A Two-Grid Stabilization Method for Solving the Steady-State Navier-Stokes Equations

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Abstract

We formulate a subgrid eddy viscosity method for solving the steady-state incompressible flow problem. The eddy viscosity does not act on the large flow structures. Optimal error estimates are obtained for velocity and pressure. The numerical illustrations agree completely with the theoretical expectations.

1 Introduction

We consider herein the approximate solution of the steady-state Navier Stokes problem equation:

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + \nabla p &= f \quad \text{in } \Omega, \\ \nabla\cdot u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^d$, with $d = 2$ or $d = 3$, $u : \Omega \to \mathbb{R}^d$ the fluid velocity, $p : \Omega \to \mathbb{R}$ the fluid pressure and f a prescribed body force. The kinematic viscosity, which is inversely proportional to the Reynolds number Re, is denoted by $\nu > 0$.

In this paper, we consider a subgrid eddy viscosity model as a numerical stabilization of a convection dominated and underresolved flow. This approach adds an artificial viscosity only on the fine scales, and is referred to as a subgrid eddy viscosity model. We consider the classical finite element method for the spatial discretization. The resulting scheme involves two grids coupled to each other through the artificial viscosity term.

The general idea of using two-grid discretization to increase the *efficiency* of numerical methods was pioneered by J. Xu (see, e.g., Marion and Xu [17]) and developed by Girault and Lions (see [6], [7]). This two-grid discretization idea and previous work [5] on stabilizations in viscoelasticity are combined with the physical ideas of eddy viscosity models. This combination of ideas lead very naturally to the presented method.

The idea of the subgrid eddy viscosity model is inspired by earlier work of Guermond [9]. In [9], subgrid scales is augmented by bubble functions. The artificial viscosity is added only on the fine scales of the problem. This concept is generalized by Layton [16] for the stationary convection diffusion problem. In the work of Kaya and Layton [14], this model has been connected with another consistent stabilization technique, also known as variational multiscale method, introduced by Hughes [12]. The model has been analyzed

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for time-dependent Navier-Stokes equations by John and Kaya [13] for the continuous finite element method and by Kaya and Rivière [15] for the discontinuous Galerkin method.

To motivate the method, we define the spaces $X := (H_0^1(\Omega))^d$ and $M := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}$ and $L := \{\mathbb{L} \in (L^2(\Omega))^{d\times d}, \mathbb{L} = \mathbb{L}^T\}$ and consider a variational formulation of (1.1): find $u \in X$, $p \in M$ and $\mathbb{G} \in L$ such that

$$\begin{aligned} a(u,v) + c(u,u,v) - b(v,p) + (\nu_T \mathbb{D}(u), \mathbb{D}(v)) - (\nu_T \mathbb{G}, \mathbb{D}(v)) &= (f,v), \quad \forall\, v \in X, \\ b(u,q) &= 0, \qquad \forall q \in M \\ (\mathbb{G} - \mathbb{D}(u), \mathbb{L}) &= 0, \qquad \forall \mathbb{L} \in L. \end{aligned} \tag{1.2}$$

where $(.,.)$ denotes the L^2 inner-product and the bilinear forms are defined below

$$\begin{aligned} a(v,w) : &= (2\nu \mathbb{D}(v), \mathbb{D}(w)), && \forall v, w \in X, \\ c(z,v,w) : &= \tfrac{1}{2}(z \cdot \nabla v, w) - \tfrac{1}{2}(z \cdot \nabla w, v), && \forall z, v, w \in X, \\ b(v,q) : &= (q, \nabla \cdot v), && \forall v \in X, \forall q \in M. \end{aligned} \tag{1.3}$$

Here, the stress tensor is defined by $\mathbb{D}(v) = 0.5(\nabla v + \nabla v^T)$ and the parameter $\nu_T > 0$ is the eddy viscosity parameter. In the continuous case, this method reduces to the standard Navier-Stokes equations. However, in the discrete case it leads to different discretizations. In this paper, we consider multiscale finite element approximation of the Navier-Stokes equation based on the formulations (1.2).

Our approach can be understood as an LES (Large Eddy Simulation) model but the point herein is to study it as a numerical stabilization. To our knowledge, this is the first paper presenting error estimates for velocity and pressure in L^2 and numerical examples for this subgrid eddy viscosity model.

The outline of the paper is as follows. In the next section, some notation and the finite element scheme are presented. In Section 3, 4 and 5, error estimates are given for velocity and pressure. The algorithm and numerical experiments are described in Section 6. Conclusions follow.

2 Notation and Scheme

We first recall some standard notation: $L^2(\Omega)$ denotes the space of square-integrable functions over Ω with norm $\|\cdot\|$ and inner-product $(\cdot,\cdot)$; $H^k(\Omega)$ denotes the standard Sobolev spaces with norm $\|\cdot\|_k$ and semi-norm $|\cdot|_k$ (Adams [1]). $H_0^1(\Omega)$ denotes the subspace of $H^1(\Omega)$ of functions whose trace is zero on $\partial\Omega$; it is a Banach space with norm $|\cdot|_1$. Finally, the space $H^{-1}(\Omega)$ is the dual space of $H^1(\Omega) \cap H_0^1(\Omega)$, and is equipped with the negative norm

$$\|z\|_{-1} = \sup_{v \in H^1(\Omega)\cap H_0^1(\Omega)} \frac{|(z,v)|}{\|v\|_1}.$$

The forms (1.3) defined in Section 1, have the following properties. The bilinear form $a(.,.)$ is clearly coercive in X: there is a constant $C_1 > 0$ such that

$$a(v,v) = 2\nu \|\mathbb{D}(v)\|^2 \geq C_1 \nu \|\nabla v\|^2, \quad \forall v \in X, \tag{2.1}$$

owing to the Korn's inequality (Duvaut and Lions [3]). Using Korn's inequality, the trilinear form $c(\cdot,\cdot,\cdot)$ satisfies the following bound (Temam [19]): there exists a constant $K > 0$ such that

$$c(z,v,w) \leq K \|\mathbb{D}(z)\| \|\mathbb{D}(v)\| \|\mathbb{D}(w)\|, \quad \forall z, v, w \in X. \tag{2.2}$$

We also recall the following property of c:

$$c(z, v, v) = 0, \quad \forall z, v \in X. \tag{2.3}$$

We now introduce the finite element discretization of (1.2). Let τ^h and τ^H be two regular triangulations of the domain Ω, such that h (resp. H) denotes the maximum diameter of the elements in τ^h (resp. τ^H) and such that $h < H$. We will refer to the mesh obtained from τ^h as the *fine mesh* and the mesh obtained from τ^H as the *coarse mesh*. Let (X^h, M^h) be a pair of conforming finite element spaces satisfying the inf-sup condition: there exists a constant β independent of h such that

$$\inf_{q^h \in M^h} \sup_{v^h \in X^h} \frac{b(v^h, q^h)}{\|q^h\| \, \|\nabla v^h\|} \geq \beta > 0. \tag{2.4}$$

Examples of such compatible spaces are the mini-element spaces (Arnold, Brezzi, Fortin [2]), the Taylor-Hood spaces (Gunzburger [10]) and the continuous piecewise quadratics for the velocity space and discontinuous piecewise constants for the pressure space (Fortin [4]). We assume that the spaces X^h and M^h contain piecewise continuous polynomials of degree k and $k-1$ respectively. We assume that $L^H \subset L$ contains piecewise polynomials (not necessarily continuous) of order $k-1$. Let $P_{L^H} : L \to L^H$ be the L^2 orthogonal projection onto L^H. Thus, we have

$$\begin{array}{rcl} (P_{L^H}\mathbb{L}, \mathbb{G}^H) & = & (\mathbb{L}, \mathbb{G}^H), \quad \forall \mathbb{G}^H \in L^H, \ \forall \mathbb{L} \in L, \\ \|\mathbb{L} - P_{L^H}\mathbb{L}\| & \leq & CH^k |\mathbb{L}|_k, \quad \forall \mathbb{L} \in L \cap (H^k(\Omega))^{d \times d}. \end{array} \tag{2.5}$$

We will also use the fact that

$$\|I - P_{L^H}\| \leq 1. \tag{2.6}$$

Furthermore, we suppose the spaces (X^h, M^h) satisfy the following approximation properties:

$$\inf_{v^h \in X^h} \left\{ \|u - v^h\| + h\|\nabla(u - v^h)\| \right\} \leq Ch^{k+1}|u|_{k+1}, \forall u \in (H^{k+1}(\Omega))^d \cap X, \tag{2.7}$$

$$\inf_{q^h \in M^h} \|p - q^h\| \leq Ch^k |p|_k, \qquad \forall p \in H^k(\Omega) \cap M. \tag{2.8}$$

We propose the following finite element approximation of (1.2): find $(u^h, p^h) \in (X^h, M^h)$ satisfying

$$\begin{array}{rcll} a(u^h, v^h) + c(u^h, u^h, v^h) - b(v^h, p^h) + g(u^h, v^h) & = & (f, v^h), & \forall v^h \in X^h, \\ b(u^h, q^h) & = & 0, & \forall q^h \in M^h, \end{array} \tag{2.9}$$

where the bilinear form g is

$$g(v^h, w^h) = (\nu_T (I - P_{L^H})\mathbb{D}(v^h), (I - P_{L^H})\mathbb{D}(w^h)), \quad \forall v^h, w^h \in X^h.$$

The eddy viscosity parameter $\nu_T > 0$ is to be defined later.

We can formulate another problem in the space of discrete divergence-free functions, denoted by V^h:

$$V^h := \left\{ v^h \in X^h : (\nabla \cdot v^h, q^h) = 0, \quad \forall q^h \in M^h \right\}. \tag{2.10}$$

Under the inf-sup condition (2.4), the formulation (2.9) is equivalent to the following problem [8]: find $u^h \in V^h$ such that

$$a(u^h, v^h) + c(u^h, u^h, v^h) + g(u^h, v^h) \quad = \quad (f, v^h), \quad \forall v^h \in V^h. \tag{2.11}$$

Our analysis is based on the assumption that the following global uniqueness condition holds:

$$K\|f\|_{-1} \leq C_1 \nu^2 \tag{2.12}$$

where K is the constant of (2.2) and C_1 is the constant of (2.1). Recall [8] that under this condition (2.12), (1.2) has a unique solution $(u, p) \in (X, M)$. It is easy to show that under the condition (2.12) and the inf-sup condition (2.4), there exists a unique solution to (2.9).

Remark 2.1. *We could also consider the following forms for the nonlinear term in (2.9):*

$$\begin{aligned} c(z, v, w) &= (z \cdot \nabla v, w) \quad \textit{(convective form)} \\ c(z, v, w) &= -(z \cdot \nabla w, v) \quad \textit{(conservation form)} \end{aligned}$$

In both cases, the analysis and error estimates remain the same.

Throughout the paper, C is a generic constant that does not depend on ν, ν_T, h and H, unless specified otherwise.

3 Error Estimate for Velocity in H_0^1

In this section, we first prove a stability result for the approximation of velocity for (2.9). We then prove an error estimate for the velocity in the energy norm.

Lemma 3.1. *The finite element approximation of velocity for (2.9) is stable, i.e. there is a constant C independent of ν, ν_T, H and h such that*

$$\nu\|\mathbb{D}(u^h)\|^2 + \frac{\nu_T}{2}\|(I - P_{L^H})\mathbb{D}(u^h)\|^2 \leq \frac{C}{\nu}\|f\|_{-1}^2. \tag{3.1}$$

Proof. The result is easily obtained by setting $v^h = u^h$ in (2.11) and using (2.3), Cauchy Schwarz, Korn's and Young's inequalities. ☐

Remark 3.1. *Lemma 3.1 directly implies that*

$$\|\mathbb{D}(u^h)\| \leq \frac{C}{\nu}\|f\|_{-1}. \tag{3.2}$$

Theorem 3.1. *Suppose the global uniqueness condition (2.12) holds. Then,*

$$\begin{aligned} &\nu\|\mathbb{D}(u - u^h)\|^2 + \frac{\nu_T}{2}\|(I - P_{L^H})\mathbb{D}(u - u^h)\|^2 \\ &\leq C \inf_{w^h \in V^h} \{\nu\|\mathbb{D}(u - w^h)\|^2 + \frac{K^2}{\nu}(\|\nabla u\| + \|\nabla u^h\|)^2\|\mathbb{D}(u - w^h)\|^2 \\ &+\nu_T\|(I - P_{L^H})\mathbb{D}(u - w^h)\|^2 + \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|^2\} + C \inf_{q^h \in M^h} \frac{1}{\nu}\|p - q^h\|^2. \end{aligned}$$

where C is independent of ν, ν_T, h and H.

Proof. We first derive an error equation by noting that the true solution satisfies

$$a(u,v^h)+c(u,u,v^h)-b(v^h,p)+g(u,v^h)=(f,v^h)+g(u,v^h)\quad \forall v^h\in X^h, \tag{3.3}$$

and by subtracting (2.9) to (3.3):

$$\begin{aligned} a(u-u^h,v^h)+c(u,u,v^h)-c(u^h,u^h,v^h)-b(v^h,p-p^h)\\ +g(u-u^h,v^h)=g(u,v^h),\quad \forall v^h\in X^h. \end{aligned}\tag{3.4}$$

We now decompose the error $u-u^h=\eta-\phi^h$, with $\eta=u-w^h$ and $\phi^h=u^h-w^h$, where w^h is any function in V^h. Rearranging the terms of (3.4), choosing $v^h=\phi^h\in V^h$, we obtain:

$$\begin{aligned} a(\phi^h,\phi^h)+g(\phi^h,\phi^h)=a(\eta,\phi^h)+c(u,u,\phi^h)-c(u^h,u^h,\phi^h)+g(\eta,\phi^h)\\ -b(\phi^h,p-q^h)+g(u,\phi^h),\quad \forall q^h\in M^h. \end{aligned}\tag{3.5}$$

To bound the linear terms in the right-hand side of (3.5), we simply use Cauchy Schwarz inequality and Young's inequality. To bound the nonlinear convective terms we rewrite these terms as follows:

$$c(u,u,\phi^h)-c(u^h,u^h,\phi^h)=c(u,\eta,\phi^h)+c(\eta,u^h,\phi^h)-c(\phi^h,u^h,\phi^h).$$

Then we use the bound (2.2) and Young's inequality. From (2.6), the last term in the right-hand side of (3.5), which characterizes the inconsistency error, is bounded by

$$|g(u,\phi^h)|\le \nu_T\|(I-P_{L^H})\mathbb{D}(u)\|\|(I-P_{L^H})\mathbb{D}(\phi^h)\|. \tag{3.6}$$

The final result is easily obtained by combining all the bounds above and by using the triangle inequality

$$\begin{aligned} &\nu\|\mathbb{D}(u-u^h)\|^2+\frac{\nu_T}{2}\|(I-P_{L^H})\mathbb{D}(u-u^h)\|^2\\ &\le C(\nu\|\mathbb{D}(u-w^h)\|^2+\nu_T\|(I-P_{L^H})\mathbb{D}(u-w^h)\|^2\\ &\quad+\nu\|\mathbb{D}(\phi^h)\|^2+\nu_T\|(I-P_{L^H})\mathbb{D}(\phi^h)\|^2). \end{aligned}$$

□

By appropriately choosing the parameters ν_T, H and h, one can obtain an optimal error estimate, as stated in the following corollary.

Corollary 3.1. *Under the assumption of Theorem 3.1, and under the regularity assumptions $u\in H^{k+1}(\Omega)^d\cap X$ and $p\in H^k(\Omega)\cap M$, there is a constant C independent of ν_T, h and H such that:*

$$\begin{aligned} &\nu\|\mathbb{D}(u-u^h)\|^2+\frac{\nu_T}{2}\|(I-P_{L^H})\mathbb{D}(u-u^h)\|^2\\ &\le Ch^{2k}|u|^2_{k+1}(\nu+\frac{1}{\nu}(1+\frac{1}{\nu})^2+\nu_T)+C\nu h^{2k}|p|^2_k+C\nu_T H^{2k}|u|^2_{k+1}. \end{aligned}$$

In particular,

$$\|\mathbb{D}(u-u^h)\|=\mathcal{O}(h^k)\ \textit{if}\ \begin{cases} k=1 & (\nu_T,H)=(h,h^{1/2}),\\ k=2 & (\nu_T,H)=(h,h^{1/2})\quad \textit{or}\ (\nu_T,H)=(h^2,h^{1/2})\\ k=3 & (\nu_T,H)=(h,h^{5/6})\quad \textit{or}\ (\nu_T,H)=(h^2,h^{2/3}). \end{cases}$$

Corollary 3.2. *Suppose we use the mini-element spaces i.e., continuous polynomial of degree k plus bubbles functions for X^h and piecewise continuous polynomial of degree $k-1$ for M^h. If $\nu_T = h^\alpha$ and $H = h^\beta$ with $\alpha + 2\beta \geq 2k$, then the error in the energy norm is bounded as*

$$\|\mathbb{D}(u - u^h)\| \leq Ch^k.$$

For instance, one may choose $\nu_T = h^{2k-1}$ and $H = h^{1/2}$.

4 Error Estimate for Pressure

This section is devoted to the estimation of the discrete pressure.

Theorem 4.1. *Suppose that the hypotheses of Theorem 3.1 hold. Then the pressure error satisfies*

$$\begin{aligned}\|p - p^h\| \leq C((\nu + 1)\|\mathbb{D}(u - u^h)\| + \|\mathbb{D}(u - u^h)\|^2 + \nu_T\|(I - P_{L^H})\mathbb{D}(u - u^h)\| \\ + \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|) + C \inf_{q^h \in M^h} \|p - q^h\|,\end{aligned}$$

where C is independent of ν, ν_T, h and H.

Proof. The proof follows the approach given by Rannacher and Heywood [11]. Denoting the error in velocity $e = u - u^h$ and introducing an approximation $\tilde{p} \in M^h$ of the pressure in the error equation (3.4), we obtain:

$$\begin{aligned}b(v^h, p^h - \tilde{p}) = b(v^h, p - \tilde{p}) - a(e, v^h) - (c(u, u, v^h) - c(u^h, u^h, v^h)) \\ - g(e, v^h) + g(u, v^h), \quad \forall v^h \in X^h.\end{aligned} \tag{4.1}$$

To bound the linear terms in the right-hand side of (4.1), we apply Cauchy Schwarz inequality, and Korn's inequality and (2.6). The inconsistency term $g(u, v^h)$ is bounded as in (3.6). In view of Lemma 2.2 and Korn's inequality, the nonlinear terms are bounded as:

$$\begin{aligned}|c(u, u, v^h) - c(u^h, u^h, v^h)| = | - c(e, e, v^h) + c(e, u, v^h) + c(u, e, v^h)| \\ \leq C(\|\mathbb{D}(e)\| + K\|\nabla u\|)\|\mathbb{D}(e)\|\|\nabla v^h\|.\end{aligned}$$

Combining all the bounds, then we have

$$\begin{aligned}|b(p^h - \tilde{p}, v^h)| \leq C\{\|p - \tilde{p}\| + \nu\|\mathbb{D}(e)\| + (\|\mathbb{D}(e)\| + \|\nabla u\|)\|\mathbb{D}(e)\| \\ + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\| + \nu_T\|(I - P_{L^H})\mathbb{D}(u))\|\}\|\nabla v^h\|.\end{aligned} \tag{4.2}$$

On the other hand, the inf-sup condition (2.4) implies that there exists a nontrivial $v^h \in X^h$, such that

$$(p^h - \tilde{p}, \nabla \cdot v^h) \geq \beta\|\nabla v^h\|\|p^h - \tilde{p}\|. \tag{4.3}$$

In view of (4.3), we have

$$\|p - p^h\| \leq \|p - \tilde{p}\| + \beta^{-1}\frac{|b(v, p^h - \tilde{p})|}{\|\nabla v^h\|}. \tag{4.4}$$

We conclude our proof by inserting (4.2) into (4.4):

$$\begin{aligned}\|p - p^h\| \leq C\|p - \tilde{p}\| + C(\nu\|\mathbb{D}(e)\| + \|\mathbb{D}(e)\|^2 + \|\mathbb{D}(e)\|\|\nabla u\| \\ + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\| + \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|).\end{aligned} \tag{4.5}$$

□

Corollary 4.1. *The statement of Theorem 3.1, the approximation results (2.7) and (2.8), and Corollary 3.1 imply that*

$$\|p - p^h\| \le C(h^k + \nu_T H^k + \nu_T^{1/2}(\nu_T + 1)(h^k + H^k) + \nu_T h^k)$$

where C is independent of ν_T, h and H.

Therefore, if $\nu_T = h^\alpha$, $H = h^\beta$ and $\alpha + 2\beta \ge 2k$, the error in the pressure is bounded by

$$\|p - p^h\| \le Ch^k.$$

For instance, one can choose for $k = 1$, $(\nu_T, H) = (h, h^{1/2})$ or for $k = 2$, $(\nu_T, H) = (h^2, h^{1/2})$.

5 Error Estimate for Velocity in L^2

We now give an error estimate in L^2 for the velocity by using the duality argument [8]. We first consider the linearized adjoint problem of the Navier-Stokes equations: given $\xi \in L^2(\Omega)$, find $(\phi, \chi) \in (X, Q)$ with

$$a(\phi, v) + c(u, v, \phi) + c(v, u, \phi) + g(\phi, v) - b(v, \chi) + b(\phi, q) = (\xi, v), \quad \forall (v, q) \in (X, Q). \tag{5.1}$$

Since (u, p) is a nonsingular solution of (1.1) and the boundary $\partial\Omega$ is smooth enough, there exists a unique (ϕ, χ), ([8]) solution to (5.1). We also assume that the linearized adjoint problem is $H^2(\Omega)$ regular. This means that for any $\xi \in L^2(\Omega)$ there exists a unique pair (ϕ, χ) in $(X \cap H^2(\Omega)^d) \times (M \cap H^1(\Omega))$ such that the following inequality holds

$$\|\phi\|_2 + \|\chi\|_1 \le C\|\xi\|. \tag{5.2}$$

We now state the L^2 error estimate.

Theorem 5.1. *Assume that the assumptions of Theorem 3.1 and Theorem 4.1 hold and the solution of the dual problem (5.1) satisfies the stability estimate (5.2). Then, there exists a constant C such that*

$$\|u - u^h\| \le Ch^{k+1}(1 + \nu_T^{1/2} + \nu_T + \nu_T^{3/2}) + C\nu_T H^{k+1} + ChH^k \nu_T^{1/2}(1 + \nu_T^{1/2} + \nu_T),$$

where C is independent of ν_T, h, H.

Proof. Subtracting (3.3) from (2.9), and denoting $e = u - u^h$, gives the following error equation:

$$\begin{aligned} &a(e, v^h) + c(u, u, v^h) - c(u^h, u^h, v^h) - b(v^h, p - p^h) \\ &-b(e, q^h + g(e, v^h) - g(u, v^h) = 0, \quad \forall v_h \in X^h, \forall q^h \in M^h. \end{aligned} \tag{5.3}$$

On the other hand, consider the dual problem (5.1) with $\xi = e$, choose $v = e$, $q = p^h - p$ and subtract (5.3) to the resulting equation

$$\begin{aligned} \|e\|^2 &\le |a(\phi - v^h, e)| + |c(u, e, \phi) + c(e, u, \phi) - c(u, u, v^h) + c(u^h, u^h, v^h)| \\ &\quad + |b(e, \chi - q^h)| - |b(\phi - v^h, p - p^h)| + |g(\phi - v^h, e) + g(u, v^h)| \\ &\le C(\nu\|\mathbb{D}(e)\| + \|p - p^h\| + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\|)\|\mathbb{D}(\phi - v^h)\| \\ &\quad + C\|\chi - q^h\|\|\mathbb{D}(e)\| + \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|\|(I - P_{L^H})\mathbb{D}(v^h)\| \\ &\qquad + |c(u, e, \phi) + c(e, u, \phi) - c(u, u, v^h) + c(u^h, u^h, v^h)|, \end{aligned} \tag{5.4}$$

owing to Cauchy-Schwarz, Korn's inequality and (2.6). We then choose $(v^h, q^h) = (\tilde{\phi}, \tilde{\chi})$ where $\tilde{\phi}, \tilde{\chi}$ are the best approximation of (ϕ, χ) in (X^h, M^h). Using the approximation properties we have:

$$\|\phi - \tilde{\phi}\|_1 \leq Ch\|\phi\|_2,$$
$$\|\chi - \tilde{\chi}\| \leq Ch\|\chi\|_1.$$

The equation (5.4) becomes

$$\begin{aligned}\|e\|^2 \leq Ch(\nu\|\mathbb{D}(e)\| + \|p - p^h\| + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\|)\|\phi\|_2 \\ +Ch\|\chi\|_1\|\mathbb{D}(e)\| + \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|\|(I - P_{L^H})\mathbb{D}(\tilde{\phi})\| \\ +|c(u, e, \phi) + c(e, u, \phi) - c(u, u, \tilde{\phi}) + c(u^h, u^h, \tilde{\phi})|.\end{aligned} \tag{5.5}$$

The consistency error term in the right-hand side of (5.5) is bounded by using (2.5):

$$\begin{aligned}\nu_T\|(I - P_{L^H})\mathbb{D}(u)\|\|(I - P_{L^H})\mathbb{D}(\tilde{\phi})\| &\leq \nu_T H^k|u|_{k+1}H\|\mathbb{D}(\tilde{\phi})\|_1 \\ &\leq C\nu_T H^{k+1}|u|_{k+1}\|\tilde{\phi}\|_2 \\ &\leq C\nu_T H^{k+1}|u|_{k+1}(\|\tilde{\phi} - \phi\|_2 + \|\phi\|_2) \\ &\leq C\nu_T H^{k+1}|u|_{k+1}\|\phi\|_2.\end{aligned}$$

We now consider the nonlinear terms in (5.5). Adding and subtracting u^h, gives

$$\begin{aligned}c(u, e, \phi) + c(e, u, \phi) - c(u, u, \tilde{\phi}) + c(u^h, u^h, \tilde{\phi}) = c(e, e, \phi) + c(u, e, \phi - \tilde{\phi}) \\ +c(e, u, \phi - \tilde{\phi}) + c(e, e, \tilde{\phi} - \phi).\end{aligned}$$

Using the Lemma 2.2 and Korn's inequality, we have

$$\begin{aligned}&|c(u, e, \phi) + c(e, u, \phi) - c(u, u, \tilde{\phi}) + c(u^h, u^h, \tilde{\phi})| \\ \leq\ & C\|\mathbb{D}(e)\|^2\|\phi\|_1 + C\|\nabla u\|\|\mathbb{D}(e)\|\|\phi - \tilde{\phi}\|_1 + C\|\mathbb{D}(e)\|^2\|\phi - \tilde{\phi}\|_1 \\ \leq\ & C(\|\mathbb{D}(e)\| + h)\|\mathbb{D}(e)\|\|\phi\|_2.\end{aligned}$$

Combining all bounds and using the stability property (5.2) give:

$$\|e\| \leq C(\nu + \nu_T)\|\mathbb{D}(e)\| + C\|p - p^h\| + C\nu_T H^{k+1}|u|_{k+1} + C\|\mathbb{D}(e)\|(h + \|\mathbb{D}(e)\|)$$

The final result is obtained by using Corollary 3.1 and Corollary 4.1. □

Corollary 5.1. *Suppose we use the mini-element spaces described in Corollary 3.2. If one chooses $(\nu_T, H) = (h, h^{1/2})$ in the case $k = 1$ and $(\nu_T, H) = (h^{2k}, h^{1/k})$ in the case $k \geq 2$, we obtain $h^{1/2})$ the optimal error:*

$$\|u - u^h\| \leq Ch^{k+1}.$$

6 Numerical Experiments

We first describe the algorithm used for handling the nonlinearity and the subgrid eddy viscosity term. We then present two numerical examples: one with a known analytical solution that allows for a numerical study of the convergence rates; and one benchmark problem. In both cases, the mini-element spaces ($k = 1$) are used.

6.1 Algorithm

To solve the nonlinear system a Newton method is used. Given (u^{m-1}, p^{m-1}), we find (u^m, p^m) satisfying

$$\begin{aligned}
a(u^m, v^h) + \frac{1}{2}c(u^{m-1}, u^m, v^h) + \frac{1}{2}c(u^m, u^{m-1}, v^h) - \frac{1}{2}c(u^{m-1}, v^h, u^m) \\
-\frac{1}{2}c(u^m, v^h, u^{m-1}) - b(v^h, p^m) \\
= (f, v^h) + \frac{1}{2}c(u^{m-1}, u^{m-1}, v^h) - \frac{1}{2}c(u^{m-1}, v^h, u^{m-1}) - g(u^{m-1}, v^h),\ \forall v^h \in X^h, \\
b(u^m, q^h) = 0,\ \forall q^h \in M^h.
\end{aligned} \tag{6.1}$$

This algorithm leads to a linear system of the form $Ax = b$ with A nonsymmetric. To solve this linear system we use the iterative conjugate gradient squared method of [18]. The stopping criteria of this Newton method is based on the absolute residual.

We now show that the extra stabilization term $g(u^{m-1}, v^h)$ requires a modification of the right-hand side of the linear system, that can be computed locally.

First, from (2.5), we can write

$$g(u^{m-1}, v^h) = \nu_T(\mathbb{D}(u^{m-1}), \mathbb{D}(v^h)) - \nu_T(P_{L^H}\mathbb{D}(u^{m-1}), \mathbb{D}(v^h)).$$

In this decomposition, adding the first term is straight-forward, as it is similar to the diffusive term $a(u^{m-1}, v^h)$. The difficulty is to incorporate the second term, since it couples coarse and fine meshes. Denoting a basis of X^h by $\{\phi_j^h\}_{j=1}^{N^h}$, we want to compute $(P_{L^H}\mathbb{D}(u^{m-1}), \mathbb{D}(\phi_j^h))$, for all j. Denoting a basis of L^H by $\{\psi_j^H\}_{j=1}^{N^H}$, we can write

$$P_{L^H}\mathbb{D}(u^{m-1}) = \sum_{j=1}^{N^H} \beta_j \psi_j^H, \tag{6.2}$$

where the β_j's are unknown coefficients, uniquely defined. Thus, we have

$$(P_{L^H}\mathbb{D}(u^{m-1}), \mathbb{D}(\phi_i^h))_i = (\sum_{j=1}^{N^H} \beta_j \psi_j^H, \mathbb{D}(\phi_i^h))_i = R \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_{N^H} \end{bmatrix},$$

where R is the matrix that couples the fine and large scales: $R_{ij} = (\psi_j^H, \mathbb{D}(\phi_i^h))$. To determine the unknown coefficients β_j's, it suffices to take the inner product to both sides of (6.2) with ψ_i^H:

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_{N^H} \end{bmatrix} = S^{-1}(P_{L^H}\mathbb{D}(u^{m-1}), \psi_i^H) = S^{-1}(\mathbb{D}(u^{m-1}), \psi_i^H), \tag{6.3}$$

where $S = (\psi_j^H, \psi_i^H)$ is the mass matrix associated to L^H. Thus, we have so far

$$(P_{L^H}\mathbb{D}(u^{m-1}), \mathbb{D}(\phi_i^h))_i = RS^{-1}(\mathbb{D}(u^{m-1}), \psi_i^H)_i. \tag{6.4}$$

To conclude, we decompose $\mathbb{D}(u^{m-1})$ as

$$\mathbb{D}(u^{m-1}) = \sum_{j=1}^{N^h} \alpha_j^{m-1} \phi_j^h,$$

and substitute this into (6.4):

$$(P_{L^H}\mathbb{D}(u^{m-1}), \mathbb{D}(\phi_i^h))_i = RS^{-1}R^T \left[\begin{array}{c} \alpha_1^{m-1} \\ \alpha_2^{m-1} \\ \cdots \\ \alpha_{N^h}^{m-1} \end{array} \right].$$

Since the α_j^{m-1}'s are known, it suffices to compute R and S. We note that if one chooses discontinuous piecewise polynomial basis functions for L^H, the matrix S is the block diagonal and then computing $RS^{-1}R^T$ can be done locally on each element in the coarse mesh τ^H.

6.2 Convergence Rates

We consider the equation (1.1) on the domain $\Omega = [0,1] \times [0,1]$, with a body force obtained such that the true solution is given by $u = (u_1, u_2)$,

$$u_1 = 2x^2(x-1)^2 y(y-1)(2y-1), \quad u_2 = -y^2(y-1)^2 2x(x-1)(2x-1),$$
$$p = y.$$

The fluid viscosity is $\nu = 10^{-2}$, which gives a Reynolds number of the order 10^2. From Corollary 3.2, we choose $\nu_T = h$ and H such that $H^2 \leq h$. The theoretical analysis then predicts a convergence rate of $\mathcal{O}(h)$ for the velocity in the energy norm, $\mathcal{O}(h^2)$ for the velocity in the L^2 norm, and $\mathcal{O}(h)$ for the pressure. The domain is subdivided into triangles. First, the coarse mesh is chosen such that $H = 1/2$ and the fine mesh is a refinement of the coarse mesh, so that $h = 1/4$ (here, $h = H^2$). Other pairs of meshes are obtained by successive uniform refinements (see Figure 1 for the case $H = 1/8$ and $h = 1/16$). We choose for basis functions of L^H, discontinuous piecewise constants and two quadratics defined the reference elements. If $\mathcal{F}$ denote the affine mapping from the reference element to the physical element, we have:

$$L^H = \{\mathbb{L} : \mathbb{L}|_E = \mathcal{F}\hat{\mathbb{L}},\ \forall\, \hat{\mathbb{L}} \in \hat{L}^H,\ \forall E \in \tau^H\},$$

$$\hat{L}^H = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{\partial b}{\partial x} & \frac{1}{2}\frac{\partial b}{\partial y} \\ \frac{1}{2}\frac{\partial b}{\partial y} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial b}{\partial x} \\ \frac{1}{2}\frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{pmatrix} \right\},$$

where b denotes the bubble function defined as

$$b(x,y) = 27xy(1-x-y).$$

Table 1 gives the errors and convergence rates for $u - u^h$ and $p - p^h$ in different norms. These numerical results confirm the theoretical error estimates: the convergence rates are optimal. Figure 2 shows both computed solution and exact solution for the case $(H,h) = (1/8, 1/16)$.

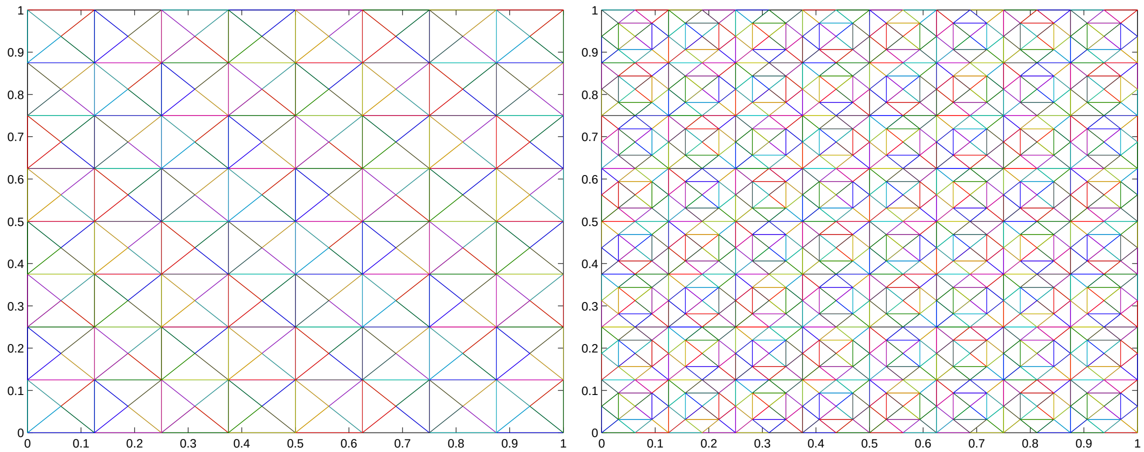
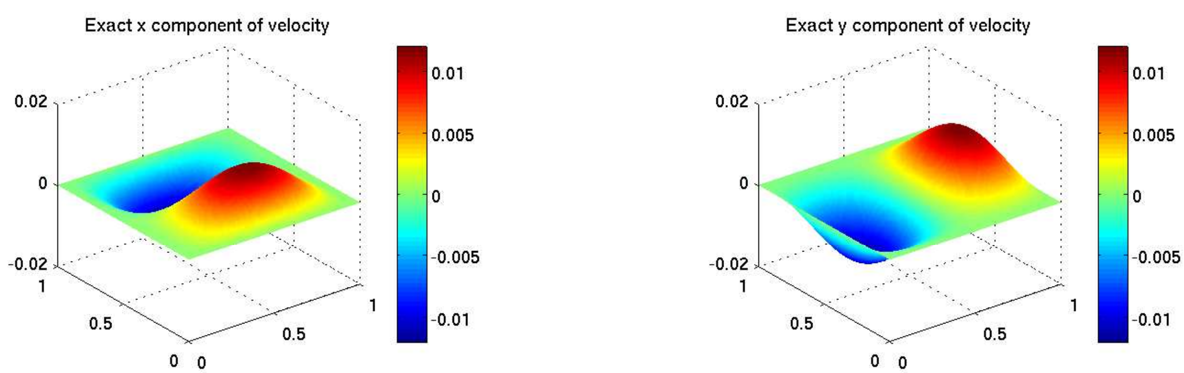


Figure 1: $H = 1/8$ with one refinement $h = 1/16$

Table 1: Numerical errors and degrees of freedom.

meshes	N^h	L^2	Rate	H_0^1	Rate	L^2 pressure	Rate
H=1/2, h=1/4	218	0.0069		0.0509		4.3269e-04	
H=1/4, h=1/8	882	0.0017	2.0211	0.0241	1.0786	2.4448e-04	0.8236
H=1/8, h=1/16	3554	3.9446e-04	2.1076	0.0108	1.1580	9.6978e-05	1.3340
H=1/16, h=1/32	14274	8.1066e-05	2.2827	0.0046	1.2313	3.3879e-05	1.5173
H=1/32, h=1/64	57218	1.6313e-05	2.3131	0.0020	1.2016	1.1026e-05	1.6195



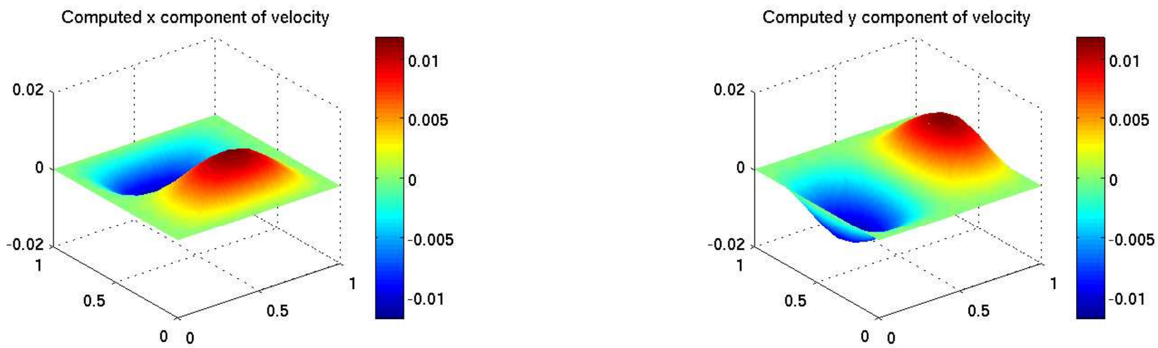


Figure 2: Comparison between the true solution and computed solution $(H, h) = (1/8, 1/16)$.

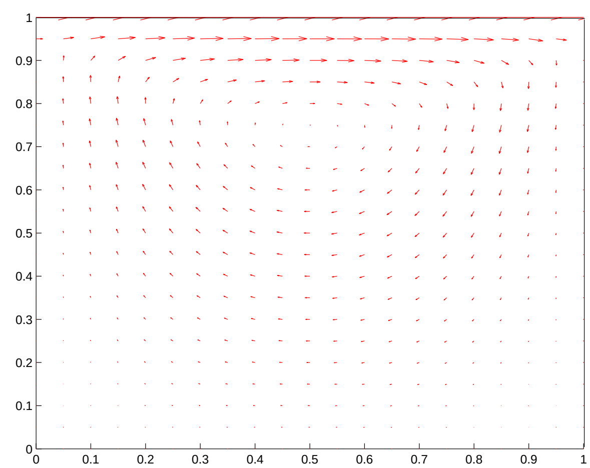


Figure 3: Velocity streamlines for $Re = 1$.

6.3 Driven Cavity Problem

The second problem is the driven cavity problem, in which fluid is enclosed in a square box, with an imposed velocity of unity in the horizontal direction on the top boundary, and a no slip condition on the remaining walls.

We consider the flow for different Reynolds number for fixed mesh where $H = 1/8$, $h = 1/16$. The same basis functions $\hat{L}^H$ are chosen as Section 6.1.The computational results for a set of different Reynolds numbers (Re= 1, 100, 2500) are shown below. In these numerical tests, we observe the effect of Reynolds number on the flow pattern. For the low Reynolds number (Re= 1), the flow has only one vortex located above the center (Figure 3). When Re= 100, the flow pattern starts to form reverse circulation cells in lower corners (Figure 4). In addition, for Re= 2500, we compare the velocity streamline behavior for Navier-Stokes equation, subgrid eddy viscosity and artificial viscosity model. Figure 5 shows that the main eddy of artificial eddy viscosity model is too small and its center is too close to the upper lid. On the other hand, with the higher Reynolds number the subgrid eddy viscosity model reproduce the main eddy well and steady flow pattern becomes more complex with reverse circulation cells in both lower corners.

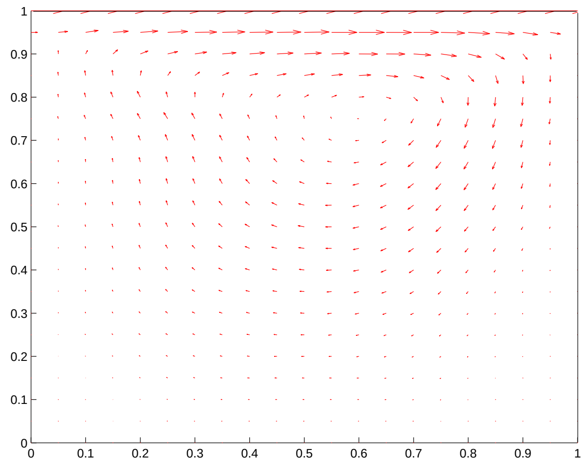


Figure 4: Velocity streamlines for $Re = 100$.

7 Concluding Remarks

In this article, we presented and analyzed a two-grid method for solving the steady-state Navier-Stokes equations. This method has the advantage of adding diffusion only on the large scales. Numerical tests showed that the new stabilization technique is robust and efficient way of solving Navier-Stokes equations for a wide range of Reynolds numbers. The simulation of this model applied to the time dependent Navier-Stokes is currently under investigation.

Acknowledgements. The authors would like to thank Professor Vince Ervin from Clemson University for his advice and for providing them with a software solving the Navier-Stokes equations, and Professor William Layton from University of Pittsburgh for his constructive comments.

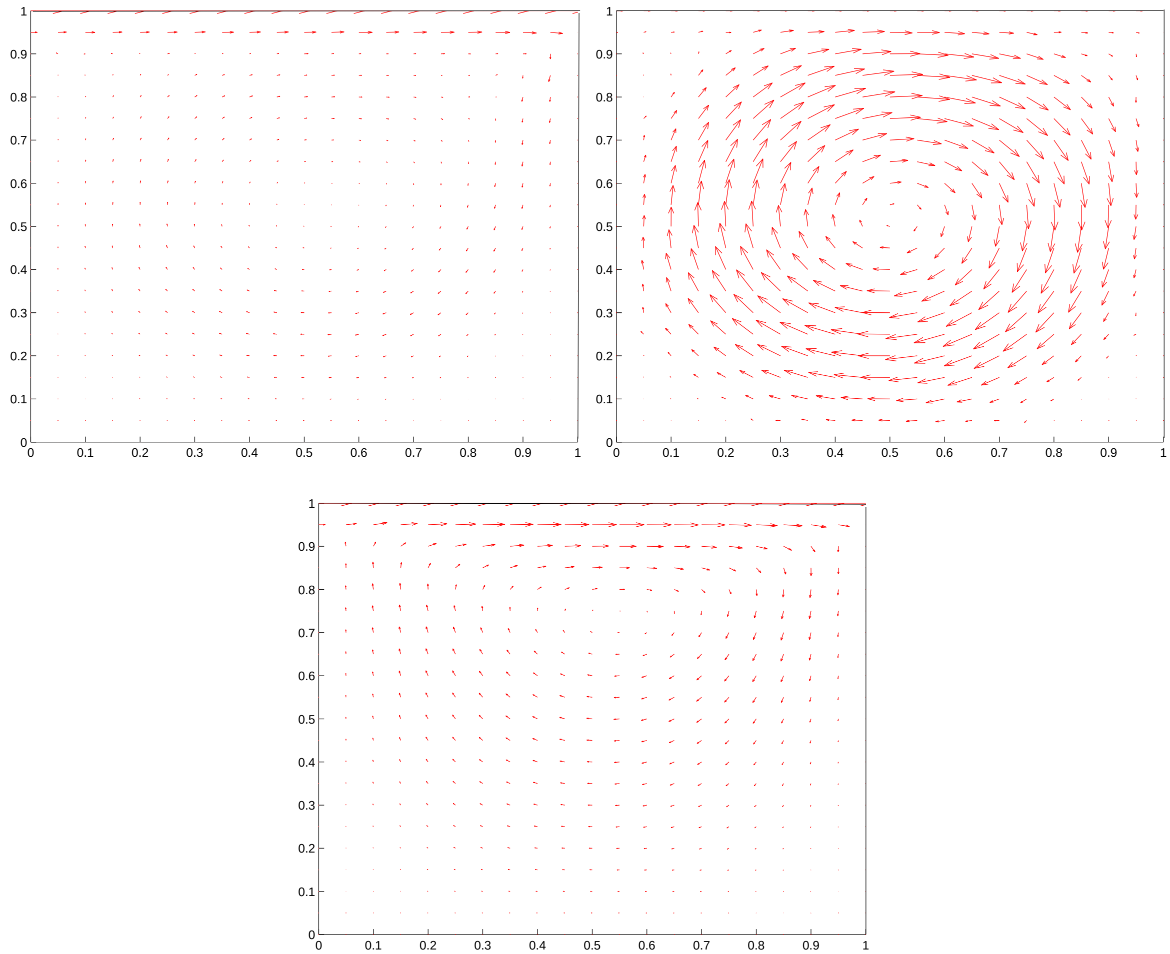


Figure 5: Velocity streamlines for $Re = 2500$ for Navier-Stokes Equation, Subgrid Eddy Viscosity Model, Artificial Viscosity Model, from left to right $(H, h) = (1/8, 1/16)$.

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