Local flux mimetic finite difference methods

Konstantin Lipnikov** Mikhail Shashkov* Ivan Yotov†

November 4, 2005

Abstract

We develop a local flux mimetic finite difference method for second order elliptic equations with full tensor coefficients on polyhedral grids. To approximate the flux (vector variable), the method uses two degrees of freedom per element edge in two dimensions and n degrees of freedom per (n-gon) element face in three dimensions. To approximate the pressure (scalar variable), the method uses one degree of freedom per element. A specially chosen inner product in the space of discrete fluxes allows for local flux elimination and reduction of the method to a symmetric cell-centered finite difference scheme for the pressure. In the case of simplicial grids, optimal first-order convergence is proved for both variables, as well as second-order convergence for the scalar variable. Numerical results confirm the theory.

Keywords: mimetic finite differences, multipoint flux approximation, cell centered discretization, tensor coefficient, error estimates

AMS Subject Classification: 65N06, 65N12, 65N15, 65N30

1 Introduction

The mimetic finite difference (MFD) method has been successfully employed for solving problems of continuum mechanics [19], electromagnetics [13], gas dynamics [7], and linear diffusion on polygonal and polyhedral meshes in both the Cartesian and polar coordinates [14, 20, 18]. The MFD method mimics essential properties of the continuum equations, such as conservation laws, solution symmetries, and the fundamental identities and theorems of vector and tensor calculus. For second-order elliptic problems, the MFD method mimics the Gauss divergence theorem, preserves the null space of the gradient operator,

*Corresponding author

[1]Mathematical Modeling and Analysis Group, Theoretical Division, Mail Stop B284, Los Alamos National Laboratory, Los Alamos, NM 87545, Email: {lipnikov, shashkov}@lanl.gov

[2]Department of Mathematics, 301 Thackeray Hall, University of Pittsburgh, Pittsburgh, PA 15260, Email: yotov@math.pitt.edu

and keeps the adjoint relationship between the gradient and the divergence operators. This leads to a symmetric and locally conservative finite difference scheme. However, the resulting algebraic system is of saddle-point type and couples the flux (vector variable) and the pressure (scalar variable) unknowns. The elimination of the flux results in a cell-centered discretization scheme with a non-local stencil.

In this paper, we develop a new MFD method which results in a *symmetric* cell-centered discretization scheme with a *local* stencil. To approximate the flux, the method uses two degrees of freedom per element edge in two dimensions and n degrees of freedom per element face (which is n-gon) in three dimensions, thus associating one flux unknown with each vertex (corner). To approximate the pressure, the method uses one degree of freedom per element. These choices are similar to the degrees of freedom in the multipoint flux approximation (MPFA) method [2, 1, 9]. A specially chosen flux inner product couples only the flux degrees of freedom associated with each mesh vertex and allows for *local* flux elimination, reducing the method to a symmetric cell-centered finite difference scheme for the pressure.

In the case of simplicial meshes, we prove optimal first-order convergence for both the flux and the pressure variables, as well as superconvergence of the pressure in discrete L_2 norms. Our analysis can be extended to smooth quadrilateral meshes. Recent results [15, 16, 22] provide analysis for the MPFA method and some related mixed finite element methods by employing finite element techniques. Our approach is based on estimating the errors directly in the norms of the discrete mimetic spaces and does not utilize finite element polynomial extensions, except in the pressure superconvergence proof.

The paper outline is as follows. The new MFD method is developed in Section 2. In Section 3, we prove optimal convergence estimates for the pressure and the velocity and superconvergence for the pressure. Results of numerical experiments confirming the theoretical estimates are presented in Section 4.

2 Mimetic finite difference method

Let X_1 and X_2 be Hilbert spaces and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two linear operators, $\mathcal{L}_i\colon X_i \to Y_i$, $i = 1, 2$, which satisfy some fundamental identity:

$$\mathcal{I}(\mathcal{L}_1, \mathcal{L}_2; f_1, f_2) = 0 \qquad \forall f_1 \in X_1,\ f_2 \in X_2.$$

Suppose that discrete approximation spaces X_{ih}, Y_{ih}, $i = 1, 2$, and the discrete operator $\mathcal{L}_{1h}$ are given. The idea of the mimetic discretization is to find a discrete operator $\mathcal{L}_{2h}$ such that a discrete analog of the fundamental identity holds, i.e

$$\mathcal{I}_h(\mathcal{L}_{1,h}, \mathcal{L}_{2,h}; f_{1h}, f_{2h}) = 0 \qquad \forall f_{1h} \in X_{1h}, f_{1h} \in X_{2h}. \tag{2.1}$$

This implies that operators $\mathcal{L}_1$ and $\mathcal{L}_2$ cannot be discretized independently from each other. In the MFD method, formula (2.1) is the implicit definition of the operator $\mathcal{L}_{2,h}$.

We consider the second order elliptic problem written as a system of two first order equations

$$
\begin{aligned}
\vec{u} &= -K \operatorname{grad} p \qquad \text{in} \quad \Omega, \\
\operatorname{div} \vec{u} &= f \qquad\qquad\quad \text{in} \quad \Omega,
\end{aligned}
\tag{2.2}
$$

subject to appropriate boundary conditions. For simplicity, we consider the homogeneous Dirichlet boundary conditions (see [12] for more general boundary conditions):

$$
p = 0 \quad \text{on} \quad \partial\Omega. \tag{2.3}
$$

We consider a polygonal domain $\Omega \subset \mathbf{R}^d$, $d = 2$ or 3, with boundary $\partial\Omega$ and outward unit normal $\vec{n}$. The coefficient K is a symmetric and uniformly positive definite tensor satisfying

$$
k_0 \xi^T \xi \le \xi^T K(x) \xi \le k_1 \xi^T \xi \qquad \forall x \in \Omega,\ \forall \xi \in \mathbf{R}^d, \tag{2.4}
$$

for some positive constants k_0 and k_1. Following the terminology established in porous media applications, we refer to p as the pressure, to $\vec{u}$ as the velocity, and to K as the permeability tensor.

In the problem of interest (2.2), the operators are $\mathcal{L}_1 = \operatorname{div}$ and $\mathcal{L}_2 = -K \operatorname{grad}$, the spaces are $X_1 = H(\operatorname{div};\Omega)$, $Y_1 = L_2(\Omega)$, $X_2 = H_0^1(\Omega)$ and $Y_2 = (L_2(\Omega))^d$, and $\mathcal{I}$ is the Green formula,

$$
\mathcal{I}(\mathcal{L}_1, \mathcal{L}_2; \vec{u}, p) = \int_\Omega p \operatorname{div} \vec{u} \,\mathrm{d}x + \int_\Omega \vec{u} \cdot K^{-1}(K \operatorname{grad} p) \,\mathrm{d}x. \tag{2.5}
$$

2.1 The local flux MFD method

The MFD method has four steps. First, we define degrees of freedom for the pressure and the velocity. Second, we discretize the easiest of the two operators; depending on the chosen degrees of freedom, it could be either of them. Third, we discretize the Green formula using quadrature rules for each of the integrals in (2.5). Some minimal approximation properties for these quadratures are required to prove the optimal convergence rates. Fourth, we derive a discrete formula for the other operator.

Let Ω_h be a conforming shape-regular partition (see [8]) of the computational domain into polygonal elements. Let

$$
h = \max_{E \in \Omega_h} h_E,
$$

where h_E is the diameter of element E. We assume that each vertex of E is shared by exactly d edges (faces in 3D) of that element. In two dimensions, we split each edge into two *sub-edges* using the mid-point. In three dimensions, we split each face into several quadrilateral *facets* by connecting the face center of mass with the edge midpoints. To simplify the presentation, we shall refer to the sub-edges as facets. We denote the area (volume in 3D) of an element E by $|E|$. Similarly, for each facet e, we denote by $|e|$ its length (area in 3D).

For each element E, we denote by n_E the number of its vertices and by k_E the number of its facets. The boundaries of facets are marked by thin lines in Fig. 1. In the following ∂E denotes either the union of all edges (faces in 3D) or the union of all facets of E, depending on the context.

The discrete pressure space Q_h consists of one degree of freedom per element corresponding to the pressure value at the center of mass. The dimension of Q_h equals the number of elements. For $\mathbf{q} \in Q_h$, we shall denote by q_E (or $(\mathbf{q})_E$) its (constant) value on element E.

The discrete velocity space X_h consists of one degree of freedom per facet, which corresponds to the average normal flux. The location of velocity degrees of freedom is shown in Fig. 1. For $\mathbf{v} \in X_h$, we shall denote by $\mathbf{v}_E$ the restriction of $\mathbf{v}$ to element E, and by v_E^e (or $(\mathbf{v})_E^e$) its (constant) value on facet e. The total number, N_X, of the velocity degrees of freedom equals the number of boundary facets plus twice the number of interior facets. We define X_h as the subspace of $\mathbf{R}^{N_X}$ which satisfies the *continuity* property

$$v_{E_1}^e = -v_{E_2}^e \tag{2.6}$$

for each facet e shared by elements E_1 and E_2. Note that the dimension of X_h equals the number of facets.

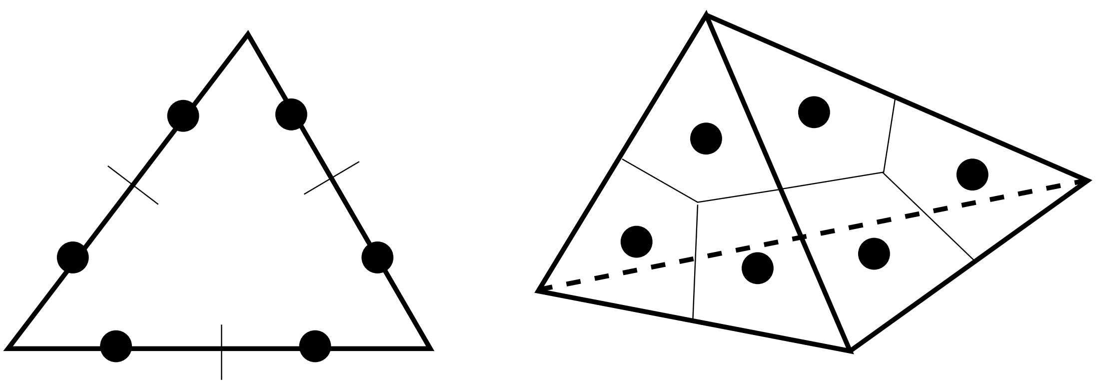


Figure 1: Velocity degrees of freedom marked by solid circles for a triangle ($n_E = 3$, $k_E = 6$) and a tetrahedron ($n_E = 4$, $k_E = 12$). The boundaries of the facets are marked by thin lines.

The normal velocity components result in a simple discretization of the divergence operator. Integrating $\operatorname{div} \vec{u}$ over element E, applying the divergence theorem, and using the definition of discrete velocity unknowns, we get

$$(\mathcal{DIV}\,\mathbf{u})_E = \frac{1}{|E|} \sum_{e \in \partial E} |e|\, u_E^e. \tag{2.7}$$

Let us discretize each term in (2.5). For any $q \in L^1(\Omega)$, we define $q^I \in Q_h$ such that

$$(q^I)_E = \frac{1}{|E|} \int_E q(x)\,\mathrm{d}x \qquad \forall E \in \Omega_h. \tag{2.8}$$

The first integral in (2.5) is approximated by

$$\int_\Omega p(x)\, q(x)\, \mathrm{d}x \approx \sum_{E\in\Omega_h} |E|\, p_E\, q_E \equiv [\mathbf{p},\, \mathbf{q}]_Q, \tag{2.9}$$

where $\mathbf{p} = p^I$ and $\mathbf{q} = q^I$. Note that (2.9) is an inner product in Q_h.

The discretization of the second integral in (2.5) requires some additional notation. Given $\mathbf{v} \in X_h$, let $\vec{v}_E(\mathbf{r}_s) \in \mathbf{R}^d$ be a vector associated with vertex $\mathbf{r}_s$ of E such that its normal component on any facet e that shares $\mathbf{r}_s$ is equal to v_E^e. Since each vertex is shared by exactly d facets, then the vector $\vec{v}_E(\mathbf{r}_s)$ is uniquely determined. We refer to $\vec{v}_E(\mathbf{r}_s)$ as the *recovered* vector. The expression for $\vec{v}_E(\mathbf{r}_i)$ can be found in [18].

We approximate K by a symmetric and positive definite piecewise constant tensor $\overline{K}$ that equals the mean value K_E of K on E. The Taylor theorem implies that

$$\max_{\mathbf{x}\in E} |K_{ij}(\mathbf{x}) - K_{E,ij}| \le C\|K_{ij}\|_{1,\infty,E}\, h_E, \qquad 1 \le i,j \le d, \tag{2.10}$$

where $\|\cdot\|_{1,\infty}$ is the norm in the Sobolev space W^1_∞. We will also use the notation

$$\|K\|_\sigma = \max_{i,j} \|K_{ij}\|_\sigma$$

for tensor valued functions. In (2.10) and throughout the paper C denotes a generic positive constant, which is independent of h. Now, the second integral in (2.5) is approximated element-by-element by

$$\int_\Omega K^{-1}\vec{u}(x)\cdot\vec{v}(x)\, \mathrm{d}x \approx \sum_{E\in\Omega_h} [\mathbf{u},\, \mathbf{v}]_{X,E} \equiv [\mathbf{u},\, \mathbf{v}]_X, \tag{2.11}$$

where

$$[\mathbf{u},\, \mathbf{v}]_{X,E} = \gamma_E \sum_{i=1}^{n_E} w_i K_E^{-1} \vec{u}_E(\mathbf{r}_i)\cdot\vec{v}_E(\mathbf{r}_i), \qquad \gamma_E^{-1} = \frac{1}{|E|}\sum_{i=1}^{n_E} w_i, \tag{2.12}$$

w_i are some positive weights, and $\vec{u}_E(\mathbf{r}_i)$ and $\vec{v}_E(\mathbf{r}_i)$ are the recovered vectors. For simplicial elements, the weights are equal to $|E|/(d+1)$ and $\gamma_E = 1$. Later, we will show (see Lemma 2.2) that (2.11) is an inner product in X_h.

The discrete gradient operator is derived from the discrete Green formula

$$[\mathbf{q},\, \mathcal{DIV}\,\mathbf{v}]_Q + [\mathbf{v},\, \mathcal{GRAD}\,\mathbf{q}]_X = 0, \qquad \forall\, \mathbf{q} \in Q_h,\ \forall\, \mathbf{v} \in X_h. \tag{2.13}$$

This formula gives a unique definition for operator $\mathcal{GRAD}$. The local flux MFD method reads: find $\mathbf{u}_h \in X_h$ and $\mathbf{p}_h \in Q_h$ such that

$$\begin{aligned} \mathbf{u}_h &= -\mathcal{GRAD}\,\mathbf{p}_h, \\ \mathcal{DIV}\,\mathbf{u}_h &= \mathbf{f}, \end{aligned} \tag{2.14}$$

where $\mathbf{f} = f^I$.

2.2 Well-posedness of the method

The following interpolant will be used in the analysis. For any $\vec{v} \in (L^s(\Omega))^d$, $s > 2$, we define $\vec{v}^I \in X_h$ such that

$$(\vec{v}^I)^e_E = \frac{1}{|e|} \int_e \vec{v} \cdot \vec{n}^e_E \, \mathrm{d}s \qquad \forall E \in \Omega_h, \quad \forall e \subset \partial E. \tag{2.15}$$

Note that $\vec{v}^I$ satisfies the continuity property (2.6).

The definitions (2.8) and (2.15) of the interpolation operators and the divergence theorem imply the following simple result.

Lemma 2.1 *For sufficiently smooth vector functions $\vec{v}$, we have*

$$(\mathcal{DIV}\, \vec{v}^I)_E = (\mathrm{div}\ \vec{v})^I_E \tag{2.16}$$

for every element $E \in \Omega_h$.

The next lemma shows that $[\cdot, \cdot]_X$ is a norm in X_h.

Lemma 2.2 *There exist two positive constants α_0 and α_1 independent of h such that*

$$\forall E \in \Omega_h, \quad \alpha_0 |E| \sum_{e \in \partial E} |v^e_E|^2 \le [\mathbf{v}, \mathbf{v}]_{X,E} \le \alpha_1 |E| \sum_{e \in \partial E} |v^e_E|^2 \tag{2.17}$$

for any $\mathbf{v} \in X_h$.

Proof. For any element E and its vertex $\mathbf{r}_i$, let $e_{i,j}$, $j = 1, \ldots, d$, be the facets that share $\mathbf{r}_i$ and let $\vec{v}(\mathbf{r}_i)$ be the recovered vector. Furthermore, let $\vec{n}_{i,j}$ be the outward normal to $e_{i,j}$. It is easy to see that $\vec{v}(\mathbf{r}_i) = N_i^{-T}(v_E^{e_{i,j}}, \ldots, v_E^{e_{i,d}})^T$ where N_i is the $d \times d$ matrix whose columns are the normals $\vec{n}_{i,j}$ (see [18] for more detail).

The definition (2.12) implies that

$$\alpha_0 = k_0 \min_{1 \le i \le n_E} \lambda_{min}(N_i^{-1} N_i^{-T}).$$

A similar estimate holds for α_1. The spectral properties of the matrix N_i depend only on the mesh regularity constants. This proves the assertion of the lemma. □

We are now ready to prove the solvability of (2.14).

Lemma 2.3 *The discrete problem (2.14) has a unique solution.*

Proof. It is convenient to rewrite (2.14) in the equivalent variational form

$$\begin{aligned} [\mathbf{u}_h, \mathbf{v}]_X - [\mathbf{p}_h, \mathcal{DIV}\, \mathbf{v}]_Q &= 0, & \forall\, \mathbf{v} \in X_h, \\ [\mathcal{DIV}\, \mathbf{u}_h, \mathbf{q}]_Q &= [\mathbf{f}, \mathbf{q}]_Q, & \forall\, \mathbf{q} \in Q_h, \end{aligned} \tag{2.18}$$

where we have used the discrete Green formula (2.13). Since (2.18) is a square system, it suffices to show uniqueness for the homogeneous problem. Letting $\mathbf{f} = 0$, $\mathbf{v} = \mathbf{u}_h$, and $\mathbf{q} = \mathbf{p}_h$, we conclude that $[\mathbf{u}_h, \mathbf{u}_h]_X = 0$. Hence, due to (2.17), $\mathbf{u}_h = 0$. Let p_h be a piecewise constant function such that $p_h|_E = (\mathbf{p}_h)_E$.

Let us consider again (2.18) and take $\mathbf{v} = (\,\mathrm{grad}\,\phi)^I$, where ϕ is the solution to

$$\begin{aligned} \Delta\phi = p_h \qquad &\text{in} \quad \Omega, \\ \phi = 0 \qquad &\text{on} \quad \partial\Omega. \end{aligned}$$

Using (2.16), we have that

$$\mathcal{DIV}\,\mathbf{v} = \mathcal{DIV}\,(\,\mathrm{grad}\,\phi)^I = (\mathrm{div}\ \mathrm{grad}\,\phi)^I = (p_h)^I = \mathbf{p}_h,$$

which implies that $[\mathbf{p}_h, \mathbf{p}_h]_Q = 0$, therefore $\mathbf{p}_h = 0$. □

2.3 Reduction to a cell-centered scheme

In order to derive the explicit formula for $\mathcal{GRAD}$, we consider an auxiliary inner product $<\cdot\,,\,\cdot>$ and relate it to inner products (2.9) and (2.11). Let $<\cdot\,,\,\cdot>$ be the usual vector dot product. Then

$$[\mathbf{p},\,\mathbf{q}]_Q =< D\,\mathbf{p},\,\mathbf{q} > \qquad \text{and} \qquad [\mathbf{u},\,\mathbf{v}]_X =< M\,\mathbf{u},\,\mathbf{v} >, \tag{2.19}$$

where D is a diagonal matrix, $D = \mathrm{diag}\{|E_1|,\ldots,|E_{N_Q}|\}$, and M is a block-diagonal matrix. Since $[\cdot,\cdot]_X$ is an inner product, M is symmetric and positive definite.

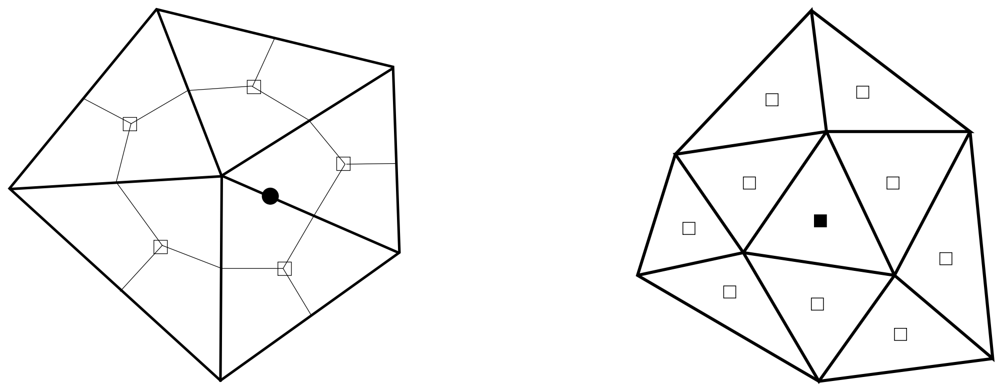
To incorporate the continuity conditions, we write $\mathbf{u} = C\hat{\mathbf{u}}$ where C is the rectangular matrix with one non-zero element in each row (which equals to 1) and the entries of vector $\hat{\mathbf{u}}$ are independent degrees of freedom. Thus, the size of vector $\hat{\mathbf{u}}$ equals to the number of mesh faces. Similarly, we write $\mathbf{v} = C\hat{\mathbf{v}}$. The matrix $C^T MC$ is also block diagonal with as many blocks as there are mesh nodes. Thus, the discrete Green formula yields

$$\mathcal{GRAD} = -(C^T MC)^{-1}(\mathcal{DIV}\,C)^T D.$$

In two dimensions, each block of $C^T MC$ is a tridiagonal cyclic matrix whose non-zero entries describe interaction of neighboring velocity unknowns on edges sharing a mesh node. The block corresponding to the interior node shown in Fig. 2 is a 5×5 matrix. Therefore, the inverse of this block can be easily computed which gives us an explicit local formula for each component of $\mathbf{u}_h$ and thus reduces (2.14) to a cell-centered discretization

$$-\mathcal{DIV}\ C\,\mathcal{GRAD}\,\mathbf{p}_h = \mathbf{f}. \tag{2.20}$$

Examples of the stencils for the operators $\mathcal{GRAD}$ and $\mathcal{DIV}\ C\,\mathcal{GRAD}$ are shown in Figure 2(a) and Figure 2(b), respectively.



a. $\mathcal{GRAD}$ stencil   b. $\mathcal{DIV}\,\mathcal{GRAD}$ stencil

Figure 2: Stencils for operators $\mathcal{GRAD}$ and $\mathcal{DIV}\,\mathcal{GRAD}$ on a triangular mesh. On the left, the equation for the velocity unknown at the position marked by a solid circle involves pressure unknowns at the positions marked by squares. On the right, the pressure marked by a solid square is coupled with the pressures marked by squares.

The coefficient matrix of problem (2.20) is symmetric with respect to inner product (2.9):

$$[-\mathcal{DIV}\; C\,\mathcal{GRAD}\,\mathbf{p},\mathbf{q}]_Q = < D\,\mathcal{DIV}\,C(C^TMC)^{-1}(\mathcal{DIV}\,C)^T D\,\mathbf{p},\mathbf{q} > .$$

Moreover, since $\mathcal{DIV}^T\mathbf{q} = 0$ implies $\mathbf{q} = 0$, as shown in the proof of Lemma 2.3, the resulting algebraic system has a symmetric and positive definite matrix.

3 Convergence analysis

In this section, we prove convergence estimates for the velocity and pressure in the case of simplicial meshes ($n_E = d + 1$).

We begin with the proof of a discrete Green formula for linear functions. In two dimensions, for each edge with end points $\mathbf{a}_1$ and $\mathbf{a}_2$, we define two new points

$$\mathbf{a}_{12} = (2\mathbf{a}_1 + \mathbf{a}_2)/3 \qquad \text{and} \qquad \mathbf{a}_{21} = (\mathbf{a}_1 + 2\mathbf{a}_2)/3$$

which are interior points of the two facets, see Figure 3(a). In three dimensions, for each face (with is a triangle) with vertices $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, we define three new points

$$\mathbf{a}_{123} = (2\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)/4, \qquad \mathbf{a}_{231} = (\mathbf{a}_1+2\mathbf{a}_2+\mathbf{a}_3)/4, \quad \text{and} \quad \mathbf{a}_{312} = (\mathbf{a}_1+\mathbf{a}_2+2\mathbf{a}_3)/4$$

which are interior points of the three facets, see Figure 3(b). Note that d new points are the projections of the center of mass, $\mathbf{c}_E$, onto the edge (the face in 3D) along directions parallel to the other d edges. We use notation $\mathbf{c}_e$ for the new point inside facet e.

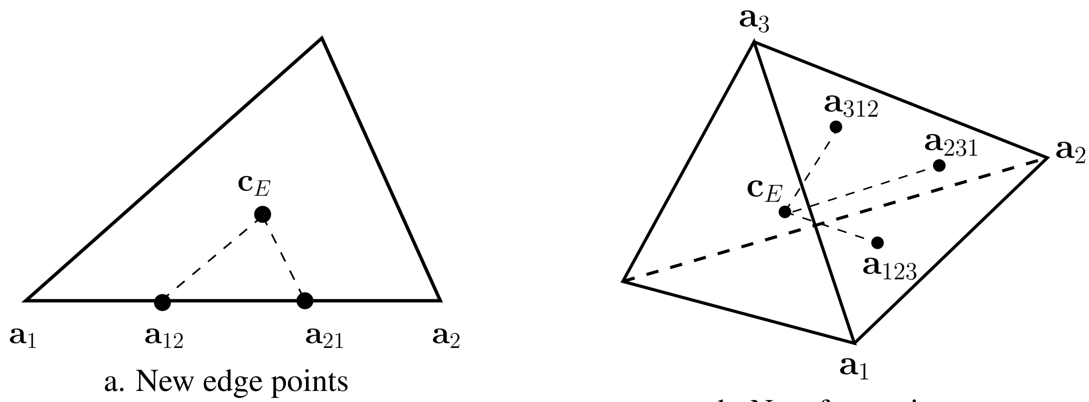


Figure 3: Auxiliary edge and face points.

Lemma 3.1 *For every E in Ω_h, the inner product (2.12) satisfies*

$$[\mathbf{v},\, (K_E\,\mathrm{grad}\,q^1)^I]_{X,E} = \sum_{e\in\partial E} |e|\, q^1(\mathbf{c}_e)\, v_E^e - [\mathcal{DIV}\,\mathbf{v},\, (q^1)^I]_{Q,E}, \qquad \forall\,\mathbf{v}\in X_h, \quad (3.1)$$

for any linear function q^1.

Proof. Let M_E be the symmetric positive definite $k_E \times k_E$ matrix defined by the inner product $[\cdot,\cdot]_{X,E}$, see (2.19). Since the vectors recovered at different vertices use separate degrees of freedom, the matrix M_E is block-diagonal with $d+1$ blocks and each block is a $d\times d$ matrix. The result of this special structure of M_E is that the proof of (3.1) is reduced to proving $d+1$ independent identities associated with the vertices of E.

Let $\mathbf{r}$ be a vertex of E and let e_i, $i = 1,\ldots,d$, be the facets that share $\mathbf{r}$. Furthermore, let $\vec{n}_i$, be the outward normal to e_i and let $\vec{v}$ be the vector recovered at vertex $\mathbf{r}$. Since the constant vector is recovered exactly, (3.1) reduces to

$$\frac{|E|}{d+1}(K_E^{-1}\vec{v})\cdot(K_E\,\mathrm{grad}\,q^1) = \sum_{i=1}^{d} |e_i|\,(q^1(\mathbf{c}_{e_i}) - q^1(\mathbf{c}_E))\, v_E^{e_i}, \quad (3.2)$$

where $\mathbf{c}_E$ is the center of mass of E. Since $\vec{v} = N^{-T}(v_E^{e_1},\ldots,v_E^{e_d})^T$, where N is the $d\times d$ matrix with columns $\vec{n}_i$, (3.2) is equivalent to

$$\frac{|E|}{d+1}\,\mathrm{grad}\,q^1 = \sum_{i=1}^{d} |e_i|\,\vec{n}_i\, q^1(\mathbf{c}_{e_i} - \mathbf{c}_E) \quad (3.3)$$

To prove (3.3), it is sufficient to check that

$$\frac{|E|}{d+1}\,\mathrm{grad}\,q^1\cdot\vec{w} = \sum_{i=1}^{d} |e_i|\,(\vec{w}\cdot\vec{n}_i)\, q^1(\mathbf{c}_{e_i} - \mathbf{c}_E), \qquad \forall\vec{w}\in\mathbf{R}^d. \quad (3.4)$$

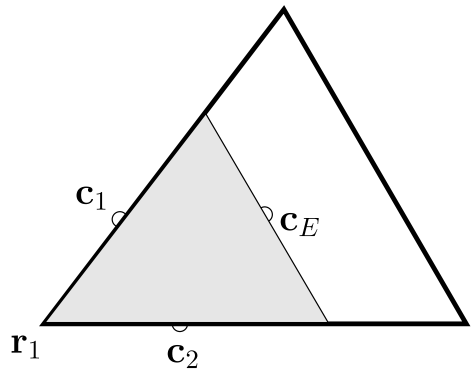


Figure 4: The congruent triangles E and $\hat{E}$ (shaded).

Let us consider the triangular element E shown in Fig. 4. The shaded triangle $\hat{E}$ is congruent to E and $|\hat{E}| = d/(d+1)|E|$. The points $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_E$ are the mid-points of the edges of $\hat{E}$. Since the midpoint rule is exact for linear functions, the right hand side of (3.4) is

$$\sum_{i=1}^{d} |e_i| \, (\vec{w} \cdot \vec{n}_i) \, q^1(\mathbf{c}_{e_i} - \mathbf{c}_E) = \frac{1}{d} \int_{\partial \hat{E}} (\vec{w} \cdot \vec{n}_{\hat{E}}) \, q^1(\mathbf{s} - \mathbf{c}_E) \, \mathrm{d}s \tag{3.5}$$

Using the Green formula, we get

$$\frac{1}{d} \int_{\partial \hat{E}} (\vec{w} \cdot \vec{n}_{\hat{E}}) \, q^1(\mathbf{s} - \mathbf{c}_E) \, \mathrm{d}s = \frac{1}{d} \int_{\hat{E}} \vec{w} \cdot \operatorname{grad} q^1 \, \mathrm{d}x = \frac{|E|}{d+1} \, \vec{w} \cdot \operatorname{grad} q^1. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain (3.4). The same argument proves (3.4) in the case of tetrahedral elements. $\square$

We will also use repeatedly the following approximation result [4, Lemma 4.3.8]. For every element E, if $\phi \in W_p^{m+1}$, $p \geq 1$, there exists ϕ^m, a polynomial of degree at most m, such that

$$|\phi - \phi^m|_{W_p^k(E)} \leq C h^{m+1-k} |\phi|_{W_p^{m+1}(E)}, \quad k = 0, \ldots, m+1. \tag{3.7}$$

In particular, there exists a linear function p_E^1 such that

$$\|p - p_E^1\|_{L^2(E)} \leq \, C \, h_E^2 \, \|p\|_{H^2(E)}, \qquad \|p - p_E^1\|_{H^1(E)} \leq \, C \, h_E \, \|p\|_{H^2(E)}. \tag{3.8}$$

For the error on the edges (faces in 3D), we have [3]

$$\|\chi\|^2_{L^2(\tilde{e})} \leq \, C \left(h_E^{-1} \|\chi\|^2_{L^2(E)} + h_E \, |\chi|^2_{H^1(E)} \right), \qquad \forall \chi \in H^1(E), \tag{3.9}$$

where $\tilde{e}$ is any edge (face) of E. The constant C in (3.8) and (3.9) depends only on the shape-regularity constants of E. Applying (3.9) to the difference $p - p_E^1$ and using (3.8), we have

$$\|p - p_E^1\|^2_{L^2(\tilde{e})} + h_E^2 \|\nabla(p - p_E^1)\|^2_{L^2(\tilde{e})} \leq \, C \, h_E^3 \, \|p\|^2_{H^2(E)}. \tag{3.10}$$

It is obvious that a similar estimate holds for any facet e of E.

3.1 Optimal velocity estimate

We are now ready to prove optimal error estimates for both the scalar and vector variables. These estimates are derived for the mesh dependent norms induced by the inner products:

$$|||\mathbf{q}|||_Q = [\mathbf{q},\, \mathbf{q}]_Q^{1/2} \qquad \text{and} \qquad |||\mathbf{v}|||_X = [\mathbf{v},\, \mathbf{v}]_X^{1/2}.$$

Theorem 3.1 *For the solutions $(p,\, \vec{u})$ and $(\mathbf{p}_h,\, \mathbf{u}_h)$ of problems (2.2) and (2.14), respectively, there exists a constant C independent of h such that*

$$|||\vec{u}^I - \mathbf{u}_h|||_X \leq C\, h\, \|p\|_{H^2(\Omega)}.$$

Proof. Let $\mathbf{v} \in X_h$ be such that $\mathcal{DIV}\,\mathbf{v} = 0$. Then, using the discrete Green formula (2.13), we get

$$[\vec{u}^I - \mathbf{u}_h,\, \mathbf{v}]_X = [(K\,\mathrm{grad}\, p)^I,\, \mathbf{v}]_X + [\mathcal{GRAD}\,\mathbf{p}_h,\, \mathbf{v}]_X = [(K\,\mathrm{grad}\, p)^I,\, \mathbf{v}]_X.$$

Let p^1 be a discontinuous piecewise linear function satisfying (3.8) on every element E. Adding and subtracting terms $(K\,\mathrm{grad}\, p^1)^I$ and $(\overline{K}\,\mathrm{grad}\, p^1)^I$, where $\overline{K}$ is the piecewise constant approximation to K defined in Section 2 and satisfying (2.10), we have

$$\begin{aligned}[\vec{u}^I - \mathbf{u}_h,\, \mathbf{v}]_X &= [(K\,\mathrm{grad}\, p)^I - (K\,\mathrm{grad}\, p^1)^I,\, \mathbf{v}]_X + [(K\,\mathrm{grad}\, p^1)^I - (\overline{K}\,\mathrm{grad}\, p^1)^I,\, \mathbf{v}]_X \\ &+ [(\overline{K}\,\mathrm{grad}\, p^1)^I,\, \mathbf{v}]_X \equiv \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.\end{aligned}$$

Terms similar to $\mathcal{I}_1$ and $\mathcal{I}_2$ appear in [6]. Using the Cauchy-Schwarz inequality, we bound $\mathcal{I}_1$ as

$$\begin{aligned}|\mathcal{I}_1| &\leq |||(K\,\mathrm{grad}\, p - K\,\mathrm{grad}\, p^1)^I|||_X |||\mathbf{v}|||_X \\ &\leq \left(\alpha_1 \sum_{E\in\Omega_h} \sum_{e\in\partial E} \left(((K\,\mathrm{grad}\, p - K\,\mathrm{grad}\, p^1)^I)_E^e\right)^2 |E|\right)^{1/2} |||\mathbf{v}|||_X \\ &= \left(\alpha_1 \sum_{E\in\Omega_h} \sum_{e\in\partial E} \left(\frac{1}{|e|}\int_e K\,\mathrm{grad}\,(p - p^1)\cdot \vec{n}_E^e\,\mathrm{d}s\right)^2 |E|\right)^{1/2} |||\mathbf{v}|||_X \qquad (3.11) \\ &\leq Ch\|p\|_{H^2(\Omega)} |||\mathbf{v}|||_X,\end{aligned}$$

where we have used (2.17) in the second inequality and (3.10) in the last inequality. For term $\mathcal{I}_2$, using (2.10), we have

$$|\mathcal{I}_2| \leq Ch|||(\,\mathrm{grad}\, p^1)^I|||_X |||\mathbf{v}|||_X. \qquad (3.12)$$

Since the inner product (2.12) is exact for constant vectors, we get

$$|||(\,\mathrm{grad}\, p^1)^I|||_{X,E} = \|\,\mathrm{grad}\, p_E^1\|_{L^2(E)} \leq \|\,\mathrm{grad}\, p\|_{L^2(E)} + \|\,\mathrm{grad}\,(p - p_E^1)\|_{L^2(E)} \leq C\|p\|_{H^2(E)},$$

using (3.8). The above inequality and (3.12) imply that

$$|\mathcal{I}_2| \le Ch\|p\|_{H^2(\Omega)} |\!|\!| \mathbf{v} |\!|\!|_X. \tag{3.13}$$

To estimate the remaining term, we apply Lemma 3.1 and use $\mathcal{DIV}\,\mathbf{v} = 0$ to obtain

$$\mathcal{I}_3 = \sum_{E\in\Omega_h} \sum_{e\in\partial E} |e|\, p_E^1(\mathbf{c}_e)\, v_E^e.$$

Recall that $\mathbf{c}_e$ is the mid-point of one of the edges (faces in 3D) of the shadow element $\hat{E}$ (see Figure 4). Denoting the corresponding edge (face) by $\hat{e}(e)$, we get

$$p_E^1(\mathbf{c}_e) = \frac{1}{|\hat{e}(e)|} \int_{\hat{e}(e)} p_E^1(\mathbf{s})\,\mathrm{d}s.$$

Using the continuity of p and the approximation result (3.10), we have

$$\begin{aligned}
|\mathcal{I}_3| &= \left| \sum_{E\in\Omega_h} \sum_{e\in\partial E} v_E^e \frac{|e|}{|\hat{e}(e)|} \int_{\hat{e}(e)} (p_E^1 - p)\,\mathrm{d}s \right| \\
&\le C \sum_{E\in\Omega_h} \sum_{e\in\partial E} |e|^{1/2} |v_E^e|\, \|p_E^1 - p\|_{L_2(\hat{e}(e))} \\
&\le C \sum_{E\in\Omega_h} h_E \left(|E| \sum_{e\in\partial E} |v_E^e|^2 \right)^{1/2} \|p\|_{H^2(E)} \quad \le C\, h\, \|p\|_{H^2(\Omega)}\, |\!|\!| \mathbf{v} |\!|\!|_X
\end{aligned} \tag{3.14}$$

We next note that Lemma 2.1 implies that

$$\mathcal{DIV}\,(\vec{u}^I - \mathbf{u}_h) = f^I - f^I = 0;$$

hence we can take $\mathbf{v} = \vec{u}^I - \mathbf{u}_h$ in the above estimates. Combining estimates for $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$, we prove the assertion of the theorem. □

3.2 Optimal pressure estimate

To prove optimal convergence for the pressure variable, we first show that an *inf-sup* condition holds. Let us define the mesh dependent H_{div} norm:

$$|\!|\!| \mathbf{v} |\!|\!|_{div}^2 = |\!|\!| \mathbf{v} |\!|\!|_X^2 + |\!|\!| \mathcal{DIV}\,\mathbf{v} |\!|\!|_Q^2.$$

Lemma 3.2 *There exist a positive constant β independent of h such that for any $\mathbf{q} \in Q_h$*

$$\sup_{\mathbf{v}\in X_h,\ \mathbf{v}\neq 0} \frac{[\mathcal{DIV}\,\mathbf{v},\, \mathbf{q}]_Q}{|\!|\!| \mathbf{v} |\!|\!|_{div}} \ge \beta |\!|\!| \mathbf{q} |\!|\!|_Q. \tag{3.15}$$

Proof. Let $\mathbf{q} \in Q_h$ and let q be the piecewise-constant function which is equal to $(\mathbf{q})_E$ on E. We will construct $\vec{v} \in (H^1(\Omega))^d$ such that $\operatorname{div} \vec{v} = q$ and

$$\|\vec{v}\|_{(H^1(\Omega))^d} \leq C_1 \|q\|_{L^2(\Omega)}, \tag{3.16}$$

where C_1 is a positive constant independent of h. Let q_0 be the integral average of q,

$$q_0 = \frac{1}{|\Omega|} \int_\Omega q \, \mathrm{d}x.$$

We define $\vec{v} = \vec{v}_1 + \vec{v}_2$ where $\vec{v}_1$ is a solution to

$$\begin{aligned} \operatorname{div} \vec{v}_1 &= q - q_0 \quad \text{in} \quad \Omega, \\ \vec{v}_1 &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

and $\vec{v}_2$ is a solution to

$$\begin{aligned} \operatorname{div} \vec{v}_2 &= q_0 \quad \text{in} \quad \Omega, \\ \vec{v}_2 &= \vec{g} \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\vec{g} \in (H^{1/2}(\partial\Omega))^d$ and satisfies the compatibility condition

$$\int_{\partial\Omega} \vec{g} \cdot \vec{n} \, \mathrm{d}s = q_0 |\Omega|.$$

The above problems are known to have solutions [10] satisfying

$$\|\vec{v}_1\|_{(H^1(\Omega))^d} \leq C \|q\|_{L^2(\Omega)} \quad \text{and} \quad \|\vec{v}_2\|_{(H^1(\Omega))^d} \leq C \left(\|q_0\|_{L^2(\Omega)} + \|\vec{g}\|_{(H^{1/2}(\partial\Omega))^d} \right).$$

We choose $\vec{g} = |\Omega| q_0 \phi \vec{n}$, where ϕ is a smooth function with support contained within one side of Ω such that $\int_{\partial\Omega} \phi \, \mathrm{d}s = 1$. It is easy to see that $\|\vec{g}\|_{(H^{1/2}(\partial\Omega))^d} \leq C \|q_0\|_{L^2(\Omega)}$; therefore $\vec{v}$ satisfies (3.16).

Let $\mathbf{v} = \vec{v}^I$. Using Lemma 2.2, inequality (3.9), and the assumption of mesh regularity, we get

$$\begin{aligned} [\mathbf{v}, \mathbf{v}]_{X,E} &\leq \alpha_1 |E| \sum_{e \in \partial E} |v_E^e|^2 \\ &\leq C \sum_{e \in \partial E} \frac{|E|}{|e|} \left((h_E^{-1} \|\vec{v}\|^2_{(L^2(E))^d} + h_E |\vec{v}|^2_{(H^1(E))^d}) \right) \\ &\leq C \sum_{e \in \partial E} \left(\|\vec{v}\|^2_{(L^2(E))^d} + h_E^2 \, |\vec{v}|^2_{(H^1(E))^d} \right) \\ &\leq C_2 \|\vec{v}\|^2_{(H^1(E))^d}. \end{aligned}$$

Therefore, using (3.16),

$$|||\mathbf{v}|||_X^2 \leq C_2 \|\vec{v}\|^2_{(H^1(\Omega))^d} \leq C_1^2 C_2 |||\mathbf{q}|||_Q^2.$$

Further, Lemma 2.1 implies

$$\mathcal{DIV}\,\mathbf{v} = (\mathrm{div}\,\vec{v})^I = q^I = \mathbf{q}.$$

The last two estimates imply that

$$|\!|\!|\mathbf{v}|\!|\!|_{div} \le \sqrt{1 + C_1^2 C_2}\, |\!|\!|\mathbf{q}|\!|\!|_Q,$$

thus the assertion of the lemma follows with $\beta = 1/\sqrt{1 + C_1^2 C_2}$. □

Theorem 3.2 *For the solutions $(p,\, \vec{u})$ and $(\mathbf{p}_h,\, \mathbf{u}_h)$ of problems (2.2) and (2.14), respectively, there exists a constant C independent of h such that*

$$|\!|\!|p^I - \mathbf{p}_h|\!|\!|_Q \le C\, h\, \|p\|_{H^2(\Omega)}.$$

Proof. Using Lemma 3.2, we have

$$|\!|\!|p^I - \mathbf{p}_h|\!|\!|_Q \le \frac{1}{\beta} \sup_{\mathbf{v}\in X_h, \mathbf{v}\neq 0} \frac{[\mathcal{DIV}\,\mathbf{v},\, p^I - \mathbf{p}_h]_Q}{|\!|\!|\mathbf{v}|\!|\!|_{div}} \tag{3.17}$$

To estimate the denominator, we first add and subtract $(p^1)^I$ where p^1 is the discontinuous piecewise linear approximation to p satisfying (3.8), and then apply Lemma 3.1:

$$\begin{aligned}
[\mathcal{DIV}\,\mathbf{v},\, p^I - \mathbf{p}_h]_Q &= [\mathcal{DIV}\,\mathbf{v},\, (p - p^1)^I]_Q + [\mathcal{DIV}\,\mathbf{v},\, (p^1)^I]_Q + [\mathbf{u}_h,\, \mathbf{v}]_X \\
&= [\mathcal{DIV}\,\mathbf{v},\, (p - p^1)^I]_Q + \sum_{E\in\Omega_h} \sum_{e\in\partial E} |e|\, p_E^1(\mathbf{c}_e) v_E^e \\
&\quad - \sum_{E\in\Omega_h} [(K_E\, \mathrm{grad}\, p_E^1)^I,\, \mathbf{v}]_{X,E} + [\mathbf{u}_h,\, \mathbf{v}]_X \\
&\equiv \mathcal{I}_4 + \mathcal{I}_5 - \mathcal{I}_6 + \mathcal{I}_7.
\end{aligned}$$

The term $\mathcal{I}_4$ is estimated using (3.8):

$$|\mathcal{I}_4| \le Ch^2\, |\!|\!|\mathbf{v}|\!|\!|_{div}\, \|p\|_{H^2(\Omega)}. \tag{3.18}$$

The second term is estimated as the similar term in the proof of Theorem 3.1:

$$|\mathcal{I}_5| \le C\, h\, |\!|\!|\mathbf{v}|\!|\!|_X\, \|p\|_{H^2(\Omega)}. \tag{3.19}$$

The last two terms are treated by adding and subtracting $(K\,\mathrm{grad}\, p^1)^I$ and $(K\,\mathrm{grad}\, p)^I$:

$$\begin{aligned}
\mathcal{I}_6 - \mathcal{I}_7 &= [(\overline{K}\,\mathrm{grad}\, p^1)^I - (K\,\mathrm{grad}\, p^1)^I,\, \mathbf{v}]_X \\
&\quad + [(K\,\mathrm{grad}\, p^1)^I - (K\,\mathrm{grad}\, p)^I,\, \mathbf{v}]_X + [\vec{u}^I - \mathbf{u}_h,\, \mathbf{v}]_X \\
&\equiv \mathcal{I}_{67}^a + \mathcal{I}_{67}^b + \mathcal{I}_{67}^c.
\end{aligned}$$

The first two terms appeared in the proof of Theorem 3.1; therefore

$$|\mathcal{I}_{67}^{a}| + |\mathcal{I}_{67}^{b}| \leq C\,h\,|||\mathbf{v}|||_X\,\|p\|_{H^2(\Omega)}. \tag{3.20}$$

The term $\mathcal{I}_{67}^{c}$ is estimated using Theorem 3.1:

$$|\mathcal{I}_{67}^{c}| \leq |||\vec{u}^I - \mathbf{u}_h|||_X\,|||\mathbf{v}|||_X \leq C\,h\|p\|_{H^2(\Omega)}\,|||\mathbf{v}|||_X. \tag{3.21}$$

The proof is completed by combining (3.17)–(3.21). □

3.3 Superconvergence of the pressure

In this section we prove a second-order convergence estimate for the pressure. We denote the original edges (faces in 3D) by $\tilde{e}$ to distinguish them from facets e.

Let us introduce two additional interpolation operators. Let $\mathbf{V}_h$ be the lowest order Brezzi-Douglas-Marini BDM_1 mixed finite element space on Ω_h, consisting of piecewise linear vector functions with continuous normal components [5]. For any $\vec{v} \in (L^s(\Omega))^d$, $s > 2$, let $\Pi\vec{v} \in \mathbf{V}_h$ be its finite element interpolant satisfying for every element edge (face in 3D) $\tilde{e} \in \partial E$

$$\int_{\tilde{e}} (\Pi\vec{v} - \vec{v}) \cdot \vec{n}_E\, p_1\, \mathrm{d}s = 0 \quad \text{for every linear function } p_1. \tag{3.22}$$

This implies that

$$\int_E \mathrm{div}\,(\Pi\vec{v} - \vec{v})\,\mathrm{d}x = 0. \tag{3.23}$$

It has been shown in [5] that for any smooth enough vector $\vec{v}$,

$$\|\vec{v} - \Pi\vec{v}\|_{(L^2(\Omega))^d} \leq Ch^k\|\vec{v}\|_{(H^k(\Omega))^d}, \quad 1 \leq k \leq 2. \tag{3.24}$$

It is also easy to see that for all elements E

$$\|\Pi\vec{v}\|_{(H^1(E))^d} \leq C\|\vec{v}\|_{(H^1(E))^d}. \tag{3.25}$$

For any $\vec{v} \in \mathbf{V}_h$, define an interpolant $\vec{v}^{\tilde{I}} \in X_h$ such that, for every facet $e \in \partial E$,

$$(\vec{v}^{\tilde{I}})_E^e = \vec{v}(\mathbf{r}_e) \cdot \vec{n}_E,$$

where $\mathbf{r}_e$ is the vertex of E shared by e. Note that $\vec{v}^{\tilde{I}}$ satisfies the continuity condition (2.6).

Lemma 3.3 *For every $\vec{v} \in \mathbf{V}_h$,*

$$\mathcal{DIV}\,\vec{v}^{\tilde{I}} = (\mathrm{div}\,\vec{v})^I.$$

Proof. For any E in Ω_h, we have

$$(\mathcal{DIV}\,\vec{v}^{\tilde{I}})_E = \frac{1}{|E|}\sum_{e\in\partial E}|e|\,(\vec{v}^{\tilde{I}})^e_E = \frac{1}{|E|}\sum_{\tilde{e}\in\partial E}\frac{|\tilde{e}|}{d}\sum_{i=1}^{d}\vec{v}(\mathbf{r}^i_e)\cdot\vec{n}_E,$$

where $\mathbf{r}^i_e$, $i = 1,\dots,d$, are the vertices of e. The last sum is the quadrature rule for exact integration of linear functions. Therefore,

$$(\mathcal{DIV}\,\vec{v}^{\tilde{I}})_E = \frac{1}{|E|}\sum_{\tilde{e}\in\partial E}\int_{\tilde{e}}\vec{v}\cdot\vec{n}_E\,\mathrm{d}s = (\mathrm{div}\,\vec{v})^I_E.$$

□

We are now ready to prove second-order convergence for the pressure.

Theorem 3.3 *Assume that problem* (2.2) *is* H^2*-elliptic regular. Then, for the solutions* $(p,\,\vec{u})$ *and* $(\mathbf{p}_h,\,\mathbf{u}_h)$ *of problems (2.2) and (2.14), respectively, there exists a constant* C *independent of* h *such that*

$$|||p^I - \mathbf{p}_h|||_Q \le C\,h^2\,(\|\vec{u}\|_{(H^2(\Omega))^d} + \|p\|_{H^2(\Omega)}).$$

Proof. The proof is based on a duality argument. Let φ be the solution to

$$-\mathrm{div}\,K\,\mathrm{grad}\,\varphi = R(p^I - \mathbf{p}_h) \quad \text{in } \Omega,$$
$$\varphi = 0 \quad \text{on } \partial\Omega,$$

where $R(p^I - \mathbf{p}_h)$ is the piecewise constant function equal to $(p^I - \mathbf{p}_h)_E$ on each element E. The regularity assumption implies

$$\|\varphi\|_{H^2(\Omega)} \le C\|R(p^I - \mathbf{p}_h)\|_{L^2(\Omega)}; \tag{3.26}$$

see [11, 17] for sufficient conditions. Let $\vec{\psi} = -K\,\mathrm{grad}\,\varphi$. Let $(\cdot,\cdot)$ denote the L^2 inner product over Ω. Using Lemma 3.3 and (2.18), we get

$$\begin{aligned}\|R(p^I - \mathbf{p}_h)\|^2_{L^2(\Omega)} &= (R(p^I - \mathbf{p}_h), \mathrm{div}\,\Pi\vec{\psi}) \\ &= (p, \mathrm{div}\,\Pi\vec{\psi}) - [\mathbf{p}_h, \mathcal{DIV}\,(\Pi\vec{\psi})^{\tilde{I}}]_Q \\ &= (K^{-1}\vec{u}, \Pi\vec{\psi}) - [\mathbf{u}_h, (\Pi\vec{\psi})^{\tilde{I}}]_X \\ &= (K^{-1}(\vec{u} - \Pi\vec{u}), \Pi\vec{\psi}) + \sigma(K^{-1}\Pi\vec{u}, \Pi\vec{\psi}) + [(\Pi\vec{u})^{\tilde{I}} - \mathbf{u}_h, (\Pi\vec{\psi})^{\tilde{I}}]_X,\end{aligned} \tag{3.27}$$

where

$$\sigma(K^{-1}\vec{u}, \vec{v}) \equiv (K^{-1}\vec{u}, \vec{v}) - [\vec{u}^{\tilde{I}}, \vec{v}^{\tilde{I}}]_X$$

represents the error in integrating the dot product of two vector-valued functions. The first term on the right in (3.27) can be bounded using (3.24) and (3.25):

$$|(K^{-1}(\vec{u} - \Pi\vec{u}), \Pi\vec{\psi}) \le Ch^2\|\vec{u}\|_{(H^2(\Omega))^d}\|\varphi\|_{H^2(\Omega)}. \tag{3.28}$$

The second term on the right in (3.27) can be bounded using Lemma 3.4 (which we shall prove below) and (3.25):

$$|\sigma(K^{-1}\Pi\vec{u},\Pi\vec{\psi})| \le Ch^2\|\vec{u}\|_{(H^1(\Omega))^d}\|\varphi\|_{H^2(\Omega)}. \tag{3.29}$$

Let $\mathbf{v} = (\Pi\vec{u})^{\tilde{I}} - \mathbf{u}_h$. Then, for the last term on the right in (3.27), we have

$$\begin{aligned}
[(\Pi\vec{u})^{\tilde{I}} - \mathbf{u}_h, (\Pi\vec{\psi})^{\tilde{I}}]_X &= [\mathbf{v}, (-\Pi K\,\mathrm{grad}\,\varphi)^{\tilde{I}}]_X \\
&= [\mathbf{v}, (-\Pi K\,\mathrm{grad}\,\varphi + \Pi K\,\mathrm{grad}\,\varphi^1)^{\tilde{I}}]_X \\
&\quad + [\mathbf{v}, (-\Pi K\,\mathrm{grad}\,\varphi^1 + \Pi\overline{K}\,\mathrm{grad}\,\varphi^1)^{\tilde{I}}]_X - [\mathbf{v}, (\overline{K}\,\mathrm{grad}\,\varphi^1)^{\tilde{I}}]_X \\
&\equiv \mathcal{I}_8 + \mathcal{I}_9 - \mathcal{I}_{10},
\end{aligned} \tag{3.30}$$

where φ_1 is the piecewise linear approximation to φ satisfying (3.8) on every element E, and $\overline{K}$ is the piecewise constant approximation to K defined in Section 2. For $\mathcal{I}_8$, we have

$$\begin{aligned}
|\mathcal{I}_8| &\le |\!|\!|(\Pi K\,\mathrm{grad}\,(\varphi - \varphi^1))^{\tilde{I}}|\!|\!|_X |\!|\!|\mathbf{v}|\!|\!|_X \\
&\le \left(\alpha_1 \sum_{E\in\Omega_h}\sum_{e\in\partial E}(\Pi K\,\mathrm{grad}\,(\varphi-\varphi^1)(\mathbf{r}_e)\cdot\vec{n}_E)^2|E|\right)^{1/2} |\!|\!|\mathbf{v}|\!|\!|_X \\
&\le C\left(\sum_{E\in\Omega_h}\sum_{\tilde{e}\in\partial E}\frac{|E|}{|\tilde{e}|}\int_{\tilde{e}}(K\,\mathrm{grad}\,(\varphi-\varphi^1)\cdot\vec{n}_E)^2\,\mathrm{d}s\right)^{1/2} |\!|\!|\mathbf{v}|\!|\!|_X \\
&\le Ch\|\varphi\|_{H^2(\Omega)}|\!|\!|\mathbf{v}|\!|\!|_X,
\end{aligned} \tag{3.31}$$

where we have used (2.17) in the second inequality, (3.22) in the third inequality, and (3.10) in the last inequality. Using the above argument, the term $\mathcal{I}_9$ can be bounded as follows:

$$\begin{aligned}
|\mathcal{I}_9| &\le |\!|\!|(\Pi(K-\overline{K})\,\mathrm{grad}\,\varphi^1)^{\tilde{I}}|\!|\!|_X |\!|\!|\mathbf{v}|\!|\!|_X \\
&\le C\left(\sum_{E\in\Omega_h}\sum_{\tilde{e}\in\partial E}\frac{|E|}{|\tilde{e}|}\int_{\tilde{e}}(\Pi(K-\overline{K})\,\mathrm{grad}\,\varphi^1\cdot\vec{n}_E)^2\,\mathrm{d}s\right)^{1/2} |\!|\!|\mathbf{v}|\!|\!|_X \\
&\le C\left(\sum_{E\in\Omega_h} h_E^2\|K\|_{1,\infty,E}^2\|\,\mathrm{grad}\,\varphi^1\|_{L^2(E)}^2\right)^{1/2} |\!|\!|\mathbf{v}|\!|\!|_X \\
&\le Ch\|\varphi\|_{H^2(\Omega)}|\!|\!|\mathbf{v}|\!|\!|_X,
\end{aligned} \tag{3.32}$$

where we have used (2.10) and (3.9) in the third inequality and (3.8) in the fourth inequality.

To estimate $\mathcal{I}_{10}$, we note that $(\overline{K}\,\mathrm{grad}\,\varphi^1)^{\tilde{I}} = (\overline{K}\,\mathrm{grad}\,\varphi^1)^I$ and $\mathcal{DIV}\,\mathbf{v} = 0$. Applying Lemma 3.1 and the argument from (3.14), we get

$$|I_{10}| = \left|\sum_{E\in\Omega_h}\sum_{e\in\partial E}|e|\varphi^1(\mathbf{c}_e)v_E^e\right| \le Ch\|\varphi\|_{H^2(\Omega)}|\!|\!|\mathbf{v}|\!|\!|_X. \tag{3.33}$$

Next, the triangle inequality gives

$$\|\mathbf{v}\|_X \leq \|(\Pi\vec{u})^{\tilde{I}} - \vec{u}^I\|_X + \|\vec{u}^I - \mathbf{u}_h\|_X. \tag{3.34}$$

The second term on the right is bounded in Theorem 3.1. To bound the first term, we choose $\vec{u}_0$ as the piecewise constant approximation to $\vec{u}$ that satisfies (3.7). The triangle inequality gives

$$\|(\Pi\vec{u})^{\tilde{I}} - \vec{u}^I\|_X \leq \|(\Pi\vec{u})^{\tilde{I}} - (\Pi\vec{u}_0)^{\tilde{I}}\|_X + \|(\Pi\vec{u}_0)^{\tilde{I}} - \vec{u}_0^I\|_X + \|\vec{u}_0^I - \vec{u}^I\|_X. \tag{3.35}$$

The second term on the right above is zero. The first term is bounded using the argument from (3.31):

$$\|(\Pi\vec{u})^{\tilde{I}} - (\Pi\vec{u}_0)^{\tilde{I}}\|_X \leq C\left(\sum_{E\in\Omega_h}\sum_{\tilde{e}\in\partial E}\frac{|E|}{|\tilde{e}|}\int_{\tilde{e}}(\Pi(\vec{u}-\vec{u}_0)\cdot\vec{n}_E)^2\,\mathrm{d}s\right)^{1/2} \leq Ch\|\vec{u}\|_{(H^1(\Omega))^d}, \tag{3.36}$$

using (3.7) and (3.9) in the last inequality. The last term in (3.35) is bounded in a similar way,

$$\|\vec{u}_0^I - \vec{u}^I\|_X \leq Ch\|\vec{u}\|_{(H^1(\Omega))^d}. \tag{3.37}$$

The proof is completed by combining (3.27)–(3.37), Theorem 3.1, and (3.26). □

It remains to establish the bound (3.29).

Lemma 3.4 *Let $K^{-1} \in W_\infty^2(E)$ for all elements E. Then, for all $\vec{u}_h, \vec{v}_h \in \mathbf{V}_h$, there exists a constant C independent of h such that*

$$|\sigma(K^{-1}\vec{u}_h, \vec{v}_h)| \leq C\sum_{E\in\Omega_h} h^2\|\vec{u}_h\|_{(H^1(E))^d}\|\vec{v}_h\|_{(H^1(E))^d}.$$

Proof. We first note that for all $\vec{u}_h \in \mathbf{V}_h$ and for all piecewise constant vectors $\vec{v}_0$,

$$\sigma(\vec{u}_h, \vec{v}_0) = 0, \tag{3.38}$$

which follows from

$$[\vec{u}_h^{\tilde{I}}, \vec{v}_0^{\tilde{I}}]_{X,E} = \frac{|E|}{d+1}\sum_{i=1}^{d+1}\vec{u}_h(\mathbf{r}_i)\cdot\vec{v}_0(\mathbf{r}_i) = (\vec{u}_h, \vec{v}_0)_E,$$

using that the middle term is the quadrature rule for exact integration of linear functions. Next, using (3.38), we write

$$\begin{aligned}\sigma_E(K^{-1}\vec{u}_h, \vec{v}_h) = {} & \sigma_E((K^{-1} - K_E^{-1})(\vec{u}_h - \vec{u}_{h,0}), \vec{v}_h) + \sigma_E((K^{-1} - K_E^{-1})\vec{u}_{h,0}, \vec{v}_h - \vec{v}_{h,0}) \\ & + \sigma_E(K^{-1}\vec{u}_{h,0}, \vec{v}_{h,0}) + \sigma_E(K_E^{-1}(\vec{u}_h - \vec{u}_{h,0}), \vec{v}_h - \vec{v}_{h,0}),\end{aligned} \tag{3.39}$$

where $\vec{u}_{h,0}$ and $\vec{v}_{h,0}$ are the constant approximations on E to $\vec{u}_h$ and $\vec{v}_h$, respectively, satisfying (3.7). The first, second, and fourth terms above are bounded by

$$Ch^2\|K^{-1}\|_{1,\infty,E}\|\vec{u}_h\|_{(H^1(E))^d}\|\vec{v}_h\|_{(H^1(E))^d}. \tag{3.40}$$

Since the third term on the right in (3.39) is zero for linear tensors, letting $(K^{-1})^1_E$ be the linear approximation to K^{-1} on E satisfying (3.7), we have

$$\begin{aligned}|\sigma_E(K^{-1}\vec{u}_{h,0},\vec{v}_{h,0})| &= |\sigma_E((K^{-1}-(K^{-1})^1_E)\vec{u}_{h,0},\vec{v}_{h,0})| \\ &\le Ch^2|K^{-1}|_{2,\infty,E}\|\vec{u}_h\|_{(L^2(E))^d}\|\vec{v}_h\|_{(L^2(E))^d}\end{aligned} \tag{3.41}$$

A combination of (3.39)–(3.41) completes the proof of the lemma. □

4 Numerical experiments

In this section, we present results of numerical experiments. As we mention in Sec. 2, the velocity unknown can be eliminated from the discrete system resulting in a cell-centered discretization with a symmetric positive definite matrix. This problem is solved with the preconditioned conjugate gradient (PCG) method. In the numerical experiments, we used one V-cycle of the algebraic multigrid method [21] as a preconditioner. The stopping criterion for the PCG method is the relative decrease in the residual norm by a factor of 10^{-12}.

Let us consider the 2D problem (2.2) in the unit square with the known analytical solution

$$p(x,\,y) = x^3y^2 + x\sin(2\pi xy)\sin(2\pi y)$$

and the tensor coefficient

$$K = \begin{pmatrix}(x+1)^2+y^2 & -xy \\ -xy & (x+1)^2\end{pmatrix}.$$

In the first set of experiments, we consider the sequence of smooth triangular meshes generated from uniform square meshes by splitting each square cell into four equal triangles. The convergence rates are shown in Table 1 for the discrete L_2 norms defined earlier, as well as in discrete L_∞ norms equal the maximum component absolute values of the algebraic vectors. We use the linear regression algorithm to estimate the convergence rates. We observe second-order convergence rate (superconvergence) of the pressure variable and first-order convergence rate of the flux variable in the discrete L_2 norms. The slightly faster convergence rate in the discrete L_∞ norm for the flux (see the last column) is due to faster convergence on coarse meshes.

In the second set of experiments, we take the meshes generated above and perturb randomly positions of the mesh nodes. More precisely, we move each of the mesh nodes into a random position inside a square of size $h/2$ centered at the node. The convergence rates are shown in Table 2. As in the first example, we observe second-order convergence of the pressure and first-order convergence of the flux.

Both experiments confirm the theoretical results proved in the previous sections.

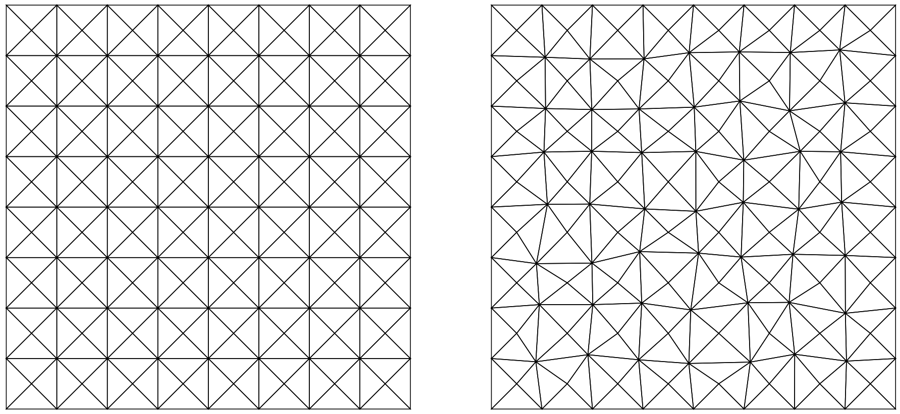


Figure 5: Examples of meshes used in experiments 1 and 2.

Table 1: Convergence rates in the first set of experiments.

$1/h$	$\|\!\|p^I - \mathbf{p}_h\|\!\|_Q$	$\|\!\|p^I - \mathbf{p}_h\|\!\|_\infty$	$\|\!\|\vec{u}^I - \mathbf{u}_h\|\!\|_X$	$\|\!\|\vec{u}^I - \mathbf{u}_h\|\!\|_\infty$
8	1.08e-2	4.06e-2	2.55e-1	2.60e-0
16	2.75e-3	1.18e-2	9.14e-2	1.04e-0
32	6.92e-4	3.18e-3	4.03e-2	3.94e-1
64	1.73e-4	8.17e-4	1.95e-2	1.58e-1
128	4.34e-5	2.07e-4	9.56e-3	7.94e-2
Rate	1.99	1.91	1.17	1.28

5 Conclusion

We develop a local flux mimetic finite difference method, which reduces to cell-centered finite differences for the pressure. Borrowing an idea from the MPFA method, we introduce *facet fluxes*, which are eliminated from the algebraic system by solving small local systems for each mesh vertex. The method is defined on general polyhedral elements. We present analysis for simplicial elements, showing optimal convergence for both variables and superconvergence for the pressure at the element centers. Our analysis is based on discrete space arguments and does not rely on finite element polynomial extensions, with the exception of the pressure superconvergence proof. The analysis can be extended to smooth quadrilateral meshes.

Table 2: Convergence rates in the second set of experiments.

$1/h$	$\|\|\|p^I - \mathbf{p}_h\|\|\|_Q$	$\|\|\|p^I - \mathbf{p}_h\|\|\|_\infty$	$\|\|\|\vec{u}^I - \mathbf{u}_h\|\|\|_X$	$\|\|\|\vec{u}^I - \mathbf{u}_h\|\|\|_\infty$
8	1.14e-2	4.06e-2	2.94e-1	2.67e-0
16	2.93e-3	1.18e-2	1.24e-1	1.14e-0
32	7.13e-4	3.23e-3	5.97e-2	5.12e-1
64	1.77e-4	9.49e-4	3.01e-2	3.56e-1
128	4.48e-5	2.58e-4	1.52e-2	1.98e-1
Rate	2.00	1.82	1.06	0.92

Acknowledgments

The work was partly performed at Los Alamos National Laboratory, which is operated by the University of California for the US Department of Energy under contract W-7405-ENG-36 (LAUR-05-8364). The authors acknowledge the partial support of the DOE/ASCR Program in the Applied Mathematical Sciences and DOE's Accelerated Strategic Computing Initiative (ASC). The last author was partially supported by NSF grant DMS 0411694, by DOE grant DE-FG02-04ER25618, and by the Los Alamos National Laboratory through a visitor research support.

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