An Energy and Helicity Conserving Finite Element Scheme for the Navier-Stokes Equations

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Abstract

We present a new finite element scheme for solving the Navier-Stokes equations that exactly conserves both energy ($\int_\Omega u^2$) and helicity ($\int_\Omega u \cdot (\nabla \times u)$) in the absence of viscosity and external force. We prove

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stability, exact conservation, and convergence for the scheme. Energy and helicity are exactly conserved by using a combination of the usual (convective) form with the rotational form of the nonlinearity, and solving for both velocity and a projected vorticity.

1 Introduction

It is well known that the Navier-Stokes equations (NSE) preserve energy ($E = \frac{1}{2}\int_\Omega |u|^2$) in the absence of viscosity and external force. Conserving energy in numerical schemes for the NSE not only leads to stability for the scheme, but also is necessary for physical relevance of solutions. In rotational flows, however, other integral invariants are also important. In two dimensions enstrophy ($Ens = \frac{1}{2}\int_\Omega |\nabla \times u|^2$) and in three dimensions helicity ($H = \int_\Omega u \cdot (\nabla \times u)$) are also conserved quantities of the NSE when viscosity and external force are not present. In three dimensions, helicity also admits a topological interpretation for a flow in terms of its reflectional symmetry [11]. Hence accurate helicity prediction should be a goal in schemes for three dimensional rotational flows.

For two dimensional flows, schemes such as the classical Arakawa [1] have existed for over forty years which conserve both energy and enstrophy (this and all future references to E/H/Ens conservation implicitly refer to the case of no viscosity or external force). For three dimensional flows, however, it was not until 2004 that Liu and Wang developed the first scheme that conserves both energy and helicity. In [10], they present an energy and helicity preserving scheme for axisymmetric flows, and show that this dual conservation eliminates the need for excessive numerical viscosity. It is their work which motivated this article.

In this report, we present a new finite element scheme that globally conserves both energy and helicity for general flows. Our development of the scheme herein is for periodic boundaries (and hence we use a box for the domain Ω); for non-periodic boundary conditions, helicity is not necessarily globally conserved (on the other hand, helicity generation near walls and helicity flux away from walls are equally important for non-periodic problems). The key features that allow the scheme to conserve both energy and helicity is the use of the projection of the vorticity in the scheme, and a new variational formulation of the nonlinearity that vanishes when tested against either the velocity or projected vorticity. We present the scheme in Section 3, after providing the necessary notation in Section 2. Section 4 gives a rigorous numerical analysis for the scheme, and Section 5 presents conclusions.

2 Notation and Preliminaries

$(\cdot,\cdot)$ and $\|\cdot\|$ denote the usual L^2 inner product and norm, respectively, and $\|\cdot\|_k$ for the $H^k(\Omega)$ norm. $\|\cdot\|_\infty$ will denote the usual $L^\infty(\Omega)$ norm, and all other norms that appear in this article will be clearly labeled with subscripts. The domain Ω we use, as stated above, will be the box $(0,L)^3$.

Definition 2.1. *The Hilbert space $H^1_\#(\Omega)$ will be defined as*

$$H^1_\# := (v \in H^1 \ : \ v \text{ periodic on } \Omega, \ \int_\Omega v \ dx = 0).$$

This is the natural velocity space for the NSE with periodic boundary conditions, as discussed in [8] and [9]. Note that velocities in this space automatically conserve momentum ($\int_\Omega u$), i.e. if $u \in H^1_\#$, $\frac{d}{dt}\int_\Omega u = 0$. This is physically important because the Navier-Stokes equations (with periodic boundary conditions) also conserve momentum [5].

The following lemma (which can be found in [3] for example) will be used frequently throughout the analysis in this article.

Lemma 2.2. *For $u, v \in H^1_\#(\Omega)$,*

$$(\nabla \times u, v) = (u, \nabla \times v)$$

Proof. The proof of this well known lemma follows immediately from integrating by parts since u, v are periodic. □

Let $T^h = T^h(\Omega)$ be a conforming finite element mesh on Ω. Define the spaces $(X^h, Q^h) \subset (H^1_\#, L^2_0)$ to be conforming velocity, pressure finite element spaces (see, e.g. [2],[4] or [6] for examples) that satisfy the LBB^h condition

$$0 < \beta \leq \inf_{q \in Q^h} \sup_{v \in X^h} \frac{(q, \nabla \cdot v)}{\|v\|_1 \|q\|}. \tag{1}$$

Define V^h to be the space of discretely divergence free, zero-mean, periodic functions.

$$V^h = \{v \in X^h \ : \ (\nabla \cdot v, q) = 0 \ \forall q \in Q^h\},$$

Since V^h is a closed subspace of $H^1_{\#}(\Omega)$, we have also that V^h is a Hilbert space, and thus the following result.

Lemma 2.3. *Let $u^h \in V^h$. Then there exists a unique $w^h \in V^h$ satisfying*

$$(w^h, v) = (\nabla \times u^h, v) \ \forall v \in V^h \tag{2}$$

Proof. Since $u^h \in V^h \subset H^1(\Omega)$, it follows that $\nabla \times u^h \in L^2(\Omega)$. Since V^h is a closed subset of the Hilbert space $L^2(\Omega)$, the Riesz representation theorem implies the existence and uniqueness of a solution w^h to (2). $\square$

The next lemma shows how an elementary property of the cross product can be used for double skew symmetry of a trilinear term.

Lemma 2.4. *Let $u^h, w^h \in X^h$. Then*

$$(u^h \times w^h, u^h) = (u^h \times w^h, w^h) = 0.$$

Proof. This follows from an elementary property of the cross product; the cross product of two vectors is perpendicular to each of them. $\square$

The significance of this lemma is that in a finite element scheme, the trilinear form $c(u^h, w^h, v^h) := (u^h \times w^h, v^h)$ will vanish when $v^h = u^h$ or w^h. Such a trilinear form has significance in the NSE if the rotational form of the nonlinearity is used (see, e.g. [5] p.461 or [12]). Our scheme uses this form, and exploits the double skew symmetry to show the scheme conserves both energy and helicity.

The discrete Gronwall lemma will also be an essential tool in the error analysis; we present it now.

Lemma 2.5. *(Discrete Gronwall) Let Δt, H, and a_n, b_n, c_n, d_n (for integers $n \geq 0$) be nonnegative numbers such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l} d_n a_n + \Delta t \sum_{n=0}^{l} c_n + H \quad for\ l \geq 0. \tag{3}$$

Suppose that $\Delta t d_n < 1\ \forall n$. Then,

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq exp\left(\Delta t \sum_{n=0}^{l} \frac{d_n}{1-\Delta t d_n}\right)\left(\Delta t \sum_{n=0}^{l} c_n + H\right) \quad for\ l \geq 0. \tag{4}$$

Proof. See [7], for example, for proof of this well known lemma. ☐

We end this section with definitions for discrete energy and helicity.

Definition 2.6. *We define the discrete energy E and helicity H to be, at time t_k,*

$$\begin{aligned} E(t_k) &= \frac{1}{2}\|u_h^k\|^2, \\ H(t_k) &= (u_h^k, \nabla \times u_h^k). \end{aligned}$$

We are now ready to present the scheme.

3 An energy and helicity preserving scheme for periodic flows

The energy and helicity preserving finite element scheme we study is composed of a trapezoidal time discretization with a nonlinearity which is doubly skew-symmetic. Let Δt denote the timestep, $t_k = k\Delta t$, $t_{k+1/2} = (k + \frac{1}{2})\Delta t$, and u_h^k the approximation to $u(x, t_k)$. $u_h^{k+1/2}$ will denote

$$u_h^{k+1/2} := \frac{1}{2}(u_h^{k+1} + u_h^k),$$

and $f^{n+1/2}(x) := f(t_{n+1/2}, x) \in V^{h,*}$. $T = Nk$ denotes the final time. Given $u_h^0 \in V^h$, define w_h^0 to be the (unique in V^h by Lemma 2.3) solution of $(w_h^0, v) = (\nabla \times u_h^0, v)\ \forall v \in V^h$, and find $(u_h^k, w_h^k, p_h^k) \in (X_h, V_h, Q_h)$ for $k = 1..N$, satisfying

$$\frac{1}{\Delta t}(u_h^{n+1}, v) + (u_h^{n+1/2} \times w_h^{n+1/2}, v) - (p_h^{n+1/2}, \nabla \cdot v) + \frac{\nu}{2}(\nabla u_h^{n+1/2}, \nabla v) + \frac{\nu}{2}(w_h^{n+1/2}, \nabla \times v) = (f^{n+1/2}, v) + \frac{1}{\Delta t}(u_h^n, v)\ \forall v \in X^h \quad (5)$$

$$(\nabla \cdot u_h^{n+1}, q) = 0\ \forall q \in Q^h \quad (6)$$

$$(w_h^{n+1} - \nabla \times u_h^{n+1}, \chi) = 0\ \forall \chi \in V^h \quad (7)$$

We now prove the conservation properties of the scheme; energy and helicity are exactly conserved in the absence and viscosity and external force. As a remark, we note that even though the scheme (5)-(7) conserves helicity, the analogous, continuous in time form of the scheme does not. This is due to an inability to identify helicity with the the integral of the scalar product of velocity and the projected vorticity, as $((u_h^k)_t, w_h^k)$ is not necessarily equal to $((u_h^k)_t, \nabla \times u_h^k)$ because $(u_h^k)_t$ need not be in V^h. Since the scheme (5)-(7) solves for the projected vorticity, such as identification is critical to prove helicity conservation for the scheme, but only holds in the discrete time case for our formulation.

Lemma 3.1. *The scheme (5)-(7) conserves energy and helicity in the absence of viscosity and body force, that is,* $E(t_n) = E(t_0)$ *and* $H(t_n) = H(t_0)$ $\forall n \leq N$ *provided* $\nu = f = 0$.

Proof. For the conservation of energy, set $v = u_h^{n+1/2}$ and $\nu = f = 0$ in (5). This gives

$$(u_h^{n+1}, u_h^{n+1/2}) = (u_h^n, u_h^{n+1/2}). \tag{8}$$

By expanding the $u_h^{n+1/2}$ terms in (8), we have

$$\begin{aligned} \|u_h^{n+1}\|^2 + \frac{1}{2}(u_h^{n+1}, u_h^n) &= \frac{1}{2}\|u_h^n\|^2 + \frac{1}{2}(u_h^n, u_h^{n+1}), & (9) \\ E(t_{n+1}) &= E(t_n), & (10) \end{aligned}$$

which implies that $E(t_n) = E(t_0)$.

For helicity conservation, set $v = w_h^{n+1/2}$ in (5). The pressure term vanishes since $w_h^n, w_h^{n+1} \in V^h$, and so after setting $\nu = f = 0$, we are left with

$$\frac{1}{2}(u_h^{n+1}, w_h^{n+1}) + \frac{1}{2}(u_h^{n+1}, w_h^n) = \frac{1}{2}(u_h^n, w_h^n) + \frac{1}{2}(u_h^n, w_h^{n+1}). \tag{11}$$

Using equation (7) and Lemma 2.2, we have the following identities for the terms in (11).

$$\begin{aligned} (u_h^{n+1}, w_h^{n+1}) &= (u_h^{n+1}, \nabla \times u_h^{n+1}) = H(t_{n+1}) & (12) \\ (u_h^n, w_h^n) &= (u_h^n, \nabla \times u_h^n) = H(t_n) & (13) \\ (u_h^{n+1}, w_h^n) &= (u_h^n, w_h^{n+1}) & (14) \end{aligned}$$

Thus (11) can be rewritten as

$$H(t_{n+1}) = H(t_n), \tag{15}$$

which implies that $H(t_n) = H(t_0)$. □

The following lemma shows that the energy and helicity conserving scheme is also stable.

Lemma 3.2. *Solutions to the discrete scheme (5)-(7) satisfy*

$$\|u_h^N\|^2 + \Delta t \sum_{n=0}^{N-1} \left(\frac{\nu}{2} \|\nabla u_h^{n+1/2}\|^2 + \nu \|w_h^{n+1/2}\|^2 \right) \leq \|u_h^0\|^2 + \frac{2\Delta t}{\nu} \sum_{n=0}^{N-1} \|f^{n+1/2}\|_*^2 \quad (16)$$

Proof. Set $v = u_h^{n+1/2}$ in (5), $q = p_h^{n+1/2}$ in (6) and add the equations. This gives

$$\frac{1}{2\Delta t}\|u_h^{n+1}\|^2 + \frac{1}{2\Delta t}(u_h^{n+1}, u_h^n) + \frac{\nu}{2}\|\nabla u_h^{n+1/2}\|^2 + \frac{\nu}{2}(w_h^{n+1/2}, \nabla \times u^{n+1/2}) = (f^{n+1/2}, u_h^{n+1/2}) + \frac{1}{2\Delta t}\|u_h^n\|^2 + \frac{1}{2\Delta t}(u_h^n, u_h^{n+1}). \quad (17)$$

Note that $(w_h^{n+1/2}, \nabla \times u_h^{n+1/2}) = \|w_h^{n+1/2}\|^2$ since (7) must hold for $(n+1)$ replaced by (n), and thus also for $(n+1)$ replaced by $(n+1/2)$. By making this substitution, (17) reduces to

$$\frac{1}{2\Delta t}\|u_h^{n+1}\|^2 + \frac{\nu}{2}\|\nabla u_h^{n+1/2}\|^2 + \frac{\nu}{2}\|w_h^{n+1/2}\|^2 = (f^{n+1/2}, u_h^{n+1/2}) + \frac{1}{2\Delta t}\|u_h^n\|^2. \quad (18)$$

Next we use the bound $(f^{n+1/2}, u_h^{n+1/2}) \leq \frac{\nu}{4}\|\nabla u_h^{n+1/2}\|^2 + \frac{1}{\nu}\|f^{n+1/2}\|_*^2$, and sum from $n = 0..(N-1)$, yielding

$$\frac{1}{2\Delta t}\|u_h^N\|^2 + \sum_{n=0}^{N-1} \left(\frac{\nu}{2}\|\nabla u_h^{n+1/2}\|^2 + \nu\|w_h^{n+1/2}\|^2 \right) \leq \frac{1}{2\Delta t}\|u_h^0\|^2 + \frac{1}{\nu}\sum_{n=0}^{N-1}\|f^{n+1/2}\|_*^2 \quad (19)$$

Now multiplying both sides by $(2\Delta t)$ proves the lemma. □

3.1 Existence of solutions for the scheme

Given $u_h^n, w_h^n \in V^h$, a nonlinear system must be solved for the approximations at time level $n+1$. The question arises: does that system have a solution? In other words, does imposing two integral invariants overdetermine the system for u_h^{n+1}, w_h^{n+1}? The answer is that solutions to (5)-(7) do exist, as we will show in this section.

For clarity, we show existence for the equivalent nonlinear problem: Given $\nu, \Delta t > 0$, $f^{n+1/2} \in V^{h,*}$, and $u_h^n \in V^h$, find u_h, w_h satisfying

$$\frac{2}{\Delta t}(u_h, v) + (u_h \times w_h, v) + \frac{\nu}{2}(\nabla u_h, \nabla v) + \frac{\nu}{2}(w_h, \nabla \times v) = (f^{n+1/2}, v) + \frac{2}{\Delta t}(u_h^n, v) \ \forall v \in V^h, \quad (20)$$

$$(w_h - \nabla \times u_h, \chi) = 0 \ \forall \chi \in V^h. \quad (21)$$

This form of the scheme is derived from (5)-(7) by defining $u := u_h^{n+1/2}$, $w := w_h^{n+1/2}$, and restricting the test functions to V^h. The equations (20)-(21) are equivalent (5)-(7). To show solutions exist, we formulate (20)-(21) as a fixed point problem, $y = F(y)$, and use the Leray-Schauder fixed point theorem. We will first prove several preliminary lemmas, followed by a theorem which proves solution to (20)-(21) exist.

Lemma 3.3. *For $\nu, \Delta t > 0$, there exists a unique solution (u_h, w_h) to: Given $g \in V^{h,*}$, find $(u_h, w_h) \in V^h \times V^h$ satisfying*

$$\frac{2}{\Delta t}(u_h, v) + \frac{\nu}{2}(\nabla u_h, \nabla v) + \frac{\nu}{2}(w_h, \nabla \times v) = (g, v) \ \forall v \in V^h, \quad (22)$$

$$(w_h - \nabla \times u_h, \chi) = 0 \ \forall \chi \in V^h. \quad (23)$$

Proof. We will prove uniqueness of solutions to (22)-(23) by showing only the trivial solution solves the homogeneous problem, which will also imply the existence of solutions to the finite dimensional problem. Choose $v = u_h$ in (22), $\chi = w_h$ in (23) and substitute (23) into (22). This gives

$$\frac{2}{\Delta t}\|u_h\|^2 + \frac{\nu}{2}\|\nabla u_h\|^2 + \frac{\nu}{2}\|w_h\|^2 = 0, \tag{24}$$

which implies $u_h = w_h = 0$, i.e. uniqueness. □

This lemma allows us to define a solution operator to (22)-(23).

Definition 3.4. *We define the solution operator $T : V^{h,*} \to (V^h \times V^h)$, to be the solution operator of (22)-(23): if $g \in V^{h,*}$, $T(g) = (u_h, w_h)$ solves (22)-(23).*

We have that T is well defined by the previous lemma, and we now prove it is also bounded and linear.

Lemma 3.5. *The solution operator T is linear, bounded, and continuous.*

Proof. The linearity of T follows from the fact that T is a solution operator to a linear problem. To see that T is bounded (and thus continuous since it is linear), we let $v = u_h$, $\chi = w_h$ in (22)-(23), multiply (23) by $\frac{\nu}{2}$, and add the equations. This gives

$$\frac{2\|u_h\|^2}{\Delta t} + \frac{\nu}{4}\|\nabla u_h\|^2 + \frac{\nu}{2}\|w_h\|^2 \leq \frac{1}{\nu}\|g\|_*^2$$

Then since u_h, w_h are finite dimensional, $\|u_h, w_h\|_{V^h \times V^h} \leq C\|g\|_*$. Hence,

$$\|T\| = \sup_{g \in V_*^h} \frac{\|T(g)\|}{\|g\|_*} = \sup_{g \in V_*^h} \frac{\|u_h, w_h\|_{V^h \times V^h}}{\|g\|_*} \leq C.$$

□

We next define the operator N. The function F that will be used in the formulation of the fixed point problem will be a composition of T and N.

Definition 3.6. *We define the operator N on $(V^h \times V^h)$ by*

$$N(u_h, w_h) := f^{n+1/2} + \frac{2}{\Delta t} u_h^n + u_h \times w_h$$

We now prove properties for N necessary for use in Leray-Schauder.

Lemma 3.7. *For the nonlinear operator N, we have that $N : V^h \times V^h \to V^{h,*}$, N is bounded, and N is continuous.*

Proof. To show N maps as stated, we let $(u_h, w_h) \in V^h \times V^h$ and write

$$\|N(u_h, w_h)\|_* = \sup_{v \in V^h} \frac{(N(u_h, w_h), v)}{\|v\|_1}.$$

From the definition of N, we have that $\frac{(f^{n+1/2}, v) + (2(\Delta t)^{-1} u_h^n, v)}{\|v\|_1} \leq \|f\|_* + C_1 \|u_h^n\| \leq C_2$, and that

$$\frac{(u_h \times w_h, v)}{\|v\|_1} \leq \|u_h\|_\infty \|w_h\| \leq C_3$$

since $u_h.w^h$ was given to be in V^h and all norms are equivalent in finite dimension. Hence $\|N(u, w)\|_* < C$, and so N maps as stated. Note we have also proven that N is bounded.

The equivalence of norms in finite dimension is also key in showing N is continuous, as

$$\begin{aligned} \|N(u, w) - N(u_k, w_k)\|_* &\leq \|u \times (w - w_k)\|_* + \|(u - u_k) \times w_k\|_*, && (25) \\ &\leq \|u\|_\infty \|w - w_k\| + \|w_k\|_\infty \|u - u_k\|, && (26) \end{aligned}$$

and thus $\to$ 0 as $\|(u, w) - (u_k, w_k)\| \to 0$. □

We are now ready to define the operator F, which will formulate (20)-(21) as a fixed point problem.

Definition 3.8. *Define the operator* $F : (V^h \times V^h) \to (V^h \times V^h)$ *to be composition of* T *and* N*:* $F(y) = T(N(Y))$.

Lemma 3.9. F *is well defined and compact, and a solution to* $y = F(y)$ *solves (20)-(21).*

Proof. F is well defined because N and T are. The fact that F is compact follows from the fact that both N and T are continuous and bounded. It can easily be seen that a fixed point of F solves (20)-(21) by expanding F. $\square$

We are now ready to prove existence to (20)-(21).

Theorem 3.10. *Let* $y_\lambda = (u_\lambda, w_\lambda) \in V^h$ *and consider the family of fixed point problems* $y_\lambda = \lambda F(y_\lambda)$, $0 \le \lambda \le 1$. *A solution* y_λ *to any of these fixed point problems satisfies* $\|y_\lambda\| < K$*, independent of* λ*. Since* F *is compact, and fixed points of* F *solve (20)-(21), by the Leray-Schauder theorem there exist solutions to (20)-(21).*

Proof. All we have to show to prove this theorem is that solutions to $y_\lambda = \lambda F(y_\lambda)$ are bounded independent of λ. Using the definition of F and the linearity of T we have that

$$y_\lambda = \lambda F(y_\lambda) = \lambda T(N(y_\lambda)) = T(\lambda N(y_\lambda)) = T(\lambda(f^{n+1/2} + \frac{2}{\Delta t} u_h^n + u_\lambda \times w_\lambda)),$$

which implies that

$$\frac{2}{\Delta t}(u_\lambda, v) - \lambda(u_\lambda \times w_\lambda, v) + \frac{\nu}{2}(\nabla u_\lambda, \nabla v)$$
$$+ \frac{\nu}{2}(w_\lambda, \nabla \times v) = (\lambda f^{n+1/2}, v) + \frac{2\lambda}{\Delta t}(u_h^n, v) \ \forall v \in V^h, \quad (27)$$

$$(w_\lambda - \nabla \times u_\lambda, \chi) = 0 \ \forall \chi \in V^h. \quad (28)$$

Multiply (28) by $\frac{\nu}{2}$, let $\chi = w_\lambda$ in (28), $v = u_\lambda$ in (27), and add the equations. Similar to the stability estimate, this gives

$$\frac{1}{\Delta t}\|u_\lambda\|^2 + \frac{\nu}{4}\|\nabla u_\lambda\|^2 + \frac{\nu}{2}\|w_\lambda\|^2$$
$$\leq \lambda^2\left(\frac{1}{\nu}\|f^{n+1/2}\|^2 + \frac{1}{\Delta t}\|u_h^n\|^2\right) \leq \left(\frac{1}{\nu}\|f^{n+1/2}\|^2 + \frac{1}{\Delta t}\|u_h^n\|^2\right) \leq C \quad (29)$$

which is a bound independent of λ. Thus the theorem is proven. □

We have now shown that the scheme (5)-(7) preserves energy and helicity when $\nu = f = 0$, is stable, and admits solutions. The final step is an error analysis for the scheme.

4 Error analysis of the scheme

This section presents a theorem and corollary for the convergence of the scheme, followed by the proof. The restriction that the theorem places on the time step is for the use of the discrete Gronwall lemma. Although we found its use necessary in the proof, it is widely believed that it gives a gross underestimate of the largest timestep one can use and expect the same asymptotic error results.

Theorem 4.1. *Let $C_* = C_*(\Omega)$ be an interpolation constant satisfying*

$$\|u\|_{1/2}^2 \leq C_* \|u\| \|\nabla u\| \ \forall u \in \Omega.$$

Assume $\|u\|_{L_\infty(\Omega\times(0,T))}$, $\|w\|_{L^\infty(\Omega\times(0,T)}$ $< M$, and $\|u_{ttt}\|_{L_\infty(\Omega\times(0,T))} < \infty$. Select $\Delta t < (\frac{6M^2}{\nu} + \frac{32C_*^2}{\nu^3}\sup_{t<T}\inf_{v\in V^h}\|u(t)-v\|_1^4)^{-1}$, and set $N = \frac{T}{\Delta t}$ and $\rho^{n+1/2} := u_t^{n+1/2} - \frac{u^{n+1}-u^n}{\Delta t}$. Then the error $(u-u^h)$ satisfies

$$\|(u-u_h)^N\|^2 + \nu\Delta t \sum_{n=0}^{N-1} \left(\|\nabla(u-u_h)^{n+1/2}\|^2 + \|(w-w_h)^{n+1/2}\|^2\right) \leq$$

$$\inf_{v\in V^h}\|u^N - v\|^2 + C(\nu,M)\Delta t \sum_{n=0}^{N-1}(\inf_{v\in V^h}\|\nabla(u^{n+1/2}-v)\|^2 + \inf_{v\in V^h}\|w^{n+1/2}-v\|^2$$

$$+ \|\rho^{n+1/2}\|^2 + \inf_{q\in Q^h}\|p^{n+1/2}-q\|^2) \quad (30)$$

.

It is important to note, with smoothness assumptions on the true solution, the restriction of the time step Δt (arising from the use of Gronwall's lemma) reduces to

$$\Delta t < (\frac{6M^2}{\nu} + \frac{32C_*^2}{\nu^3}\max_{t<T}\inf_{v\in V^h}\|u(t)-v\|_1^4)^{-1} \approx \frac{\nu}{6M^2} \quad (31)$$

for fine meshes since $\inf_{v\in V^h}\|u(t)-v\|_1^4 = O(h^{4k})$. Hence the restriction is of the order ν^{-1} and not ν^{-3} for fine meshes.

Before proving this theorem, we give as a corollary the asymptotic convergence rate of the scheme for a usual choice of elements. This corollary follows immediately from the theorem.

Corollary 4.2. *Assuming smoothness of true solutions and that the scheme is solved with (P_k, P_{k-1}) velocity-pressure elements, the asymptotic convergence rate of the scheme is*

$$\|(u-u_h)^N\| + \nu\Delta t \sum_{n=0}^{N-1} \left(\|\nabla(u-u_h)^{n+1/2}\| + \|(w-w_h)^{n+1/2}\|\right) = O(h^k + (\Delta t)^2) \quad (32)$$

Proof. The proof of this theorem is divided into the following parts. We first develop the error equations by subtracting our scheme from the NSE (with periodic boundary conditions). The error is then split into parts in and out of the finite element spaces. This is followed by bounding the error in the space by the interpolation error, and the proof concludes by bounding the total error. Note we assume $(u_h^0, v) = (u^0, v)\ \forall v \in V^h$.

Using the identity $u \cdot \nabla u = \frac{1}{2}\nabla(u^2) - u \times (\nabla \times u)$, and grouping the usual pressure gradient with the $\frac{1}{2}\nabla(u^2)$ term to form the Bernoulli pressure, a periodic solution (u, p) and $w := \nabla \times u$ of the NSE satisfies

$$\frac{1}{\Delta t}(u^{n+1} - u^n, v) - (u^{n+1/2} \times w^{n+1/2}, v) - (p^{n+1/2}, \nabla \times v) + \frac{\nu}{2}(\nabla u^{n+1/2}, \nabla v) + \frac{\nu}{2}(w^{n+1/2}, v) = (f^{n+1/2}, v) + (\rho^{n+1/2}, v)\ \forall v \in V^h, \quad (33)$$

where $\rho^{n+1/2} = u_t^{n+1/2} - \frac{(u^{n+1}-u^n)}{\Delta t}$. Subtracting the scheme (5)-(7) from (33) and the identity $w(t) = \nabla \times u(t)$ and restricting v to be in V^h, we have the error equations

$$\frac{1}{\Delta t}(e^{n+1} - e^n, v) - (u^{n+1/2} \times E^{n+1/2}, v) + (e^{n+1/2} \times w_h^{n+1/2}, v) - (p^{n+1/2}, \nabla \times v) + \frac{\nu}{2}(\nabla e^{n+1/2}, \nabla v) + \frac{\nu}{2}(E^{n+1/2}, \nabla \times v) = (\rho^{n+1/2}, v)\ \forall v \in V^h, \quad (34)$$

$$(E^{n+1/2},\chi) = (\nabla \times e^{n+1/2},\chi)\ \forall \chi \in V^h, \tag{35}$$

where $E^i = w^i - w_h^i$ and $e^i = u^i - u_h^i$. We split the errors into parts inside the finite element space and interpolation error by defining W^i, U^i to be the L^2 projections of w^i, u^i onto V^h, and rewrite

$$E^i = (w^i - W^i) - (w_h^i - W^i) \quad =: \quad r^i - s_h^i, \tag{36}$$

$$e^i = (u^i - U^i) - (u_h^i - U^i) \quad =: \quad \eta^i - \phi_h^i \tag{37}$$

We now rewrite (34)-(35) as one equation after expanding each E^i and e^i, letting $\chi = s_h^{n+1/2}$ and $v = \phi_h^{n+1/2}$ in (34)-(35), and subtracting the equations, which yields

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\phi_h^{n+1}\|^2 - \|\phi_h^n\|^2) + \frac{\nu}{2}\|\nabla \phi_h^{n+1/2}\|^2 + \frac{\nu}{2}\|s_h^{n+1/2}\|^2 = \frac{1}{\Delta t}(\eta^{n+1} - \eta^n, \phi_h^{n+1/2}) \\
&\quad - (p^{n+1/2}, \nabla \cdot \phi_h^{n+1/2}) + \frac{\nu}{2}(\nabla \eta^{n+1/2}, \nabla \phi_h^{n+1/2}) + \frac{\nu}{2}(r^{n+1/2}, \nabla \times \phi_h^{n+1/2}) \\
&\quad + \frac{\nu}{2}(r^{n+1/2}, s_h^{n+1/2}) - \frac{\nu}{2}(\nabla \times \eta^{n+1/2}, s_h^{n+1/2}) - (\rho^{n+1/2}, \phi_h^{n+1/2}) \\
&- (u^{n+1/2} \times r^{n+1/2}, \phi_h^{n+1/2}) + (u^{n+1/2} \times s_h^{n+1/2}, \phi_h^{n+1/2}) + (\eta^{n+1/2} \times w_h^{n+1/2}, \phi_h^{n+1/2}).
\end{aligned} \tag{38}$$

We first note that since η^i is the difference between u^i and its projection of u^i onto V^h, η^i must be orthogonal to any element of V^h. Hence $(\eta^{n+1} - \eta^n, \phi_h^{n+1/2})$ vanishes. Cauchy-Schwarz and Young's inequalities give the following bounds for terms on the right hand side of (38).

$$\begin{aligned}
\frac{\nu}{2}(\nabla\eta^{n+1/2},\nabla\phi_h^{n+1/2}) &\leq \frac{\nu}{2}\|\nabla\eta^{n+1/2}\|^2+\frac{\nu}{8}\|\nabla\phi_h^{n+1/2}\|^2 && (39)\\
\frac{\nu}{2}(r^{n+1/2},\nabla\times\phi_h^{n+1/2}) &\leq \frac{\nu}{2}\|r^{n+1/2}\|^2+\frac{\nu}{8}\|\nabla\phi_h^{n+1/2}\|^2 && (40)\\
\frac{\nu}{2}(r^{n+1/2},s_h^{n+1/2}) &\leq \frac{\nu}{2}\|r^{n+1/2}\|^2+\frac{\nu}{8}\|s_h^{n+1/2}\|^2 && (41)\\
-\frac{\nu}{2}(\nabla\times\eta^{n+1/2},s_h^{n+1/2}) &\leq \frac{\nu}{2}\|\nabla\eta^{n+1/2}\|^2+\frac{\nu}{8}\|s_h^{n+1/2}\|^2 && (42)
\end{aligned}$$

Thus the error equation becomes

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\phi_h^{n+1}\|^2-\|\phi_h^n\|^2)+\frac{\nu}{4}\|\nabla\phi_h^{n+1/2}\|^2+\frac{\nu}{4}\|s_h^{n+1/2}\|^2\leq\\
&\quad\nu\|\nabla\eta^{n+1/2}\|^2+\nu\|r^{n+1/2}\|^2-(p^{n+1/2},\nabla\cdot\phi_h^{n+1/2})-(\rho^{n+1/2},\phi_h^{n+1/2})\\
&\quad-(u^{n+1/2}\times r^{n+1/2},\phi_h^{n+1/2})+(u^{n+1/2}\times s_h^{n+1/2},\phi_h^{n+1/2})\\
&\quad+(\eta^{n+1/2}\times w_h^{n+1/2},\phi_h^{n+1/2}). \qquad (43)
\end{aligned}$$

Recall that since $\phi_h^{n+1/2}\in V^h$, $(q,\nabla\cdot\phi_h^{n+1/2})=0\ \forall q\in Q^h$. Hence we make this substitution into (43) and use Cauchy-Schwarz on this pressure term, and bound the ρ term using Cauchy-Schwarz, which gives

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\phi_h^{n+1}\|^2-\|\phi_h^n\|^2)+\frac{\nu}{4}\|\nabla\phi_h^{n+1/2}\|^2+\frac{\nu}{4}\|s_h^{n+1/2}\|^2\leq\nu\|\nabla\eta^{n+1/2}\|^2\\
&\quad+\nu\|r^{n+1/2}\|^2+\inf_{q\in Q^h}\|p^{n+1/2}-q\|\|\nabla\phi_h^{n+1/2}\|+\|\rho^{n+1/2}\|\|\phi_h^{n+1/2}\|\\
&\quad-(u^{n+1/2}\times r^{n+1/2},\phi_h^{n+1/2})+(u^{n+1/2}\times s_h^{n+1/2},\phi_h^{n+1/2})\\
&\quad+(\eta^{n+1/2}\times w_h^{n+1/2},\phi_h^{n+1/2}). \qquad (44)
\end{aligned}$$

To get upper bounds on the trilinear terms, we must use the bounds, $(a\times b,c)\leq\|a\|_\infty\|b\|\|c\|$ and, since the scheme is for three dimensions, $(a\times b,c)\leq$

$\|a\|_0\|b\|_{1/2}\|c\|_1$. The first bound will be applied to the first two trilinear terms, and the infinity norm will be used on the $u^{n+1/2}$ term, since by assumption, $\|u(t)\|_\infty < M$. The third trilinear term will be rewritten by adding and subtracting w to w_h. Then using the assumption that $\|w(t)\|_\infty < M$ and the bounds stated above, we have

$$\begin{aligned} &-(u^{n+1/2}\times r^{n+1/2},\phi_h^{n+1/2}) + (u^{n+1/2}\times s_h^{n+1/2},\phi_h^{n+1/2}) \\ &\qquad + (\eta^{n+1/2}\times w_h^{n+1/2},\phi_h^{n+1/2}) \le M\|r^{n+1/2}\|\|\phi_h^{n+1/2}\| \\ &+ M\|s_h\|\|\phi_h^{n+1/2}\| + M\|\eta^{n+1/2}\|\|\phi_h^{n+1/2}\| + \|\eta^{n+1/2}\|_1\|E^{n+1/2}\|\|\phi_h^{n+1/2}\|_{1/2}. \end{aligned} \tag{45}$$

Cauchy-Schwarz, Young, and Sobolev imbedding inequalities now give the upper bound

$$\begin{aligned} &-(u^{n+1/2}\times r^{n+1/2},\phi_h^{n+1/2}) + (u^{n+1/2}\times s_h^{n+1/2},\phi_h^{n+1/2}) \\ &\qquad + (\eta^{n+1/2}\times w_h^{n+1/2},\phi_h^{n+1/2}) \le \frac{\nu}{4}\|r^{n+1/2}\|^2 \\ &+ \frac{\nu}{16}\|s_h\|^2 + \frac{5M^2}{\nu}\|\phi_h^{n+1/2}\|^2 + \frac{\nu}{4}\|\eta^{n+1/2}\|^2 + \frac{\nu}{16}\|E^{n+1/2}\|^2 \\ &\qquad + \frac{4C_*}{\nu}(\|\eta^{n+1/2}\|_1^2\|\nabla\phi_h^{n+1/2}\|\|\phi_h^{n+1/2}\|). \end{aligned} \tag{46}$$

Young's inequality now gives

$$\begin{aligned} &-(u^{n+1/2}\times r^{n+1/2},\phi_h^{n+1/2}) + (u^{n+1/2}\times s_h^{n+1/2},\phi_h^{n+1/2}) \\ &+ (\eta^{n+1/2}\times w_h^{n+1/2},\phi_h^{n+1/2}) \le \frac{5\nu}{16}\|r^{n+1/2}\|^2 + \frac{\nu}{8}\|s_h\|^2 + \frac{5M^2}{\nu}\|\phi_h^{n+1/2}\|^2 \\ &\qquad + \frac{\nu}{4}\|\eta^{n+1/2}\|^2 + \frac{\nu}{8}\|\nabla\phi_h^{n+1/2}\|^2 + \frac{32C_*^2}{\nu^3}\|\eta^{n+1/2}\|_1^4\|\phi_h^{n+1/2}\|^2. \end{aligned} \tag{47}$$

Inserting (47) back into (44) yields

$$\frac{1}{2\Delta t}(\|\phi_h^{n+1}\|^2 - \|\phi_h^n\|^2) + \frac{\nu}{8}\|\nabla\phi_h^{n+1/2}\|^2 + \frac{\nu}{8}\|s_h^{n+1/2}\|^2 \leq \nu\|\nabla\eta^{n+1/2}\|^2 + \frac{21\nu}{16}\|r^{n+1/2}\|^2 + \inf_{q\in Q^h}\|p^{n+1/2} - q\|\|\nabla\phi_h^{n+1/2}\| + \|\rho^{n+1/2}\|\|\phi_h^{n+1/2}\| + \|\phi_h^{n+1/2}\|^2(\frac{5M^2}{\nu} + \frac{32C_*^2}{\nu^3}\|\eta^{n+1/2}\|_1^4). \quad (48)$$

Next we use Young's inequality on the ρ term and the pressure term. Thus we now have

$$\frac{1}{2\Delta t}(\|\phi_h^{n+1}\|^2 - \|\phi_h^n\|^2) + \frac{\nu}{16}\|\nabla\phi_h^{n+1/2}\|^2 + \frac{\nu}{8}\|s_h^{n+1/2}\|^2 \leq \nu\|\nabla\eta^{n+1/2}\|^2 + \frac{21\nu}{16}\|r^{n+1/2}\|^2 + \frac{4}{\nu}\inf_{q\in Q^h}\|p^{n+1/2} - q\|^2 + \frac{\nu}{4M^2}\|\rho^{n+1/2}\|^2 + \|\phi_h^{n+1/2}\|^2(\frac{6M^2}{\nu} + \frac{32C_*^2}{\nu^3}\|\eta^{n+1/2}\|_1^4). \quad (49)$$

Decompose the $\|\phi_h^{n+1/2}\|^2$ term, and multiply through by $2\Delta t$. Then using the assumption on Δt, we apply the discrete Gronwall lemma to get

$$\|\phi_h^N\|^2 + \frac{\nu\Delta t}{8}\sum_{n=0}^{N-1}\left(\|\nabla\phi_h^{n+1/2}\|^2 + 2\|s_h^{n+1/2}\|^2\right) \leq C(\nu, M)\Delta t\sum_{n=0}^{N-1}(\|\nabla\eta^{n+1/2}\|^2 + \|r^{n+1/2}\|^2 + \inf_{q\in Q^h}\|p^{n+1/2} - q\|^2 + \|\rho^{n+1/2}\|^2). \quad (50)$$

By the triangle inequality, we can now bound the total error as

$$\|(u-u_h)^N\|^2 + \frac{\nu\Delta t}{8}\sum_{n=0}^{N-1}\left(\|\nabla(u-u_h)^{n+1/2}\|^2 + 2\|(w-w_h)^{n+1/2}\|^2\right) \leq$$

$$\|\eta^N\|^2 + C(\nu,M)\Delta t\sum_{n=0}^{N-1}(\|\nabla\eta^{n+1/2}\|^2 + \|r^{n+1/2}\|^2$$

$$+ \|\rho^{n+1/2}\|^2 + \inf_{q\in Q^h}\|p^{n+1/2}-q\|^2). \quad (51)$$

Hence,

$$\|(u-u_h)^N\|^2 + \frac{\nu\Delta t}{8}\sum_{n=0}^{N-1}\left(\|\nabla(u-u_h)^{n+1/2}\|^2 + 2\|(w-w_h)^{n+1/2}\|^2\right) \leq$$

$$\inf_{v\in V^h}\|u^n - v\|^2 + C(\nu,M)\Delta t\sum_{n=0}^{N-1}(\inf_{v\in V^h}\|\nabla(u^{n+1/2}-v)\|^2 + \inf_{v\in V^h}\|s^{n+1/2}-v\|^2$$

$$+ \|\rho^{n+1/2}\|^2 + \inf_{q\in Q^h}\|p^{n+1/2}-q\|^2). \quad (52)$$

This proves the theorem. We now prove the corollary. Recalling the definitions of V^h and Q^h, and from approximation theory that $\|\rho^{n+1/2}\|^2 \leq \|u_{ttt}\|_{L^\infty(\Omega\times(t_n,t_{n+1}))}(\Delta t)^4$, we can bound the error in terms of the mesh size h and Δt. Thus, assuming (P_k, P_{k-1}) velocity pressure elements (with $k \geq 2$ to satisfy LBB^h), we have

$$\|(u-u_h)^N\|^2 + \frac{\nu\Delta t}{8}\sum_{n=0}^{N-1}\left(\|\nabla(u-u_h)^{n+1/2}\|^2 + 2\|(w-w_h)^{n+1/2}\|^2\right) \leq$$

$$h^{2(k+1)}\left|u\right|_k^2 + C(\nu,M,T)(\Delta t^4 + h^{2k}\left|u\right|_k^2 + h^{2(k+1)}\left|w\right|_k^2 + h^{2k}\left|p\right|_{k-1}^2)$$

$$= C(\nu,M,T,\left|u\right|_k,\left|p\right|_{k-1})\left(\Delta t^4 + h^{2k}\right). \quad (53)$$

Taking square roots finishes the proof.

□

5 Conclusions

We have developed a new energy and helicity preserving scheme for periodic flows which is second order in time and converges optimally in space. The scheme was able to conserve two inviscid invariants by using the rotational form of the nonlinearity, and solving for a projected vorticity. The scheme does not lose asymptotic convergence rates in velocity from the usual Crank-Nicholson finite element method for the NSE. For a given mesh, each linear system that needs to be solved (in a Newton iteration, for example) is roughly double the size of the resulting linear systems in a scheme that solves for only velocity. However, for higher Reynolds number flows, our scheme offers a more physically meaningful solution.

6 References

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