A SIMILARITY THEORY OF APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE*

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Abstract. We apply the phenomenology of homogeneous, isotropic turbulence to the family of approximate deconvolution models proposed by Stolz and Adams. In particular, we establish that the models themselves have an energy cascade with two asymptotically different inertial ranges. Delineation of these gives insight into the resolution requirements of using approximate deconvolution models. The approximate deconvolution model's energy balance contains both an enhanced energy dissipation and a modification to the model's kinetic energy. The modification of the model's kinetic energy induces a secondary energy cascade which accelerates scale truncation. The enhanced energy dissipation completes the scale truncation by reducing the model's micro-scale from the Kolmogorov micro-scale.

AMS subject classification.

Key words. energy cascade, large eddy simulation, turbulence, deconvolution

1. Introduction. Turbulent flows consist of complex, interacting three dimensional eddies of various sizes. In 1941 Kolmogorov gave a remarkable, universal description of the eddies in turbulent flow by combining a judicious mix of physical insight, conjecture, mathematical and dimensional analysis. In his description, the largest eddies are deterministic in nature. Those below a critical size are dominated by viscous forces, and die very quickly due to these forces. This critical length, the Kolmogorov microscale, is $\eta = O(Re^{-3/4})$ in $3d$, so the persistent eddies in a $3d$ flow requires taking

$$\Delta x = \Delta y = \Delta z = O(Re^{-3/4})$$

giving $O(Re^{+9/4})$ mesh points in space per time step. Therefore, direct numerical simulation of turbulent flows (down to the Kolmogorov microscale) is often not computationally economical or even feasible. On the other hand, the largest structures in the flow (containing most of the flow's energy) are responsible for much of the mixing and most of the flow's momentum transport. Thus, various turbulence models are used for simulations seeking to predict a flow's large structures.

One promising approach to the simulation of turbulent flows is called *Large Eddy Simulation* or *LES*. In LES the evolution of local, spatial averages over length scales $l \geq \delta$ is sought where δ is user selected. The selection of this averaging radius δ is determined typically by three factors: computational resources i.e. δ must be related to the finest computationally feasible mesh, turnaround time needed for the calculation, and estimates of the scales of the persistent eddies needed to be resolved for an accurate simulation. On the face of it, LES seems feasible since the large eddies are believed to be deterministic. The small eddies (accepting Kolmogorov's

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description) have a universal structure so, in principle, their mean effects on the large eddies should be model-able. The crudest estimate of cost is

$$\Delta x = \Delta y \ = \Delta z = O(\delta),$$

with thus $O(\delta^{-3})$ storage required in space per time step. On the other hand, it is entirely possible that the computational mesh must be smaller than $O(\delta)$ to predict the $O(\delta)$ structures correctly. It is also entirely possible that, since LES models are themselves inexact and uncertain, solutions to an LES model contain persistent energetic structures smaller than $O(\delta)$. Even with an accurate closure model (in the sense of consistency error or residual stress), the resulting LES model must be solved to find the LES velocity. The nonlinear interactions and the sensitivity to perturbations of the LES model might also introduce unintended and persistent small scales.

To begin, consider the Navier-Stokes equations in a periodic box in $\mathbb{R}^3$:

$$\begin{gathered} u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = f \quad \text{in } \Omega = (0, L)^3, \ t > 0, \\ \nabla \cdot u = 0 \quad \text{in } (0, L)^3, \end{gathered} \tag{1.1}$$

subject to periodic (with zero mean) conditions

$$\begin{gathered} u(x + Le_j, t) = u(x, t) \quad j = 1, 2, 3 \quad \text{and,} \\ \int_\Omega \phi dx = 0 \ \text{ for } \ \phi = u, \ u_0, \ f, \ p. \end{gathered} \tag{1.2}$$

LES computes an approximation to local spatial averages of solutions to (1.1)-(1.2). Many averaging operators are used in LES. Herein we choose a differential filter, [11], associated with a length-scale $\delta > 0$. (The case of other filters is summarized in section 5.1.) Given $\phi(x)$, $\overline{\phi(x)}$ is the unique L-periodic solution of

$$A\overline{\phi} := -\delta^2 \triangle \overline{\phi} + \overline{\phi} = \phi\, , \ \text{in } \Omega.$$

Averaging the NSE (meaning: applying A^{-1} to (1.1)) gives the exact space filtered NSE for $\overline{u}$

$$\begin{gathered} \overline{u}_t + \overline{u \cdot \nabla u} - \nu \triangle \overline{u} + \nabla \overline{p} = \overline{f} \text{ and} \\ \nabla \cdot \overline{u} = 0. \end{gathered}$$

This is not closed since (noting that $\overline{u \cdot \nabla u} = \nabla \cdot (\overline{uu})$)

$$\overline{u\, u} \ \neq \overline{u}\, \overline{u}.$$

There are many closure models used in LES, see [24], [13], [4] for a surveys. Approximate de-convolution models, studied herein, are used, with success, in many simulations of turbulent flows, e.g., [1], [2], [25]. They are among the most accurate of turbulence models, [6], [18] and one of the few for which a mathematical confirmation of their effectiveness is known. Briefly, an approximate deconvolution operator (constructed in section 2) denoted by D_N is an operator satisfying

$$\phi = D_N(\overline{\phi}) + O(\delta^{2N+2}) \quad \text{for smooth } \ \phi.$$

Since $D_N \overline{u}$ approximates u to accuracy $O(\delta^{2N+2})$ in the smooth flow regions it is justified to consider the closure approximation:

$$\overline{uu} \simeq \overline{D_N \overline{u} D_N \overline{u}} + O(\delta^{2N+2}). \tag{1.3}$$

Using this closure approximation results in an LES model whose solutions are intended to approximate the true flow averages, $w \approx \overline{u}, q \approx \overline{p}$. The resulting models, introduced by Adams and Stolz [1], [2], [25], are given by

$$w_t + \nabla\cdot(\overline{D_N w\ D_N w}) - \nu \triangle w + \nabla q + \chi\,(w - \overline{w}) = \overline{f},\text{ and } \nabla\cdot w = 0,\ N = 0, 1, 2, \cdots. \quad (1.4)$$

The time relaxation term $\chi\,(w - \overline{w})$ is included in numerical simulations of (1.4) to damp strongly the temporal growth of the fluctuating component of w driven by noise, numerical errors, inexact boundary conditions and so on. It can be used as a numerical regularization in any model and is studied in [2] [19], [23]. In this report we study the parameters-free deconvolution model that results by setting $\chi = 0$.

This report investigates the following questions for the family of approximate deconvolution model (1.4): *What is the length scale of the smallest persistent eddy in the model's solution?* (This length scale corresponds to the Kolmogorov dissipation length scale for a turbulent flow.) *Do solutions of the LES model exhibit an energy cascade and, if so, what are its details? How does the model act to truncate the small eddies?* Inspired by Muschinsky's study of the Smagorinsky model [20], the answers to these questions will come from two simple but powerful tools: a precise energy balance for the models themselves in [15], [16], [18], and [6][1] and Kolmogorov's similarity theory[2], e.g., [4], [9], [22], [17], [24], suitably adapted.

1.1. Summary of results. By adapting the reasoning of Richardson and Kolmogorov, we establish the model's energy cascade. The micro-scale of the model (the length-scale of the smallest persistent structure in the model's solution) is shown to be

$$\eta_{model} \simeq Re^{-\frac{3}{10}} L^{\frac{2}{5}} \delta^{\frac{3}{5}}.$$

This depends upon the filter chosen (see section 5.1). For the second order differential filter (the case above), it is typically smaller than the desired cutoff length-scale of $O(\delta)$. In fact it is easy to calculate that $\delta = \eta_{model} \ \Leftrightarrow\ \delta \simeq Re^{-3/4}$ and the flow is fully resolved. Thus the behavior of the model in the intermediate range $\delta \geq l \geq \eta_{model}$ is critical. By examining the details of the energy cascade of the model, we see a second mechanism for fast but not exponential truncation of the number of scales of the model's solution. Over the wave numbers corresponding to the resolved scales, $0 \leq k \leq \frac{1}{\delta}$, i.e. over length scales: $L \geq l \geq \delta$ we see that the model correctly predicts an energy spectrum of the form $\alpha_{\ model} \varepsilon_{model}^{2/3} k^{-5/3}$. Above the cutoff frequency and down to the model's micro-scale, the kinetic energy in the model's solution drops algebraically almost like k^{-4} according to $\alpha_{\ model} \varepsilon_{model}^{2/3} \delta^{-2} k^{-11/3}$. The model thus algebraically truncates the effective scales present. The derivation of these results involves the classical dimensional analysis arguments of Kolmogorov coupled with precise mathematical knowledge of the model's kinetic energy balance.

2. Approximate deconvolution models of turbulence.

2.1. The van Cittert Algorithm. The basic problem in approximate deconvolution is: given $\overline{u}$ find *useful* approximations $D_N(\bar{u})$ of u that lead to *accurate*

[1] To keep this report as self-contained as posible, we have included the key ideas in the proofs of the results used from these papers.

[2] See also the supplement to this report at www.math.pitt.edu/techreports.html.

and *stable* LES models. In other words, solve the equation below for an approximation which is appropriate for the application at hand

$$A^{-1}u = \overline{u}, \text{ solve for } u.$$

The de-convolution algorithm we consider was studied by van Cittert in 1931 and its use in LES pioneered by Stolz and Adams [1], [25]. For each $N = 0, 1, ...$ it computes an approximate solution u_N to the above de-convolution equation by N steps of a fixed point iteration, [3], for the fixed point problem:

$$\text{given } \overline{u} \text{ solve } u = u + \{\overline{u} - A^{-1}u\} \text{ for } u.$$

The de-convolution approximation is then computed as follows.

ALGORITHM 2.1 (van Cittert approximate de-convolution algorithm). $u_0 = \overline{u}$,
for n=1,2,...,N-1, perform
$u_{n+1} = u_n + \{\overline{u} - A^{-1}u_n\}$
Call $u_N = D_N\overline{u}$ *.*

By eliminating the intermediate steps, it is easy to find an explicit formula for the N^{th} de-convolution operator D_N :

$$D_N\phi := \sum_{n=0}^{N}(I - A^{-1})^n\phi. \tag{2.1}$$

For example, the approximate de-convolution operator corresponding to $N = 0, 1, 2$ are:

$$\begin{aligned}
&D_0\overline{u} = \overline{u},\\
&D_1\overline{u} = 2\overline{u} - \overline{\overline{u}},\\
&D_2\overline{u} = 3\overline{u} - 3\overline{\overline{u}} + \overline{\overline{\overline{u}}}.
\end{aligned}$$

We begin by reviewing a result of Stolz, Adams and Kleiser [26] and Dunca and Epshteyn [6].

LEMMA 2.1. *[Error in approximate de-convolution] For any* $\phi \in L^2(\Omega)$,

$$\begin{aligned}
\phi - D_N\overline{\phi} &= (I - A^{-1})^{N+1}\phi\\
&= (-1)^{N+1}\delta^{2N+2}\triangle^{N+1}A^{-(N+1)}\phi.
\end{aligned}$$

Proof. Let $B = I - A^{-1}$. Since $\overline{\phi} = A^{-1}\phi$, $\overline{\phi} = (I - B)\phi$. Since $D_N := \sum_{n=0}^{N} B^n$, a geometric series calculation gives

$$(I - B)D_N\overline{\phi} = (I - B^{N+1})\overline{\phi}.$$

Subtraction gives

$$\phi - D_N\overline{\phi} = AB^{N+1}\overline{\phi} = B^{N+1}A\overline{\phi} = B^{N+1}\phi.$$

Finally, $B = I - A^{-1}$, so rearranging terms gives the claimed result:

$$\begin{aligned}
\phi - D_N\overline{\phi} &= (A - I)^{N+1}A^{-(N+1)}\phi\\
&= A^{-(N+1)}((-1)^{N+1}\delta^{2N+2}\triangle^{N+1})\phi.
\end{aligned}$$

□

The simplest example of an approximate deconvolution model (1.4) arises when $N = 0$ and $\chi = 0$. This zeroth order model also arises as the zeroth order model in many different families of LES models and has been studied carefully in [15], [16], and [21].

$$w_t + \nabla \cdot (\overline{w\, w}) - \nu \triangle w + \nabla q = \overline{f}, \text{ and } \nabla \cdot w = 0. \tag{2.2}$$

To see the mathematical key to the energy cascade that follows we first recall from [15] the energy equality for (2.2).

PROPOSITION 2.2. *Let $u_0 \in L_0^2(\Omega), f \in L^2(\Omega \times (0,T))$. If w is a weak or strong solution of (2.2), w satisfies*

$$\frac{1}{2}\left\|w(t)\right\|^2 + \frac{\delta^2}{2}\left\|\nabla w(t)\right\|^2 + \int_0^t \nu \left\|\nabla w(t')\right\|^2 + \nu\delta^2 \left\|\triangle w(t')\right\|^2 dt' =$$
$$= \frac{1}{2}\left\|\overline{u}_0\right\|^2 + \frac{\delta^2}{2}\left\|\nabla \overline{u}_0\right\|^2 + \int_0^t (f(t'), w(t'))dt'. \tag{2.3}$$

Proof. For strong solutions, multiplying (2.2) by $Aw := (-\delta^2\triangle + 1)w$ and integrating over the flow domain gives

$$\int_\Omega w_t \cdot Aw + \nabla \cdot (\overline{w\, w}) \cdot Aw - \nu\triangle w \cdot Aw + \nabla q \cdot Aw d\mathbf{x} = \int_\Omega \overline{f} \cdot Aw d\mathbf{x}.$$

The nonlinear term exactly vanishes because

$$\int_\Omega \nabla \cdot (\overline{w\, w}) \cdot Aw d\mathbf{x} = \int_\Omega A^{-1}(\nabla \cdot (w\ w)) \cdot Aw d\mathbf{x} =$$
$$= \int_\Omega \nabla \cdot (w\ w) \cdot w d\mathbf{x} = 0.$$

Integrating by parts the remaining terms gives

$$\frac{d}{dt}\frac{1}{2}\{||w(t)||^2 + \delta^2||\nabla w(t)||^2\} + \nu\{||\nabla w(t)||^2 + \delta^2||\triangle w(t)||^2\} = \int_\Omega f(t) \cdot w(t)dx.$$

The results follows by integrating this from 0 to t. For weak solutions a more precise version of this argument, [15], is used. □

DEFINITION 2.3. *The deconvolution weighted inner product and norm, $(\cdot,\cdot)_N$ and $||\cdot||_N$ are*

$$(u,v)_N := (u, D_N v)\,, \quad ||u||_N := (u,u)_N^{\frac{1}{2}}$$

LEMMA 2.4. *Consider the approximate deconvolution operator D_N as defined above. Then*

$$||\phi||^2 \le ||\phi||_N \le (N+1)||\phi||^2, \forall \phi \in L^2(\Omega)\ .$$

Proof. By the spectral mapping theorem we have

$$\lambda(D_N) = \sum_{n=0}^{N} \lambda(I - A^{-1})^n = \sum_{n=0}^{N} (1 - \lambda(A^{-1}))^n, \text{ and}$$
$$0 < \lambda(A^{-1}) \leq 1 \text{ by the definition of operator A.}$$

Thus, $1 \leq \lambda(D_N) \leq N+1$. Since $\lambda(D_N)$ is a self-adjoint operator, this proves the above equivalence of norms. □

DEFINITION 2.5. *Given two quantities A and B (such as E_{model}, ε_{model}) we shall write $A \sim B$ if there are positive constants $C_1(N)$, $C_2(N)$ with*

$$C_1(N)A \leq B \leq C_2(N)A$$

PROPOSITION 2.6. *Suppose $\chi = 0$ in the ADM (1.4). Then, if w is a strong solution of (1.4), w satisfies*

$$\frac{1}{2}[||w(t)||_N^2 + \delta^2||\nabla w(t)||_N^2] + \int_0^t \nu||\nabla w(t')||_N^2 + \nu\delta^2||\triangle w(t')||_N^2 dt' =$$
$$= \frac{1}{2}[||\overline{u}_0(t)||_N^2 + \delta^2||\nabla \overline{u}_0(t)||_N^2] + \int_0^t (f(t'), w(t'))_N \, dt'.$$

Proof. Let (w, q) denote a periodic solution of the Nth order model with $\chi = 0$. Multiplying (1.4) by $AD_N w$ and integrating over the flow domain gives

$$\int_\Omega w_t \cdot AD_N w + \nabla \cdot (\overline{D_N w \; D_N w}) \cdot AD_N w - \nu \triangle w \cdot AD_N w + \nabla q \cdot AD_N w d\mathbf{x} =$$
$$= \int_\Omega \overline{f} \cdot AD_N w d\mathbf{x}.$$

The nonlinear term exactly vanishes exactly as in the zeroth order case because

$$\int_\Omega \nabla \cdot (\overline{D_N w \; D_N w}) \cdot AD_N w d\mathbf{x} = \int_\Omega A^{-1}(\nabla \cdot (D_N w \; D_N w)) \cdot AD_N w d\mathbf{x} =$$
$$= \int_\Omega \nabla \cdot (D_N w \; D_N w) \cdot D_N w d\mathbf{x} = 0.$$

Integrating by parts the remaining terms gives

$$\frac{d}{dt}\frac{1}{2}\{||w(t)||_N^2 + \delta^2||\nabla w(t)||_N^2\} + \nu\{||\nabla w(t)||_N^2 + \delta^2||\triangle w(t)||_N^2\} = (f(t) \cdot w(t))_N \, .$$

The results follows by integrating this from 0 to t. □

REMARK 2.1. *We can clearly identify three physical quantities of kinetic energy, energy dissipation rate and power input. Let L =the selected length-scale; then these are given by*

$$\textit{Model's energy: } E_{model}(w)(t) := \frac{1}{2L^3}\{||w(t)||_N^2 + \delta^2||\nabla w(t)||_N^2\}, \quad (2.4)$$
$$\textit{Model's dissipation rate: } \varepsilon_{model}(w)(t) := \frac{\nu}{L^3}\{||\nabla w(t)||_N^2 + \delta^2||\triangle w(t)||_N^2\}, \quad (2.5)$$
$$\textit{Model's power input: } P_{model}(w)(t) := \frac{1}{L^3}(f(t), w(t))_N. \quad (2.6)$$

Remark 2.2. *The ADM thus has two terms which reflect extraction of energy from resolved scales. The energy dissipation in the model (2.5) is enhanced by the extra term which is equivalent to* $\nu\delta^2 \|\triangle w(t)\|^2$ *(by Lemma 2.4). Thus, this term dissipates energy locally where large curvatures in the velocity* w *occur, rather than large gradients. This term thus acts as an irreversible energy drain localized at large local fluctuations. The second term, which is uniformly equivalent to* $\delta^2 \|\nabla w(t)\|^2$, *(by Lemma 2.4) occurs in the models kinetic energy given by (2.4). The true kinetic energy* ($\frac{1}{2}\|w(t)\|^2$) *in regions of large deformations is thus extracted, conserved and stored in the kinetic energy penalty term* $\delta^2 \|\nabla w(t)\|^2$. *Thus, this reversible term acts as a kinetic "Energy sponge". Both terms have to an obvious regularizing effect.*

Lemma 2.7. *As* $\delta \to 0$,

$$
\begin{aligned}
E_{model}(w)(t) &\to E(w)(t) = \frac{1}{2L^3}||w(t)||^2,\\
\varepsilon_{model}(w)(t) &\to \varepsilon(w)(t) = \frac{\nu}{2L^3}||\nabla w(t)||^2, \ \textit{and}\\
P_{model}(w)(t) &\to P(w)(t) = \frac{1}{L^3}(f(t), w(t)).
\end{aligned}
$$

Proof. As $\delta \to 0$ all the δ^2 terms drop out in the definitions above, $D_N \to I$ and $||\phi||_N \to ||\phi||$. □

3. Energy Cascades of Approximate Deconvolution Models. If we apply A to the model (1.4) (with $\chi = 0$) it becomes:

$$\frac{\partial}{\partial t}\left[w - \delta^2 \triangle w\right] + D_N(w) \cdot \nabla D_N(w) - \nu\left[\triangle w - \delta^2 \triangle^2 w\right] + \nabla P = f\,, \quad \text{in } \Omega \times (0, T).$$

Since D_N is spectrally equivalent to the identity (uniformly in k, δ, nonuniformly in N) the nonlinear interaction $D_N(w) \cdot \nabla D_N(w)$ (like those in the NSE) will pump energy from large scales to small scales. The viscous terms in the above equation will damp energy at the small scales (more strongly than in the NSE in fact). Lastly, when $\nu = 0$, $f \equiv 0$ the model's kinetic energy is exactly conserved (Remark 2.1 and Proposition 2.6)

$$E_{model}(w)(t) = E_{model}(\overline{u}_0).$$

Thus, (1.4) satisfies all the requirements for the existence of a Richardson - like energy cascade for E_{model}. We thus proceed to develop a similarity theory for ADM's (paralleling the K-41 theory of turbulence) using the Π-theorem of dimensional analysis, recalled next. We stress that the Π-theorem is a rigorous mathematical theorem. The only phenomenology or physical intuition involved is the selection of variables and assumptions of dimensional homogeneity.

Theorem 3.1 (The Π-theorem). *If it is known that a physical process is governed by a dimensionally homogeneous relation involving n dimensional parameters, such as*

$$x_1 = f(x_2, x_3, ...x_n), \tag{3.1}$$

where the x's are dimensional variables, there exists an equivalent relation involving a smaller number, $(n - k)$, *of dimensionless parameters, such that*

$$\Pi_1 = F(\Pi_2, \Pi_3, ..., \Pi_{n-k}), \tag{3.2}$$

where the Π's are dimensionless groups constructed from the x's. The reduction, k, is usually equal, but never more than, the number of fundamental dimensions involved in the x's.

Proof. The proof can be found in [5]. $\square$

Let $< \cdot >$ denote long time averaging

$$< \phi > (\mathbf{x}) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \phi(\mathbf{x},t)dt. \tag{3.3}$$

To define the kinetic energy distribution recall that the total kinetic energy in a velocity w (assuming unit density) at time t is, $E(w)(t) := \int_\Omega \frac{1}{2}|w(\mathbf{x},t)|^2 d\mathbf{x}$. Thus, the time averaged kinetic energy distribution in physical space (at the point $\mathbf{x}$ in space) is given by $E(\mathbf{x}) :=< \frac{1}{2}|w(\mathbf{x},t)|^2 >$. We will similarly define a distribution in wave number space. Expand the velocity w in a Fourier series

$$w(\mathbf{x},t) = \sum_{\mathbf{k}} \widehat{w}(\mathbf{k},t)e^{-i\mathbf{k}\cdot\mathbf{x}},\ \text{where } \mathbf{k} = \frac{2\pi\mathbf{n}}{L} \text{ is the wave number and } \mathbf{n}\in\mathbb{Z}^3.$$

The Fourier coefficients are given by

$$\widehat{w}(\mathbf{k},t) = \frac{1}{L^3}\int_\Omega w(x,t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}.$$

The magnitude of $\mathbf{k},\mathbf{n}$ are defined by

$$|\mathbf{n}| = \{|n_1|^2+|n_2|^2+|n_3|^2\}^{1/2}, |\mathbf{k}| = \frac{2\pi|\mathbf{n}|}{L}.$$

$$|\mathbf{n}|_\infty = \max\{|n_1|,|n_2|,|n_3|\}, |\mathbf{k}|_\infty = \frac{2\pi|\mathbf{n}|}{L}.$$

The length-scale of the wave number $\mathbf{k}$ is defined by $l = \frac{2\pi}{|\mathbf{k}|_\infty}$. In studies of the periodic problem the wave-number vector $\mathbf{k} = (k_1,k_2,k_3)$ is often called a *triad*. Begin by recalling Parseval's equality.

Lemma 3.2 (Parseval's equality). *For $w \in L^2(\Omega)$,*

$$\frac{1}{L^3}\int_\Omega \frac{1}{2}|w(x,t)|^2dx = \sum_{\mathbf{k}}\frac{1}{2}|\widehat{w}(\mathbf{k},t)|^2 =$$

$$= \sum_k \left(\sum_{|\mathbf{k}|=k}\frac{1}{2}|\widehat{w}(\mathbf{k},t)|^2\right), \textit{where } \mathbf{k} = \frac{2\pi\mathbf{n}}{L} \textit{ is the wave number and } \mathbf{n}\in\mathbb{Z}^3.$$

Definition 3.3. *The kinetic energy distribution functions are defined by*

$$E(k,t) = \frac{L}{2\pi}\sum_{|\mathbf{k}|=k}\frac{1}{2}|\widehat{w}(\mathbf{k},t)|^2, \quad E_{model}(k,t) = \frac{L}{2\pi}\sum_{|\mathbf{k}|=k}\frac{1}{2}\left(\widehat{D_N}(k)+\delta^2k^2\widehat{D_N}(k)\right)|\widehat{w}(k,t)|^2$$

$$E(k) = < E(k,t) >, \quad E_{model}(k) =< E_{model}(k,t) >$$

Parseval's equality thus can be rewritten as

$$\frac{1}{L^3}\int_\Omega \frac{1}{2}|w(x,t)|^2 dx = \frac{2\pi}{L}\sum_k E(k,t), \text{ and}$$
$$< \frac{1}{L^3}\int_\Omega \frac{1}{2}|w(x,t)|^2 dx >= \frac{2\pi}{L}\sum_k E(k).$$

The units of a variable will be denoted by $[\cdot]$. Thus, for example, $[\text{velocity}] = L/T$. We start the dimensional analysis for the approximate deconvolution model following Kolmogorov's analysis of the NSE by selecting the variables:

- E_{model} - energy spectrum of model with $[E_{model}(k)] = [L]^3[T]^{-2}$,
- ε_{model} - time averaged energy dissipation rate of the model's solution with $[\varepsilon_{model}(k)] = [L]^2[T]^{-3}$,
- k - wave number with $[k] = [L]^{-1}$ and
- δ - averaging radius with $[\delta] = [L]$.

Choosing the set of fundamental or primary dimensions M, L and T, we then work with to 2 dimensionless ratios, Π_1 and Π_2 . Choosing ε and k for the repeating variables (note that ε and k cannot form a dimensionless group) we obtain $\Pi_1 = \varepsilon^a_{model}k^bE_{\ model}$ and $\Pi_2 = \varepsilon^c_{model}k^d\delta$ for some a, b, c, d real numbers. Equating the exponents of the corresponding dimensions in both dimensionless groups gives us:

$$\Pi_1 = \varepsilon^{-2/3}_{model}k^{5/3}E_{\ model} \quad \text{and} \quad \Pi_2 = k\delta$$

The Π-theorem implies that there is a functional relationship between Π_1 and Π_2 , i.e., $\Pi_1 = f(\Pi_2)$, or

$$E_{\ model}\varepsilon^{-2/3}_{model}k^{5/3} = f(k\delta) \text{ or } E_{\ model} = \varepsilon^{2/3}_{model}k^{-5/3}f(k\delta).$$

The simplest case[3] is when $f(\Pi_2) = \alpha_{\ model}$. In this case we have

$$E_{model}(k) = \alpha_{\ model}\varepsilon^{2/3}_{model}k^{-5/3}.$$

It is not surprising that, since the ADM is dimensionally consistent with the Navier-Stokes equations, dimensional analysis would reveal a similar energy cascade for the model's kinetic energy. However, interesting conclusions result from the difference between $E(w)$ and $E_{model}(w)$.

$$\begin{aligned} E_{\ model}(w) &:= < \frac{1}{2L^3}(||w||^2_N + \delta^2||\nabla w||^2_N) > \\ &\sim < \frac{1}{2L^3}[||w||^2 + \delta^2||\nabla w||^2] > \quad \text{by Lemma (2.4)} \\ &\simeq \sum_k (1+\delta^2\, k^2)E(k) \quad \text{using Parceval's equality.} \end{aligned}$$

Further, since $E_{\ model}(k) \simeq \alpha_{model}\varepsilon^{2/3}_{model}k^{-5/3}$ we have:

$$E(k) \simeq \frac{\alpha_{\ model}\varepsilon^{2/3}_{model}k^{-5/3}}{1+\delta^2k^2}. \tag{3.4}$$

[3] We shall show in subsection (3.1) that this case is implied by Kraichnan's dynamic argument.

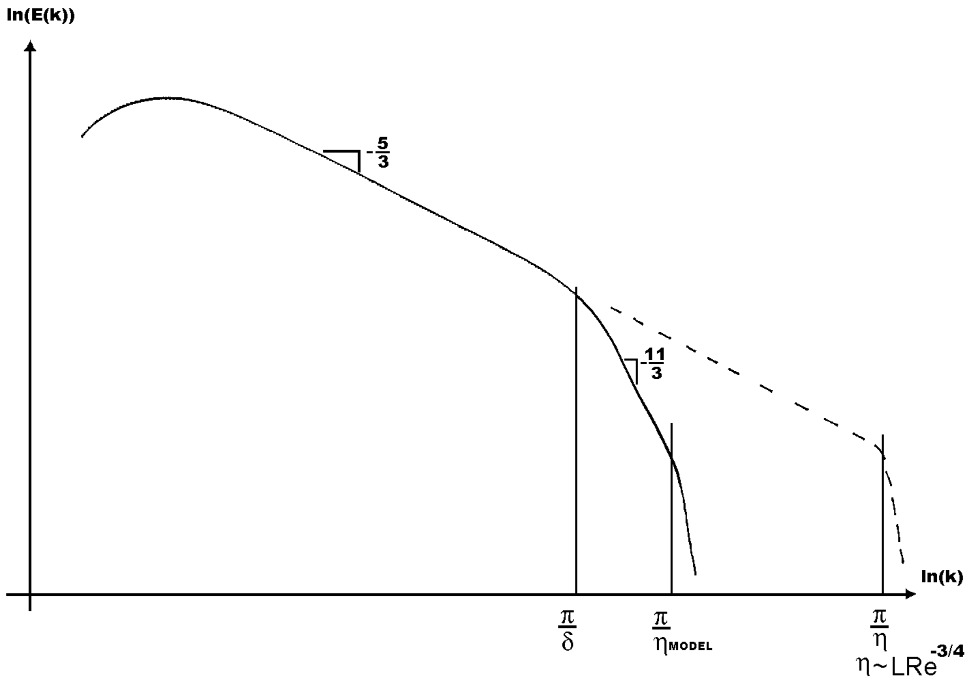

FIG. 3.1. *Kinetic Energy Spectrum of the Model*

Equation (3.4) gives precise information about how small scales are truncated by the ADM. Indeed, there are two wave number regions depending on which term in the denominator is dominant: 1 or $\delta^2 k^2$. The transition point is the cutoff wave number $k = \frac{1}{\delta}$. We thus have:

$$E(k) \simeq \alpha_{\ model} \varepsilon_{model}^{2/3} k^{-5/3}, \quad \text{for} \quad k \leq \frac{1}{\delta},$$
$$E(k) \simeq \alpha_{\ model} \varepsilon_{model}^{2/3} \delta^{-2} k^{-11/3}, \quad \text{for } k \geq \frac{1}{\delta} \ .$$

This asymptotic behavior is depicted in the figure.

3.1. Kraichnan's Dynamic Analysis Applied to ADM's. The energy cascade will now be investigated more closely using the dynamical argument of Kraichnan, [14]. Let $\Pi_{model}(k)$ be defined as the total rate of energy transfer from all wave numbers $< k$ to all wave numbers $> k$. Following the Kraichnan [14] we assume that $\Pi_{model}(k)$ is proportional to the total energy $(\, kE_{model}(k)\,)$ in wave numbers of the order k and to some effective rate of shear $\sigma(k)$ which acts to distort flow structures of scale $1/k$. That is:

$$\Pi_{model}(k) \simeq \sigma(k)\, k\, E_{model}(k) \tag{3.5}$$

Furthermore, we expect

$$\sigma(k)^2 \ \simeq \ \int_0^k p^2 E_{model}(p) dp \tag{3.6}$$

The major contribution to (3.6) is from $p \simeq k$, in accord with Kolmogorov's localness assumption. This is because all wave numbers $\leq k$ should contribute to the effective mean-square shear acting on wave numbers of order k, while the effects of all wave numbers $\gg k$ can plausibly be expected to average out over the scales of order $1/k$ and over times the order of the characteristic distortion time $\sigma(k)^{-1}$.

We shall say that there is an energy cascade if in some "inertial" range, $\Pi_{model}(k)$ is independent of the wave number, i.e., $\Pi_{model}(k) = \varepsilon_{model}$. Using the equations (3.5) and (3.6) we get:

$$E_{model}(k) \simeq \varepsilon_{model}^{2/3} k^{-5/3}$$

Then, using the relation $E\ _{model}(k) \simeq (1+\delta^2 k^2)E(k)$ we have:

$$E(k) \simeq \varepsilon_{model}^{2/3} k^{-5/3}, \text{ for } \ k \leq \frac{1}{\delta},$$
$$E(k) \simeq \varepsilon_{model}^{2/3} \delta^{-2} k^{-11/3} \text{ , for } k \geq \frac{1}{\delta} \ .$$

This is consistent with our previous derived result using dimensional analysis.

3.2. The micro-scale of approximate deconvolution models. The model's Reynolds numbers with respect to the model's largest and smallest scales are

$$\text{Large scales: } Re_{model-large} = \frac{UL}{\nu(1+(\frac{\delta}{L})^2)}$$
$$\text{Small scales: } Re_{model-small} = \frac{w_{small}\eta_{model}}{\nu(1+(\frac{\delta}{\eta_{model}})^2)}.$$

As in the Navier-Stokes equations, the ADM's energy cascade is halted by viscosity grinding down eddies exponentially fast when

$$Re_{model-small} \simeq O(1), \text{ i.e., when}$$
$$\frac{w_{small}\eta_{model}}{\nu(1+(\frac{\delta}{\eta_{model}})^2)} \simeq 1.$$

This last equation allows us to determine the characteristic velocity of the model's smallest persistent eddies w_{small} and eliminate it from subsequent equations. This gives

$$w_{small} \simeq \nu(1+(\frac{\delta}{\eta_{model}})^2)/\eta_{model}.$$

The second important equation determining the model's micro-scale comes from matching energy in to energy out. The rate of energy input to the largest scales is the energy over the associated time scale

$$\frac{E_{model}}{(\frac{L}{U})} = \frac{U^2(1+(\frac{\delta}{L})^2)}{(\frac{L}{U})} = \frac{U^3}{L}(1+(\frac{\delta}{L})^2).$$

When the model reaches statistical equilibrium, the energy input to the largest scales must match the energy dissipation at the model's micro-scale which scales like $\varepsilon_{small} \simeq \nu(|\nabla w_{small}|^2 + \delta^2|\triangle w_{small}|^2) \simeq \nu(\frac{w_{small}}{\eta_{model}})^2(1+(\frac{\delta}{\eta_{model}})^2)$. Thus we have

$$\frac{U^3}{L}(1+(\frac{\delta}{L})^2) \simeq \nu(\frac{w_{small}}{\eta_{model}})^2(1+(\frac{\delta}{\eta_{model}})^2).$$

Inserting the above formula for the micro-eddies characteristic velocity w_{small} gives

$$\frac{U^3}{L}(1+(\frac{\delta}{L})^2) \simeq \frac{\nu^3}{\eta_{model}^4}(1+(\frac{\delta}{\eta_{model}})^2)^3.$$

First note that the expected case in LES is when $(\frac{\delta}{L})^2 << 1$ (otherwise the procedure should be considered a VLES[4]). In this case the LHS simplifies to just $\frac{U^3}{L}$. Next, with this simplification, the solution to this equation depends on which term in the numerator of the RHS is dominant: 1 or $(\frac{\delta}{\eta_{model}})^2$. The former case occurs when the averaging radius δ is so small that the model is very close to the NSE so the latter is the expected case. In this case we have $\eta_{model} \simeq Re^{-\frac{3}{4}}L$, when $\delta < \eta_{model}$. In the expected case, solving for the micro-scale gives

$$\eta_{model} \simeq Re^{-\frac{3}{10}}L^{\frac{2}{5}}\delta^{\frac{3}{5}} \text{ , when } \delta > \eta_{model}.$$

4. Design of an Experimental Test of the Model's Energy Cascade. The main open question not resolved in the similarity theory pertains to the unknown, non-dimensional function $f(\Pi_2)$. The principle of economy of explanation suggests that $f(\Pi_2)$ is constant, and this is supported, strongly by Kraichnan's dynamic theory of turbulence, subsection (3.1). This question can be resolved by numerical experiments on the model itself (not on the Navier-Stokes equations) establishing the curve between the Π 's. Having this curve we can get complete quantitative information. Suppose that the E_{model} is desired for conditions k_a and δ_a. The dimensionless group $(\Pi_2)_a$ can be immediately evaluated as $k_a\delta_a$. Corresponding to this value of $(\Pi_2)_a$, the value of $(\Pi_1)_a$ is read off the plot. $(E_{model})_a$ is then computed.

5. Conclusions and open problems. The basic Approximate Deconvolution Model possesses an energy cascade that truncates the true energy spectrum in two ways. First, there is an enhanced viscosity acting in the model. This enhanced viscosity does not dissipate energy for laminar shear flows and its amount is related to the local curvature of the velocity field. Further, it disappears when $\nu = 0$. The action of this enhanced viscosity is to trigger exponential decay of eddies at the model's micro-scale of

$$\eta_{model} \simeq Re^{-\frac{3}{10}}L^{\frac{2}{5}}\frac{\delta^{\frac{3}{5}}}{(1+(\frac{\delta}{L})^2)^{\frac{1}{10}}}(>> \eta_{NSE}).$$

The second way the ADM truncates the scales of motion is through an energy sponge in the model's kinetic energy. The extra term triggers an accelerated energy decay of $O(k^{-\frac{11}{3}})$ at the cutoff length scale. *Above the cutoff length scale the ADM predicts the correct energy cascade!*

This analysis presupposes two things. First, the relaxation term in the original model is zero. Its effects were studied separately in [19] where a similarity theory was developed for the model: Navier-Stokes + time relaxation term. It was showed that the action of this relaxation term is to induce a micro-scale, analogous to the Kolmogorov micro-scale in turbulence, and to trigger decay of eddies at the model's micro-scale. Based on this, the intent of adding the time relaxation term is clearly to further truncate the energy cascade of deconvolution models. The result of combining ADM and time relaxation is currently under study.

[4] Very Large Eddy Simulation. The estimates of the micro-scale are easily extended to this case too.

5.1. Other Filters. Tracking the effects of the choice of filter backward through the analysis leads to a very simple conclusion. The secondary cascade ($k^{-11/3}$) in the energy cascade of the model's solution results because the filter decays like

$$\widehat{A^{-1}}(k) \simeq k^{-2} \quad \text{ and } -5/3 + (-2) = -11/3$$

It is easy to check, for example, tracking forward that if the filter arises from $4th$ order (hyperviscosity like) operator with symbol decaying like k^{-4} then the secondary cascade will have exponent $k^{-17/3}$ (i.e. -17/3=-5/3+(-4)). Continuing, if a gaussian filter (which has exponential decay in wave number space) is used, then exponential decay of the energy spectrum begins at the cutoff frequency. This immediate truncation might compensate in some calculation for its extra complexity.

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