, $\mathbf{w}-\mathbf{w}^h$, is considered, where $\mathbf{w}^h$ is an approximation to $\mathbf{w}$. We analyze two types of schemes, depending on the discretization of the differential filter equation (which can be done in the same mesh as the solution of the problem or in a finer one). When the filtering operation is performed on the same mesh, $\mathbf{w}^h$ is stable and we can prove that an optimal error estimate holds, whereas if it is performed in a finer mesh, we can show only that $\mathbf{w}^h$ is stable for a small finite interval of time. In addition, the convergence results assume strong regularity properties on the true solution $\mathbf{w}$ and require strong conditions on the body forces and the mesh size h. Computationally, this means that solving the filtering equation in the same mesh used to compute the solution is the best approach, since it is both more economical and stable.

The Zeroth Order Model is the lowest order model in a family of Approximate Deconvolution Models pioneered by Stolz and Adams [1]. The strategies adopted in [17, 18] were successfully extended to analyze the whole family of models [6]. Therefore, despite being the simplest example of the family, it is the key to understanding the higher order members. This supports our belief that understanding how the finite element method must be applied to the Zeroth Order Model is the key in determining the right discretization for the entire class of models and possibly many other LES models as well.

This paper is organized as follows. Section 2 introduces notations and mathematical preliminaries. In Section 3, we give a brief derivation of the model. Properties of the differential filter are presented in Section 4. The stability and error analysis of the model with respect to both discretizations of the filtering equations are studied in Section 5. We also include some numerical experiments regarding the kinetic energy computed with each scheme in Section 6. Section 7 presents some conclusion remarks.

2 Notation and Mathematical Preliminaries

Throughout this paper, we use standard notation for Lebesgue and Sobolev spaces (Adams [2]). The $L^2(\Omega)$ norm and inner product are denoted by $||\cdot||$ and $(\cdot,\cdot)$, respectively. For the Hilbert space $H^k(\Omega)$, the norm is denoted by $||\cdot||_k$. For Y a function space and time-dependent functions $\mathbf{v}:[0,\infty)\rightarrow Y$ we use the notation

$$L^p(0,T;Y)=\left\{\mathbf{v}:\mathbf{v}(t):(0,T)\rightarrow Y,\text{measurable and }\int_0^T ||\,\mathbf{v}(t)\,||_Y^p\,dt<\infty\right\},$$

with $1 \leq p < \infty$, and the usual modification if $p = \infty$. It is natural to define the following velocity and pressure spaces (for $d = 2, 3$), respectively:

$$\begin{aligned} \mathbf{X} : &= H_0^1(\Omega)^d = \{ \mathbf{v} \in L^2(\Omega)^d : \nabla \mathbf{v} \in L^2(\Omega)^{d \times d} \quad \text{and} \quad \mathbf{v} = 0 \ \text{ on } \ \partial\Omega \}, \\ \mathbb{Q} : &= L_0^2(\Omega) = \left\{ q : q \in L^2(\Omega), \int_\Omega q \, d\mathbf{x} = 0 \right\}. \end{aligned}$$

For any $\boldsymbol{\phi}$ in the dual space of $\mathbf{X}$, its norm is defined by

$$\|\boldsymbol{\phi}\|_{-1} = \sup_{\mathbf{v} \in \mathbf{X}} \frac{|(\boldsymbol{\phi}, \mathbf{v})|}{\|\mathbf{v}\|_1},$$

and the space of divergence free functions is defined as follows:

$$\mathbf{V} = \{ \mathbf{v} \in \mathbf{X} : (\nabla \cdot \mathbf{v}, q) = 0 \text{ for all } q \in \mathbb{Q} \}.$$

We often use the following inequalities:
Young's Inequality:

$$ab \leq \frac{\epsilon}{p} a^p + \frac{\epsilon^{-q/p}}{q} b^q, \quad 1 < q,\, p < \infty, \quad \frac{1}{q} + \frac{1}{p} = 1, \quad a, b \in \mathbb{R}.$$

Poincaré-Friedrichs' Inequality:

$$\|\mathbf{v}\| \leq C_{PF} \|\nabla \mathbf{v}\| \quad \forall \mathbf{v} \in \mathbf{X},$$

where C_{PF} is a constant depending on Ω.

We consider the following trilinear form on $\mathbf{X} \times \mathbf{X} \times \mathbf{X}$ for the convective term, defined by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2} \int_\Omega \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x} - \frac{1}{2} \int_\Omega \mathbf{u} \cdot \nabla \mathbf{w} \cdot \mathbf{v} \, d\mathbf{x}.$$

Note that the skew symmetrized trilinear form $b(\cdot, \cdot, \cdot)$ has the following properties:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \qquad \text{and} \qquad b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}.$$

Lemma 2.0.1. *Let $\Omega \subset \mathbb{R}^d, d = 2$ or 3. Then there exists a constant $M = M(\Omega) < \infty$ such that*

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq M || \nabla \mathbf{u} || \, || \nabla \mathbf{v} || \, || \nabla \mathbf{w} ||, \qquad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}. \tag{2.0.1}$$

When $d = 3$, this can be improved to

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq M \sqrt{|| \mathbf{u} || \, || \nabla \mathbf{u} ||} \, || \nabla \mathbf{v} || \, || \nabla \mathbf{w} ||, \qquad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}. \tag{2.0.2}$$

or, equivalently, to

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq M || \nabla \mathbf{u} || \, || \nabla \mathbf{v} || \sqrt{|| \mathbf{w} || \, || \nabla \mathbf{w} ||}, \qquad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}. \tag{2.0.3}$$

Proof. We refer to [11] for the proof of inequality (2.0.1). To prove (2.0.2), we first use Lemma 2.1 p.12 of Temam [20]:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C(\Omega)\|\mathbf{u}\|_{1/2}\|\mathbf{v}\|_1\|\mathbf{w}\|_1.$$

Then, using Poincaré-Friedrichs and Korn's inequality,

$$\|\mathbf{v}\| \leq C_{PF}\|\nabla\mathbf{v}\|, \ \|\mathbf{w}\|_1 \leq C\|\nabla\mathbf{w}\|.$$

Lastly, an interpolation inequality between $L^2(\Omega)$ and $H^1(\Omega)$ (see [2]) gives

$$\|\mathbf{u}\|_{1/2} \leq C(\Omega)\|\mathbf{u}\|^{1/2}\|\mathbf{u}\|_1^{1/2} \leq C(\Omega)\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}.$$

Similarly, (2.0.3) follows from

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C(\Omega)\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_{1/2}.$$

□

We construct conforming finite element approximations of velocity-pressure spaces $\mathbf{X}^h$, $\mathbb{Q}^h$ with $\mathbf{X}^h \subset \mathbf{X}$ and $\mathbb{Q}^h \subset \mathbb{Q}$. We assume that these spaces satisfy the discrete inf-sup condition, i.e. there exists a constant $\beta^h > 0$, bounded away from zero, uniformly in h such that

$$\inf_{q^h\in\mathbb{Q}^h}\sup_{\mathbf{v}^h\in\mathbf{X}^h}\frac{(q^h,\,\nabla\cdot\mathbf{v}^h)}{||\,\nabla\mathbf{v}^h\,||\,||\,q^h\,||} \geq \beta^h > 0, \tag{2.0.4}$$

Examples of such spaces are given in Gunzburger [13], Brezzi and Fortin [4] and Girault and Raviart [11]. The space of discretely divergence free functions can be defined as

$$\mathbf{V}^h = \{\mathbf{v}^h \in \mathbf{X}^h : (q^h,\,\nabla\cdot\mathbf{v}^h) = 0,\,\forall\, q^h \in \mathbb{Q}^h\}.$$

We assume that the following approximation assumptions, typical of piecewise polynomial velocity-pressure finite element spaces of degree $(k, k-1)$, hold: there is $k \geq 1$ such that for any $\mathbf{u} \in (H^{k+1}(\Omega))^d \cap \mathbf{X}$ and $p \in (H^k(\Omega) \cap \mathbb{Q})$:

$$\inf_{\mathbf{v}^h\in\mathbf{X}^h}\left\{\|\mathbf{u}-\mathbf{v}^h\| + h\|\nabla(\mathbf{u}-\mathbf{v}^h)\|\right\} \leq Ch^{k+1}\,\|\mathbf{u}\|_{k+1}\,, \tag{2.0.5}$$

$$\inf_{q^h\in Q^h}\|p-q^h\| \leq Ch^k\,\|p\|_k\,. \tag{2.0.6}$$

We also introduce the discrete Laplace operator. For $\boldsymbol{\zeta}^h \in \mathbf{X}^h$, $\Delta^h : \mathbf{X}^h \to \mathbf{X}^h$ is typically defined as

$$(\Delta^h\boldsymbol{\zeta}^h, \mathbf{v}) = -(\nabla\boldsymbol{\zeta}^h, \nabla\mathbf{v}), \qquad \forall\mathbf{v} \in \mathbf{X}^h,$$

and assume that $\mathbf{X}^h$ and $\mathbb{Q}^h$ are such that an inverse inequality holds:

$$||\,\nabla\boldsymbol{\zeta}^h\,|| \leq C\,h^{-1}||\,\boldsymbol{\zeta}^h\,||, \qquad \forall\,\boldsymbol{\zeta}^h \in \mathbf{X}^h.$$

Throughout this paper, C is a generic constant that does not depend on the mesh size h or the filter width δ.

We next prove a *Continuation Lemma*, useful in the proof of Theorem 5.2.1. It allows us to conclude that the solution to a certain nonlinear ordinary differential equation is bounded for a finite interval of time in terms of the problem data. We will elaborate on this later, but this type of argument, somewhat foreign to this kind of analysis, will be valuable when the usual Gronwall's lemma cannot be used.

Lemma 2.0.2 (Continuation Lemma). *Let $y(t) \in C^1[0,1]$ be a non negative function satisfying*

$$\begin{aligned} y' + \alpha y &\leq \beta y^3 + \gamma \\ 0 \leq y(0) &\leq \gamma, \end{aligned} \tag{2.0.7}$$

where $\alpha \in L^1(0,1)$, $\beta > 0$ and $\gamma > 0$ are constants.

Then, there exists $\gamma_0 > 0$ and a constant $M \geq 1$ such that for $\gamma < \gamma_0$, y satisfies $y \leq M\gamma$, for $0 \leq t \leq 1$.

Proof. Let $I = \{t \in [0,1] : y \leq M\gamma\}$. We show that $I = [0,1]$ by showing that I is both closed and open in $[0,1]$ for some γ_0 small enough and M large enough.

First, observe that I is nonempty, since $M \geq 1$ implies that $0 \in I$. Also, I is closed because it is the pre-image of a closed set under a continuous mapping. Next, we still need to show that I is open.

Let $[0,t^*] \subset I$. We show that for ϵ small enough, $t^* + \epsilon \in I$, i.e. $y(t^*) < M\gamma$.

Using an integrating factor and integrating (2.0.7) from 0 to t^* gives

$$y(t^*) \leq \int_0^{t^*} e^{\int_{t^*}^{t} \alpha(t')\,dt'} (\beta y^3 + \gamma) dt.$$

Let $K = e^{||\alpha||_{L^1(0,1)}}$. Since $[0,t^*] \subset I$ and $t^* < 1$,

$$y(t^*) < K\,(\gamma + \beta M^3 \gamma^3).$$

Let $M > 2K$ so that $K\gamma < M\frac{\gamma}{2}$ and let $K\beta M^2\gamma^2 < \frac{1}{2}$. Under these conditions, $y(t^*) < M\gamma$ and, by continuity, $y(t) \leq M\gamma$ for $t^* \leq t \leq t^* + \epsilon$, showing that I is open. □

Remark 2.0.1. *Lemma 2.0.2 can be extended for the interval $0 \leq t \leq T$, for fixed T and the result is valid for exponents other than 3 on the right-hand side of (2.0.7).*

3 Derivation of the model

The Zeroth Order Model is potentially the simplest LES model for the incompressible Navier-Stokes equations, presented below:

$$\begin{aligned} \mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } (0,T] \times \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } [0,T] \times \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{in } [0,T] \times \partial\Omega, \\ \mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega \\ \textstyle\int_\Omega p\, d\mathbf{x} &= 0, \end{aligned} \tag{3.0.8}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2,3$ is a bounded, regular domain, $\mathbf{u}$ is the fluid velocity, p is the fluid pressure, $\mathbf{f}$ is the body force driving the flow and ν is the kinematic viscosity. The Reynolds number, Re, is inversely proportional to ν. For the finite element error analysis, we need some assumptions on the regularity of the data. We assume that $\mathbf{f} \in L^2(0,T; L^2(\Omega))$ and $\mathbf{u}_0 \in \mathbf{X}$.

Spacial filtering (usually denoted by *overbar*) is achieved by an averaging process (here, the solution of a Poisson problem) that requires that $\mathbf{u}$ be defined in terms of $\overline{\mathbf{u}}$ (closure problem), i.e. the action of the small scales on the large scales must be modeled. We shall work with a filter that (minimally) satisfies

$$\mathbf{u} = \overline{\mathbf{u}} + O(\delta^2),$$

for smooth $\mathbf{u}$. There are various models of $\mathbf{u}$ which are used in the literature; see e.g. Aldama [3], Sagaut [19], John [15] for an overview. This is the first, simplest member of a family models introduced by Stolz and Adams [1], called Approximate Deconvolution Models (ADM). Other examples are $\mathbf{u} \approx 2\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}}$ (first order extrapolation) and $\mathbf{u} \approx 3\overline{\mathbf{u}} - 3\overline{\overline{\mathbf{u}}} + \overline{\overline{\overline{\mathbf{u}}}}$ (second order extrapolation).

Applying this spacial averaging operator to (3.0.8) gives the space filtered Navier-Stokes equations (imposition of zero boundary condition is explained in Section 4)

$$\begin{aligned} \overline{\mathbf{u}}_t + \overline{\nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}} + O(\delta^2))} - \nu\Delta\overline{\mathbf{u}} + \overline{\nabla p} &= \overline{\mathbf{f}} && \text{in } (0,T] \times \Omega, \\ \nabla \cdot \overline{\mathbf{u}} + O(\delta^2) &= 0 && \text{in } [0,T] \times \Omega, \\ \overline{\mathbf{u}} &= 0 && \text{on } [0,T] \times \partial\Omega, \\ \overline{\mathbf{u}}(0,\mathbf{x}) &= \overline{\mathbf{u}}_0(\mathbf{x}) && \text{in } \Omega. \end{aligned} \tag{3.0.9}$$

Letting $\mathbf{w}$ denote the approximation to $\overline{\mathbf{u}}$ induced by this closure model, and dropping the $O(\delta^2)$ terms, system (3.0.9) gives that $(\mathbf{w}, p)$ satisfies

$$\begin{aligned} \mathbf{w}_t + \overline{\nabla \cdot (\mathbf{w}\mathbf{w})} - \nu\Delta\mathbf{w} + \overline{\nabla p} &= \overline{\mathbf{f}} && \text{in } (0,T] \times \Omega, \\ \nabla \cdot \mathbf{w} &= 0 && \text{in } [0,T] \times \Omega, \\ \mathbf{w} &= 0 && \text{on } [0,T] \times \partial\Omega, \\ \mathbf{w}(0,\mathbf{x}) &= \overline{\mathbf{u}}_0(\mathbf{x}) && \text{in } \Omega. \end{aligned} \tag{3.0.10}$$

Once again, we remark that, for this model, all operators we consider commute up to the modeling error of $O(\delta^2)$. We commute operators or not, according to which yields a stable model. For example, we can impose $\nabla \cdot \mathbf{w} = 0$ as in (3.0.10) (to preserve incompressibility), or $\nabla\overline{p}$ instead of $\overline{\nabla p}$, (see Section 5 for details).

4 Properties of Differential Filters

The selection of an appropriate filter is fundamental. A good survey of the spatial filters commonly used in LES is given in [3, 19], but perhaps the most unanimous choice is the Gaussian filter. Its application to the Stokes and the steady state Navier-Stokes equations has been reviewed in [9, 16]. Another example, which is precisely the one we adopted in this manuscript, is Germano's idea [10] of a differential filter. This seems to be a natural extension of filtering on the whole space to a bounded domain, since the differential equation can be supplemented with appropriate boundary conditions.

We work with the following Poisson problem: given $\boldsymbol{\phi}$, its differential filter $\overline{\boldsymbol{\phi}}$ is the solution of

$$-\delta^2\Delta\overline{\boldsymbol{\phi}} + \overline{\boldsymbol{\phi}} = \boldsymbol{\phi}$$

subject to zero boundary condition.

Remark 4.0.1. *From here on, as an alternative to the usual notation of overbar to indicate averaging, we introduce a new variable. For clarity, instead of denoting the differential filter of $\boldsymbol{\phi}$ by $\overline{\boldsymbol{\phi}}$, we will denote it by $\boldsymbol{\psi}$.*

Given $\boldsymbol{\phi} \in L^2(\Omega)^d$, $\boldsymbol{\psi} \in \mathbf{X}$ satisfies

$$\delta^2(\nabla\boldsymbol{\psi}, \nabla\mathbf{v}) + (\boldsymbol{\psi}, \mathbf{v}) = (\boldsymbol{\phi}, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X}, \tag{4.0.11}$$

with solution operator $T : L^2(\Omega)^d \to \mathbf{X}$ such that $T\boldsymbol{\phi} = \boldsymbol{\psi}$. It is well known that given $\boldsymbol{\phi} \in L^2(\Omega)^d$, (4.0.11) has a unique solution and that if Ω is a convex polygon, then $\boldsymbol{\psi} \in H^2(\Omega)^d$. In general, if $\boldsymbol{\phi} \in H^k(\Omega)^d$, then $\boldsymbol{\psi} \in H^{k+2}(\Omega)^d$ [12].

Similarly, the discrete filter $\boldsymbol{\psi}^h \in \mathbf{X}^h$ is given by

$$\delta^2(\nabla\boldsymbol{\psi}^h, \nabla\mathbf{v}) + (\boldsymbol{\psi}^h, \mathbf{v}) = (\boldsymbol{\phi}, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X}^h, \tag{4.0.12}$$

with solution operator $T^h : L^2(\Omega) \to \mathbf{X}^h$ satisfying $T^h\boldsymbol{\phi} = \boldsymbol{\psi}^h$.

Next we describe some of the properties of differential filters to be used in the error estimation in Section 5.1. Similar properties are given in [7] and are included here for completeness.

Lemma 4.0.3. *If $\boldsymbol{\phi} \in L^2(\Omega)^d$, the following stability estimate for problem (4.0.11) holds:*

$$\delta^2 || \nabla\boldsymbol{\psi} ||^2 + \frac{1}{2} || \boldsymbol{\psi} ||^2 \leq \frac{1}{2} || \boldsymbol{\phi} ||^2.$$

Proof. In (4.0.11), choose $\mathbf{v} = \boldsymbol{\psi}$, then apply the Cauchy-Schwarz and Young's inequality on the right-hand side. □

Lemma 4.0.4. *The operator $T : L^2(\Omega)^d \to \mathbf{X}$ is self-adjoint.*

Proof. Let $\mathbf{v} \in \mathbf{X}$. Then $T\mathbf{v} \in \mathbf{X}$ and from (4.0.11) and symmetry of inner products, we have

$$(\boldsymbol{\phi}, T\mathbf{v}) = \delta^2(\nabla(T\boldsymbol{\phi}), \nabla(T\mathbf{v})) + (T\boldsymbol{\phi}, T\mathbf{v}) = (T\boldsymbol{\phi}, \mathbf{v}).$$

□

Lemma 4.0.5. *If $\nabla\boldsymbol{\phi} \in L^2(\Omega)^d$ and $\boldsymbol{\psi}$ satisfies (4.0.11), then*

$$\frac{\delta^2}{2} || \nabla(\boldsymbol{\phi} - \boldsymbol{\psi}) ||^2 + || \boldsymbol{\phi} - \boldsymbol{\psi} ||^2 \leq \frac{\delta^2}{2} || \nabla\, \boldsymbol{\phi} ||^2, \tag{4.0.13}$$

If, additionally, $\Delta\boldsymbol{\phi} \in L^2(\Omega)^d$, then

$$\delta^2 || \nabla(\boldsymbol{\phi} - \boldsymbol{\psi}) ||^2 + \frac{1}{2} || \boldsymbol{\phi} - \boldsymbol{\psi} ||^2 \leq \frac{\delta^4}{2} || \Delta\, \boldsymbol{\phi} ||^2. \tag{4.0.14}$$

Proof. Add and subtract $\delta^2(\nabla\boldsymbol{\phi}, \nabla\mathbf{v})$ to (4.0.11), then choose $\mathbf{v} = \boldsymbol{\psi} - \boldsymbol{\phi}$. Applying the Cauchy-Schwarz and Young's inequalities proves the first assertion.

Note that for $\Delta\boldsymbol{\phi} \in L^2(\Omega)^d$, integration by parts implies that

$$\delta^2(\nabla\boldsymbol{\phi}, \nabla\mathbf{v}) + (\boldsymbol{\phi}, \mathbf{v}) = (-\delta^2\Delta\boldsymbol{\phi} + \boldsymbol{\phi}, \mathbf{v}) \quad \forall\mathbf{v} \in \mathbf{X}. \tag{4.0.15}$$

Subtracting (4.0.11) for $\mathbf{v} \in \mathbf{X}$ from (4.0.15),

$$\delta^2(\nabla(\boldsymbol{\phi} - \boldsymbol{\psi}), \nabla\mathbf{v}) + (\boldsymbol{\phi} - \boldsymbol{\psi}, \mathbf{v}) = -\delta^2(\Delta\boldsymbol{\phi}, \mathbf{v}). \tag{4.0.16}$$

Letting $\mathbf{v} = \boldsymbol{\phi} - \boldsymbol{\psi}$ and using Cauchy Schwarz, followed by Young's inequality, gives the second claim. □

Lemma 4.0.6. *The operator $T^h : L^2(\Omega) \to \mathbf{X}^h$ is self-adjoint and positive semi-definite on $L^2(\Omega)$ and positive definite on $\mathbf{X}^h$.*

Proof. Symmetry follows as in the continuous case. T^h is positive semi-definite on $L^2(\Omega)$, since

$$(\boldsymbol{\phi}, T^h\boldsymbol{\phi}) = \delta^2 || \nabla(T^h\boldsymbol{\phi}) ||^2 + || T^h\boldsymbol{\phi} ||^2 \geq 0.$$

Now, for $\boldsymbol{\phi}^h \in \mathbf{X}^h$ with $T^h\boldsymbol{\phi}^h = 0$, we have

$$(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) = \delta^2(\nabla(T^h\boldsymbol{\phi}^h), \nabla\boldsymbol{\phi}^h) + (T^h\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) = 0,$$

which proves the last claim. □

This guarantees that $T^h : L^2(\Omega) \to \mathbf{X}^h$ is invertible on $\mathbf{X}^h$. Let $A^h : \mathbf{X}^h \to \mathbf{X}^h$ be the inverse of T^h on $\mathbf{X}^h$. Then, it is easy to show that $A^h := -\delta^2\Delta^h + I$.

Next, we prove an error estimate for problems (4.0.11) and (4.0.12).

Theorem 4.0.1. *Let $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^h$ be solutions of problems (4.0.11) and (4.0.12), respectively and assume that approximation property (2.0.5) holds. Then,*

$$\delta|| \nabla(\boldsymbol{\psi} - \boldsymbol{\psi}^h) || + || \boldsymbol{\psi} - \boldsymbol{\psi}^h || \leq Ch^k(\delta + h)|| \boldsymbol{\psi} ||_{k+1}. \tag{4.0.17}$$

Proof. From the usual finite element techniques, we get

$$\delta|| \nabla(\boldsymbol{\psi} - \boldsymbol{\psi}^h) || + || \boldsymbol{\psi} - \boldsymbol{\psi}^h || \leq C \inf_{\tilde{\boldsymbol{\psi}} \in \mathbf{X}^h} \left(\delta|| \nabla(\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}) || + || \boldsymbol{\psi} - \tilde{\boldsymbol{\psi}} ||\right),$$

where $\tilde{\boldsymbol{\psi}}$ is an approximation to $\boldsymbol{\psi} \in \mathbf{X}^h$. □

Examining the right hand side, the optimal parameter selection is $\delta = O(h)$. In this case, we have the following.

Corollary 4.0.1. *If, in addition to the assumptions of Theorem 4.0.1, we choose $\delta = O(h)$, the following is true:*

$$\delta|| \nabla(\boldsymbol{\psi} - \boldsymbol{\psi}^h) || + || \boldsymbol{\psi} - \boldsymbol{\psi}^h || \leq Ch^{k+1}|| \boldsymbol{\psi} ||_{k+1}. \tag{4.0.18}$$

Before going on to the numerical analysis, let us first say a few words on how we define averaging.

Remark 4.0.2. *Recall formulations (4.0.11) and (4.0.12). These may seem like odd choices for many terms in the filtered equations of Section 5. For example, even if $\boldsymbol{\phi} \neq 0$ on $\partial\Omega$, we would still have $T^h\boldsymbol{\phi} = 0$ and $T\boldsymbol{\phi} = 0$ on $\partial\Omega$. The reasons are that in a weak formulation, these terms occur as $(T\boldsymbol{\phi}, \mathbf{v})$, $\mathbf{v} \in \mathbf{X}$ and $(T^h\boldsymbol{\phi}, \mathbf{v}^h)$, $\mathbf{v}^h \in \mathbf{X}^h$. Thus, the component of $\boldsymbol{\phi}$ (for example) outside of $\mathbf{X}$ or $\mathbf{X}^h$, respectively, will not influence the weak formulation. In other words, if $-\Delta^h : \mathbf{X}^h \to \mathbf{X}^h$ denotes the discrete laplacian and Π^h, the L^2 projection into $\mathbf{X}^h$, (4.0.12), for example, implies that $T^h\boldsymbol{\phi} = T^h(\Pi^h\boldsymbol{\phi})$. Moreover, for stability, it is important that all averaging operators have common domains, and then the same boundary conditions.*

Remark 4.0.3. *We have also considered filtering as the solution of a Stokes problem, rather than a Poisson problem. The motivation for that was to preserve incompressibility exactly, not up to $O(\delta^2)$. Although it is not clear yet, this type of averaging seems to be unstable and deserves to be further investigated.*

5 Stability and Error Analysis of the Model

In this section, we suggest two semi discretizations of (3.0.10) and discuss their stability and convergence properties. The basic difference between the two formulations is the manner in which the filtering operation is performed.

Consider the problem: Find $(\mathbf{w}, p) \in (\mathbf{X}, \mathbb{Q})$ such that $\mathbf{w}(0, \mathbf{x})$ is an approximation of $\overline{\mathbf{u}}_0(\mathbf{x})$ and

$$\begin{aligned} &(\mathbf{w}_t, \mathbf{v}) + (\overline{\nabla \cdot (\mathbf{w}\mathbf{w})}, \mathbf{v}) + \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + (\overline{\nabla p}, \mathbf{v}) = (\overline{\mathbf{f}}, \mathbf{v}), \qquad \forall\, \mathbf{v} \in \mathbf{X} \\ &(\nabla \cdot \mathbf{w}, q) = 0, \quad \forall\, q \in \mathbb{Q} \end{aligned} \tag{5.0.19}$$

The discretization of (5.0.19) calls for a decision on whether the computation of $\overline{\nabla \cdot (\mathbf{w}\,\mathbf{w})}$, for example, should be performed in the same mesh used to calculate $\mathbf{w}$. In other words, should the filtering operation be carried out in the same mesh used to approximate the solution of the problem, or in a finer mesh? We investigate the two possibilities. Ultimately, since we are performing a numerical analysis, all filters must be discrete. Nevertheless, we regard the case in which filtering/averaging is performed by solving the filtering problem on a finer mesh as "exact filter". When filtering and computation of $\mathbf{w}$ are performed in the same mesh, we call it "discrete filter". In order to keep the notation clear, we use the operators T and T^h (as defined in Section 4) instead of *overbar*. If $\boldsymbol{\zeta}$ is the quantity to be filtered, then $T(\boldsymbol{\zeta})$ means that $\overline{\boldsymbol{\zeta}}$ is computed in a finer mesh and $T^h(\boldsymbol{\zeta})$ means that $\overline{\boldsymbol{\zeta}}$ is computed on the same mesh. Based on Remark 4.0.2, we will omit the use of projections into $\mathbf{X}$ and $\mathbf{X}^h$ from now on. For instance, we will write $T^h(\boldsymbol{\zeta})$ instead of $T^h(\Pi^h \boldsymbol{\zeta})$, since these are equivalent.

Different filters will lead to different schemes. Note that Cases I and II (discrete and exact filter, respectively, discussed below) require (for stability) a different formulation of the pressure term in the momentum equation, $(\overline{\nabla p^h}, \mathbf{v}^h)$ versus $(\lambda^h, \nabla \cdot \mathbf{v}^h)$. In the two formulations, p^h is an approximation to p (Navier-Stokes pressure), while λ^h is an approximation to $\overline{p}$. This means that, in Case II, we are introducing a commutation error (within $O(\delta^2)$ of the modeling error, due to the non commutativity of filtering and differentiation in bounded domains).

We need to introduce two new skew symmetric forms on $\mathbf{X} \times \mathbf{X} \times \mathbf{X}$. For $\mathbf{u} \cdot \nabla \mathbf{v} \in L^2(\Omega)^d$ and $(\nabla \cdot \mathbf{u})\, \mathbf{v} \in L^2(\Omega)^d$, the bilinear forms are:

$$B(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (T(\mathbf{u} \cdot \nabla \mathbf{v}), \mathbf{w}) + \frac{1}{2}(T((\nabla \cdot \mathbf{u})\, \mathbf{v}), \mathbf{w}) \tag{5.0.20}$$

and

$$B^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (T^h(\mathbf{u} \cdot \nabla \mathbf{v}), \mathbf{w}) + \frac{1}{2}(T^h((\nabla \cdot \mathbf{u})\, \mathbf{v}), \mathbf{w}). \tag{5.0.21}$$

These bilinear forms have some key properties:

Lemma 5.0.7. *For all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$,

$$B(\mathbf{u}, \mathbf{v}, \mathbf{w}) = b(\mathbf{u}, \mathbf{v}, T(\mathbf{w})).$$

Proof. Let $\hat{\mathbf{w}} = T(\mathbf{w})$ and since $\mathbf{w} \in \mathbf{X}$, $\hat{\mathbf{w}} \in \mathbf{X}$. From (5.0.20) and the self-adjointness of the operator T, we can write

$$B(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \hat{\mathbf{w}}) + \frac{1}{2}((\nabla \cdot \mathbf{u})\mathbf{v}, \hat{\mathbf{w}}),$$

and integration by parts gives that

$$((\nabla \cdot \mathbf{u})\mathbf{v}, \hat{\mathbf{w}}) = -(\mathbf{u} \cdot \nabla \mathbf{v}, \hat{\mathbf{w}}) - (\mathbf{u} \cdot \nabla \hat{\mathbf{w}}, \mathbf{v}).$$

□

Lemma 5.0.8. *For all* $\mathbf{u}, \mathbf{v} \in \mathbf{X}$ *and* $\mathbf{w}^h \in \mathbf{X}^h$,

$$B^h(\mathbf{u}, \mathbf{v}, A^h \mathbf{w}^h) = b(\mathbf{u}, \mathbf{v}, \mathbf{w}^h).$$

Proof. From (5.0.21) and the self-adjointness of $T^h : L^2(\Omega) \to \mathbf{X}^h$, we have

$$B^h(\mathbf{u}, \mathbf{v}, A^h \mathbf{w}^h) \quad = \quad (\mathbf{u} \cdot \nabla \mathbf{v}, T^h A^h \mathbf{w}^h) + \frac{1}{2}((\nabla \cdot \mathbf{u})\mathbf{v}, T^h A^h \mathbf{w}^h).$$

Since $T^h A^h \mathbf{w}^h = \mathbf{w}^h$ on $\mathbf{X}^h$, integration by parts gives

$$(\mathbf{u} \cdot \nabla \mathbf{v}, T^h A^h \mathbf{w}^h) + \frac{1}{2}((\nabla \cdot \mathbf{u})\mathbf{v}, T^h A^h \mathbf{w}^h) = b(\mathbf{u}, \mathbf{v}, \mathbf{w}^h).$$

□

Corollary 5.0.2. *For all* $\mathbf{u}^h, \mathbf{v}^h \in \mathbf{X}^h$,

$$B^h(\mathbf{u}^h, \mathbf{v}^h, A^h \mathbf{v}^h) = 0.$$

Proof. Follows directly from Lemma 5.0.8 and the property that $b(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h) = 0$. □

Using (5.0.20), the Zeroth Order Model (3.0.10) can be rewritten as: Find $(\mathbf{w}, p) \in (\mathbf{X}, \mathbb{Q})$ such that $\mathbf{w}(0, \mathbf{x}) = \overline{\mathbf{u}}_{\mathbf{0}}(\mathbf{x})$

$$\begin{aligned} &(\mathbf{w}_t, \mathbf{v}) + B(\mathbf{w}, \mathbf{w}, \mathbf{v}) + \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + (T(\nabla p), \mathbf{v}) = (T(\mathbf{f}), \mathbf{v}), \qquad \forall\, \mathbf{v} \in \mathbf{X} \\ &(\nabla \cdot \mathbf{w}, q) = 0, \quad \forall\, q \in \mathbb{Q} \end{aligned} \tag{5.0.22}$$

In the next pages, we show that the solution computed with the discrete filter is stable and convergent. However, in the exact filter case, it appears that the solution is stable only for a finite time, which raises some issues on how well the computed solution approximates the exact solution.

5.1 Case I: Discrete Differential Filter

Consider a semi-discretization of the Zeroth Order Model (5.0.19): Find $\mathbf{w}^h : [0, T] \to \mathbf{X}^h$, $p^h : (0, T] \to \mathbb{Q}^h$ satisfying

$$\begin{aligned} (\mathbf{w}_t^h, \mathbf{v}^h) \;\; &+ \;\; B^h(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) \\ &+ \;\; \nu(\nabla \mathbf{w}^h, \nabla \mathbf{v}^h) + (T^h(\nabla p^h), \mathbf{v}^h) = (T^h(\mathbf{f}), \mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{X}^h \\ &\quad\;\; (\nabla \cdot \mathbf{w}^h, q^h) = 0, \quad \forall\, q^h \in \mathbb{Q}^h \end{aligned} \tag{5.1.1}$$

where $\mathbf{w}^h(0, \mathbf{x})$ is an approximation of $\overline{\mathbf{u}}_0(\mathbf{x})$.

Lemma 5.1.1 (Stability of the semi-discrete solution). *Let $\mathbf{w}^h$ be the solution of (5.1.1). Then,*

$$\begin{aligned}\frac{1}{2}\|\mathbf{w}^h\|^2_{L^\infty(0,T,L^2(\Omega))} &+ \frac{\delta^2}{2}\|\nabla\mathbf{w}^h\|^2_{L^\infty(0,T,L^2(\Omega))}\\ &+ \frac{\nu}{2}\|\nabla\mathbf{w}^h\|^2_{L^2(0,T,L^2(\Omega))} + \nu\delta^2\|\Delta^h\mathbf{w}^h\|^2_{L^2(0,T,L^2(\Omega))}\\ &\le \frac{1}{2}(\|\mathbf{w}^h(0)\|^2 + \delta^2\|\nabla\mathbf{w}^h(0)\|^2) + \frac{1}{2\nu}\|\mathbf{f}\|^2_{L^2(0,T;H^{-1}(\Omega))}.\end{aligned}$$

Proof. Consider the variational formulation (5.1.1) and take $\mathbf{v}^h = A^h\mathbf{w}^h$ and $q^h = p^h$. Adding the two equations, using the self-adjointness of T^h, the definitions of A^h and Π^h and Corollary 5.0.2, this is the same as

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{w}^h\|^2 + \delta^2\|\nabla\mathbf{w}^h\|^2) + \nu(\nabla\mathbf{w}^h, \nabla(A^h\mathbf{w}^h)) = (\mathbf{f}, \mathbf{w}^h)$$

From this, the Cauchy-Schwarz and Young's inequalities give

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{w}^h\|^2 + \delta^2\|\nabla\mathbf{w}^h\|^2) + \frac{\nu}{2}\|\nabla\mathbf{w}^h\|^2 + \nu\delta^2\|\Delta^h\mathbf{w}^h\|^2 \le \frac{1}{2\nu}\|\mathbf{f}\|^2_{-1}.$$

The result follows by integrating in time. □

In what follows we give an estimate for the difference between the exact solution, $\mathbf{w}$, and the semi-discrete, $\mathbf{w}^h$.

Theorem 5.1.1 (Accuracy of Discretization). *Let $\mathbf{w}$ be the solution of (5.0.22) and $\mathbf{w}^h$, the solution of (5.1.1). Assume that $\mathbf{w}\cdot\nabla\mathbf{w}$ and $\nabla p \in L^2(\Omega)^d$ for every $t \in (0,T)$ and that $\nabla\mathbf{w} \in L^4(0,T,L^2(\Omega))$. Then, the numerical error satisfies:*

$$\begin{aligned}\|\mathbf{w}-\mathbf{w}^h\|^2_{L^\infty(0,T,L^2(\Omega))} &+ \delta^2\|\nabla(\mathbf{w}-\mathbf{w}^h)\|^2_{L^\infty(0,T,L^2(\Omega))} + \nu\|\nabla(\mathbf{w}-\mathbf{w}^h)\|^2_{L^2(0,T,L^2(\Omega))}\\ &\le C^*\inf_{\tilde{\mathbf{w}}\in\mathbf{X}^h}(\|(\mathbf{w}-\tilde{\mathbf{w}})(0)\|^2 + \delta^2\|\nabla(\mathbf{w}-\tilde{\mathbf{w}})(0)\|^2)\\ &+C^*\nu^{-1}\inf_{\tilde{\mathbf{w}}\in\mathbf{X}^h p^h\in Q^h}\Big\{(1+\delta^2)\Big(\|(\mathbf{w}-\tilde{\mathbf{w}})_t\|^2_{L^2(0,T;L^2(\Omega))}\\ &+\|p-p^h\|^2_{L^2(0,T;L^2(\Omega))} + \|(T-T^h)(\mathbf{w}\cdot\nabla\mathbf{w})\|^2_{L^2(0,T;L^2(\Omega))}\\ &+\|(T-T^h)(\mathbf{f})\|^2_{L^2(0,T;L^2(\Omega))} + \|(T-T^h)(\nabla p)\|^2_{L^2(0,T;L^2(\Omega))}\Big)\\ &+(1+\delta^2h^{-2})\|\nabla(\mathbf{w}-\tilde{\mathbf{w}})\|^2_{L^4(0,T;L^2(\Omega))}\\ &+\|\nabla\mathbf{w}\|^2_{L^4(0,T;L^2(\Omega))}\|\nabla(\mathbf{w}-\tilde{\mathbf{w}})\|^2_{L^4(0,T;L^2(\Omega))}\\ &+\|\mathbf{w}^h\|_{L^\infty(0,T;L^2(\Omega))}\|\nabla\mathbf{w}^h\|_{L^2(0,T;L^2(\Omega))}\|\nabla(\mathbf{w}-\tilde{\mathbf{w}})\|^2_{L^4(0,T;L^2(\Omega))}\Big\}\end{aligned}$$

where $C^* = e^{C\nu^{-3}\|\nabla\mathbf{w}\|^4_{L^4(0,T;L^2(\Omega))}}$.

Proof. Subtracting (5.1.1) from (5.0.22), we have

$$\begin{aligned}&((\mathbf{w}-\mathbf{w}^h)_t, \mathbf{v}^h) + \nu(\nabla(\mathbf{w}-\mathbf{w}^h), \nabla\mathbf{v}^h) + B(\mathbf{w},\mathbf{w},\mathbf{v}^h)\\ &-B^h(\mathbf{w}^h,\mathbf{w}^h,\mathbf{v}^h) + (T(\nabla p) - T^h(\nabla p^h), \mathbf{v}^h) = (T(\mathbf{f}) - T^h(\mathbf{f}), \mathbf{v}^h), \qquad \forall\,\mathbf{v}^h \in \mathbf{X}^h.\end{aligned}$$

Adding and subtracting $(T^h(\mathbf{w}\cdot\nabla\mathbf{w}),\mathbf{v}^h)$ to the nonlinear terms, we get

$$\begin{aligned} B(\mathbf{w},\mathbf{w},\mathbf{v}^h)-B^h(\mathbf{w}^h,\mathbf{w}^h,\mathbf{v}^h) &= ((T-T^h)(\mathbf{w}\cdot\nabla\mathbf{w}),\mathbf{v}^h) \\ &+ B^h(\mathbf{w},\mathbf{w},\mathbf{v}^h)-B^h(\mathbf{w}^h,\mathbf{w}^h,\mathbf{v}^h), \end{aligned}$$

since $\nabla\cdot\mathbf{w}=0$. Similarly, adding and subtracting $T^h(\nabla p)$ to the pressure term, we get

$$(T(\nabla p)-T^h(\nabla p^h),\mathbf{v}^h)=((T-T^h)(\nabla p),\mathbf{v}^h)+(T^h(\nabla p-\nabla p^h),\mathbf{v}^h)$$

Then, the error equation becomes

$$\begin{aligned} ((\mathbf{w}-\mathbf{w}^h)_t,\mathbf{v}^h) &+ \nu(\nabla(\mathbf{w}-\mathbf{w}^h),\nabla\mathbf{v}^h) \\ &+ ((T-T^h)(\mathbf{w}\cdot\nabla\mathbf{w}),\mathbf{v}^h)+B^h(\mathbf{w},\mathbf{w},\mathbf{v}^h)-B^h(\mathbf{w}^h,\mathbf{w}^h,\mathbf{v}^h) \\ &+ ((T-T^h)(\nabla p),\mathbf{v}^h)+(T^h(\nabla p-\nabla p^h),\mathbf{v}^h)=((T-T^h)(\mathbf{f}),\mathbf{v}^h). \end{aligned}$$

Choose an interpolant $\tilde{\mathbf{w}}\in\mathbf{V}^h$ and set $\mathbf{e}=\mathbf{w}-\mathbf{w}^h=(\mathbf{w}-\tilde{\mathbf{w}})-(\mathbf{w}^h-\tilde{\mathbf{w}})=\boldsymbol{\eta}-\boldsymbol{\chi}^h$. Then,

$$\begin{aligned} (\boldsymbol{\chi}_t^h,\mathbf{v}^h) &+ \nu(\nabla\boldsymbol{\chi}^h,\nabla\mathbf{v}^h)+((T-T^h)(\mathbf{w}\cdot\nabla\mathbf{w}),\mathbf{v}^h) \\ &+ B^h(\mathbf{w},\mathbf{w},\mathbf{v}^h)-B^h(\mathbf{w}^h,\mathbf{w}^h,\mathbf{v}^h)+((T-T^h)(\nabla p),\mathbf{v}^h)+(T^h(\nabla p-\nabla p^h),\mathbf{v}^h) \\ &= (\boldsymbol{\eta}_t,\mathbf{v}^h)+\nu(\nabla\boldsymbol{\eta},\nabla\mathbf{v}^h)+((T-T^h)(\mathbf{f}),\mathbf{v}^h). \end{aligned} \tag{5.1.2}$$

Since $\tilde{\mathbf{w}}\in\mathbf{X}^h$, we can set $\mathbf{v}^h=A^h\boldsymbol{\chi}^h$. From Lemma 5.0.8, we write

$$B^h(\mathbf{w},\mathbf{w},A^h\boldsymbol{\chi}^h)-B^h(\mathbf{w}^h,\mathbf{w}^h,A^h\boldsymbol{\chi}^h)=b(\mathbf{w},\mathbf{w},\boldsymbol{\chi}^h)-b(\mathbf{w}^h,\mathbf{w}^h,\boldsymbol{\chi}^h).$$

and we also have

$$(T^h(\nabla p-\nabla p^h),A^h\boldsymbol{\chi}^h)=(\nabla p-\nabla p^h,T^hA^h\boldsymbol{\chi}^h)=-(p-p^h,\nabla\cdot\boldsymbol{\chi}^h).$$

For the other terms, we have to use the fact that $A^h=-\delta^2\Delta^h+I$.

Thus, equation (5.1.2) becomes

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(||\,\boldsymbol{\chi}^h\,||^2 &+ \delta^2||\,\nabla\boldsymbol{\chi}^h\,||^2\Big)+\nu||\,\nabla\boldsymbol{\chi}^h\,||^2+\nu\delta^2||\,\Delta^h\boldsymbol{\chi}^h\,||^2 \\ &=(\boldsymbol{\eta}_t,\boldsymbol{\chi}^h)-\delta^2(\boldsymbol{\eta}_t,\Delta^h\boldsymbol{\chi}^h) \\ &+\nu(\nabla\boldsymbol{\eta},\nabla\boldsymbol{\chi}^h)-\nu\delta^2(\nabla\boldsymbol{\eta},\nabla(\Delta^h\boldsymbol{\chi}^h)) \\ &+(p-p^h,\nabla\cdot\boldsymbol{\chi}^h) \\ &-b(\mathbf{w},\mathbf{w},\boldsymbol{\chi}^h)+b(\mathbf{w}^h,\mathbf{w}^h,\boldsymbol{\chi}^h) \\ &-((T-T^h)(\mathbf{w}\cdot\nabla\mathbf{w}),\boldsymbol{\chi}^h)+\delta^2((T-T^h)(\mathbf{w}\cdot\nabla\mathbf{w}),\Delta^h\boldsymbol{\chi}^h) \\ &-((T-T^h)(\nabla p),\boldsymbol{\chi}^h)+\delta^2((T-T^h)(\nabla p),\Delta^h\boldsymbol{\chi}^h) \\ &+((T-T^h)(\mathbf{f}),\boldsymbol{\chi}^h)-\delta^2((T-T^h)(\mathbf{f}),\Delta^h\boldsymbol{\chi}^h). \end{aligned} \tag{5.1.3}$$

Next, each of the terms in (5.1.3) is bounded. First, we examine the linear terms (we use an inverse inequality of the form $||\,\nabla(\Delta^h \boldsymbol{\chi}^h)\,|| \le C\,h^{-1}||\,\Delta^h\boldsymbol{\chi}^h\,||$):

$$(\boldsymbol{\eta}_t, \boldsymbol{\chi}^h) \le \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 + \frac{C}{\nu}||\,\boldsymbol{\eta}_t\,||^2$$

$$\delta^2(\boldsymbol{\eta}_t, \Delta^h\boldsymbol{\chi}^h) \le \frac{\nu\delta^2}{10}||\,\Delta^h\boldsymbol{\chi}^h\,||^2 + \frac{C\delta^2}{\nu}||\,\boldsymbol{\eta}_t\,||^2$$

$$\nu(\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\chi}^h) \le \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 + C\,\nu\,||\,\nabla\boldsymbol{\eta}\,||^2$$

$$\nu\delta^2(\nabla\boldsymbol{\eta}, \nabla(\Delta^h\boldsymbol{\chi}^h)) \le \frac{\nu\delta^2}{10}||\,\Delta^h\boldsymbol{\chi}^h\,||^2 + \frac{C\,\nu\,\delta^2}{h^2}||\,\nabla\boldsymbol{\eta}\,||^2$$

$$(p - p^h, \nabla\cdot\boldsymbol{\chi}^h) \le \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 + \frac{C}{\nu}||\,p - p^h\,||^2$$

$$((T - T^h)(\nabla p), \boldsymbol{\chi}^h) \le \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 + \frac{C}{\nu}||\,(T - T^h)(\nabla p)\,||^2$$

$$\delta^2((T - T^h)(\nabla p), \Delta^h\boldsymbol{\chi}^h) \le \frac{\nu\delta^2}{10}||\,\Delta^h\boldsymbol{\chi}^h\,||^2 + \frac{C\delta^2}{\nu}||\,(T - T^h)(\nabla p)\,||^2$$

$$((T - T^h)(\mathbf{f}), \boldsymbol{\chi}^h) \le \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 + \frac{C}{\nu}||\,(T - T^h)(\mathbf{f})\,||^2$$

$$\delta^2((T - T^h)(\mathbf{f}), \Delta^h\boldsymbol{\chi}^h) \le \frac{\nu\delta^2}{10}||\,\Delta^h\boldsymbol{\chi}^h\,||^2 + \frac{C\delta^2}{\nu}||\,(T - T^h)(\mathbf{f})\,||^2$$

For the nonlinear terms, we have:

$$((T - T^h)(\mathbf{w}\cdot\nabla\mathbf{w}), \boldsymbol{\chi}^h) \le \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 + \frac{C}{\nu}||\,(T - T^h)(\mathbf{w}\cdot\nabla\mathbf{w})\,||^2$$

$$\delta^2((T - T^h)(\mathbf{w}\cdot\nabla\mathbf{w}), \Delta^h\boldsymbol{\chi}^h) \le \frac{\nu\delta^2}{10}||\,\Delta^h\boldsymbol{\chi}^h\,||^2 + \frac{C\delta^2}{\nu}||\,(T - T^h)(\mathbf{w}\cdot\nabla\mathbf{w})\,||^2$$

Lastly, we look at the term $b(\mathbf{w}, \mathbf{w}, \boldsymbol{\chi}^h) - b(\mathbf{w}^h, \mathbf{w}^h, \boldsymbol{\chi}^h)$. Adding and subtracting $b(\mathbf{w}^h, \mathbf{w}, \boldsymbol{\chi}^h)$, it can be rewritten as

$$b(\mathbf{w}, \mathbf{w}, \boldsymbol{\chi}^h) - b(\mathbf{w}^h, \mathbf{w}^h, \boldsymbol{\chi}^h) = b(\boldsymbol{\eta}, \mathbf{w}, \boldsymbol{\chi}^h) - b(\boldsymbol{\chi}^h, \mathbf{w}, \boldsymbol{\chi}^h) + b(\mathbf{w}^h, \boldsymbol{\eta}, \boldsymbol{\chi}^h),$$

where each of the terms is bounded as follows:

$$\begin{aligned} b(\boldsymbol{\eta}, \mathbf{w}, \boldsymbol{\chi}^h) &\le M||\,\boldsymbol{\eta}\,||^{1/2}||\,\nabla\boldsymbol{\eta}\,||^{1/2}||\,\nabla\mathbf{w}\,||\,||\,\nabla\boldsymbol{\chi}^h\,|| \\ &\le \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 + \frac{C}{\nu}||\,\nabla\boldsymbol{\eta}\,||^2||\,\nabla\mathbf{w}\,||^2 \end{aligned}$$

$$\begin{aligned} b(\boldsymbol{\chi}^h, \mathbf{w}, \boldsymbol{\chi}^h) &\le M||\,\boldsymbol{\chi}^h\,||^{1/2}||\,\nabla\boldsymbol{\chi}^h\,||^{1/2}||\,\nabla\mathbf{w}\,||\,||\,\nabla\boldsymbol{\chi}^h\,|| \\ &\le \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 + \frac{C}{\nu^3}||\,\nabla\mathbf{w}\,||^4||\,\boldsymbol{\chi}^h\,||^2 \end{aligned}$$

$$\begin{aligned} b(\mathbf{w}^h, \boldsymbol{\eta}, \boldsymbol{\chi}^h) &\le M||\,\mathbf{w}^h\,||^{1/2}||\,\nabla\mathbf{w}^h\,||^{1/2}||\,\nabla\boldsymbol{\eta}\,||\,||\,\nabla\boldsymbol{\chi}^h\,|| \\ &\le \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 + \frac{C}{\nu}||\,\mathbf{w}^h\,||\,||\,\nabla\mathbf{w}^h\,||\,||\nabla\boldsymbol{\eta}\,||^2 \end{aligned}$$

Thus,

$$
\begin{aligned}
b(\mathbf{w},\mathbf{w},\boldsymbol{\chi}^h)-b(\mathbf{w}^h,\mathbf{w}^h,\boldsymbol{\chi}^h) &\le \frac{3\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2+\frac{C}{\nu}||\,\nabla\boldsymbol{\eta}\,||^2||\,\nabla\mathbf{w}\,||^2\\
&+ \frac{C}{\nu^3}||\,\nabla\mathbf{w}\,||^4||\,\boldsymbol{\chi}^h\,||^2+\frac{C}{\nu}||\,\mathbf{w}^h\,||\,||\,\nabla\mathbf{w}^h\,||\,||\,\nabla\boldsymbol{\eta}\,||^2
\end{aligned}
$$

Putting all the estimates together, equation (5.1.3) gives

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(||\,\boldsymbol{\chi}^h\,||^2 &+ \delta^2||\,\nabla\boldsymbol{\chi}^h\,||^2\Big)+\nu||\,\nabla\boldsymbol{\chi}^h\,||^2+\nu\delta^2||\,\Delta^h\boldsymbol{\chi}^h\,||^2\\
&\le C\nu^{-1}(1+\delta^2)||\,\boldsymbol{\eta}_t\,||^2+C\nu^{-1}||\,p-p^h\,||^2\\
&+C\nu^{-1}(\nu+\nu\delta^2h^{-2}+||\,\nabla\mathbf{w}\,||^2+||\,\mathbf{w}^h\,||\,||\,\nabla\mathbf{w}^h\,||)\,||\nabla\boldsymbol{\eta}\,||^2\\
&+C\nu^{-1}(1+\delta^2)||(T-T^h)(\mathbf{w}\cdot\nabla\mathbf{w})\,||^2+C\nu^{-1}(1+\delta^2)||\,(T-T^h)(\nabla p)\,||^2\\
&+C\nu^{-1}(1+\delta^2)||(T-T^h)(\mathbf{f})\,||^2+C\nu^{-3}||\,\nabla\mathbf{w}\,||^4||\,\boldsymbol{\chi}^h\,||^2
\end{aligned}
$$

Using Gronwall's inequality, this becomes

$$
\begin{aligned}
\frac{1}{2}||\,\boldsymbol{\chi}^h\,||^2+\frac{\delta^2}{2}||\,\nabla\boldsymbol{\chi}^h\,||^2 &+ \frac{\nu}{2}\int_0^T(||\,\nabla\boldsymbol{\chi}^h\,||^2+\delta^2||\,\Delta^h\boldsymbol{\chi}^h\,||^2)\\
&\le \frac{C^*}{2}(||\,\boldsymbol{\chi}^h(0)\,||^2+\delta^2||\,\nabla\boldsymbol{\chi}^h(0)\,||^2)\\
&+C^*\nu^{-1}\int_0^T\Big\{(1+\delta^2)||\,\boldsymbol{\eta}_t\,||^2+||\,p-p^h\,||^2\\
&+(\nu+\nu\delta^2h^{-2}+||\,\nabla\mathbf{w}\,||^2+||\,\mathbf{w}^h\,||\,||\,\nabla\mathbf{w}^h\,||)\,||\nabla\boldsymbol{\eta}\,||^2\\
&+(1+\delta^2)||(T-T^h)(\mathbf{w}\cdot\nabla\mathbf{w})\,||^2\\
&+(1+\delta^2)||\,(T-T^h)(\nabla p)\,||^2+(1+\delta^2)||(T-T^h)(\mathbf{f})\,||^2\Big\}\,dt,
\end{aligned}
$$

where $C^*=e^{C\nu^{-3}\int_0^T||\,\nabla\mathbf{w}\,||^4dt}$. Drop the term that contains the operator Δ^h and the triangle inequality gives the result. □

Corollary 5.1.1. *Let $\delta=O(h)$. If $\mathbf{w}$, $\mathbf{f}$ and p are regular enough and satisfy the assumptions of Theorem 5.1.1, then*

$$
\begin{aligned}
||\,\mathbf{w}-\mathbf{w}^h\,||^2_{L^\infty(0,T,L^2(\Omega))}+\delta^2||\,\nabla(\mathbf{w}-\mathbf{w}^h)\,||^2_{L^\infty(0,T,L^2(\Omega))}+\nu||\,\nabla(\mathbf{w}-\mathbf{w}^h)\,||^2_{L^2(0,T,L^2(\Omega))}&\\
\le C(\mathbf{w},\mathbf{f},p,\nu)\,h^{2k}.&
\end{aligned}
$$

Proof. Follows from the estimate in Theorem 5.1.1, Corollary 4.0.1 (with $\boldsymbol{\psi}=T(\mathbf{w}\cdot\nabla\mathbf{w})$ and $\boldsymbol{\psi}^h=T^h(\mathbf{w}\cdot\nabla\mathbf{w})$, for instance) and the approximation properties (2.0.5) and (2.0.6). □

5.2 Case II: Exact Differential Filter

In this case, we show that the semi-discrete scheme is stable only under some conditions. Due to the fact that the nonlinear term does not vanish, this extra term will impose restrictions on the stability of the scheme. Briefly, it is a cubic term, which can only be dominated by quadratic terms for a finite time interval; eventually the higher order term will dominate. This also means that the kinetic energy of the model is not likely to be monotonic decreasing. In this context, it is natural to raise questions on the convergence properties of the scheme. According to the subsequent analysis, the scheme does converge over small time intervals at least, provided some regularity properties are considered.

The semi-discrete formulation of (5.0.19) now reads as follows: Find $\mathbf{w}^h : [0,T] \to \mathbf{X}^h$, $\lambda^h : (0,T] \to \mathbb{Q}^h$ satisfying $\mathbf{w}^h(0,\mathbf{x}) \approx \overline{\mathbf{u}}_\mathbf{0}(\mathbf{x})$ and

$$\begin{array}{rcll} (\mathbf{w}^h_t, \mathbf{v}^h) + B(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) + \nu(\nabla \mathbf{w}^h, \nabla \mathbf{v}^h) - (\lambda^h, \nabla \cdot \mathbf{v}^h) & = & (T(\mathbf{f}), \mathbf{v}^h), & \forall\, \mathbf{v}^h \in \mathbf{X}^h, \\ (\nabla \cdot \mathbf{w}^h, q^h) & = & 0, & \forall\, q^h \in \mathbb{Q}^h. \end{array} \tag{5.2.1}$$

We now present a basic a priori estimate for the solution of (5.2.1). The following stability theorem is proven by using an idea similar to the one in [20] (see Lemma 3.2, p. 21).

Lemma 5.2.1. *(Stability of $\mathbf{w}^h$) The solution of (5.2.1) satisfies*

$$\|\mathbf{w}^h(t)\|^2 \le 2(1 + \|\mathbf{w}^h(0)\|^2)$$

with

$$t \le T^* = C(f, \nu, T)\frac{\delta^2 h^2}{(1 + \|\mathbf{w}^h(0)\|^2)}.$$

Proof. Restrict $\mathbf{v}^h \in \mathbf{V}^h$ in (5.2.1). Choose $\mathbf{v}^h = \mathbf{w}^h$ and use Cauchy-Schwarz and Young's inequalities

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}^h\|^2 + \frac{\nu}{2}\|\nabla \mathbf{w}^h\|^2 \le \frac{1}{2\nu}\|T(\mathbf{f})\|^2_{-1} - B(\mathbf{w}^h, \mathbf{w}^h, \mathbf{w}^h). \tag{5.2.2}$$

We first consider the nonlinear term. The first step is to use Lemma 5.0.7, followed by equation (2.0.3) in Lemma 2.0.1. Then, use Lemma 4.0.3 and an inverse inequality of the form $||\,\nabla \mathbf{w}^h\,|| \le Ch^{-1}||\,\mathbf{w}^h\,||$. The last step is to apply Young's inequality with conjugate exponents 4 and 4/3.

$$\begin{array}{rcl} B(\mathbf{w}^h, \mathbf{w}^h, \mathbf{w}^h) & = & b(\mathbf{w}^h, \mathbf{w}^h, T(\mathbf{w}^h)) \\ & \le & M||\,\nabla \mathbf{w}^h\,||\,||\,\nabla \mathbf{w}^h\,||\,||\,T(\mathbf{w}^h)\,||^{1/2}\,||\,\nabla(T\mathbf{w}^h)\,||^{1/2} \\ & \le & C\delta^{-1/2}h^{-1/2}||\,\nabla \mathbf{w}^h\,||^{3/2}||\,\mathbf{w}^h\,||^{3/2} \\ & \le & \frac{\nu}{2}||\,\nabla \mathbf{w}^h\,||^2 + C\delta^{-2}h^{-2}\,\nu^{-3}||\,\mathbf{w}^h\,||^6. \end{array}$$

Using this last inequality, rewrite (5.2.2), multiply it by 2 and drop the term $\nu\|\nabla \mathbf{w}^h\|^2$. In the resulting differential inequality, set $z(t) = 1 + \|\mathbf{w}^h\|^2$ to get the following:

$$\frac{dz}{dt} \le C^*\delta^{-2}h^{-2}z^3,$$

where

$$C^* = \max(\nu^{-1}\sup_{t\in[0,T]}\|T(\mathbf{f})\|^2_{-1}, C\,\nu^{-3}).$$

If we solve the differential inequality and integrate from 0 to t, we derive

$$z(t) \le \frac{z(0)}{\sqrt{1 - 2C^*h^{-2}\delta^{-2}z^2(0)t}} \tag{5.2.3}$$

with $t < 1/(2C^* h^{-2}\delta^{-2}z^2(0))$. We can verify that $\frac{1}{\sqrt{1-2C^*h^{-2}\delta^{-2}z^2(0)t}} \leq 2$ and then equation (5.2.3) becomes

$$1 + \|\mathbf{w}^h\|^2 \leq 2(1 + \|\mathbf{w}^h(0)\|^2), \quad \text{with} \quad 0 \leq t \leq \frac{3}{8C^*}\frac{\delta^2 h^2}{(1+\|\mathbf{w}^h(0)\|^2)^2}.$$

This concludes the proof of the lemma. □

Lemma 5.2.1 confirms what was expected, according to our discussion in the beginning of this section: the approximate solution $\mathbf{w}^h$ is bounded in terms the problem data, $\mathbf{w}^h(0)$ and $T(\mathbf{f})$, for a bounded time interval.

The next natural step is to analyze convergence properties of the scheme. The following theorem gives an estimate of the error between $\mathbf{w}$ and its finite element approximation, $\mathbf{w}^h$. We assume that the true solution $\mathbf{w}$ is very regular, in a way that would probably not be representative of a turbulent flow. In addition, we also have a condition relating the mesh size h and the filter width δ to the viscosity of the fluid, which can be restrictive for high Reynolds number flows. This may not be the ideal situation, but in view of the fact that the solution is stable only for small time intervals, it is probably the best we can get.

Theorem 5.2.1. *Let* $\mathbf{w}$ *and* $\mathbf{w}^h$ *be solutions to (5.0.22) and (5.2.1), respectively and assume that* $\mathbf{w} \in L^\infty(0,T,H^1(\Omega))$. *Then the error satisfies*

$$\begin{aligned}||\,\mathbf{w}-\mathbf{w}^h\,||^2_{L^\infty(0,T;L^2)} \leq C \inf_{\tilde{\mathbf{w}}\in\mathbf{V}^h,\lambda^h\in\mathbb{Q}^h} \Big[&\nu^{-1}||\,\mathbf{w}_t-\tilde{\mathbf{w}}_t\,||^2_{L^\infty(0,T;H^{-1})} + \nu||\,\nabla(\mathbf{w}-\tilde{\mathbf{w}})\,||^2_{L^\infty(0,T;L^2)} \\ &+\nu^{-1}||\,\lambda-\lambda^h\,||^2_{L^\infty(0,T;L^2)} + C(\nu)\,\delta^{-2}||\,\nabla(\mathbf{w}-\tilde{\mathbf{w}})\,||^4_{L^\infty(0,T;L^4)} \\ &+ C(\nu)||\,\nabla(\mathbf{w}-\tilde{\mathbf{w}})\,||^2_{L^\infty(0,T;L^2)}\left(\delta^2||\,\mathbf{w}\,||^2_{L^\infty(0,T;L^\infty)} + ||\,\nabla\mathbf{w}\,||^2_{L^\infty(0,T;L^2)}\right)\Big].\end{aligned}$$

Proof. In order to get an error equation, we subtract (5.2.1) from (5.0.22) for $\mathbf{v}^h \in \mathbf{V}^h$ and set $\mathbf{e} = \mathbf{w} - \mathbf{w}^h$. Then, the equation for $\mathbf{e}$ becomes

$$(\mathbf{e}_t, \mathbf{v}^h) + \nu(\nabla\mathbf{e}, \nabla\mathbf{v}^h) + B(\mathbf{w},\mathbf{w},\mathbf{v}^h) - B(\mathbf{w}^h,\mathbf{w}^h,\mathbf{v}^h) - (\lambda, \nabla\cdot\mathbf{v}^h) = 0 \quad \mathbf{v}^h \in \mathbf{V}^h.$$

By picking $\tilde{\mathbf{w}}$ to be the best approximation of $\mathbf{w}$ in $\mathbf{V}^h$, we can decompose error in two parts as: $\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\chi}^h$ where $\boldsymbol{\eta} = \mathbf{w} - \tilde{\mathbf{w}}$ and $\boldsymbol{\chi}^h = \mathbf{w}^h - \tilde{\mathbf{w}}$. Thus, using the fact that $(\lambda^h, \nabla\cdot\mathbf{v}^h) = 0$ for all $q^h \in \mathbb{Q}^h$ and the decomposition of $\mathbf{e}$, the error equation can be reformulated as

$$\begin{aligned}(\boldsymbol{\chi}^h_t, \mathbf{v}^h) + \nu(\nabla\boldsymbol{\chi}^h, \nabla\mathbf{v}^h) &= (\boldsymbol{\eta}_t, \mathbf{v}^h) - \nu(\nabla\boldsymbol{\eta}, \nabla\mathbf{v}^h) + (\lambda-\lambda^h, \nabla\cdot\mathbf{v}^h) \\ &\quad -B(\mathbf{w},\mathbf{w},\mathbf{v}^h) + B(\mathbf{w}^h,\mathbf{w}^h,\mathbf{v}^h).\end{aligned}$$

Setting $\mathbf{v}^h = \boldsymbol{\chi}^h$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\,\boldsymbol{\chi}^h\,||^2 + \nu||\,\nabla\boldsymbol{\chi}^h\,||^2 &= (\boldsymbol{\eta}_t, \boldsymbol{\chi}^h) - \nu(\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\chi}^h) + (\lambda-\lambda^h, \nabla\cdot\boldsymbol{\chi}^h) \\ &\quad -B(\mathbf{w},\mathbf{w},\boldsymbol{\chi}^h) + B(\mathbf{w}^h,\mathbf{w}^h,\boldsymbol{\chi}^h). \qquad (5.2.4)\end{aligned}$$

We now analyze the nonlinear terms on the right hand side of (5.2.4). Using the self adjointness property of differential filter, Lemma 5.0.7 and the skew-symmetry of the trilinear form yields

$$\begin{aligned} B(\mathbf{w},\mathbf{w},\boldsymbol{\chi}^h)-B(\mathbf{w}^h,\mathbf{w}^h,\boldsymbol{\chi}^h) &= b(\mathbf{w},\mathbf{w},T(\boldsymbol{\chi}^h))-b(\mathbf{w}^h,\mathbf{w}^h,T(\boldsymbol{\chi}^h)) \\ &= b(\mathbf{w},\mathbf{e},T(\boldsymbol{\chi}^h))-b(\mathbf{e},\mathbf{e},T(\boldsymbol{\chi}^h))+b(\mathbf{e},\mathbf{w},T(\boldsymbol{\chi}^h)) \\ &= b(\mathbf{w},\boldsymbol{\eta},T(\boldsymbol{\chi}^h))-b(\mathbf{w},\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h))-b(\boldsymbol{\chi}^h,\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h)) \\ &\quad -b(\boldsymbol{\eta},\boldsymbol{\eta},T(\boldsymbol{\chi}^h))+b(\boldsymbol{\chi}^h,\boldsymbol{\eta},T(\boldsymbol{\chi}^h))+b(\boldsymbol{\eta},\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h)) \\ &\quad -b(\boldsymbol{\chi}^h,\mathbf{w},T(\boldsymbol{\chi}^h))+b(\boldsymbol{\eta},\mathbf{w},T(\boldsymbol{\chi}^h)). \end{aligned}$$

With the aid of this result, equation (5.2.4) becomes

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\,\boldsymbol{\chi}^h\,||^2+\nu||\,\nabla\boldsymbol{\chi}^h\,||^2 &= (\boldsymbol{\eta}_t,\boldsymbol{\chi}^h)-\nu(\nabla\boldsymbol{\eta},\nabla\boldsymbol{\chi}^h)+(\lambda-\lambda^h,\nabla\cdot\boldsymbol{\chi}^h) \\ &\quad -b(\mathbf{w},\boldsymbol{\eta},T(\boldsymbol{\chi}^h))+b(\mathbf{w},\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h))+b(\boldsymbol{\chi}^h,\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h)) \\ &\quad +b(\boldsymbol{\eta},\boldsymbol{\eta},T(\boldsymbol{\chi}^h))-b(\boldsymbol{\chi}^h,\boldsymbol{\eta},T(\boldsymbol{\chi}^h))-b(\boldsymbol{\eta},\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h)) \\ &\quad +b(\boldsymbol{\chi}^h,\mathbf{w},T(\boldsymbol{\chi}^h))-b(\boldsymbol{\eta},\mathbf{w},T(\boldsymbol{\chi}^h)). \end{aligned} \tag{5.2.5}$$

We wish to bound the terms on the right hand side of (5.2.5). Therefore, we use the Cauchy-Schwarz inequality followed by Young's inequality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\,\boldsymbol{\chi}^h\,||^2+\nu||\,\nabla\boldsymbol{\chi}^h\,||^2 &\leq \frac{C}{\nu}||\,\boldsymbol{\eta}_t\,||_{-1}^2+C\,\nu||\,\nabla\boldsymbol{\eta}\,||^2+\frac{C}{\nu}||\,\lambda-\lambda^h\,||^2+\frac{3\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2 \\ &\quad +|b(\mathbf{w},\boldsymbol{\eta},T(\boldsymbol{\chi}^h))+b(\mathbf{w},\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h))+b(\boldsymbol{\chi}^h,\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h)) \\ &\quad +b(\boldsymbol{\eta},\boldsymbol{\eta},T(\boldsymbol{\chi}^h))-b(\boldsymbol{\chi}^h,\boldsymbol{\eta},T(\boldsymbol{\chi}^h))-b(\boldsymbol{\eta},\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h)) \\ &\quad +b(\boldsymbol{\chi}^h,\mathbf{w},T(\boldsymbol{\chi}^h))-b(\boldsymbol{\eta},\mathbf{w},T(\boldsymbol{\chi}^h))|. \end{aligned} \tag{5.2.6}$$

Next, we use standard bounds on each of the trilinear forms (as per Lemma 2.0.1) on the right hand side of (5.2.6). We also make frequent use of Lemma 4.0.3, Lemma 4.0.5 and Young's inequality (with conjugate exponents 2 and 2, or 4/3 and 4); in some cases, we apply an inverse inequality.

$$\begin{aligned} b(\mathbf{w},\boldsymbol{\eta},T(\boldsymbol{\chi}^h)) &= b(\mathbf{w},\boldsymbol{\eta},T(\boldsymbol{\chi}^h)-\boldsymbol{\chi}^h)+b(\mathbf{w},\boldsymbol{\eta},\boldsymbol{\chi}^h) \\ &\leq M||\,\nabla\mathbf{w}\,||\,||\,\nabla\boldsymbol{\eta}\,||\,||\,\nabla(T(\boldsymbol{\chi}^h)-\boldsymbol{\chi}^h)\,||+M||\,\nabla\mathbf{w}\,||\,||\,\nabla\boldsymbol{\eta}\,||\,||\,\nabla\boldsymbol{\chi}^h\,|| \\ &\leq C||\,\nabla\mathbf{w}\,||\,||\,\nabla\boldsymbol{\eta}\,||\,||\,\nabla\boldsymbol{\chi}^h\,|| \\ &\leq \frac{\nu}{14}\,||\,\nabla\,\boldsymbol{\chi}^h\,||^2+\frac{C}{\nu}\,||\,\nabla\mathbf{w}\,||^2\,||\,\nabla\boldsymbol{\eta}\,||^2 \end{aligned}$$

$$b(\boldsymbol{\eta},\mathbf{w},T(\boldsymbol{\chi}^h)) \leq \frac{\nu}{14}\,||\,\nabla\,\boldsymbol{\chi}^h\,||^2+\frac{C}{\nu}\,||\,\nabla\mathbf{w}\,||^2\,||\,\nabla\boldsymbol{\eta}\,||^2$$

$$\begin{aligned} b(\boldsymbol{\chi}^h,\boldsymbol{\chi}^h,T(\boldsymbol{\chi}^h)) &\leq M\,||\,\boldsymbol{\chi}^h\,||^{1/2}\,||\,\nabla\boldsymbol{\chi}^h\,||^{3/2}\,||\,\nabla T(\boldsymbol{\chi}^h)\,|| \\ &\leq \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2+\frac{C}{\nu^3}\,||\,\boldsymbol{\chi}^h\,||^2\,||\,\nabla T(\boldsymbol{\chi}^h)\,||^4 \\ &\leq \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2+\frac{C}{\nu^3\,\delta^4}||\,\boldsymbol{\chi}^h\,||^6 \end{aligned}$$

$$\begin{aligned} b(\boldsymbol{\eta},\boldsymbol{\eta},T(\boldsymbol{\chi}^h)) &= b(\boldsymbol{\eta},\boldsymbol{\eta},T(\boldsymbol{\chi}^h)-\boldsymbol{\chi}^h)+b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{\chi}^h) \\ &\leq M\,||\,\nabla\boldsymbol{\eta}\,||^2\,||\,T(\boldsymbol{\chi}^h)-\boldsymbol{\chi}^h\,||^{1/2}\,||\,\nabla(T(\boldsymbol{\chi}^h)-\boldsymbol{\chi}^h)\,||^{1/2}+M||\,\nabla\boldsymbol{\eta}\,||^2\,||\,\nabla\boldsymbol{\chi}^h\,|| \\ &\leq C\,\delta^{1/2}||\,\nabla\boldsymbol{\eta}\,||^2\,||\,\nabla\boldsymbol{\chi}^h\,||+C||\,\nabla\boldsymbol{\eta}\,||^2\,||\,\nabla\boldsymbol{\chi}^h\,||, \\ &\leq \frac{\nu}{14}||\,\nabla\boldsymbol{\chi}^h\,||^2+\frac{C}{\nu}(\delta+1)||\,\nabla\boldsymbol{\eta}\,||^4, \end{aligned}$$

$$
\begin{aligned}
b(\boldsymbol{\eta}, \boldsymbol{\chi}^h, T(\boldsymbol{\chi}^h)) &\leq M \|\nabla \boldsymbol{\eta}\| \, \|\nabla \boldsymbol{\chi}^h\| \, \|T(\boldsymbol{\chi}^h)\|^{1/2} \, \|\nabla T(\boldsymbol{\chi}^h)\|^{1/2} \\
&\leq C h^{-1} \delta^{-1/2} \|\nabla \boldsymbol{\eta}\| \, \|\boldsymbol{\chi}^h\|^2
\end{aligned}
$$

$$
b(\boldsymbol{\chi}^h, \boldsymbol{\eta}, T(\boldsymbol{\chi}^h)) \leq C h^{-1} \delta^{-1/2} \|\nabla \boldsymbol{\eta}\| \, \|\boldsymbol{\chi}^h\|^2
$$

$$
\begin{aligned}
b(\mathbf{w}, \boldsymbol{\chi}^h, T(\boldsymbol{\chi}^h)) &\leq M \|\nabla \mathbf{w}\| \, \|\nabla \boldsymbol{\chi}^h\| \, \|T(\boldsymbol{\chi}^h)\|^{1/2} \, \|\nabla T(\boldsymbol{\chi}^h)\|^{1/2} \\
&\leq C h^{-1} \delta^{-1/2} \|\nabla \mathbf{w}\| \, \|\boldsymbol{\chi}^h\|^2
\end{aligned}
$$

$$
b(\boldsymbol{\chi}^h, \mathbf{w}, T(\boldsymbol{\chi}^h)) \leq M \|\nabla \mathbf{w}\| \, \|\nabla \boldsymbol{\chi}^h\| \, \|T(\boldsymbol{\chi}^h)\|^{1/2} \, \|\nabla T(\boldsymbol{\chi}^h)\|^{1/2}
$$

Putting everything together and using Poincarè's inequality on the left hand side we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\chi}^h\|^2 &+ C_{PF}^{-2}\frac{\nu}{2}\|\boldsymbol{\chi}^h\|^2 \leq \frac{C}{\nu}\|\boldsymbol{\eta}_t\|_{-1}^2 \\
&+ C\nu\|\nabla\boldsymbol{\eta}\|^2 + \frac{C}{\nu}\|\lambda - \lambda^h\|^2 + \frac{C}{\nu}(\delta+1)\|\nabla\boldsymbol{\eta}\|^4 + \frac{C}{\nu}\|\nabla\mathbf{w}\|^2\|\nabla\boldsymbol{\eta}\|^2 \\
&+ C\delta^{-1/2}h^{-1}(\|\nabla\boldsymbol{\eta}\| + \|\nabla\mathbf{w}\|)\|\boldsymbol{\chi}^h\|^2 + \frac{C}{\nu^3\,\delta^4}\|\boldsymbol{\chi}^h\|^6. \qquad (5.2.7)
\end{aligned}
$$

Setting $y(t) = \|\boldsymbol{\chi}^h\|^2$ and combining terms, this equation can be rewritten as

$$
\frac{d}{dt}y(t) + \alpha(t)\,y(t) \leq \beta\,y(t)^3 + \gamma,
$$

where $\alpha(t)$ is the coefficient of $\|\boldsymbol{\chi}^h\|^2$, β is the coefficient of $\|\boldsymbol{\chi}^h\|^6$ and γ is the maximum over $[0,T]$ of all the remaining terms on the right hand side of (5.2.7). With a suitable choice of $\tilde{\mathbf{w}}$, we also have $0 \leq \|\boldsymbol{\chi}^h(0)\| \leq \gamma$.

The Continuation Lemma (Lemma 2.0.2) implies that there exists a constant $M \geq 1$ and $\gamma_0 > 0$ such that for $\gamma \leq \gamma_0$,

$$
y(t) \leq M\gamma, \qquad (5.2.8)
$$

for $0 \leq t \leq T$. In other words,

$$
\begin{aligned}
\|\boldsymbol{\chi}^h\|^2 \leq C \max_{0\leq t\leq T}\Big[&\nu^{-1}\|\boldsymbol{\eta}_t\|_{-1}^2 + \nu\|\nabla\boldsymbol{\eta}\|^2 + \nu^{-1}\|\lambda-\lambda^h\|^2 + (\delta+1)\nu^{-1}\|\nabla\boldsymbol{\eta}\|^4 \\
&+ \nu^{-1}\|\nabla\mathbf{w}\|^2\|\nabla\boldsymbol{\eta}\|^2\Big].
\end{aligned}
$$

Applying the triangle inequality and taking the infimum over $\tilde{\mathbf{w}} \in \mathbf{V^h}$ and $\lambda^h \in \mathbb{Q}^h$, gives the required result. □

6 Numerical Experiments

In this section, we investigate the kinetic energy of the exact filter versus discrete filter discretization, given respectively by:

$$
E_E(\mathbf{w}^h) = \frac{1}{2}\|\mathbf{w}^h\|^2, \quad \text{for } t \in [0,T].
$$

and

$$E_D(\mathbf{w}^h) = \frac{1}{2}|| \mathbf{w}^h ||^2 + \frac{\delta^2}{2}|| \nabla \mathbf{w}^h ||^2, \quad \text{for } t \in [0, T].$$

The kinetic energy is one of the indicators of whether a model is useful for turbulent flow computations [15]. Notice that E_D has an extra term, justified by the energy inequality in Lemma 5.1.1. In order to be able to compare these two cases, we normalize the results and present graphs of $E_E/E_E^{initial}$ and $E_D/E_D^{initial}$ in Figures 1 and 2.

We used the software *FreeFem++* [14] to run the numerical tests. The time stepping scheme is Backward Euler; in space, we use the well known Taylor-Hood element (continuous piecewise quadratic polynomials for the velocity and linear for the pressure).

Our test problem is determined by the following choices:

$$\Omega = (0,1) \times (0,1) \qquad \mathbf{w}^h|_{\partial\Omega} = 0 \qquad \overline{\mathbf{f}} = \mathbf{0}.$$

A nonzero divergence free initial condition is obtained by construction. Let $\psi(x,y) = 10\sin(100xy^2)x^2(1-x)^2y^2(1-y)^2$ and set

$$\mathbf{w}_0 = \begin{bmatrix} \psi_y \\ -\psi_x \end{bmatrix}.$$

Then, $\nabla \cdot \mathbf{w}_0 = 0$.

We have proven that the kinetic energy for the exact filter scheme is bounded for a finite time interval. Here, we show numerically that it actually blows up after a certain time by computing the total kinetic energy of the approximated velocity.

Let n_Z be the number of mesh points used in the discretization of the Zeroth Order Model equation and n_F, the corresponding number for the filtering problem. We obtained the following results:

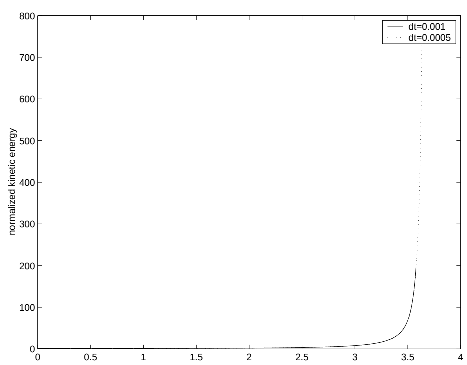

Figure 1: Time vs. $E_E/E_E^{initial}$, $n_F = 16$,$n_Z = 8$,$Re = 100000$

Considering that the boundary conditions and the forcing term, $\overline{\mathbf{f}}$, are zero, one would expect that after some transient, where the effects of the nonzero initial condition are still important, the solution would tend to zero. Figure 1 shows that the kinetic energy computed with the exact filter does not correspond to the expected true kinetic energy. It not only does not go to zero, but actually blows up. This can be verified by dividing the time step by 2, which gives the same qualitative result.

On the other hand, the kinetic energy of the discrete filter scheme is consistent with what we expect and tends to zero asymptotically and monotonically, as shown in Figure 2:

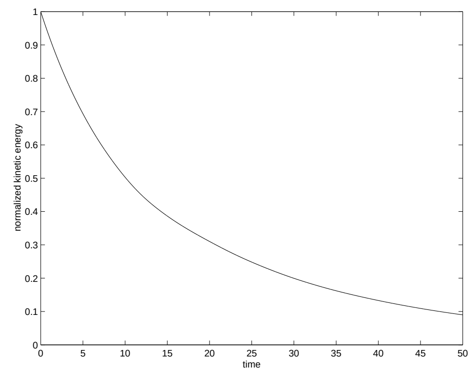

Figure 2: Time vs. $E_D/E_D^{initial}$, $n_F = 8, n_Z = 8, Re = 100000, dt = 0.001$

7 Remarks and Conclusions

The scheme analyzed in this work can be easily incorporated into widely known and reliable finite element codes for solving the time dependent Navier-Stokes equations.

We have shown that the precise implementation of the filtering operation plays a major role in the stability and convergence properties of the scheme. When the discretization of the LES model and the filtering are performed in the same mesh, the resulting scheme is stable and convergent. However, when a finer mesh is used for filtering, stability of the resulting scheme becomes much less clear. It might even be weakly unstable over long time intervals. Further, convergence over small time intervals requires extra regularity assumptions on the solution. In terms of computations, this means that solving the filter problem in a finer mesh does not improve the overall performance of the scheme and may even produce an unstable solution.

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