Convergence of Time Averaged Statistics of Finite Element Approximations of the Navier-Stokes equations

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Abstract

When discussing numerical solutions of the Navier-Stokes equations, especially when turbulent flows are concerned, there are at least two questions that can be raised. *What is meaningful to compute? How to determine the fidelity of the computed solution with respect to the true solution?* This paper takes a step to the answer of these questions for turbulent flows. We consider long time averages of weak solutions of the Navier-Stokes equations, rather than strong solutions. We present time averaged error estimates for the time averaged energy dissipation rate, drag and lift. For shear flows, we address the question of fidelity of the computed solution with respect to the true solution, in view of Kolmogorov's energy cascade theory.

1 Introduction

The motion of an incompressible fluid is governed by the incompressible Navier-Stokes equations. A fundamental problem of fluid motion is turbulence and a fundamental problem in the Navier-Stokes equations is that of uniqueness of weak solutions in the general case of no assumed extra regularity or small data. The Leray conjecture [14] is that these two problems are connected: the lack of uniqueness of weak solutions (which he called "turbulent solutions") is not an artifact of imperfect mathematical techniques but it reflects fundamental physical mechanisms of turbulence.

The numerical analysis of turbulent flows is caught between the gaps in the physical understanding of turbulence and those in the mathematical foundations of the Navier-Stokes equations. For example, smooth strong solutions are not expected while, if the uniqueness of the weak solution is unknown, bounding the error in a numerical simulation is currently not possible without assuming extra regularity on the solution, or without assuming both the initial data u_0 and the body force $f(x,t)$ are very small.

On the other hand, computational simulations are carried out and statistics of computed fluid velocities and pressures often reflect rather accurately statistics of physical flows even in the absence of mathematical justification for this accuracy. Further, statistics (by which we shall mean long time averages) are often smooth, behave deterministically (often in accord with the Kolmogorov theory [12]) in both numerical simulations and in physical experiments. From this situation, a challenge for the numerical analysis of fluid motion arises: develop a rigorous understanding of how statistics computed from numerical simulations reflect those for the unknown solution of the Navier-Stokes equations.

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The incompressible Navier-Stokes equations are given by

$$\begin{aligned} & u_t + u\cdot\nabla u - \nu\nabla u + \nabla p = f(x,t), \quad x\in\Omega, \quad 0<t<\infty \\ & \nabla\cdot u = 0, \quad x\in\Omega, \quad 0<t<\infty \\ & u(x,0) = u_0(x), \quad x\in\Omega \\ & u = 0 \quad \text{on} \quad \partial\Omega, \quad t\geq 0. \end{aligned} \tag{1.1}$$

Here, Ω denotes a bounded and regular flow domain in $\mathbb{R}^d$ (d=2 or 3), $u(x,t)$, $p(x,t)$ denotes the fluid velocity and pressure, ν is the viscosity, $f(x,t)\in L^\infty(0,\infty;L^2(\Omega))$ are the body forces and $u_0\in L^2(\Omega)$ is a weakly divergence free initial condition. The Reynolds number Re is the inverse of the viscosity.

We will study statistics of the energy dissipation rate and the total kinetic energy of the flow. The energy dissipation rate of the flow at time t is given by

$$\varepsilon(u) := \frac{\nu}{|\,\Omega\,|}||\,\nabla u(\cdot,t)\,||^2,$$

where $|\,\Omega\,|$ is the measure of Ω and $||\cdot||$ denotes the $L^2(\Omega)$-norm, and its total kinetic energy is

$$k(u) := \frac{1}{2}||\,u(\cdot,t)\,||^2.$$

The time average $< q >$ of a quantity q is defined by

$$< q >= \limsup_{T\to\infty}\frac{1}{T}\int_0^T q(t)\,dt,$$

such that time average of the energy dissipation rate is

$$< \varepsilon(u) >= \limsup_{T\to\infty}\frac{1}{T}\int_0^T \varepsilon(u)\,dt = \limsup_{t\to\infty}\frac{1}{T}\int_0^T \frac{\nu}{|\,\Omega\,|}||\nabla u||^2\,dt,$$

and the time average of the kinetic energy is

$$< k(u) >= \limsup_{T\to\infty}\frac{1}{T}\int_0^T k(u)\,dt.$$

In practical simulations of turbulent flows, it is typical to output time averaged flow statistics (which are then matched against benchmark averages, e.g. [1, 11, 9, 10, 15, 4]). However, there is very little numerical analysis in support of these calculations. Of course, if the error in certain norms of the velocity and the pressure is provably optimal over $0\leq t<\infty$ then time averages involving these norms are convergent as well. But, the practical case is complementary: time averages seem to be predictable even when dynamic flow behavior over bounded time intervals is irregular. This is the case we aim to study. However, a complete analysis seems to be still beyond the present mathematical tools.

We consider herein as a first step the case of arbitrary initial data u_0 and asymptotically small body forces which converge to a stationary limit $f^*(x) = \lim_{t\to\infty} f(x,t)$. Let (u^h, p^h) be a finite element approximation of the velocity field and the pressure and assume that a small data condition is satisfied. Let u^* be the solution of the stationary Navier-Stokes equations with body force f^*. We show that

$$< \varepsilon(u-u^*) >= 0, \quad < \varepsilon(u^h - u^{*\,h}) >= 0.$$

and we prove, see Theorem 3.2, an error estimate which shows that the problem of estimating $< \varepsilon(u-u^h) >$ reduces to the one of estimating $||\nabla(u^* - u^{*\,h})||^2$. So the error goes to zero optimally as the mesh width $h\to 0$. This result is plausible because the possible irregularities caused by large initial data are washed out by the time averaging.

Section 4 studies the flow around a body. The results of Section 3 are used to give estimates for the time averaged drag and the lift coefficients at the body.

In Section 5, we consider the complementary situation of a flow driven by a large and persistent boundary condition. We are not (yet) able to perform a complete error analysis in this case. However, following the important work of Constantin and Doering [3] in the continuous case, we show that provided the first mesh line in the finite element mesh is within $O(1/Re)$ of the moving wall which drives the flow, then the computed time averaged energy dissipation rate for the shear flow scales as predicted for the continuous flow by the Kolmogorov theory:

$$< \varepsilon(u^h) > \quad \leq \quad C\,\frac{U^3}{L}.$$

This restriction on the mesh size arises from mathematical analysis of constructible background flows in finite element spaces and their subsequent analysis. However, it is in accordance with entirely different observation of the thickness of time averaged turbulent boundary layers [19].

2 Mathematical Preliminaries

The velocity at a given time t is sought in the space

$$\mathbb{X} = H_0^1(\Omega)^d = \{v \in L^2(\Omega)^d : \nabla\, v \in L^2(\Omega)^{d\times d} \quad \text{and} \quad v = 0 \ \text{ on } \ \partial\Omega\}$$

equipped with the norm $\|v\|_{\mathbb{X}} = \|\nabla v\|$. The norm of the dual space of $\mathbb{X}$ is denoted by by $||\cdot||_{-1}$. The pressure at time t space is sought in

$$\mathbb{Q} = L_0^2(\Omega) = \left\{ q : \, q \in L^2(\Omega)\,, \int_\Omega q\,dx = 0 \right\}.$$

In addition, the space of weakly divergence free functions is denoted by

$$\mathbb{V} = \{v \in \mathbb{X} : (\nabla\cdot v, q) = 0 \text{ for all } q \in \mathbb{Q}\}.$$

For Y being a function space of functions $v : [0,\infty) \to Y$, we use the notation

$$L^p(0,T;Y) = \left\{ v : v(t) : (0,T) \to Y, \text{strongly measurable and } \int_0^T ||\, v(t)\,||_Y^p\, dt < \infty \right\},$$

with $1 \leq p < \infty$, and the usual modification is made if $p = \infty$.

Define the trilinear forms on $\mathbb{X}\times\mathbb{X}\times\mathbb{X}$:

$$b(u,v,w) = \int_\Omega u\cdot\nabla v\cdot w\,dx, \qquad \text{and} \qquad b^*(u,v,w) = \frac{1}{2}b(u,v,w) - \frac{1}{2}b(u,w,v).$$

Let $M \geq N$ denote the finite constants

$$M = \sup_{u,v,w\in\mathbb{X}} \frac{b^*(u,v,w)}{||\,\nabla u\,||\;||\,\nabla v\,||\;||\,\nabla w\,||} < \infty \qquad \text{and} \qquad N = \sup_{u,v,w\in\mathbb{V}} \frac{b^*(u,v,w)}{||\,\nabla u\,||\;||\,\nabla v\,||\;||\,\nabla w\,||} < \infty. \quad (2.1)$$

Corresponding constants M^h and N^h are defined by replacing $\mathbb{X}$ by the finite element space $\mathbb{X}^h \subset \mathbb{X}$ and $\mathbb{V}$ by $\mathbb{V}^h \subset \mathbb{X}$ (defined below). Note that $M \geq M^h, N^h, N$ and that as $h \to 0$, $N^h \to N$ (see [6]) and by the same argument $M^h \to M$.

We shall assume throughout the paper that the velocity-pressure finite element spaces $\mathbb{X}^h \subset \mathbb{X}$ and $\mathbb{Q}^h \subset \mathbb{Q}$ are conforming, have approximation properties typical of finite element spaces commonly in use and satisfy the discrete inf-sup condition,

$$\inf_{q^h\in\mathbb{Q}^h} \sup_{v^h\in\mathbb{X}^h} \frac{(q^h,\,\nabla\cdot v^h)}{||\,\nabla v^h\,||\;||\,q^h\,||} \geq \beta^h > 0,$$

where β^h is bounded away from zero uniformly in h. The space of discretely divergence free functions is defined as

$$\mathbb{V}^h = \{v^h \in \mathbb{X}^h : (q^h,\, \nabla \cdot v^h) = 0,\, \forall\, q^h \in \mathbb{Q}^h\}.$$

For examples of such spaces see, e.g., Gunzburger [8], Brezzi and Fortin [2] and Girault and Raviart [6].

2.1 The Continuous in Time Finite Element Discretization

Consider the standard finite element discretization of the Navier-Stokes equations. The semi-discrete (continuous in time) finite element approximations $u^h = u^h(\cdot, t)$ and $p^h = p^h(\cdot, t)$ are maps $u^h : [\,0, \infty\,) \to \mathbb{X}^h$, $p^h : [\,0, \infty\,) \to \mathbb{Q}^h$ satisfying

$$\begin{aligned}
(u^h_t, v^h) + \nu(\nabla u^h, \nabla v^h) + b^*(u^h, u^h, v^h) - (p^h, \nabla \cdot v^h) &= (f, v^h) \qquad \forall\, v^h \in \mathbb{X}^h && (2.2)\\
(\nabla \cdot u^h, q^h) &= 0 \quad \forall\, q^h \in \mathbb{Q}^h && (2.3)\\
(u^h(\cdot, 0) - u_0, v^h) &= 0 \quad \forall\, v^h \in \mathbb{X}^h.
\end{aligned}$$

Under the inf-sup condition (2), this is equivalent to: find $u^h : [\,0, \infty\,) \to V^h$ satisfying

$$\begin{aligned}
(u^h_t, v^h) + \nu(\nabla u^h, \nabla v^h) + b^*(u^h, u^h, v^h) &= (f, v^h), \qquad \forall\, v^h \in \mathbb{V}^h, && (2.4)\\
(u^h(\cdot, 0) - u_0, v^h) &= 0 \quad \forall\, v^h \in \mathbb{V}^h.
\end{aligned}$$

2.2 The Associated Equilibrium Problem

Consider the Navier-Stokes (1.1). When $f(x,t) \to f^*(x)$ as $t \to \infty$, we can associate with (1.1) the following equilibrium problem: find $u^*(x)$, $p^*(x)$ satisfying

$$\begin{aligned}
-\nu \Delta u^* + u^* \cdot \nabla u^* + \nabla p^* &= f^* \qquad \text{in } \Omega\\
\nabla \cdot u^* &= 0 \qquad \text{in } \Omega && (2.5)\\
u^* &= 0, \quad \text{on } \partial\Omega, \qquad \text{and} \qquad \int_\Omega p^* dx = 0.
\end{aligned}$$

The variational formulation of the equilibrium problem is: Find $u^* \in \mathbb{X}$ and $p^* \in \mathbb{Q}$ such that

$$\begin{aligned}
\nu(\nabla u^*, \nabla v) + b^*(u^*, u^*, v) - (p^*, \nabla \cdot v) &= (f^*, v) \quad \forall\, v \in \mathbb{X} && (2.6)\\
(\nabla \cdot u^*, q) &= 0 \quad \forall\, q \in \mathbb{Q} && (2.7)
\end{aligned}$$

or, equivalently: Find $u^* \in \mathbb{V}$ such that

$$\nu(\nabla u^*, \nabla v) + b^*(u^*, u^*, v) = (f^*, v) \qquad \forall\, v \in \mathbb{V}. \tag{2.8}$$

The finite element approximations $u^{*\,h}$ and $p^{*\,h}$ satisfy the equations

$$\begin{aligned}
\nu(\nabla u^{*\,h}, \nabla v^h) + b^*(u^{*\,h}, u^{*\,h}, v^h) - (p^{*\,h}, \nabla \cdot v^h) &= (f^*, v^h) \qquad \forall\, v^h \in \mathbb{X}^h && (2.9)\\
(\nabla \cdot u^{*\,h}, q^h) &= 0 \qquad\qquad \forall\, q^h \in \mathbb{Q}^h.
\end{aligned}$$

This becomes in $\mathbb{V}^h$: Find $u^{*\,h} \in \mathbb{V}^h$ such that

$$\nu(\nabla u^{*\,h}, \nabla v^h) + b^*(u^{*\,h}, u^{*\,h}, v^h) = (f^*, v^h) \qquad \forall\, v^h \in \mathbb{V}^h. \tag{2.10}$$

It is known that solutions to of the equilibrium problem are nonsingular for small data, generically nonsingular for large data and optimally approximated by $u^{*\,h}$ when nonsingular, [7]. Setting $v = u^*$ in (2.6) and $v^h = u^{*h}$ in (2.9), it is easy to check the a priori bounds

$$||\, \nabla u^* \,|| \le \nu^{-1} ||\, f^* \,||_{-1}, \qquad ||\, \nabla u^{*\,h} \,|| \le \nu^{-1} ||\, f^* \,||_{-1}. \tag{2.11}$$

Both bounds can be sharpened slightly. In the continuous case, $\| f^* \|_{-1}$ can be replaced by the dual norm of $\mathbb{V}$,

$$\| f^* \|_* := \sup_{v \in \mathbb{V}} \frac{(f^*, v)}{\| \nabla v \|},$$

and in the discrete case by the dual norm of $\mathbb{V}^h$,

$$\| f^* \|_{*\,h} := \sup_{v^h \in \mathbb{V}^h} \frac{(f^*, v^h)}{\| \nabla v^h \|}.$$

It is known that if the problem data is small enough, concretely if

$$N\nu^{-2} \| f^* \|_{-1} < 1, \tag{2.12}$$

then u^* is unique. If additionally $f(x,t) \equiv f^*(x)$, $u(x,t) \to u^*(x)$ in $L^2(\Omega)$ exponentially fast as $t \to \infty$ and (u^h, p^h) approximates (u, p) optimally, [6, 13, 8].

2.3 Weak Solutions of the Navier-Stokes Equations

The problem of turbulence is perhaps intimately connected with questions about weak solutions vs. strong solutions of the Navier Stokes equations. It is well-known that weak solutions exist but it is not known if they are unique. (Thus, different methods of proving existence might possibly lead to different solutions.) A strong solution is generally defined as a weak solution which has enough extra regularity to ensure global uniqueness, i.e., which fulfills Serrin's condition [20]. In 3d, it is also unknown if strong solutions exist, [5, 21]. But if a strong solution exists, it is unique. Strong solutions might conceivably describe all fluid motion. However, in at least one conjecture about turbulence the case of strong solutions is associated with laminar flow.

For clearness of notation, we will give the definition of a weak and a strong solution, following [5]:

Definition 2.1. *Let*

1. $\mathbf{D}_T = \{v \in C^\infty(\Omega \times [0,T]) : v(t) \in C_0^\infty(\Omega) \text{ for each } t\}^d$,
2. $\mathcal{D} = \left\{\psi \in C^\infty(\Omega)^d : \psi \text{ has compact support in } \Omega \text{ and } \nabla \cdot \psi = 0 \text{ in } \Omega\right\}$,
3. $H(\Omega) \equiv \left\{v \in L^2(\Omega)^d : \nabla \cdot v = 0 \ \text{ and } \ v \cdot \hat{n} \text{ on } \partial\Omega\right\}$,
4. $\mathcal{D}_T$=$\{\ \phi(x,t) \in C^\infty(\Omega \times [0,T]) : \phi(x,t) \in \mathcal{D} \ \text{ for each } t, \ \ 0 \le t \le T\}$.

Let $u_0 \in H(\Omega)$, $f \in L^2(0,T; L^2(\Omega))$. *A measurable function* $u(x,t) : \Omega \times [0,T] \to \mathbb{R}^d$ *is a weak solution of the Navier-Stokes equations if, for all* $T > 0$

1. $u \in L^2(0,T;V) \cap L^\infty(0,T;H(\Omega))$,
2. *u satisfies the integral relation*

$$(u(T), \phi(T)) - \int_0^T \left[\left(u, \frac{\partial \phi}{\partial t}\right) - \nu\, (\nabla u, \nabla \phi) - (u \cdot \nabla u, \phi) \right] dt = (u(0), \phi(0)) + \int_0^T (f, \phi)\, dt \tag{2.13}$$

for all $\phi \in \mathcal{D}_T$, *which is equivalent to*

$$\frac{d}{dt}(u,v) + \nu(\nabla u, \nabla v) + (u \cdot \nabla u, v) - (f, v) = 0 \tag{2.14}$$

for all $v \in \mathbb{V}$.

3. *u is a strong solution if u is a weak solution and* $u \in L^\infty(0,T;\mathbb{V})$ *for any* $T > 0$.

We note that if Ω is a bounded domain with $\partial\Omega$ satisfying a cone condition, then it is known that, given a weak solution u, there exists a pressure $p(x,t) \in L^\infty(0,T;L_0^2(\Omega))$ (see, e.g. Galdi [5], Remark 2.5) satisfying

$$(u(T),\phi(T)) - \int_0^T \left(u, \frac{\partial \phi}{\partial t}\right) - \nu\,(\nabla u, \nabla\phi) - (u\cdot\nabla u,\phi) + (p,\nabla\cdot\phi)\,dt$$
$$= \quad (u(0),\phi(0)) + \int_0^T (f,\phi)\,dt \quad \forall\,\phi \in \mathbf{D}_T. \tag{2.15}$$

This is equivalent to

$$\frac{d}{dt}(u,v) + \nu\,(\nabla u,\nabla v) + (u\cdot\nabla u,\nabla v) - (p,\nabla\cdot v) = (f,v) \qquad \forall\, v \in \mathbb{X}. \tag{2.16}$$

Lemma 2.1. *Let (u,p) be a weak solution of the Navier-Stokes equations and (u^h,p^h) its finite element approximation in finite dimensional subspaces $\mathbb{X}^h \subset \mathbb{X}$ and $\mathbb{Q}^h \subset \mathbb{Q}$. Let $e = u - u^h$. Then, for any C^1 maps $v^h : [0,T] \to \mathbb{X}^h$, $q^h : (0,T] \to \mathbb{Q}^h$ (for each T, $0 < T < \infty$),*

$$(e(T),v^h(T)) - \int_0^T \Bigg[\left(e, \frac{\partial v^h}{\partial t}\right) - \nu(\nabla e,\nabla v^h) - b^*(u,u,v^h) + b^*(u^h,u^h,v^h)$$
$$+(p-p^h,\nabla\cdot v^h) + (\nabla\cdot(u-u^h),q^h)\Bigg]\,dt = (e(0),v^h(0)) \tag{2.17}$$

which is equivalent to

$$\frac{d}{dt}(e,v^h) - \left(e,\frac{\partial v^h}{\partial t}\right) + \nu\,(\nabla e,\nabla v^h) + b^*(u,u,v^h) - b^*(u^h,u^h,v^h) - (p-p^h,\nabla\cdot v^h) = 0 \tag{2.18}$$
$$(\nabla\cdot(u-u^h),q^h) = 0$$

Proof. We shall prove (2.18). The connection between (2.18) and (2.17) is the same as (2.15) and (2.16).

First, note that both follow by subtraction provided (2.15) can be shown to hold for $\phi \in C^1(0,T;\mathbb{X}^h)$ or (2.16) can be shown for $v \in C^1(0,T;\mathbb{X}^h)$ (since $\mathbb{X}^h \subset \mathbb{X}$). We show the latter.

Since $\mathbb{X}^h \subset \mathbb{X}$ (2.16) holds for all $v^h(x) \in \mathbb{X}^h$. Next, let $A(t)$ be a $C^1(0,T)$ function. Multiplication of (2.16) by $A(t)$ and using

$$A(t)\frac{d}{dt}(u,v^h) = \frac{d}{dt}(u,A(t)v^h) - (u,A'(t)v^h)$$

gives that u and p satisfy

$$\frac{d}{dt}(u,v^h) - \left(u,\frac{\partial v^h}{\partial t}\right) + \nu\,(\nabla u,\nabla v^h) + (u\cdot\nabla u,v^h) + (p,\nabla\cdot v^h) = (f,v^h)$$

with $v^h = A(t)v^h(x)$. The same equation holds for u^h and p^h. Subtracting gives (2.18) for any v^h of the form $v^h = A(t)v^h(x)$.

Since (2.18) is linear in v^h, it also follows for any v^h which is a finite finite linear combination of such function,

$$v^h(x,t) = \sum_{i=1}^N A_i(t)v_i^h(x).$$

Picking $v_i^h(x)$ to be a basis for $\mathbb{X}^h$ completes the proof. □

It is also known, [5], that weak solutions satisfy the energy inequality: for any $t \in [0,T]$,

$$\frac{1}{2}||\, u(T)\,||^2 + \nu \int_0^T ||\,\nabla\, u(t)\,||^2\, dt \le \frac{1}{2}||\, u_0\,||^2 + \int_0^T (u(t), f(t))\ dt. \tag{2.19}$$

Strong solutions satisfy even an energy equality, i.e., (2.19) with "$\le$" replaced by "$=$".

2.4 Properties of the Time Averaging Operator

For any function $q(t)$ holds

$$\begin{aligned} |<q>| &= \left|\limsup_{T\to\infty} \frac{1}{T}\int_0^T q(t)\,dt\right| \le \limsup_{T\to\infty}\frac{1}{T}\left|\int_0^T q(t)\,dt\right| \\ &\le \limsup_{t\to\infty}\frac{1}{T}\int_0^T |q(t)|\,dt \\ &= <|q|>\,. \end{aligned} \tag{2.20}$$

Similarly, for any function $q(t,x)$ where $||\cdot||$ is a finite norm of $q(t,x)$ holds

$$||<q>|| \le <||q||>\,. \tag{2.21}$$

Let $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ with $p^{-1}+q^{-1}=1$, $p,q\in[1,\infty]$. Let the time averages of u in $L^p(\Omega)$ and of v in $L^q(\Omega)$ exist. Using the Hölder inequality for Lebesgue spaces and Young's inequality, one can show a Hölder inequality for the time averaging operator

$$\begin{aligned} |<(u,v)>| &= \limsup_{t\to\infty}\frac{1}{T}\int_0^T (u,v)\,dt \le \limsup_{t\to\infty}\frac{1}{T}\int_0^T ||u||_{L^p}||v||_{L^q}\,dt \\ &\le \limsup_{t\to\infty}\left(\frac{1}{T}\int_0^T ||u||_{L^p}^p\,dt\right)^{1/p}\left(\frac{1}{T}\int_0^T ||v||_{L^q}^q\,dt\right)^{1/q} \\ &= <||u||_{L^p}^p>^{1/p}<||v||_{L^q}^q>^{1/q}\,. \end{aligned} \tag{2.22}$$

With the same arguments, one obtains for $u \in H_0^1(\Omega)$ and $v \in H^{-1}(\Omega)$

$$|<(u,v)>| \le <||\nabla u||_{L^2}^2>^{1/2}<||v||_{H^{-1}}^2>^{1/2} \tag{2.23}$$

if the right hand side of this inequality is well defined.

If $\lim_{t\to\infty} q(t) = q^*$ then

$$<q>=q^*. \tag{2.24}$$

3 Analysis of Time Averaged Errors

We present some results involving the time averaged energy dissipation rate, aiming at establishing an error estimate for this quantity. For a force f independent of time, we first show that the time averaged energy dissipation rate $<\varepsilon(u)>$ is bounded by the time averaged power input to the flow through body force-flow interaction. It is significant that this upper bound is independent of the initial condition (then it is reasonable that errors in its approximation could be insensitive to the size of u_0). We establish time averaged errors for the pressure.

Lemma 3.1. *Let u be a weak solution to the Navier Stokes equations. If $f \in L^\infty(0,\infty;H^{-1}(\Omega))$, then $||\,u\,||$ is uniformly bounded*

$$\frac{1}{2}||\,u\,||^2(T) \le e^{-\nu C_{PF}^{-2}T}||\,u(0)\,||^2 + \frac{C_{PF}^2}{\nu^2}||\,f\,||^2_{L^\infty(0,\infty;H^{-1})}, \tag{3.1}$$

where C_{PF} is the Poincare-Friedrichs constant of Ω, and consequently

$$\lim_{T\to\infty}\frac{1}{T}||\,u\,||^2(T) = 0.$$

Proof. Let V_N be a span of eigenfunctions of the Stokes operator. The Leray-Hopf construction of weak solutions gives a sequence u_N in V_N satisfying

$$(u_{N,t}, v_N) + \nu(\nabla u_N, \nabla v_N) + (u_N \cdot \nabla u_N, v_N) = (f, v_N), \qquad \forall\, v_N \in V_N. \tag{3.2}$$

with $u_N \to u$, the weak solution, in $L^2(\Omega \times (0,T))$ strongly and $L^2(0,T; H^1(\Omega))$ weakly.

This is a system of ordinary differential equations; setting $v_N = u_N$ and using Cauchy-Schwarz and Young inequalities, followed by Poincare-Friedrich inequality, we have

$$\frac{d}{dt} || \, u_N(t) \, ||^2 + \nu C_{PF}^{-2} || \, u_N(t) \, ||^2 \leq \frac{1}{\nu} || \, f \, ||_{-1}^2.$$

Using an integrating factor, we obtain a differential inequality which can be integrated on $(0,T)$, yielding

$$|| \, u_N(T) \, ||^2 \leq e^{-\nu C_{PF}^{-2} T} || \, u_N(0) \, ||^2 + \frac{1}{\nu^2 C_{PF}^{-2}} || \, f \, ||_{L^\infty(0,\infty;H^{-1})}^2.$$

This shows the uniform boundedness of $|| \, u_N(T) \, ||$. Taking the limit inferior of both sides and using a weak convergence argument (which is standard for the Navier-Stokes equations and which we show in detail in the proof of Proposition 3.3), letting $N \to \infty$, we recover (3.1) for u. The second claim now follows from the first. □

Lemma 3.2. *Let $f \in L^\infty(0,\infty; H^{-1}(\Omega))$ and let u^h be the solution of (2.4). Then $|| \, u^h \, ||$ is uniformly bounded and consequently*

$$\lim_{T\to\infty} \frac{1}{T} || \, u^h \, ||^2 = 0, \qquad \textit{and} \qquad \lim_{T\to\infty} \frac{1}{T} || \, u - u^h \, ||^2 = 0.$$

Proof. Take $v^h = u^h$ in (2.4) (a step not possible in the continuous case of Lemma 3.1) and proceeding as in the proof of Lemma 3.1, we prove a similar uniform bound for $|| \, u^h \, ||$. This proves the first equation and the bound on $|| \, u - u^h \, ||$ follows by the triangle inequality and Lemma 3.1. □

We next consider time averages.

Proposition 3.1. *Let u be a weak solution of the Navier-Stokes equations satisfying the energy inequality (2.19). Then*

$$< \varepsilon(u) > \; \leq \; \limsup_{T\to\infty} \frac{1}{|\,\Omega\,|\, T} \int_0^T (\, f, u \,)\, dt = \frac{1}{|\,\Omega\,|} < (f,u) > . \tag{3.3}$$

If u satisfies the energy equality then the above inequality can be replaced by equality. Further, if $f \in L^\infty(0,\infty; H^{-1}(\Omega)) \cap L^2(0,T; L^2(\Omega))$ for every $0 < T < \infty$ then

$$< \varepsilon(u) > \; \leq \; \nu^{-1} \left\langle \frac{1}{|\,\Omega\,|} || \, f \, ||_{-1}^2 \right\rangle \; \leq \; \frac{\nu^{-1}}{|\,\Omega\,|} \, || \, f \, ||_{L^\infty(0,\infty;H^{-1}(\Omega))}^2. \tag{3.4}$$

The semidiscrete finite element approximation u^h of u fulfills also inequalities of form (3.3) and (3.4), where in (3.3) even equality holds.

Proof. Since u satisfies the energy inequality (2.19), we have

$$\frac{1}{2T} \frac{1}{|\,\Omega\,|} || \, u(T) \, ||^2 + \frac{1}{T} \int_0^T \frac{\nu}{|\,\Omega\,|} || \, \nabla \, u(t) \, ||^2 \, dt \leq \frac{1}{2T} \frac{1}{|\,\Omega\,|} || \, u_0 \, ||^2 + \frac{1}{T} \int_0^T \frac{1}{|\,\Omega\,|} (f,u) \, dt.$$

Since $\frac{1}{2T} || \, u(T) \, ||^2 \to 0$ by Lemma 3.1 and $\frac{1}{2T} || \, u_0 \, ||^2 \to 0$ as $T \to \infty$, we obtain (3.3). If we use as starting point the energy equality, the equal sign will be preserved.

For proving the last claim, we apply inequality (2.23) to (3.3)

$$\begin{aligned} < \varepsilon(u) > &\leq \frac{\nu}{2|\,\Omega\,|} < ||\,\nabla u(t)\,||^2 > + \frac{1}{2\nu|\,\Omega\,|} < ||\,f\,||_{-1}^2 > \\ &\leq \frac{1}{2} < \varepsilon(u) > + \frac{1}{2\nu|\,\Omega\,|} ||f||^2_{L^\infty(0,\infty;H^{-1})}. \end{aligned}$$

In the semidiscrete case take u^h as test function in (2.2). This gives

$$\frac{1}{2}\frac{d}{dt}||\,u^h(t)\,||^2 + \nu||\,\nabla u^h(t)\,||^2 = (u^h(t), f(t)).$$

Integration in $(0,T)$ shows that u^h fulfills an energy equality. Now, the arguments to derive the estimates of form (3.3) and (3.4) for u^h are the same as in the continuous case. □

Next, we consider the time averaged errors. It is important to note that there is a difference between $|| < \nabla(u-u^h) > ||$ and $< ||\nabla(u-u^h)|| >$. By Minkowski's inequality, the second is an upper bound for the first. Experience with turbulent flows suggests that $< \nabla u >$ might be smooth (and thus approximable). Thus, ideally we would like estimates for the first $|| < \nabla(u-u^h) > ||$. In the case of the error in the pressure, we are able to prove such a bound.

Theorem 3.1. *Let $f \in L^\infty(0,\infty;H^{-1}(\Omega))$ and let $(\mathbb{X}^h, \mathbb{Q}^h)$ satisfy the discrete inf-sup condition (2). Then,*

$$\begin{aligned} || < p - p^h > || &\leq \frac{\nu}{\beta^h}(1 + 2M\nu^{-2} < ||\,f\,||_{-1}^2 >^{1/2}) \left\langle ||\,\nabla(u-u^h)||^2 \right\rangle^{1/2} \\ &\quad + \left(1 + \frac{1}{\beta^h}\right) \inf_{q^h \in \mathbb{Q}^h} || < p - q^h > ||. \end{aligned}$$

Proof. A straightforward calculation shows that (2.17) is equivalent to

$$\begin{aligned} &-\int_0^T (p^h - q^h, \nabla\cdot v^h)\,dt = (e(T), v^h(T)) - (e(0), v^h(0)) \\ &\quad -\int_0^T \left[\left(e, \frac{\partial v^h}{\partial t}\right) - \nu(\nabla e, \nabla v^h) - b^*(u,e,v^h) - b^*(e,u^h,v^h) + (p - q^h, \nabla\cdot v^h)\right] dt \end{aligned}$$

for all $(v^h, q^h) \in \mathbb{X}^h \times \mathbb{Q}^h$. Since the velocity finite element functions are continuous in $\overline{\Omega}$, all terms are well defined. Let $v^h = v^h(x)$. Division by T and taking limit superior as $T \to \infty$ give

$$\begin{aligned} (< p^h - q^h >, \nabla\cdot v^h) &= \nu\,(<\nabla\,e>, \nabla\,v^h) + < b^*(u,e,v^h) > \\ &\quad + < b^*(e,u^h,v^h) > + (< p - q^h >, \nabla\cdot v^h). \end{aligned} \tag{3.5}$$

For estimating (3.5), one uses again that v^h does not depend on time, (2.1) and $||\nabla\cdot v^h|| \leq ||\nabla v^h||$ to obtain

$$\begin{aligned} \left|(< p^h - q^h >, \nabla\cdot v^h)\right| &\leq \nu\,||<\nabla\,e>||\,||\,\nabla\,v^h\,|| + M < ||\,\nabla\,u\,||\,||\,\nabla\,e\,|| > ||\,\nabla\,v^h\,|| \\ &\quad + M < ||\,\nabla\,e\,||\,||\,\nabla\,u^h\,|| > ||\,\nabla\,v^h\,|| + || < p - q^h > ||\,||\,\nabla\,v^h\,||. \end{aligned}$$

Dividing by $||\,\nabla\,v^h\,||$ and applying the discrete inf-sup condition (2) on the left hand side of this inequality leads to

$$\begin{aligned} \beta^h|| < p^h - q^h > || &\leq \nu\,||<\nabla\,e>|| + M < ||\,\nabla\,u\,||\,||\,\nabla\,e\,|| > \\ &\quad + M < ||\,\nabla\,e\,||\,||\,\nabla\,u^h\,|| > + || < p - q^h > ||. \end{aligned}$$

The first term on the right hand side can be estimated first with (2.21) and then the first three terms with (2.22). This gives

$$\begin{aligned} \beta^h|| < p^h - q^h > || &\leq \nu < ||\,\nabla\,e\,||^2 >^{1/2} + M < ||\,\nabla\,u\,||^2 >^{1/2} < ||\,\nabla\,e\,||^2 >^{1/2} \\ &\quad + M < ||\,\nabla\,e\,||^2 >^{1/2} < ||\,\nabla\,u^h\,||^2 >^{1/2} + || < p - q^h > ||. \end{aligned}$$

The norms of the weak and the discrete solution can be estimated with the results of Proposition 3.1 such that

$$\beta^h || < p^h - q^h > || \leq \nu(1 + 2M\nu^{-2} < || f ||_{-1}^2 >^{1/2}) \left\langle || \nabla e ||^2 \right\rangle^{1/2} + || < p - q^h > ||.$$

The proof concludes with the application of the triangle inequality and by taking the infimum over $q^h \in \mathbb{Q}^h$. □

The key idea in the above proof was to restrict $v^h \in \mathbb{V}^h$ to be time independent. Then, time averaging can be applied and brought inside upon the pressure error directly. It is interesting that the equations of motion give a different realization of time averaged error for the velocity and pressure ($< || \nabla(u - u^h) || >$ versus $|| < p - p^h > ||$). This appears also in the time averaged lift and drag error estimates in Theorem 4.1. At this point, we do not know if this disctintion has other, deeper causes or implications.

We next turn to the error inequalities for the time averaged velocity error $< \varepsilon(u - u^h) >$.

Proposition 3.2. *Let $Y = L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; V^h)$ and assume $u_t \in L^1(0, T; \mathbb{X}^*)$ for every $0 < T < \infty$. Then the time averaged velocity error satisfies the following inequalities*

$$\begin{aligned} < \varepsilon(u - u^h) > \quad &\leq \quad C \inf_{q^h \in \mathbb{Q}^h} \nu^{-1} < || p - q^h ||^2 > -2 < b^*(e, u, e) > \\ &+ \quad \inf_{\tilde{u} \in Y} \left[< \varepsilon(u - \tilde{u}) > +2 \left\langle b^*(u, e, u - \tilde{u}) + b^*(e, u, u - \tilde{u}) - b^*(e, e, u - \tilde{u}) \right\rangle \right], \end{aligned}$$

and

$$\begin{aligned} < \varepsilon(u - u^h) > \quad &\leq \quad C \inf_{q^h \in \mathbb{Q}^h} \nu^{-1} < || p - q^h ||^2 > \\ &+ \quad \inf_{\tilde{u} \in Y} \Big[< \varepsilon(u - \tilde{u}) > +\nu^{-3} \left\langle || e ||^{2/3} || \nabla u ||^{4/3} || \nabla(u - \tilde{u}) ||^{4/3} \right\rangle \\ &+ \quad \nu^{-3} \left\langle || e ||^2 || \nabla(u - \tilde{u}) ||^4 \right\rangle + C < \nu^{-1} || \nabla u ||^2 \, || \nabla(u - \tilde{u}) || \, || \nabla \, (u - \tilde{u}) || > \Big] \\ &+ \quad \nu^{-3} \left\langle || \nabla u ||^4 || e ||^2 \right\rangle. \end{aligned}$$

Remark 3.1. *If $|| \nabla u ||$ is uniformly bounded in time, then these equations can be closed provided $< || e ||^2 > \leq Ch^\alpha < || \nabla e ||^2 >$ for some $\alpha > 0$ and provided h is small enough. However, this is again the case when pointwise accuracy in time is reasonable to expect rather than accuracy in time averaged statistics. Thus, the problem of closing the circle in the velocity error equation for the time averaged statistics seems to catch at the same point as in the standard error analysis.*

We shall see that in at least one case the circle of analysis is closable. In more general cases, we believe the problem is due to the fact that we are estimating $< || \nabla(u - u^h) ||^2 >$ rather than $|| < \nabla(u - u^h) > ||^2$.

Proof. Since $u_t \in L^1(0, T; \mathbb{X}^*)$, the weak solution satisfies the variational formulation

$$(u_t, v^h) + \nu(\nabla u, \nabla v^h) + b^*(u, u, v^h) - (p, \nabla \cdot v^h) = (f, v^h) \qquad \forall \, v^h \in L^\infty(0, T; \mathbb{X}^h). \tag{3.6}$$

A similar equation holds for u^h, so subtraction and the fact that, for $q^h \in \mathbb{Q}^h$, $(q^h, \nabla \cdot v^h) = 0$, give an equation for the error $e = u - u^h$:

$$(e_t, v^h) + \nu(\nabla e, \nabla v^h) + b^*(u, u, v^h) - b^*(u^h, u^h, v^h) - (p - q^h, \nabla \cdot v^h) = 0 \qquad \forall \, v^h \in L^\infty(0, T; \mathbb{V}^h). \tag{3.7}$$

Let $\tilde{u}$ be an interpolant of u with $\tilde{u} \in L^2(0, \infty; \mathbb{V}) \cap L^\infty(0, \infty; V^h)$ and write $e = \eta - \phi^h$, where $\eta = u - \tilde{u}$ and $\phi^h = u^h - \tilde{u}$. Adding

$$-b^*(e, e, e) + b^*(u, u^h, \phi^h) - b^*(u, u^h, \phi^h),$$

where the first term vanishes, we get with $v^h = \phi^h$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} || \phi^h ||^2 + \nu \, || \nabla \phi^h ||^2 \quad &= \quad (\eta_t, \phi^h) + \nu(\nabla \eta, \nabla \phi^h) - (p - q^h, \nabla \cdot \phi^h) \\ &\quad + b^*(e, u, \eta) - b^*(e, e, \eta) - b^*(e, u, e) + b^*(u, e, \eta). \end{aligned}$$

Time averaging the equation above and observing that $||\,\phi^h\,||$ is uniformly bounded in time (since $||\,u^h\,||$ is bounded and $\tilde{u} \in L^\infty(0,\infty;L^2(\Omega))$), we have

$$\begin{aligned} < \varepsilon(\phi^h) > \;&=\; \frac{1}{|\Omega|}\Big[< (\eta_t, \phi^h) > + < \nu(\nabla\eta, \nabla\phi^h) > - < (p - q^h, \nabla\cdot\phi^h) > \\ & + < b^*(e,u,\eta) > - < b^*(e,e,\eta) > - < b^*(e,u,e) > + < b^*(u,e,\eta) > \Big]. \end{aligned}$$

The time-averaged Cauchy-Schwarz-Young inequality (2.22) now gives

$$\begin{aligned} \frac{1}{2} < \varepsilon(\phi^h) > \;&\leq\; C\nu^{-1} < ||\,\eta_t\,||_{-1}^2 > + < \varepsilon(\eta) > + C\,\nu^{-1} < ||\,p - q^h\,||^2 > \\ & + < b^*(e,u,\eta) > - < b^*(e,e,\eta) > - < b^*(e,u,e) > + < b^*(u,e,\eta) > . \end{aligned}$$

The triangle inequality then gives

$$\begin{aligned} < \varepsilon(u - u^h) > \;&\leq\; C < \varepsilon(u - \tilde{u}) > + C\nu^{-1} < ||\,\eta_t\,||_{-1}^2 > + C\,\nu^{-1} < ||\,p - q^h\,||^2 > \\ & +2 < b^*(e,u,\eta) > -2 < b^*(e,e,\eta) > -2 < b^*(e,u,e) > +2 < b^*(u,e,\eta) > . \end{aligned}$$

Inserting this into the right hand side of the last inequality gives the first claimed time-averaged error inequality.

For the second inequality we use the following bounds on the trilinear form, see e.g. [13],

$$\begin{aligned} < b^*(e,u,e) > \;&\leq\; C\; < ||\,\nabla u\,||\,||\,e\,||^{1/2}\,||\,\nabla e\,||^{3/2} > \\ &\leq\; \frac{\nu}{8} < ||\,\nabla e\,||^2 > + C\,\nu^{-3} < ||\,\nabla u\,||^4\,||\,e\,||^2 >, \end{aligned}$$

$$\begin{aligned} < b^*(e,e,\eta) > \;&\leq\; C\; < ||\,e\,||^{1/2}\,||\,\nabla e\,||^{3/2}\,||\,\nabla\eta\,|| > \\ &\leq\; \frac{\nu}{8} < ||\,\nabla e\,||^2 > + C\,\nu^{-3} < ||\,e\,||^2\,||\,\nabla\eta\,||^4 >, \end{aligned}$$

$$\begin{aligned} < b^*(e,u,\eta) > \;&\leq\; C\; < ||\,e\,||^{1/2}\,||\,\nabla e\,||^{1/2}\,||\,\nabla u\,||\,||\,\nabla\eta\,|| > \\ &\leq\; \frac{\nu}{8} < ||\,\nabla e\,||^2 > + C\,\nu^{-3} < ||\,e\,||^{2/3}\,||\,\nabla u\,||^{4/3}\,||\,\nabla\eta\,||^{4/3} > . \end{aligned}$$

$$\begin{aligned} < b^*(u,e,\eta) > \;&\leq\; C\; < ||\,\nabla\, u\,||\,||\,\nabla\, e\,||\,||\,\eta\,||^{1/2}\,||\,\nabla\,\eta\,||^{1/2} > \\ &+\; C\; < ||\,\nabla u\,||\,||\,\nabla\,\eta\,||\,||\,e\,||^{1/2}||\,\nabla\, e\,||^{1/2} > \\ &\leq\; \frac{\nu}{8} < ||\,\nabla e\,||^2 > + C < \nu^{-1}||\,\nabla u\,||^2\,||\,\eta\,||\,||\,\nabla\,\eta\,|| > \\ &+\; C\nu^{-3} < ||\,\nabla u\,||^{4/3}\,||\,e\,||^{2/3}\,||\nabla\,\eta\,||^{4/3} > \end{aligned}$$

Thus

$$\begin{aligned} < \varepsilon(u - u^h) > \;&\leq\; C < \varepsilon(u - \tilde{u}) > + C\,\nu^{-1} < ||\,p - q^h\,||^2 > + C\Big[\,\nu^{-1/3} < ||\,e\,||^{2/3}\,||\,\nabla u\,||^{4/3}\,||\,\nabla\eta\,||^{4/3} \\ &+\; \nu^{-3} < ||\,e\,||^2\,||\,\nabla\eta\,||^4 > + C < \nu^{-1}||\,\nabla u\,||^2\,||\,\eta\,||\,||\,\nabla\,\eta\,|| > + \nu^{-3} < ||\,\nabla u\,||^4\,||\,e\,||^2 > \Big], \end{aligned}$$

which is the second error inequality, completing the proof. □

3.1 The Case of Large u_0 and Small $f^*(x)$

There is at least one interesting case in which the error equations for the time-averaged velocity error, $< \varepsilon(u - u^h) >$ can be closed: the case of large initial condition u_0 and asymptotically small body force $f(x,t)$. In this subsection we assume

$$f(x,t) \in L^\infty(0,\infty;H^{-1}(\Omega)), \qquad f(x,t) \to f^*(x) \text{ as } t \to \infty$$

and

$$\nu^{-2}M\,||\,f^*\,||_{-1} =: \alpha < 1.$$

In this case, it is possible time averaging will eventually wash out the irregularities caused by the large initial condition. We show that this is indeed the case. To shorten the presentation, we shall simplify the condition on f to

$$f(x,t) \equiv f^*(x) \qquad \text{and} \qquad \nu^{-2}M\,||\,f^*\,||_{-1} = \alpha < 1. \tag{3.8}$$

Proposition 3.3. *Suppose (3.8) holds. Then*

$$< \varepsilon(u - u^*) >= 0, \tag{3.9}$$

where u^* is the solution of the equilibrium Navier-Stokes equations (2.5).

Proof. Let V_N be a span of eigenfunctions of the Stokes operator. Then, the Leray-Hopf construction gives again a sequence u_N in V_N converging to the weak solution u, as $N \to \infty$, strongly in $L^2(\Omega \times (0,T))$, and weakly in $L^2(0,T;H^1(\Omega))$, i.e, $\int_0^T (\nabla u_N, v)\,dt \to \int_0^T (\nabla u, v)\,dt$ for all $v \in L^2(\Omega \times (0,T))$. The Leray-Hopf Galerkin approximation satisfies

$$(u_{N,t}, v_N) + \nu(\nabla u_N, \nabla v_N) + (u_N \cdot \nabla u_N, v_N) = (f, v_N) \qquad \forall\, v_N \in V_N. \tag{3.10}$$

Let $u_N^* \in V_N$ be the Galerkin projection of u^* in V_N. Then $u_N^* \to u^*$ in $\mathbb{X}$ and $\mathbb{V}$ as $N \to \infty$ and u_N^* satisfies

$$\nu(\nabla u_N^*, \nabla v_N) + (u_N^* \cdot \nabla u_N^*, v_N) = (f, v_N) \qquad \forall\, v_N \in V_N. \tag{3.11}$$

Set $\phi_N = u_N(x,t) - u_N^*(x)$ and subtract (3.11) from (3.10) to get

$$(\phi_{N,t}, v_N) + \nu(\nabla \phi_N, \nabla v_N) + (u_N \cdot \nabla u_N, v_N) - (u_N^* \cdot \nabla u_N^*, v_N) = 0 \qquad \forall\, v_N \in V_N. \tag{3.12}$$

Let $v_N = \phi_N$, add and subtract $(u_N \cdot \nabla u_N^*, v_N)$ and integrate from 0 to T to get

$$\frac{1}{2}||\phi_N(T)||^2 + \int_0^T \nu||\nabla\phi_N||^2\,dt = \frac{1}{2}||\phi_N(0)||^2 + \int_0^T -b^*(\phi_N, u_N^*, \phi_N)\,dt$$

Next, using the bound on the trilinear form, the a priori bound (2.11) and the small data assumption (3.8) gives

$$\frac{1}{2}||\phi_N(T)||^2 + (1-\alpha)\nu\int_0^T ||\nabla\phi_N||^2\,dt \quad \leq \quad \frac{1}{2}||\phi_N(0)||^2$$

Thus, dropping the first term,

$$\int_0^T \nu||\,\nabla\phi_N\,||^2\,dt \leq (1-\alpha)^{-1}\frac{1}{2}||\,\phi_N(0)\,||^2. \tag{3.13}$$

Since $\liminf_{N\to\infty} \nabla\phi_N = \nabla\phi$ in the same manner as before, by classical properties of weak limits, we have

$$\liminf_{N\to\infty}\left(\int_0^T ||\,\nabla\phi_N\,||^2\,dt\right) \geq \int_0^T ||\,\nabla\phi\,||^2\,dt$$

Therefore, taking limit inferior on both sides of (3.13), gives

$$\int_0^T \nu||\,\nabla\phi\,||^2\,dt \leq (1-\alpha)^{-1}\frac{1}{2}||\,\phi(0)\,||^2.$$

Dividing by T and taking the limit superior as $T \to \infty$ proves the result. □

The next proposition is needed in order to prove the desired error estimate on $< \varepsilon(u - u^h) >$, given below, in Theorem 3.2.

Proposition 3.4. *Assume that (3.8) holds. Then*

$$< \varepsilon(u^h - u^{*\,h}) >= 0, \tag{3.14}$$

Proof. The proof works in the same way as that of Proposition 3.3. It is based on subtracting (2.4) and (2.10). □

Remark 3.2. *The statements of Propositions* 3.3 *and* 3.4 *also hold for the kinetic energy.*

Theorem 3.2. *Suppose that (3.8) holds. Then*

$$< \varepsilon(u - u^h) > \ \leq \ C\,\nu\,||\nabla(u^* - u^{*\,h})||^2, \tag{3.15}$$

where the constant C depends on the domain.

Proof. We can start by writing

$$||\nabla(u - u^h)||^2 = ||\nabla(u - u^* + u^* - u^{*\,h} + u^{*\,h} - u^h)||^2.$$

Next, we use the triangle inequality, together with Proposition 3.3 and Proposition 3.4 to get

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \nu\,||\nabla(u - u^h)||^2\,dt \leq C\,\nu\,||\nabla(u^* - u^{*\,h})||^2.$$

□

Remark 3.3. *The statement of Theorem 3.2 says that the problem of estimating $< \varepsilon(u - u^h) >$ reduces to the one of estimating $||\nabla(u^* - u^{*\,h})||^2$. Standard finite element error estimates thus immediately imply $< \varepsilon(u - u^h) >$ is optimal.*

Corollary 3.1. *Suppose the small data assumption (3.8) holds, ($\mathbb{X}^h$, $\mathbb{Q}^h$) satisfies the inf-sup condition. Then*

$$< \varepsilon(u - u^h) > \leq C\left[\inf_{v^h\in\mathbb{X}^h} \nu||\,\nabla(u^* - v^h)\,||^2 + \inf_{q^h\in\mathbb{Q}^h} \nu^{-1}||\,p^* - q^h\,||^2\right]$$

Proof. This follows by inserting the error estimates for $||\,\nabla(u^* - u^{*\,h})\,||$ from [6] into the right hand side of (3.15). □

Concerning the time averaged error in the pressure, we have the following corollary. It is a direct consequence of (2.24) and Theorem 3.1.

Corollary 3.2. *Let the assumptions of Theorem 3.1 be fulfilled. Suppose additionally that assumptions (3.8) hold. Then*

$$||< p - p^h >|| \leq \frac{3\nu}{\beta^h}\left\langle ||\,\nabla(u - u^h)\,||^2\right\rangle^{1/2} + \left(1 + \frac{1}{\beta^h}\right)\inf_{q^h\in\mathbb{Q}^h} ||< p - q^h >||.$$

4 Time Averaged Errors in Drag and Lift

Consider the flow around a body in a channel with flow region Ω and boundary Γ, which consists of Γ_b (boundary of the body) and $\Gamma_c = \Gamma_i \cup \Gamma_o \cup \Gamma_w$ (where Γ_i, Γ_o correspond to the inflow and outflow and Γ_w, to the walls).

Define

$$\sigma = -p\,\mathbb{I} + 2\,\nu\,\nabla^s u,$$

where ∇^s indicates the symmetric part of the operator ∇.

Then consider the Navier-Stokes equations written in the form:

$$\begin{aligned} \rho(u_t + u\cdot\ \nabla u) &= \nu\nabla\cdot\sigma + f && \text{in } \Omega \\ \nabla\cdot u &= 0 && \text{in } \Omega \\ u &= g && \text{on } \Gamma \\ u(x,0) &= u_0(x) && \text{in } \Omega \end{aligned} \tag{4.1}$$

satisfying the compatibility condition $\int_\Gamma g\cdot\hat{n}dS$. We assume $g = 0$ *on* Γ_w and on Γ_b.

We introduce the spaces

$$\mathbb{X}_g = \left\{v \in H^1(\Omega)^d : v|_\Gamma = g\right\}, \quad \mathbb{X}_0 = H^1_0(\Omega)^d.$$

A weak formulation of (4.1) reads: Find $u : [0,T] \to \mathbb{X}_g$ and $p : (0,T] \to \mathbb{Q}$ such that

$$\begin{aligned} \rho\,(u_t, v) \quad &+ \quad 2\,\nu(\nabla^s u, \nabla^s v) + \rho\, b(u,u,v) \\ &+ \quad (p, \nabla\cdot q) + (\nabla\cdot u, q) = (f, v) \qquad \forall (v,q) \in \mathbb{X}_0 \times \mathbb{Q}. \end{aligned} \tag{4.2}$$

Drag and lift are defined as

$$D = -\int_{\Gamma_b} \hat{e}_1\cdot\sigma\cdot\hat{n}\,d\gamma \qquad \text{and} \qquad L = -\int_{\Gamma_b} \hat{e}_2\cdot\sigma\cdot\hat{n}\,d\gamma,$$

respectively, where $\hat{e_i}$ is the unit vector in the i^{th} direction and $\hat{n}$ is the outward pointing unit normal to Γ_b. (We assume that $+\hat{e}_1$ is the direction of motion and $-\hat{e}_2$ is the direction of gravity.)

Remark 4.1. *Giving mathematical estimates of drag and lift involves some technical points necessary to ensure that the trace of $\sigma\cdot n$ on Γ is well defined in an appropriate space. Requiring $\sigma\cdot n$ to be well defined at each t requires regularity. On the other hand, time averages of $\sigma\cdot n$ seem to require less regularity. In this section, we assume that (u,p) is slightly more regular than a general weak solution to ensure $\sigma\cdot n \in L^1(0,T;H^{-1/2}(\Gamma_b))$. In particular, we assume that, for some $s > 1/2$, for a.e. $T > 0$, $u \in L^1(0,T;H^{1+s}(\Omega))$ and $p \in L^1(0,T;H^s(\Omega))$. This implies, by the Trace Theorem, that $\sigma\cdot n \in L^1(0,T;H^{s-1/2}(\Gamma_b))$.*

Lemma 4.1. *Let $u \in L^1(0,T;H^{s+1}(\Omega)) \cap \mathbb{X}_g$, $u_t \in L^1(0,T;H^{-1}(\Omega))$ and $p \in L^1(0,T;H^s(\Omega))$, for some $s > 1/2$ be solutions of (4.2). Then $\sigma\cdot n \in L^1(0,T;H^{s-1/2}(\Gamma_b))$ and*

$$\int_0^T\int_{\Gamma_b} v\cdot\sigma\cdot\hat{n}\,d\gamma\,dt = \int_0^T \Big\{\rho\,(u_t,v) + 2\,\nu(\nabla^s u, \nabla^s v) + \rho\, b(u,u,v) - (p,\nabla\cdot v) - (f,v)\Big\}\,dt \tag{4.3}$$

for any $v \in L^\infty(0,T;H^1(\Omega))$ with $v = 0$ on Γ_c.

Remark 4.2. *In particular, in (4.3), one choice of v satisfying $v = \hat{e}_1$ on Γ_b gives a formula for the drag and $v = \hat{e}_2$, one for the lift.*

Proof. The proof uses a density argument. First, let $\{v_j\}_{j=1}^\infty$ be a sequence in $C^\infty(\Omega \times [0,T])$ with $v_j|_{\Gamma_b} = 0$ for all j. Then, by the definition of distributional derivatives, equation (4.3) holds true with v replaced by v_j. For u, p with the assumed regularity, each term in (4.3) is a bounded linear functional of v in $L^\infty(0,T;H^1(\Omega))$. Thus it is continuous on $L^\infty(0,T;H^1(\Omega))$. Therefore we may let $j \to \infty$ and (4.3) holds for $v \in L^\infty(0,T;H^1(\Omega))$, since $C^\infty(\Omega\times[0,T])$ is dense in $L^\infty(0,T;H^1(\Omega))$. □

Theorem 4.1. *Let the assumptions of Lemmas 4.1 and 3.1 and Proposition 3.1 be fulfilled. The time averaged drag and lift can be estimated as*

$$\begin{aligned} |< D - D_h >| &\leq C\left(< \|\nabla (u - u^h)\|^2 >^{1/2} \; \| + \| < p - p^h > \|\right), \\ |< L - L_h >| &\leq C\left(< \|\nabla (u - u^h)\|^2 >^{1/2} \; \| + \| < p - p^h > \|\right). \end{aligned} \tag{4.4}$$

Proof. We present here a proof for the drag estimate, since the same argument follows for the lift estimate, according to an appropriate choice of w.

Let w be a smooth time-independent vector field satisfying $w = \hat{e}_1$ on Γ_b and $w = 0$ on Γ_c. If we take $v = w$ in (4.3), then we get a formula for the drag as follows

$$\int_0^T D\,dt = \int_0^T \{\rho\,(u_t, w) + 2\,\nu(\nabla^s u, \nabla^s w) + \rho\, b(u,u,w) - (p, \nabla \cdot w) - (f, w)\}\; dt.$$

Now let $i_h w$ be a finite element interpolant to w. Then

$$\begin{aligned} \int_0^T D - D_h\,dt &= \int_0^T \{\rho\,(u_t, w) + 2\,\nu(\nabla^s u, \nabla^s w) + \rho\, b(u,u,w) \\ &- (p, \nabla\cdot w) - (f,w) - \rho\,(u^h_t, i_h w) - 2\,\nu(\nabla^s u^h, \nabla^s i_h w) \\ &- \rho\, b(u^h, u^h, i_h w) + (p^h, \nabla \cdot i_h w) + (f, i_h w)\big\}\; dt. \end{aligned}$$

Adding and subtracting appropriate terms, this becomes

$$\begin{aligned} \int_0^T D - D_h\,dt &= \int_0^T \big\{\rho\,(u_t, w - i_h w) + \rho\,((u-u^h)_t, i_h w) + 2\,\nu(\nabla^s u, \nabla^s (w - i_h w)) \\ &+ 2\,\nu(\nabla^s (u - u^h), \nabla^s i_h w) + \rho\, b(u,u,w - i_h w) + \rho\, b(u,u,i_h w) \\ &- \rho\, b(u^h, u^h, i_h w) - (p, \nabla\cdot (w - i_h w)) \\ &- (p - p^h, \nabla \cdot i_h w) - (f, w - i_h w)\big\}\; dt. \end{aligned} \tag{4.5}$$

Observe now that the term containing $(f, w - i_h w)$ in (4.5) can be rewritten by multiplying (4.1) by $w - i_h w$ and integrating:

$$\begin{aligned} (f, w - i_h w) &= \rho\,(u_t, w - i_h w) + 2\,\nu(\nabla^s u, \nabla^s (w - i_h w)) \\ &+ \rho\, b(u,u,w - i_h w) - (p, \nabla \cdot (w - i_h w)), \end{aligned} \tag{4.6}$$

provided the mesh conforms to the boundary and the finite element space is conforming.

Hence, (4.5) and (4.6) together give

$$\begin{aligned} \int_0^T D - D_h\,dt &= \int_0^T \big\{\rho\,((u - u^h)_t, i_h w) + 2\,\nu(\nabla^s (u - u^h), \nabla^s i_h w) \\ &+ \rho\, b(u,u,i_h w) - \rho\, b(u^h,u^h,i_h w) + (p - p^h, \nabla\cdot i_h w)\big\}\; dt. \end{aligned} \tag{4.7}$$

Let $e = u - u^h$. Then divide each term in (4.7) by T and take the limit superior on both sides, using the fact that $i_h w$ does not depend on time. The first term on the left hand side yields

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t}(e(t), i_h w) &\leq \limsup_{t\to\infty} \frac{1}{t}\left(\|e(t)\|\|i_h w\|\right) \\ &\leq \limsup_{t\to\infty} \frac{1}{t}\left(\frac{\|e(t)\|^2}{2} + \frac{\|i_h w\|^2}{2}\right) \\ &= 0 \end{aligned}$$

by Lemma 3.2 and the time-independence of $i_h w$.

Now, by (2.22) and (2.24)

$$< (\nabla^s e, \nabla^s i_h w) > \; \leq \; < ||\, \nabla e \,||^2 >^{1/2} \; ||\, \nabla i_h w \,||.$$

Next, consider the nonlinear term

$$\begin{aligned}
& < b(u, u, i_h w) - b(u^h, u^h, i_h w) > \\
& \quad = \quad < b(u, e, i_h w) + b(e, u^h, i_h w) > \\
& \quad \leq \quad C < (\|\nabla u\| + \|\nabla u^h\|)\|\nabla e\|\|\nabla i_h w\| > \\
& \quad \overset{(2.22)}{\leq} \quad C\|\nabla i_h w\| < (\|\nabla u\| + \|\nabla^h\|)^2 >^{1/2} < \|\nabla e\|^2 >^{1/2} \\
& \quad \leq \quad C\|\nabla i_h w\| < \|\nabla u\|^2 + \|\nabla u^h\|^2 >^{1/2} < \|\nabla e\|^2 >^{1/2} \\
& \quad \overset{Prop.3.1}{\leq} \quad C\|\nabla i_h w\| \|f\|_{L^\infty(0,\infty;H^{-1}(\Omega))} < \|\nabla e\|^2 >^{1/2} .
\end{aligned}$$

For the pressure term, we obtain with the time-independency of $i_h w$ and the Cauchy-Schwarz inequality

$$< (p - p^h, \nabla \cdot i_h w) > \quad = \quad (< p - p^h >, \nabla \cdot i_h w) \leq ||\, < p - p^h > \,||\; ||\, \nabla i_h w \,||.$$

Putting everything together, equation (4.7) becomes

$$\begin{aligned}
| < D - D_h > | \quad & \leq \quad (\nu + C\|f\|_{L^\infty(0,\infty;H^{-1}(\Omega))}) < ||\, \nabla e \,||^2 >^{1/2} \; ||\, \nabla i_h w \,|| \\
& + \quad ||\, < p - p^h > \,||\; ||\, \nabla i_h w \,||,
\end{aligned}$$

which gives the statement of the theorem, since the terms multiplying $< ||\, \nabla e \,||^2 >^{1/2}$ are bounded and $||\, \nabla i_h w \,|| \leq 2\, ||\, w \,||_{H^2(\Omega)} = C$. □

Corollary 4.1. *If, in addition to the assumptions of Theorem 4.1, (3.8) holds, then (4.4) becomes*

$$| < D - D_h > |, \;\; | < L - L_h > | \quad \leq \quad C \left(||\nabla(u^* - {u^*}^h)|| + || < p - p^h > || \right). \tag{4.8}$$

In the next section we investigate properties of the approximate solution of shear flows.

5 Persistent Shear Flows

We have seen in the previous sections that, provided the portion of the body force driving the flow that persists is small, statistics, such as the time averaged energy dissipation rate, can be accurately predicted by a flow simulation. This accuracy holds quite generally without any of the further assumptions on u_0, ν and Re typically needed to prove accuracy over bounded time intervals.

The case when the persistent forces driving the flow are not small is much more difficult; we shall prove in this section that the analogous estimate of time averaged energy dissipation rate is physically reasonable under a condition on the finite element mesh near the walls. Briefly, we consider the finite element approximation to the following shear flow problem: let $\Omega = [0, L]^3$

$$\begin{aligned}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p - \nu \Delta u &= 0 && \text{in } \Omega \times (0, T] \\
\nabla \cdot u &= 0 && \text{in } \Omega \times (0, T] \\
u &= u_0 && \text{at } t = 0 \\
u(x_1, x_2, x_3, t) &= \phi(x_3) && \text{for } x_3 \in \partial\Omega \\
u(x_1, x_2, x_3, t) &= u(x_1 + L, x_2, x_3, t) && \text{for } x_1 \in \partial\Omega \\
u(x_1, x_2, x_3, t) &= u(x_1, x_2 + L, x_3, t) && \text{for } x_2 \in \partial\Omega
\end{aligned}$$

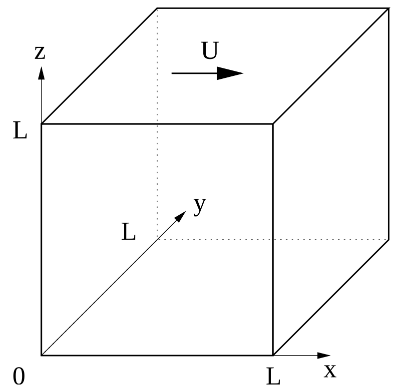

Figure 1: The shear flow problem.

where

$$\phi(x_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ if } x_3 = 0 \text{ and } \phi(x_3) = \begin{pmatrix} U \\ 0 \\ 0 \end{pmatrix} \text{ if } x_3 = L,$$

see Fig. 1.

In this problem, the persistent force driving the flow is clearly the motion of the top wall and the time averaged energy dissipation rate must balance the drag exerted by the walls on the fluid. For such problems, the Richardson-Kolmogorov energy cascade predicts quite simply [1]

$$< \varepsilon(u) > \approx \frac{U^3}{L}$$

independent of ν and Re, respectively.

Remarkably, the upper estimate has also been proven for weak solutions of the Navier-Stokes equations in full generality:

$$< \varepsilon(u) > \leq C \frac{U^3}{L}$$

by Constantin and Doering [3] and Wang [22].

Herein, we show in essence that provided the first mesh line of the finite element space is within $O(1/Re)$ of the top and bottom walls then

$$< \varepsilon(u^h) > \leq C \frac{U^3}{L},$$

i.e. the computed energy dissipation has the correct mathematical and physical scaling.

Since the proof adapts the ideas of [22] we expect that a similar analysis would hold for other variational methods as well.

[1] Briefly: the largest coherent structures are associated with the motion of the upper wall. They thus have length scale L and characteristic velocity U. Their local Reynolds number is thus $(\frac{U\,L}{\nu}(= Re))$ and viscous dissipation is negligible on them. These break up into smaller eddies (velocity u, length l, $Re\,(l) = \frac{u\,(l)\,l}{\nu}$) until $Re\,(l)$ is small enough for viscous dissipation to drive their kinetic energy to zero exponentially fast. Since viscous dissipation is negligible through this cascade, the energy dissipation rate is related then to the power input to the largest scales at the first step in the cascade. These largest eddies have energy $\frac{1}{2}U^2$ and time scale $\tau = \frac{L}{U}$ so the rate of energy transfer is $O(\frac{U^2}{\tau}) = O(\frac{U^3}{L})$.

Let

$$\begin{aligned}
\mathbb{X} &= \{v \in H^1(\Omega) \;:\; v(x_1,x_2,x_3,t) = v(x_1+L,x_2,x_3,t) \text{ for } x_1 \in \partial\Omega, \\
&\quad v(x_1,x_2,x_3,t) = v(x_1,x_2+L,x_3,t) \text{ for } x_2 \in \partial\Omega, v(x_1,x_2,x_3,t) = \phi(x_3) \text{ for } x_3 \in \partial\Omega\}, \\
\mathbb{X}_0 &= \{v \in H^1(\Omega) \;:\; v(x_1,x_2,x_3,t) = v(x_1+L,x_2,x_3,t) \text{ for } x_1 \in \partial\Omega, \\
&\quad v(x_1,x_2,x_3,t) = v(x_1,x_2+L,x_3,t) \text{ for } x_2 \in \partial\Omega, v(x_1,x_2,x_3,t) = 0 \text{ for } x_3 \in \partial\Omega\}, \\
\mathbb{Q} &= L^2_0(\Omega)
\end{aligned}$$

and denote corresponding conforming finite element spaces with a superscript h. We assume that the finite element space contains linears.

The finite element problem reads as follow: find $(u^h, p^h) \in \mathbb{X}^h \times \mathbb{Q}^h$ such that

$$(u^h_t, v^h) + \nu(\nabla u^h, \nabla v^h) + b^*(u^h, u^h, v^h) - (p^h, \nabla \cdot v^h) = 0 \quad \forall\, v^h \in \mathbb{X}^h_0 \tag{5.1}$$

$$(\nabla \cdot u^h, q^h) = 0 \quad \forall\, q^h \in \mathbb{Q}^h \tag{5.2}$$

$$(u(x,0) - u_0(x), v) = 0 \quad \forall\, v^h \in \mathbb{X}^h_0 \tag{5.3}$$

Consider the background flow (an extension of the boundary condition ϕ to the interior of Ω) given by

$$\tilde{\phi}(x_3) = \begin{cases} 0, & \text{if } x_3 \in [0, L - \beta L] \\ \frac{U}{\beta L}(x_3 - (L - \beta L)), & \text{if } x_3 \in [L - \beta L, L] \end{cases}$$

and

$$\Phi = \begin{bmatrix} \tilde{\phi}(x_3) \\ 0 \\ 0 \end{bmatrix},$$

where β is a positive number. For $\beta > 0$ this function is piecewise linear, continuous and satisfies the boundary conditions.

We shall select β appropriately so that Φ belongs to the finite element space and then take $v^h = u^h - \Phi$ in (5.1) to get

$$(u^h_t, u^h) - (u^h_t, \Phi) + \nu\, ||\, \nabla u^h\, ||^2 - \nu\, (\nabla\, u^h, \nabla\, \Phi) - b^*(u^h, u^h, \Phi) = 0 \tag{5.4}$$

since b^* can be skew symmetrized and Φ is divergence free.

Observing that $(u^h_t, \Phi) = \frac{\partial}{\partial t}(u^h, \Phi)$, since $\frac{\partial \Phi}{\partial t} = 0$, we rewrite (5.4) as

$$\frac{1}{2}\frac{d}{d\,t}||\, u^h\, ||^2 + \nu\, ||\, \nabla u^h\, ||^2 = \frac{\partial}{\partial\, t}(u^h, \Phi) + b^*(u^h, u^h, \Phi) + \nu\, (\nabla\, u^h, \nabla\, \Phi)$$

and integrate in time to get

$$\begin{aligned}
&\frac{1}{2}||\, u^h(T)\, ||^2 - \frac{1}{2}||\, u^h(0)\, ||^2 + \nu \int_0^T ||\, \nabla u^h\, ||^2\, d\,t \\
&= \quad (u^h(T), \Phi) - (u^h(0), \Phi) + \int_0^T b^*(u^h, u^h, \Phi)\, d\,t + \nu \int_0^T (\nabla\, u^h, \nabla\, \Phi)\, d\,t.
\end{aligned} \tag{5.5}$$

We need to estimate each term on the right hand side of (5.5). For some of the terms, calculated values of Φ will be needed:

$$\begin{aligned}
||\, \Phi\, ||_{L^\infty(\Omega)} &= U \\
||\, \nabla \Phi\, ||_{L^\infty(\Omega)} &= \frac{U}{\beta\, L} \\
||\, \Phi\, ||^2 &= L^2 \int_0^L |\, \tilde{\phi}(x_3)\, |^2\, d\, x_3 = L^2 \int_{L-\beta\, L}^L \frac{U^2}{(\beta\, L)^2}\, (\, x_3 - (L - \beta\, L)\,)^2\, d\, x_3 = \frac{1}{3}\, U^2\, \beta\, L^3, \\
||\, \nabla\, \Phi\, ||^2 &= \frac{U^2\, L}{\beta}.
\end{aligned}$$

For completeness, we include a short proof of the scaling of the constant in the Poincare-Friedr ichs inequality. That will be helpful for the estimation of the nonlinear term.

Lemma 5.1. *Let* $\mathcal{O}_{\beta L} = \{(x_1, x_2, x_3) \in \Omega : L - \beta L \le x_3 \le L\}$ *be the region close to the upper boundary (where the background flow* Φ *does not vanish). Then,*

$$|| u^h - \Phi ||_{L^2(\mathcal{O}_{\beta L})} \le \beta L || \nabla (u^h - \Phi) ||_{L^2(\mathcal{O}_{\beta L})}. \tag{5.6}$$

Proof. First, let $v = u^h - \Phi$ be a C^1 function. Then, componentwise ($i = 1, 2, 3$), we have

$$v_i(x_1, x_2, x_3) = v_i(x_1, x_2, L) - \int_{x_3}^{L} \frac{dv_i}{dz}(x_1, x_2, z) dz.$$

Observing that $v_i(x_1, x_2, L) = 0$, squaring both sides and using the Cauchy-Schwartz inequality, we get

$$v_i(x_1, x_2, x_3) \le \beta L \int_{L-\beta L}^{L} \left(\frac{dv_i}{dz}(x_1, x_2, z) \right)^2 dz.$$

Integrating both sides with respect to x_3 gives

$$\int_{L-\beta L}^{L} v_i(x_1, x_2, x_3) dx_3 \le (\beta L)^2 \int_{L-\beta L}^{L} \left(\frac{dv_i}{dz}(x_1, x_2, z) \right)^2 dz.$$

Then, integrating with respect to x_1 and x_2, we have

$$|| v_i ||^2_{\mathcal{O}_{\beta L}} \le (\beta L)^2 || \nabla v_i ||^2_{\mathcal{O}_{\beta L}},$$

which proves the lemma in the case of a C^1 function. The case $v \in \mathbb{X}^h \subset H^1(\Omega)$ follows by density. $\square$

Next, we use the estimates above to derive upper bounds for the following terms:

$$\begin{aligned}
(u^h(T), \Phi) &\le || u^h(T) || \, || \Phi || \le \sqrt{\frac{\beta}{3}} U L^{3/2} || u^h(T) ||, \\
(u^h(0), \Phi) &\le || u^h(0) || \, || \Phi || \le \sqrt{\frac{\beta}{3}} U L^{3/2} || u^h(0) ||, \\
\nu \int_0^T (\nabla u^h, \nabla \Phi) \, dt &\le \frac{\nu}{2} \int_0^T || \nabla u^h ||^2 \, dt + \frac{\nu}{2} \int_0^T || \nabla \Phi ||^2 \, dt \\
&\le \frac{\nu}{2} \int_0^T || \nabla u^h ||^2 \, dt + \frac{\nu}{2 \beta} L U^2 T.
\end{aligned}$$

For the nonlinear term, we add and subtract terms, and use the fact that $b^*(\cdot, \cdot, \cdot)$ is skew-symmetric, to write

$$\begin{aligned}
b^*(u^h, u^h, \Phi) &= b^*(u^h - \Phi, u^h - \Phi, \Phi) + b^*(\Phi, u^h - \Phi, \Phi) \\
&= \frac{1}{2} b(u^h - \Phi, u^h - \Phi, \Phi) - \frac{1}{2} b(u^h - \Phi, \Phi, u^h - \Phi) \\
&\quad + \frac{1}{2} b(\Phi, u^h - \Phi, \Phi) - \frac{1}{2} b(\Phi, \Phi, u^h - \Phi)
\end{aligned} \tag{5.7}$$

We use Lemma 5.1 together with the calculated values of Φ to analyze one term at a time in (5.7). In all cases, integration is restricted to $\mathcal{O}_{\beta L}$, since $\text{supp}(\Phi) \subset \overline{\mathcal{O}}_{\beta L}$.

$$\begin{aligned}
b(u^h - \Phi, u^h - \Phi, \Phi) &\le || \Phi ||_{L^\infty(\mathcal{O}_{\beta L})} || \nabla (u^h - \Phi) ||_{L^2(\mathcal{O}_{\beta L})} || u^h - \Phi ||_{L^2(\mathcal{O}_{\beta L})} \\
&\le U \beta L || \nabla (u^h - \Phi) ||^2_{L^2(\mathcal{O}_{\beta L})} \\
&\le 2 U \beta L || \nabla u^h ||^2 + 2 U^3 L^2,
\end{aligned} \tag{5.8}$$

$$
\begin{aligned}
b(u^h-\Phi,\Phi,u^h-\Phi) &\leq || \nabla\Phi ||_{L^\infty(\mathcal{O}_{\beta\, L})} || u^h-\Phi ||^2_{L^2(\mathcal{O}_{\beta\, L})} \\
&\leq U\,\beta\, L\, || \nabla(u^h-\Phi) ||^2 \\
&\leq 2\,U\,\beta\, L\, || \nabla u^h ||^2 + 2\,U^3 L^2, \qquad (5.9)
\end{aligned}
$$

$$
\begin{aligned}
b(\Phi,u^h-\Phi,\Phi) &\leq || \Phi ||_{L^\infty(\mathcal{O}_{\beta\, L})} || \Phi ||_{L^2(\mathcal{O}_{\beta\, L})} || \nabla(u^h-\Phi) ||_{L^2(\mathcal{O}_{\beta\, L})} \\
&\leq U\left(\frac{1}{3}U^2\beta L^3\right)^{1/2} || \nabla u^h || + U\left(\frac{1}{3}U^2\beta L^3\right)^{1/2}\left(\frac{U^2 L}{\beta}\right)^{1/2} \\
&\leq \frac{1}{2}U\,\beta\, L|| \nabla u^h ||^2 + \frac{5}{6}\,U^3 L^2, \qquad (5.10)
\end{aligned}
$$

$$
\begin{aligned}
b(\Phi,\Phi,u^h-\Phi) &\leq || \Phi ||_{L^\infty(\mathcal{O}_{\beta\, L})} || \nabla\Phi ||_{L^2(\mathcal{O}_{\beta\, L})} || u^h-\Phi ||_{L^2(\mathcal{O}_{\beta\, L})} \\
&\leq U\,\beta\, L\left(\frac{U^2\, L}{\beta}\right)^{1/2} || \nabla u^h || + U\,\beta\, L\left(\frac{U^2\, L}{\beta}\right)^{1/2}\left(\frac{U^2\, L}{\beta}\right)^{1/2} \\
&\leq \frac{1}{2}U\,\beta\, L|| \nabla u^h ||^2 + \frac{3}{2}\,U^3 L^2. \qquad (5.11)
\end{aligned}
$$

Putting (5.8) to (5.11) together, equation (5.7) gives

$$
b^*(u^h,u^h,\Phi) \leq \frac{5}{2}U\,\beta\, L\, || \nabla u^h ||^2 + \frac{19}{6}U^3 L^2,
$$

so that equation (5.5) becomes

$$
\begin{aligned}
\frac{1}{2}|| u^h(T) ||^2 &- \frac{1}{2}|| u^h(0) ||^2 + \nu\int_0^T || \nabla u^h ||^2\, d\,t \leq \sqrt{\frac{\beta}{3}}U L^{3/2} || u^h(T) || \\
&+ \sqrt{\frac{\beta}{3}}U L^{3/2} || u^h(0) || + \frac{\nu}{2}\int_0^T || \nabla u^h ||^2\, d\,t + \frac{\nu}{2\,\beta}L\,U^2\,T \\
&+ \frac{5}{2}\beta\, L\, U\int_0^T || \nabla\, u^h ||^2\, d\,t + \frac{19}{6}U^3 L^2\,T
\end{aligned}
$$

Divide by T, let $T\to\infty$ and use $|\Omega| = L^3$ to get

$$
L^3\left(\frac{1}{2} - \frac{5}{2}\frac{L\,\beta\, U}{\nu}\right) < \varepsilon(u^h) > \quad \leq \quad \frac{\nu}{2\,\beta}L\,U^2 + \frac{19}{6}U^3 L^2. \qquad (5.12)
$$

If

$$
\beta < \frac{1}{5\,Re}, \qquad \text{where} \quad Re = \frac{U\,L}{\nu},
$$

then equation (5.12) gives

$$
< \varepsilon(u^h) > \quad \leq \quad C\frac{U^3}{L}.
$$

6 Summary and Outlook

There are many important and interesting statistics and flow settings. In Section 5 we study a simple yet interesting case of internal flow, higher Reynolds number, large initial data and asymptotically small body force. In this setting, we prove convergence of the time averaged energy dissipation rates in 3d without assuming the solution is a strong solution, smooth, unique or the initial data is smooth.

Our long-term goal is to develop a theory paralleling and inspired by the theory of shadowing in approximation of dynamical systems [17], [16], [18], I.e., to understand when computed, time-averaged statistics from numerical simulations of the Navier-Stokes equations either reflect statistics corresponding to the exact solution of the Navier-Stokes equations with the data (which is essentially driving the flow) perturbed.

References

[1] L.C. Berselli, T. Iliescu, and W. J. Layton, *Mathematics of large eddy simulation of turbulent flows*, Scientific Computation, Springer, 2006.

[2] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer-Verlag, 1991.

[3] P. Constantin and C. Doering, *Energy dissipation in shear driven turbulence*, Physical Review Letters **69** (1992), no. 11, 1648–1651.

[4] J. W. Deardorff, *A Numerical Study of Three-Dimensional Turbulent Channel Flow of Large Reynolds Numbers*, J. Fluid Mech. **41** (1970), 453–480.

[5] G.P. Galdi, *An introduction to the Navier-Stokes initial-boundary value problem*, Fundamental Directions in Mathematical Fluid Dynamics (G.P. Galdi, J.G. Heywood, and R. Rannacher, eds.), Birkhäuser, 2000, pp. 1 – 70.

[6] V. Girault and P.-A. Raviart, *Finite element approximation of the Navier-Stokes equations*, Lecture notes in mathematics, no. 749, Springer-Verlag, 1979.

[7] ———, *Finite element methods for Navier-Stokes equations : theory and algorithms*, Springer-Verlag, 1986.

[8] M. D. Gunzburger, *Finite element methods for viscous incompressible flows : a guide to theory, practice, and algorithms*, Academic Press, 1989.

[9] T.J.R. Hughes, A. A. Oberai, and L. Mazzei, *Large Eddy Simulation of Turbulent Channel Flow by the Variational Multiscale Method*, Phys. Fluids **13** (2001), 1784–1799.

[10] T. Iliescu and P. Fischer, *Backscatter in the Rational LES model*, Computers and Fluids **33(5-6)** (2004), 783–790.

[11] V. John and S. Kaya, *A finite element variational multiscale method for the Navier-Stokes equations*, SIAM J. Sci. Comp. **26** (2005), 1485 – 1503.

[12] A. V. Kolmogorov, *The local structure of turbulence in incompressible viscous fluids for very large Reynolds number*, Dokl. Akad. Nauk. SSR **30** (1941), 9–13.

[13] W. Layton, *Numerical analysis of finite element computational fluid dynamics*, Lecture Notes, 2002.

[14] J. Leray, *Sur le mouvemennt d'un fluide visquex emplissant l'espace*, Acta. Math. **63** (1934), 193–248.

[15] R. Moser, J. Kim, and N. Mansour, *Direct Numerical Simulation of Turbulent Channel Flow up to* Re_τ*=590*, Phys. Fluids **11** (1998), 943–945.

[16] A. Ostermann and C. Palencia, *Shadowing for nonautonomous parabolic problems with applications to long-time error bounds*, SIAM J. Numer. Anal. **37** (2001), no. 5, 1399–1419.

[17] S. Yu Pilyugin, *Shadowing in dynamical systems*, Lecture Notes in Mathematics **1706** (1999).

[18] J. A. Yorke S. M. Hammel and C. Grebogi, *Numerical orbits of caotic processes represent true orbits*, American Mathematical Society **19** (1988), no. 2.

[19] H. Schlichting, *Boundary-layer theory*, McGraw-Hill, 1979.

[20] J. Serrin, *The initial value problem for the Navier-Stokes equations*, Nonlinear Probl., Proc. Sympos. Madison 1962, 69-98 , 1963.

[21] H. Sohr, *The Navier-Stokes equations, an elementary functional analytic approach*, Birhäuser Advanced Texts, Birkhäuser Verlag Basel, Boston, Berlin, 2001.

[22] X. Wang, *Time averaged energy dissipation rate for shear driven flows in R^n*, Physica D **99** (1997), 555–563.