On a Well-Posed Turbulence Model

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July 29, 2004

Abstract

This report considers mathematical properties, important for practical computations, of a model for the simulation of the motion of large eddies in a turbulent flow. In this model, closure is accomplished in the very simple way:

$$\overline{u\,u} \sim \overline{\overline{u}\,\overline{u}}, \text{ yielding the model}$$
$$\nabla \cdot w = 0,\ w_t + \nabla \cdot (\overline{w\,w}) - \nu \Delta w + \nabla q = \bar{f}.$$

In particular, we prove existence and uniqueness of strong solutions, develop the regularity of solutions of the model and give a rigorous bound on the modelling error, $||\bar{u} - w||$. Finally, we consider the question of non-physical vorticial structures (false eddies), proving that the model correctly predicts on that only a small amount of vorticity results when the total turning forces on the flow are small.

1 Introduction.

The great challenge in simulation of turbulent flows from applications ranging from geophysics to biomedical device design is that equations for the pointwise flow quantities are well-known but intractable to computational

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solution and sensitive to uncertainties and perturbation in problem data. On the other hand, closed equations for the averages of flow quantities cannot be obtained directly from the physics of fluid motion. Thus, modeling in large eddy simulation (meaning the approximation of local, spacial averages in a turbulent flow) is typically based on guesswork (phenomenology), calibration (data fitting model parameters) and (at best) approximation.

If $u(x,t), p(x,t)$ are the velocity and pressure in an incompressible turbulent flow, then u, p satisfy the Navier-Stokes equation

$$u_t + \nabla \cdot (u\ u) - \nu \Delta u + \nabla p = f \text{ and } \nabla u = 0 \text{ in } \Omega \text{ for } t > 0, \tag{1.1}$$

where Ω is the flow domain and f is the body force driving the flow. If overbar denotes a local, spacial averaging operator that commutes with differentiation, then, averaging (1.1) gives the following non-closed equations for $\overline{u}, \overline{p}$:

$$\nabla \cdot \overline{u} = 0 \text{ and } \overline{u}_t + \nabla \cdot (\overline{u\ u}) - \nu \Delta \overline{u} + \nabla \overline{p} = \overline{f}, \text{ in } \Omega \text{ for } t > 0. \tag{1.2}$$

The famous closure problem which we study herein arises because $\overline{u\ u} \neq \overline{u}\ \overline{u}$. To isolate the turbulence closure problem from the difficult question of wall laws for near wall turbulence, we study (1.2) subject to periodic boundary conditions (and zero mean)

$$\begin{cases} u(x + Le_j, t) = u(x,t) \\ \int_\Omega u_0 dx = \int_\Omega u\ dx = \int_\Omega\ f\ dx = 0 \end{cases} \tag{1.3}$$

The closure problem is to replace the tensor $\overline{u\ u}$ with a tensor $S(\overline{u},\ \overline{u})$ depending only on $\overline{u}$ (and not u). There are very many closure models proposed in large eddy simulation (or LES) (see Sagaut [Sag01] and [Joh04] for examples) reflecting the centrality of closure in turbulence simulation. Calling w, q the resulting approximations to $\overline{u}, \overline{p}$, we are led to considering the model.

$$\nabla \cdot w = 0 \text{ and } w_t + \nabla \cdot S(w,w) - \nu \Delta w + \nabla q = \overline{f}, \tag{1.4}$$

With any reasonable averaging operator, the true averages, $\overline{u}, \overline{p}$ are smoother than u, p. Thus, solutions of any derived model such as (1.4) <u>should</u> be more regular than the Navier-Stokes equations. However, in spite of the intense interest in closure models for turbulence, there are very few whose mathematical development even parallels that of the NSE, e.g., [MP94], [Joh04], [Sag01], [LL02].

In this report, we consider the simplest, accurate closure model. If u is a constant flow then $u = \overline{u}$. The simple closure model (that is exact on constant flows) is thus

$$\overline{u\,u} \cong \overline{\overline{u}\,\overline{u}}\ (=:\ S(\overline{u},\ \overline{u})), \tag{1.5}$$

leading to

$$\nabla \cdot w = 0 \text{ and } w_t + \nabla \cdot (\overline{w\,w}) - \nu \Delta w + \nabla q = \overline{f}. \tag{1.6}$$

In some sense, (1.5) is the most basic (hence zeroth) model in LES. If can arise by dropping the cross and Reynolds terms and keeping only the Leonard/resolved term [Leo74]. It is the zeroth Stolz-Adams ADM model [SA99], [Sag96]. It is the rational model [GL00] truncated to $O(\delta^2)$ terms.

We shall show that the model (1.6) has the mathematical properties which are expected of a model derived from the NSE by an averaging operation and which are important for practical computations using (1.6).

The choice averaging operator in (1.6) is a differential filter, [FM01], [DJ03], [Ger86], [LL03]. Let $\delta > 0$ denote the averaging radius (typically related to the finest computationally feasible mesh used in a simulation of (1.6)). Given a periodic function $\phi(x) \in L^2(\Omega)$, define its average $\overline{\phi}$ to be the unique periodic solution of

$$A\overline{\phi} := -\delta^2 \Delta \overline{\phi} + \overline{\phi} = \phi, \text{ in } \Omega = (0, L)^d.$$

With this averaging, the model (1.5) has consistency $O(\delta^2)$:

$$\overline{u\,u} = \overline{\overline{u}\,\overline{u}} + O(\delta^2), \text{ for smooth } u.$$

We prove that the model has a <u>unique, strong solution</u> and that the smoothness of this solution is limited only by the smoothness of the problem data u_0 and f. These properties are essential for numerical simulations. We prove that as $\delta \to 0$, the solution of the model is $w \to u$ (the solution of the NSE) in the appropriate sense. This property is critical for consistency of the <u>solution</u> of the model with the true flow averages. We also give connections of the model with the $K - 41$ theory of homogeneous isotropic turbulence. Finally, we introduce the question of spurious vorticity/eddies generated by the model and give one weakly positive result.

The development of these mathematical properties is based upon a skew symmetry property of the model's nonlinearity and the energy estimate it induces. To be specific, for sufficiently smooth functions which are periodic and divergence free

$$\int_\Omega \nabla \cdot (\overline{w\ w}) \cdot Aw dx = \int_\Omega \nabla \cdot (w\ w) \cdot A^{-1} Aw dx = \int_\Omega \nabla \cdot (w\ w) \cdot w dx = 0.$$

Thus, (loosely speaking) multiplying the model by Aw and integrating over the domain shows that w satisfies the very strong stability property

$$\sup_{0<t<T} \{||w(t)||^2 + \delta^2 ||\nabla w(t)||^2\} + \nu \int_0^T ||\nabla w||^2 + \delta^2 ||\Delta w||^2 dt \leq C(\nu, T, \overline{u_0}, \overline{f})$$

where $||\cdot||$ is the usual $L^2(\Omega)$ norm. This property is also shared by suitably defined weak solutions of the model proven to exist in [LL03]. Exploiting this strong stability property, we shall first prove existence and regularity of *strong* solutions to the model.

Theorem 1.1 *For any $u_0 \in (L^2(\Omega))^3$ with zero mean and $\nabla \cdot u_0 = 0$, $f \in L^2([0,T], (H^{-1})^3)$ the model (1.6) has a unique L-periodic weak solution*

$$(w, q) \in [L^2([0,T], (H^2)^3) \cap L^\infty([0,T], (H^1)^3)] \times L^2([0,T] \times \Omega) \tag{1.7}$$

and the energy equality holds:

$$\begin{cases} \frac{1}{2}(||w(t)||^2 + \delta^2 ||\nabla w(t)||^2) + \int_0^t \nu(||\nabla w(t)||^2 + \delta^2 ||\Delta w(t')||^2) dt' \\ = \frac{1}{2}(||\overline{u}_0||^2 + \delta^2 ||\nabla \overline{u}_0||^2) + \int_0^t (\overline{f}, w) dt'. \end{cases} \tag{1.8}$$

If $u_0 \in C^\infty_\#(\Omega \times (0,T)), \nabla \cdot u_0 = 0$ and $f \in C^\infty_\#(\Omega \times (0,T))$, then that solution is strong and smooth, $(w, q) \in [C^\infty(\Omega \times (0,T))]^4$

Remark 1.1 *On the left-hand side of (1.8) for δ fixed and the viscosity $\nu \to 0$, we retain a quite strong regularity property $w \in L^\infty(0,T; H^1(\Omega))$. Using this observation, existence can also be proven for the Euler model that arises by setting the viscosity coefficient $\nu = 0$ in (1.6). (This fact was pointed out to the author W.L. by D. D. Holm and E. Titi.)*

Corollary 1.1 *Consider the model (1.6) with $\nu = 0$ and $\delta > 0$. For any $u_0 \in L^2(\Omega), f \in L^2(\Omega \times (0,T))$, the Euler-LES model*

$$\nabla \cdot w = 0, \quad \text{and} \quad w_t + \nabla \cdot (\overline{w\, w}) + \nabla q = \overline{f}$$

has a unique weak solution. That weak solution satisfies the energy equality:

$$\begin{aligned} \frac{1}{2}(||w(t)||^2 + \delta^2 ||\nabla w(t)||^2) &= \\ = \frac{1}{2}(||u_0(t)||^2 + \delta^2 ||\nabla u_0(t)||^2) &+ \int_0^t (f, w)(t')dt. \end{aligned} \tag{1.9}$$

One of the most important criteria in evaluating a model is that it be <u>accurate</u>. Yet, there are few analytical studies of $||w - \overline{u}||$ primarily because of deficiencies in the analytical tools available. We are not able herein to give a complete and comprehensive, á priori proof of the model's accuracy. Nevertheless, we prove some partial results that confirm that the model has properties expected of one derived by averaging from the NSE. For example, we show in Section 3 that as $\delta \to 0$ there is a subsequence δ_j with

$$w \to u, \text{ a weak solution of the NSE, as } \delta_j \to 0$$

and if that weak solution u is unique

$$w \to u \quad \text{as} \quad \delta \to 0.$$

This result addresses the issue of "consistency in the limit" [Lay01] as $\delta \to 0$ of the model. The model (1.6) and Camassa-Holm model ([FHT02]) have a similar energy balance and both satisfy a limit-consistency result.

Let $\overline{\tau}$ denote the modeling consistency error tensor

$$\overline{\tau}(u,\ u) := \overline{\overline{u}\,\overline{u}} - \overline{u\, u}$$

then it is straightforward to see that the true flow averages $\overline{u}$ satisfy

$$\overline{u}_t + \nabla \cdot (\overline{\overline{u}\,\overline{u}}) + \nabla \overline{p} - \nu \Delta \overline{u} = \overline{f} + \nabla \cdot \overline{\tau}$$

and the error in the model $e = \overline{u} - w$ satisfies an equation driven only by the averaged consistency error $\nabla \cdot \overline{\tau}$:

$$e_t + \nabla \cdot (\overline{\overline{u}\,\overline{u} - w\ w}) + \nabla(\overline{p} - q) - \nu \Delta e = \nabla \cdot \overline{\tau}.$$

Thus, $||e||$ being small depends upon two factors: a small consistency error, $||\overline{\tau}||$ small, and a strong enough stability property that $||e||$ is bounded by some norm of $\overline{\tau}$. If the stability constants in this bound are to be independent of δ, then (with the analytic tools available at this time) an extra condition ensuring global uniqueness of u is necessary. In Section 3, we prove such a bound (which ensures the model is verifiable in the sense of [Lay01]). The other criteria is that $||\overline{\tau}||$ is small as $\delta \to 0$. This is often performed by computational experiments, see the discussion in [Jim99]. Herein, we give in Section 3, analytical bounds verifying that $\overline{\tau} \to 0$ (with rates) as $\delta \to 0$ for smooth enough solutions u of the NSE. One main open question is that the natural norm on $\overline{\tau}$ for verifiability is stronger than the natural norm on $\overline{\tau}$ for evaluating the model's consistency error.

Lastly, we consider the question of spurious vorticity. Our result on this question is positive but weak. We show that if $\nabla \times u_0 \equiv 0$ and $\nabla \times f \equiv 0$ then (correctly) $\nabla \times w \equiv 0$ and that the zero vorticity state is stable: if $\nabla \times u_0$ and $\nabla \times f$ are both small then $\nabla \times w$ is comparably small.

2 Uniqueness, Regularity and Stability of the Model.

2.1 Background

Let $W^{1,p}_\sharp$ denote the space of all $[0, L]^3$-periodic functions with restriction on the cell $Q = [0, L]^3$ in the Sobolev space $W^{1,p}(Q)$. By the same way one notes $H^r_\sharp = H^r = W^{1,2}_\sharp$, and for the sake of simplicity, L^p instead of $L^p_\sharp$ when no confusion occurs, and $L^p([0, T] \times Q)$ for functions space periodic, or also $L^p_{t,x}$. Recall that the averaging operator A is defined by

$$(2.1) \qquad A\overline{\phi} := -\delta^2 \Delta \overline{\phi} + \overline{\phi} = \phi, \quad \text{in } Q, \quad \text{and} \quad \phi(x + Le_j) = \phi(x).$$

defining an operator $A : W^{1,p}_\sharp \to (W^{1,p'}_\sharp)' = W^{-1,p}_\sharp$. One easily sees that A is self-adjoint and has the regularity property

$$(2.2) \qquad \forall\, r, \quad \forall\, \phi \in H^r, \quad \overline{\phi} = A^{-1}\phi \in H^{r+2}.$$

Let w be a solution of (1.6) subject to periodic boundary conditions with zero mean and the initial condition with zero divergence and mean. One notes

$$V = \{v \in (L^2_\sharp)^3 = (L^2)^3;\ \nabla \cdot v = \int_Q v = 0\}$$

It has been shown in [LL03] that when

$$u_0 \in V \quad f \in L^2([0,T],(H^{-1})^3) \tag{2.3}$$

then (1.6) has a weak solution

$$w \in L^2([0,T],(H^2)^3) \cap L^\infty([0,T],(H^1)^3)$$

Throughout the section, we assume that (2.3) holds.

2.2 Uniqueness

We first prove the following uniqueness result.

Theorem 2.1 *Assume that* (2.3) *holds. Then there exists at most one solution to (1.6).*

Proof. Let (w_1,q_1) and (w_2,q_2) be two solutions to (1.6). Write $\phi = w_2 - w_1$, $\delta q = q_2 - q_1$. Thus ϕ is solution to the problem

$$\left\{ \begin{array}{l} \phi_t + \nabla \cdot (\overline{w_2 w_2 - w_1 w_1}) - \nu \triangle \phi + \nabla \delta q = 0, \\ \nabla \cdot \phi = 0, \\ \phi_{t=0} = 0, \end{array} \right. \tag{2.4}$$

subject to periodic boundary conditions with zero mean. Notice that by using Schartz rule in the absence of boundaries one has in the sense of the distributions

$$\nabla \cdot (\overline{w_2 w_2 - w_1 w_1}) = A^{-1} \nabla \cdot (w_2 w_2 - w_1 w_1).$$

Using $A\phi$ as test function in (2.4) and integrating in space on a cell yields

$$\left\{ \begin{array}{l} \dfrac{d}{2dt} \displaystyle\int (|\phi|^2 + \delta^2 |\nabla \phi|^2) + \nu \int (|\nabla \phi|^2 + \delta^2 |\Delta \phi|)^2 = \\ -\displaystyle\int A^{-1} \nabla \cdot (w_2 w_2 - w_1 w_1). A\phi \end{array} \right. \tag{2.5}$$

We focus our attention on the r.h.s of (2.5). By using self-adjointness of A one has

$$\int A^{-1} \nabla \cdot (w_2 w_2 - w_1 w_1). A\phi = \int \nabla \cdot (w_2 w_2 - w_1 w_1). \phi.$$

Furthermore, using the incompressibilty constraint, one obtains after an easy algebraic computation and an integration by parts,

$$\int \nabla \cdot (w_2 w_2 - w_1 w_1).\, \phi = -\int (\phi \nabla)\phi.\, w_1.$$

Finally,

$$\frac{d}{2dt}\int (|\phi|^2 + \delta^2|\nabla\phi|^2) + \nu\int (|\nabla\phi|^2 + \delta^2|\Delta\phi|)^2 = \int (\phi\nabla)\phi.\, w_1. \tag{2.6}$$

By Cauchy-Schwarz inequality,

$$\left|\int (\phi\nabla)\phi \cdot\ w_1\right| \leq ||w_1||_{(L^4)^3}||\phi||_{(L^4)^3}||\nabla\phi||_{(L^2)^3}.$$

Since

$$w_1, w_2, \phi \in L^2([0,T],(H^2)^3) \cap L^\infty([0,T],(H^1)^3) \subset L^\infty([0,T],(L^4)^3),$$

(by using Sobolev embedding theorem) it follows that

$$||w_1||_{L^4}||\phi||_{(L^4)^3}||\nabla\phi||_{(L^2)3} \leq C||\nabla\phi||^2_{(L^2)^3},$$

where C is a constant which only depends on the data f and u_0. Finally, with $C' = C'(\delta)$

$$C||\nabla\phi||^2_{L^2} \leq C'(\int (|\phi|^2 + \delta^2|\nabla\phi|^2)).$$

By putting all of this together, one sees that (2.6) implies that

$$\frac{d}{2dt}\int (|\phi|^2 + \delta^2|\nabla\phi|^2) \leq C'(\int (|\phi|^2 + \delta^2|\nabla\phi|^2))$$

Since $|\phi|^2 + \delta^2|\nabla\phi|^2$ vanishes when $t = 0$, Gronwall's Lemma implies that it vanishes for almost every t. Hence, uniqueness follows and the theorem is proven.

2.3 Regularity

The aim of this subsection is the proof of the following regularity result, assuming that (2.3) holds.

Theorem 2.2 *Let $k \in \mathbf{N}$. Assume that $u_0 \in V \cap H^{k-1}$ and $f \in L^2([0,T],(H^{k-1})^3)$. Then the solution (w,q) of problem (1.6) is such that*

$$
\left\{ \begin{array}{l} w \in L^2([0,T],H^{k+2}) \cap L^\infty([0,T],H^{k+1}), \\ q \in L^2([0,T],H^k). \end{array} \right. \tag{2.7}
$$

Corollary 2.1 *When $k \geq 2$, w is continous in time and space. When f and u_0 are C^∞, then (w,q) is C^∞ in space and time.*

Proof. In the following, D^k denotes any partial derivative of total order k. The result is already proved when $k=0$. Let $k \in \mathbf{N}$. Assume

$$
u_0 \in V \cap H^{k-1} \quad \text{and} \quad f \in L^2([O,T],(H^{k-1})^3). \tag{2.8}
$$

We do the following recursive hypothesis:

$$
\left\{ \begin{array}{l} \text{for any} \quad q = 0, ..., k-1 \\ \text{and for any derivative operator of order } q,\, D^q, \\ D^q w \in L^2([0,T],(H^2)^3) \cap L^\infty([0,T],(H^1)^3). \end{array} \right. \tag{2.9}
$$

In addition to (2.9), one assumes that all the constants involved only depends on the datas u_0 and f.

It remains to prove that (2.9) implies

$$
D^k w \in L^2([0,T],(H^2)^3) \cap L^\infty([0,T],(H^1)^3). \tag{2.10}
$$

By taking the k^{th} derivative of (1.6) and using the classical Schwartz rule, one has in the sense of the distributions

$$
\left\{ \begin{array}{l} D^k w_t + \nabla \cdot (\overline{D^k(ww)}) - \nu \triangle D^k w + \nabla D^k q = D^k \overline{f}, \\ \nabla \cdot D^k w = 0, \\ D^k w_{t=0} = D^k w_0 = D^k \overline{u_0}, \end{array} \right. \tag{2.11}
$$

where boundary conditions remains periodic and still with zero mean and the initial condition with zero divergence and mean. Taking $AD^k w$ as test function in (2.11) and using the self-adjointness of the operator A yields

$$
\left\{ \begin{array}{l} \dfrac{d}{2dt} \displaystyle\int (|D^k w|^2 + \delta^2 |\nabla D^k w|^2) + \nu \int (|\nabla D^k w|^2 + \delta^2 |\Delta D^k w|)^2 \\ = < D^k f, D^k w > - \displaystyle\int \nabla \cdot (\overline{D^k(ww)}).\, AD^k w. \end{array} \right. \tag{2.12}
$$

One first notes that

$$(2.13)\qquad \begin{cases} |<D^k f, D^k w>| \leq ||D^k f||_{(H^{-1})^3}||D^k w||_{(H^1)^3} \\ \leq \dfrac{1}{2\nu}||D^k f||_{(H^{-1})^3} + \dfrac{\nu}{2}\displaystyle\int |\nabla D^k w|^2. \end{cases}$$

The second term of the r.h.s of (2.12) has to be estimated.

We show in the following that (2.9) implies

$$(2.14)\qquad \left|\int_0^T \int \nabla\cdot(\overline{D^k(ww)}).\, AD^k w\right| \leq C,$$

where the constant C involves only the datas f and u_0. Inequality (2.14) combined with (2.12) and (2.13) gives obviously (2.10).

First, by Schwartz rule,

$$\nabla\cdot(\overline{D^k(ww)}).\, AD^k w = A^{-1}\nabla\cdot(D^k(ww)).\, AD^k w.$$

Then,

$$(2.15)\qquad \int \nabla\cdot(\overline{D^k(ww)}).AD^k w = \int \nabla\cdot(D^k(ww)).\, D^k w.$$

As one notes $w = (w_1, w_2, w_3)$,

$$\nabla\cdot(D^k(ww)).\, D^k w = \partial_j(D^k(w_i w_j))D^k w_i,$$

(summation convention). Leibnitz formula reads

$$D^k(w_i w_j) = C_k^q D^q w_i D^{k-q} w_j.$$

When combining this with the constraint $\nabla\cdot D^{k-q} w = 0$, one has

$$\nabla\cdot(D^k(ww)).\, D^k w = C_k^q(\partial_j D^q w_i)D^{k-q} w_j D^k w_i.$$

Therefore, k being fixed,

$$(2.16)\qquad \int \nabla\cdot(\overline{D^k(ww)}).\, AD^k w = C_k^q \int (\partial_j D^q w_i)D^{k-q} w_j D^k w_i.$$

In the sommation with respect to the q index of the r.h.s of (2.16), one treats the case $q \geq 1$ and $q = 0$ one after each other.

Case $q \geq 1$. By the recursive assumption (2.9),

$$D^{k-q} w_j \in L^\infty([0,T], H^1) \subset L^\infty([0,T], L^6)$$

by using Sobolev imbedding theorem. Furthermore, always by (2.9), $D^k w_i \in L^2([0,T], H^1) \cap L^\infty([0,T], L^2)$. Classical interpolation inequalities using Hölder inequality (see [RL]) imply

$$D^k w_i \in L^2([0,T], H^1) \cap L^\infty([0,T], L^2) \subset L^4([0,T], L^3).$$

Finally always by (2.9),

$$\partial_j D^q w_i \in L^2([0,T], L^2).$$

Putting all together and using Hölder inequaltiy shows that for fixed $q \geq 1$, i and j,

$$(\partial_j D^q w_i) D^{k-q} w_j D^k w_i \in L^2([0,T], L^1) \subset L^1([0,T], L^1).$$

Therefore,

$$\left| \int_0^T \int (\partial_j D^q w_i) D^{k-q} w_j D^k w_i \right| \leq C, \tag{2.17}$$

for a constant C which only depends on the datas f and u_0.

Case $q = 0$. One has to consider the term $\partial_j w_i D^k w_j D^k w_i$ for fixed index. On one hand, one still has $w \in L^2([0,T], H^2) \cap L^\infty([0,T], H^1)$. Therefore,

$$\partial_j w_i \in L^\infty([0,T], L^2).$$

On the other hand, by (2.9),

$$D^k w_i, D^k w_j \in L^2([0,T], L^6).$$

Therefore,

$$\partial_j w_i D^k w_j D^k w_i \in L^1([0,T], L^{6/5}) \subset L^1([0,T], L^1),$$

yielding

$$\left| \int_0^T \int \partial_j w_i D^k w_j D^k w_i \right| \leq C, \tag{2.18}$$

for a constant C which only depends on the datas f and u_0.

When combining (2.17) and (2.18) to (2.16), one have proved (2.14) and the proof is finished.

The regularity of the pressure term is deduced from classical considerations, e.g., [AG94], [Tar78].

The corollary is a classical consequence of Theorem 2.2.

3 Accuracy of the Model.

3.1 Orientation

There are many questions that now arise. The first concern the consistancy error; we show herein that the solution of (1.6),

$$\nabla \cdot w = 0, \quad w_t + \nabla \cdot (\overline{\overline{w}\, \overline{w}}) - \nu \Delta w + \nabla q = \overline{f},$$

converges to a weak solution to the Navier-Stokes equations when δ goes to zero (stated precisely in Theorem 3.1 below) proving that the model is consistent in the limit as $\delta \to 0$.

Let τ denote the model's consistency error

$$\tau(u,\ u) := \overline{\overline{u}\ \overline{u}} - u\ u, \tag{3.1}$$

where u is a solution of the Navier-Stokes equations obtained as limit of a subsequence of the sequence $(w_\delta)_{\delta>0}$.

We also prove in Theorem 3.2 that $||\overline{u} - w||_{L^\infty(0,T;L^2(Q))}$ is bounded by $||\tau||_{L^2((0,T)\times Q)}$.

We turn to estimates of $||\tau||$ in the next section.

3.2 Limit Consistency of the Model.

Throughout the section, we assume that (2.3) holds. Let (w_δ, q_δ) the solution of (1.6) for a fixed $\delta > 0$.

Theorem 3.1 *There is a subsequence $\delta_j \to 0$ as $j \to \infty$ such that*

$$(w_{\delta_j}, q_{\delta_j}) \to (u, p) \quad as \quad \delta_j \to 0$$

where

$$(3.2)\qquad (u,p) \in [L^\infty([0,T],(L^2)^3) \cap L^2([0,T],(H^1)^3)] \times L^{\frac{4}{3}}([0,T],L^2)$$

is a weak solution of the Navier-Stokes equations. The sequence $(w_{\delta_j})_{j\in\mathbb{N}}$ *converges strongly to* u *in* $L^{\frac{4}{3}}([0,T],L^2)$ *and weakly in* $L^2([0,T];(H^1)^3)$ *while the sequence* $(q_{\delta_j})_{j\in\mathbb{N}}$ *converges weakly to* p *in the space* $L^{\frac{4}{3}}([0,T],L^2)$*.*

Before proving Theorem 3.1, we first prove the following two lemmas.

Lemma 3.1 *Let* $(f_\delta)_{\delta>0}$ *be a sequence in* $L^{\frac{4}{3}}([0,T],L^2)$*, (space periodic). Assume that* $(f_\delta)_{\delta>0}$ *converges weakly to some* f *in the space* $L^{\frac{4}{3}}([0,T],L^2)$ *when* δ *goes to zero. Then the sequence* $(\overline{f}_\delta)_{\delta>0}$ *converges weakly to* f *in the space* $L^{\frac{4}{3}}([0,T],L^2)$

Proof of Lemma 3.1. The sequence $(\overline{f}_\delta)_{\delta>0}$ is obviously bounded in the space $L^{\frac{4}{3}}([0,T],L^2)$. Thus from this sequence one can extract a subsequence (still denoted by $(\overline{f}_\delta)_{\delta>0}$) which weakly converges to some $g \in L^{\frac{4}{3}}([0,T],L^2)$. By taking $\phi \in C^\infty([0,T]\times Q)$, space periodic, one obtains after two part integrations in space in the equation satisfied by $\overline{f}_\delta$, $A\overline{f}_\delta = f_\delta$,

$$(3.3)\qquad -\delta\int_0^T\int \Delta\phi\overline{f}_\delta + \int_0^T\int \phi f_\delta = \int_0^T\int \phi f_\delta.$$

In (3.3), $\int_0^T\int \Delta\phi\overline{f}_\delta$ remains bounded because of the bound of $(\overline{f}_\delta)_{\delta>0}$ in the space $L^{\frac{4}{3}}([0,T],L^2)$. Thus, one has

$$\lim_{\delta\to 0}\delta\int_0^T\int \Delta\phi\overline{f}_\delta = 0.$$

Passing to the limit in (3.3) for $\delta \to 0$ yields

$$(3.4)\qquad \int_0^T\int \phi g = \int_0^T\int \phi f,$$

an equality which holds for each ϕ space periodic and smooth. Therefore $g = f$. The possible weak limit being unique, all the sequence converges.

Lemma 3.2 *The sequence* $(q_\delta)_{\delta>0}$ *is bounded in the space* $L^{\frac{4}{3}}([0,T],L^2)$*.*

Proof of Lemma 3.2. Taking the divergence of equation (1.6) yields

$$-\Delta q_\delta = \nabla \cdot (\nabla \cdot \overline{w_\delta w_\delta}) - \nabla \cdot f, \tag{3.5}$$

with periodic conditions and mean value equal to zero, $\int_Q q_\delta = 0$. The energy estimate for w shows that the sequence $(w_\delta)_{\delta>0}$ is bounded in the space $L^\infty([0,T],(L^2)^3) \cap L^2([0,T],(H^1)^3)$ included in $L^\infty([0,T],(L^2)^3) \cap L^2([0,T],L^6)$. Hölder inequality implies

$$L^\infty([0,T],(L^2)^3) \cap L^2([0,T],L^6) \subset L^{\frac{8}{3}}([0,T],L^4).$$

Consequently, the sequence $(w_\delta w_\delta)_{\delta>0}$ is bounded in $L^{\frac{4}{3}}([0,T],(L^2)^9)$, from which one deduces that

(3.6) The sequence $(\overline{w_\delta w_\delta})_{\delta>0}$ is bounded in $L^{\frac{4}{3}}([0,T],(L^2)^9)$

One concludes from the classical elliptic theory and from (3.5) that $(q_\delta)_{\delta>0}$ is bounded in

$$L^{\frac{4}{3}}([0,T],L^2) + L^2([0,T],L^2) \subset L^{\frac{4}{3}}([0,T],L^2)$$

and the lemma is proved.

Remark 3.1 *Note that it is easy deduced from the condiderations above that*

(3.7) *the sequence* $(\partial_t w_\delta)_{\delta>0}$ *is bounded in* $L^{\frac{4}{3}}([0,T],(H^{-1})^3)$.

Proof of Theorem 3.1. Because of the bound of the sequence $(w_\delta)_{\delta>0}$ in $L^2([0,T],(H^1)^3)$ one can extract a subsequence (still denoted $(w_\delta)_{\delta>0}$) which converges weakly to some $u \in L^2([0,T],(H^1)^3)$ when δ goes to zero. Thanks to Lemma 3.2, one can extract from the sequence $(q_\delta)_{\delta>0}$ a subsequence (still denoted by the same) which converges weakly to some p in $L^{\frac{4}{3}}([0,T],L^2)$.

We shall show that (u,p) is a weak solution to the Navier-Stokes equations by passing to the limit in each term of the equations. For this, let (v,q) be C^∞ in space and time, space periodic. One has

$$\left\{ \begin{array}{l} \int_0^T \int \partial_t w_\delta . v - \int_0^T \int \overline{w_\delta w_\delta} \, . \nabla v + \nu \int_0^T \int \nabla w_\delta . \nabla v \\ - \int_0^T \int q_\delta \, \nabla \cdot v = \int_0^T < f, v >, \\ \int_0^T \int \nabla q . w_\delta = 0. \end{array} \right. \tag{3.8}$$

Note at first that by a part integration one has

$$0=\int_0^T\int q\,\nabla\cdot w_\delta=-\int_0^T\int\nabla q.w_\delta\to-\int_0^T\int\nabla q.u=0. \tag{3.9}$$

Thus

$$\nabla\cdot u=0. \tag{3.10}$$

The weak convergence of the sequence $(q_\delta)_{\delta>0}$ in $L^{\frac{4}{3}}([0,T],L^2)$ yields

$$\lim_{\delta\to 0}\int_0^T\int q_\delta\,\nabla\cdot v=\int_0^T\int p\,\nabla\cdot v. \tag{3.11}$$

Estimate (3.7) makes sure that one can extract a subsequence from the sequence $(\partial_t w_\delta)_{\delta>0}$ which converges weakly in $L^{\frac{4}{3}}([0,T],(H^{-1})^3)$ to some g. When v has a compact support in time,

$$\int_0^T\int\partial_t w_\delta.v=-\int_0^T\int w_\delta.\partial_t v.$$

Consequently,

$$\lim_{\delta\to 0}\int_0^T\int w_\delta.\partial_t v=\int_0^T\int u.\partial_t v=-<\partial_t u,v>=-<g,v>,$$

the last brakets having to be considered in the sense of the distributions. Then $g=\partial_t u$ in the sense of the distributions. Then $\partial_t u\in L^{\frac{4}{3}}([0,T],(H^{-1})^3)$ and one have

$$\lim_{\delta\to 0}\partial_t w_\delta.v=\int_0^T\int\partial_t u.v \tag{3.12}$$

Finally one has obviously,

$$\lim_{\delta\to 0}\int_0^T\int\nabla w_\delta.\nabla v=\int_0^T\int\nabla u.\nabla v \tag{3.13}$$

It remains to pass to the limit in the nonlinear term $\nabla\cdot\overline{w_\delta w_\delta}$.

We already now that the sequence $(w_\delta w_\delta)_{\delta>0}$ is bounded in $L^{\frac{4}{3}}([0,T],(L^2)^9)$ (see the proof of lemma 3.2). Thus, up to a subsequence, it converges weakly to some Ψ in $L^{\frac{4}{3}}([0,T],(L^2)^9)$. Applying lemma 3.1, $(\overline{w_\delta w_\delta})_{\delta>0}$ converges weakly to Ψ in $L^{\frac{4}{3}}([0,T],(L^2)^9)$. We have to prove that $\Psi=uu$.
The bound of the sequence $(w_\delta)_{\delta>0}$ in $L^2([0,T],(H^1)^3)$ combined with the bound of $(\partial_t w_\delta)_{\delta>0}$ in $L^{\frac{4}{3}}([0,T],(H^{-1})^3)$ make sure that the sequence $(w_\delta)_{\delta>0}$

is compact in $L^{\frac{4}{3}}([0,T],(L^2)^3)$, and the convergence is strong in $L^{\frac{4}{3}}([0,T],(L^2)^3)$ by using Aubin-Lions's Lemma (one point that we had claim in the statement of Theorem 3.1). By classical arguments using inverse Lebesgue's Theorem, e.g., [Tar78] one can extract a subsequence (always denoted by the same) which converges *a.e.* in $[0,T]\times Q$ to w. Consequently, $(w_\delta w_\delta)_{\delta>0}$ converges *a.e* to uu and this suffises to make sure that $\Psi = uu$. Then

$$\lim_{\delta\to 0}\int_0^T\int \overline{w_\delta w_\delta}\,\nabla v = \int_0^T\int uu\,\nabla v \tag{3.14}$$

When one puts together (3.8), (3.9), (3.11), (3.12), (3.13) and (3.14), one has

$$\left\{\begin{array}{l} \int_0^T\int \partial_t u.v - \int_0^T\int uu\,.\nabla v + \nu\int_0^T\int \nabla u.\nabla v \\ -\int_0^T\int p\,\nabla\cdot v = \int_0^T < f, v >, \\ \int_0^T\int \nabla q.u = 0, \end{array}\right. \tag{3.15}$$

which implies that (u,p) is a weak solution to the Navier-Stokes equations. The proof of Theorem 3.1 is complete.

3.3 Verifiability of the Model.

Theorem 3.2 *Suppose the true solution of the NSE satisfies the regularity condition* $||\nabla u|| \in L^4(0,T)$. *Let* $\tau := \overline{u}\;\overline{u} - u\;u$. *Then* $\overline{u} - w$ *satisfies*

$$\begin{aligned} &||(\overline{u}-w)(t)||^2 + \nu\int_0^t ||\nabla(\overline{u}-w)(s)||^2 ds \\ &\leq C\nu^{-1}e^{\nu^{-3}A(t)}\int_0^t ||\tau||^2 ds \end{aligned} \tag{3.16}$$

where $A(t) := \int_0^t ||\nabla u||^4 ds$.

Remark: It is straightforward to weaken the assumption $||\nabla u|| \in L^4(0,T)$ to the Serrin [Ser63] condition that $u \in L^r(0,T;L^5(Q))$. The main problems with the estimate (3.16), however, are (1) the multiplier $e^{-\nu^{-3}A(t)}$ is huge for small ν, and (2) the natural analytic norm for measuring the consistency error is $L^2(Q\times(0,T))$. However, (as we shall see) giving analytic bounds on $||\tau||_{L^2(Q\times(0,T))}$ depends (since τ is quadratic in u) on á priori bounds on first and second derivatives of u.

3.4 Accuracy of the Model.

Proof of Theorem 3.2. As noted in the introduction $e = \overline{u} - w$ satisfies (in the sense of its variational formulation), $\nabla \cdot e = 0, e(0) = 0$ and

$$e_t + \nabla \cdot (\overline{\overline{u}\,\overline{u} - w\ w}) + \nabla(\overline{p} - q) - \nu \Delta e = \nabla \cdot \overline{\tau}.$$

where $\tau = \overline{u}\,\overline{u} - u\ u$. This is exactly the perturbation equation for the model we study. Under the stated assumption on u, u is a strong solution of the NSE and w is a strong solution of the model. Thus, this equation holds in the strong sense. Taking the inner product with Ae and following the proof of the model's basic energy estimate gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{||e||^2 + \delta^2||\nabla e||^2\} + \nu\{||\nabla e||^2 + \delta^2||\Delta e||^2\}\\ &+(\nabla \cdot (\overline{u}\ \overline{u} - w\ w), e) \leq -(\tau, \nabla e)\end{aligned}$$

Writing $\overline{u}\ \overline{u} - w\ w = we + e\overline{u}$ and using $(\nabla \cdot (we), e) = 0$ and $|(\tau, \nabla e)| \leq \frac{\nu}{2}||\nabla e||^2 + \frac{1}{2\nu}||\tau||^2$ we then have

$$\begin{aligned}&\frac{d}{dt}\{||e||^2 + \delta^2||\nabla e||^2\} + \nu\{||\nabla e||^2 + \delta^2||\Delta e||^2\} \leq \nu^{-1}||\tau||^2 +\\ &-(e \cdot \nabla\overline{u}, e) \leq \nu^{-1}||\tau||^2 + C||\nabla e||^{3/2}||e||^{1/2}||\nabla\overline{u}||.\end{aligned}$$

Thus, using $ab \leq \epsilon a^{4/3} + C\epsilon^{-3}b^4$, we obtain

$$\begin{aligned}&\frac{d}{dt}\{||e||^2 + \delta^2||\nabla e||^2\} + \nu\{||\nabla e||^2 + \delta^2||\Delta e||^2\}\\ &\leq C\nu^{-1}||\tau||^2 + C\nu^{-3}||\nabla\overline{u}||^4||e||^2.\end{aligned}$$

Gronwall's inequality then implies

$$\begin{aligned}&||e(t)||^2 + \delta^2||\nabla e(t)||^2 + \nu\int_0^t ||\nabla e||^2 + \delta^2||\Delta e||^2 ds\\ &\leq C\nu^{-1}e^{\nu^{-3}A(t)}\int_0^t ||\tau||^2 ds,\end{aligned}$$

where $A(t) = \int_0^t ||\nabla\overline{u}||^4\ ds$. Since we are searching for a result which is uniform in δ, the δ-terms on the inequalities LHS are dropped and, on the RHS, the stability bound

$$||\nabla\overline{u}|| \leq ||\nabla u||$$

(see lemma 4.1 and estimate (4.2) below) is used with the assumption that $||\nabla u|| \in L^4(0,T)$. $\square$

4 Consistency error estimates

Recall that

$$\tau(u,\ u) := \overline{u}\ \overline{u} - u\ u, \tag{4.1}$$

In this section, we shall give bounds on $||\tau||_{L^1([0,T]\times Q)}$ as $\delta \to 0$ in the general case. An estimate of $||\tau||_{L^2((0,T)\times Q)}$ will be provided under additional regularity properties.

First, classical results on singular perturbations are needed. They are recalled in the next subsection.

4.1 Some singular perturbations results

Lemma 4.1 *Let $\varphi \in L^p_x$, $1 \le p < \infty$. Then*

$$||A_\delta^{-1}\varphi||_{L^p_x} \le ||\varphi||_{L^p_x} \tag{4.2}$$

Moreover, when $p > 1$, $(A_\delta^{-1}\varphi)_{\delta>0}$ converges towards φ strongly in L^p_x.

This is a direct consequence of classical stability results in elliptic theory. For completeness, we give a short proof below, condensed from [Lew04].

Proof. Put $\overline{\varphi} = A_\delta^{-1}\varphi$. Recall that

$$-\delta^2\Delta\overline{\varphi} + \overline{\varphi} = \varphi \tag{4.3}$$

Take $\psi(\overline{\varphi}) = \overline{\varphi}|\overline{\varphi}l^{p-2}$ as test function in (4.3) when $p > 1$, (with the modification that if $p = 1$ we take $\psi(\overline{\varphi}) = \mathrm{sgn}(\overline{\varphi})$ and use some technical tools) and integrate by part. This yields

$$\delta^2 \int \psi'(\overline{\varphi})|\overline{\varphi}|^2 + \int |\overline{\varphi}|^p = \int \varphi\psi(\overline{\varphi}) \tag{4.4}$$

Because ψ is a non decreasing function, we can deduce from (4.4) that

$$\int |\overline{\varphi}|^p \le \int \varphi\psi(\overline{\varphi}) \tag{4.5}$$

Inequality (4.2) is directly deduced from (4.5) when $p = 1$. Assume now that $p > 1$. Then (4.5) yields

$$\int |\overline{\varphi}|^p \le \int |\varphi|\ |\overline{\varphi}|^{p-1} \tag{4.6}$$

We use Hölder inequality in the r.h.s of (4.6). Then (4.6) becomes

$$||\overline{\varphi}||^p_{L^p_x} \leq ||\varphi||_{L^p_x} ||\overline{\varphi}||^{p-1}_{L^p_x} \tag{4.7}$$

yielding (4.2).

The next result is easily proven as well, see e.g., [Lio68].

Lemma 4.2 *Let* $\varphi \in L^2_x$. *Then*

$$||\overline{\varphi} - \varphi||_{L^2_x} \leq \frac{\delta}{\sqrt{2}} ||\nabla \varphi||_{L^2_x} \tag{4.8}$$

4.2 L^1 Consistency erreur

Throughout this subsection, one assumes that $f \in L^2([0,T], (H^{-1}_x)^3)$. Let u be any solution to the Navier-Stokes equation. Recall that

$$\tau(u,\ u) := \overline{u}\ \overline{u} - u\ u, \tag{4.9}$$

Introduce

$$W(T) = \int_Q |u_0(x)|^2 dx + \frac{1}{\nu} \int_0^T ||f||_{(H^{-1}_x)^3} \tag{4.10}$$

Lemma 4.3 *The following holds*

$$||\tau||_{L^1([0,T],(L^1)^9)} \leq \sqrt{2}\, \delta\, T^{\frac{1}{2}} \nu^{-\frac{1}{2}} W(T) \tag{4.11}$$

Proof. Because u is a solution to the Navier-Stokes equation then the classical energy estimate holds for all $t \leq T$,

$$\int_Q |u(t,x)|^2 dx + \nu \int_0^t \int_Q |\nabla u(t',x)|^2 dx dt' \leq W(T). \tag{4.12}$$

Next write

$$\tau = (u + \overline{u})(\overline{u} - u).$$

Thus, by using (4.2), one has

$$||\tau||_{L^1([0,T],(L^1)^9)} \leq 2||u||_{L^2([0,T],(L^2)^3)} ||\overline{u} - u||_{L^2([0,T],(L^2)^3)}$$

Therefore, (4.11) is a consequence of (4.8) combined to (4.12).

4.3 L^2 Consistency erreur

Because estimate (3.16) involves L^2 norm of τ, we are lead to seek for an estimate of this quantity. As already mentionned in Theorem 3.2, a regularity assumption on the velocity is needed to derive estimate (3.16). Due to the nature of the filter, an other regularity assumption has to be introduced. This regularity is known in the 2D case, but not in the 3D case. We stay here in the 3D case.

However, we stress that such kind of estimates can be found in [Sag01] and references therein, in the 1D case and for C^∞ fields. Our result complements these since it considers solutions with the (limited) regularity typical of solutions of the NSE.

Proposition 4.1 *Let u be a solution to the NSE. Assume that*

$$u \in (L^4([0,T] \times Q))^3 \cap L^1([0,T],(H^2)^3) \tag{4.13}$$

Then one has

$$||\tau||_{L^2([0,T],(L^2)^9)} \leq C\delta, \tag{4.14}$$

where $C = C(||u||_{(L^4([0,T]\times Q))^3}, ||u||_{L^1([0,T],(H^2)^3)})$.

Proof. Observe first that by Cauchy-Scharz inequality and (4.2),

$$||\tau||_{L^2([0,T],(L^2)^9)} \leq 2||u||_{L^4_{x,t}}||\overline{u} - u||_{L^4_{x,t}}. \tag{4.15}$$

Next, it is known that

$$||\overline{u} - u||_{L^4_x} \leq C||\overline{u} - u||^{\frac{1}{4}}_{L^2_x}||\nabla(\overline{u} - u)||^{\frac{3}{4}}_{L^2_x} \tag{4.16}$$

Because we are in a periodic case, then (4.8) applies to the quantity $||\nabla(\overline{u} - u)||^{\frac{3}{4}}_{L^2_x}$. Then (4.14) is deduced from (4.15) combined to (4.16) and (4.8), a time integration and the use of Hölder inequality.

5 Vortex Structures of the Model.

In underresolved simulation of fluid flow at higher Reynolds numbers vortices often appear that seem to be spurious. Are these the result of backscatter in the true flow equations, so that the extra vortices are physically correct for a small physical perturbation? Are they non-physical vortices excited by truncation error terms? Are they non-physical vortices that appear as solutions of a turbulence model used that do not reflect appropriate averages of the true flow's eddies? Studies of the second question have been pioneered by Brown and Minon [BM95] and Drikakis and Smolarkiewicz [DS01]. Our aim here is to begin considering the third question in an (admittedly quite simplified) setting which admits analytical attack.

Taking the curl of the model we obtain an equation for the vorticity predicted by the model. If the model is appropriate, $\nabla \times w$ should be a non-spurious approximation of the true vorticity $\nabla \times u$. The question is: does the model so-derived for ω make non-physical predictions of $\nabla \times u$? Since such questions center on the relationship between $\nabla \times u$ and $\nabla \times w$ we first note that equations for the vorticity predicted by the model are easily derived. Indeed, taking the curl of the LES model shows that $\nabla \times w =: \omega$ satisfies, $\omega(0) = \overline{\nabla \times u_0}$ and

$$\omega_t + \overline{w \cdot \nabla \omega + \omega \cdot \nabla w} - \nu \Delta \omega = \overline{\nabla \times f}, \text{ if } \Omega \subset \mathbf{R}^3, \tag{5.1}$$

and

$$\omega_t + \overline{w \cdot \nabla \omega} - \nu \Delta \omega = \overline{\nabla \times f}, \text{ if } \Omega \subset \mathbf{R}^2. \tag{5.2}$$

First, we consider the simplest (and easiest) case: we show that if $\nabla \times u_0 = \nabla \times f \equiv 0$ then $\omega \equiv 0$, i.e., no spurious vorticity is generated by the model of the nonlinear interaction.

Proposition 5.1 *Let f and u_0 be smooth. If $\nabla \times u_0 \equiv 0, \nabla \times f \equiv 0$ then $\omega = \nabla \times w \equiv 0$.*

Proof: First we note that by the estimates of the previous sections, w (and hence ω) is smooth. Adapting the proof of the energy estimate of the model, take the inner product of (4.1) with $A\omega$. This gives, after integrations by parts,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{||\omega||^2 + \delta^2||\nabla\omega||^2\} + \nu\{||\nabla\omega||^2 + \delta^2||\Delta\omega||^2\} \\
&= -(\overline{w \cdot \nabla\omega}, A\omega) - (\overline{\omega \cdot \nabla w}, A\omega) \\
&= -(\omega \cdot \nabla w, \omega) \le ||\nabla w||\; ||\nabla\omega||^2.
\end{aligned}$$

Since $||\nabla w|| \in L^\infty(0,T)$, the result follows by Gronwall's inequality. □

Naturally, it is more important that small perturbations remain small. This fact also follows by essentially the same energy argument.

Proposition 5.2 *Let u_0, f be smooth and suppose*

$$||\nabla \times u_0|| \leq \epsilon, \quad ||\nabla \times f||_{L^\infty(0,T)} \leq \epsilon.$$

Then $||\nabla \times w||$ satisfies in 3d:

$$\frac{1}{2}\{||\omega(t)||^2 + \delta^2||\nabla\omega(t)||^2\} + \int_0^t \nu||\nabla\omega||^2 + \delta^2\nu||\Delta\omega||^2 ds \leq \left[(1+\frac{t}{2})e^{A(t)}\right]\epsilon^2,$$

where $A(t) = \int_0^t \frac{1}{2} + ||\nabla w||(s)ds < \infty$, and, in 2d:

$$\frac{1}{2}\{||\omega(t)||^2 + \delta^2||\nabla\omega(t)||^2\} + \int_0^t \nu||\nabla\omega||^2 + \delta^2\nu||\Delta\omega||^2 ds \leq (1+t)\epsilon^2.$$

Proof: First we note that, by integration by parts,

$$\begin{aligned}
&||\omega(0)||^2 + \delta^2||\nabla\omega(0)||^2 = (\omega(0), \omega(0) - \delta^2\Delta\omega(0)) = \\
&= (\omega(0), A\omega(0)) = (\overline{\nabla \times u_0}, A\,\overline{\nabla \times u_0}) = \\
&= (\overline{\nabla \times u_0}, \nabla \times u_0) \leq ||\nabla \times u_0||^2.
\end{aligned}$$

By the same argument as in Proposition 4.1 we have, in 3d,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{||\omega||^2 + \delta^2||\nabla\omega||^2\} + \nu\{||\nabla\omega||^2 + \delta^2||\Delta\omega||^2\} \\
&= -(\omega \cdot \nabla w, \omega) + (\nabla \times f, \omega) \\
&\leq (\frac{1}{2} + ||\nabla w||)||\omega||^2 + \frac{1}{2}||\nabla \times f||^2
\end{aligned}$$

Thus, by Gronwall's inequality

$$\begin{aligned}
&\frac{1}{2}\{||\omega(t)||^2 + \delta^2||\nabla\omega(t)||^2\} + \int_0^t \nu\{||\nabla\omega||^2 + \delta^2||\Delta\omega||^2\}ds \\
&\leq e^{A(t)}||\nabla \times u_0||^2 + \frac{1}{2}e^{A(t)}\int_0^t ||\nabla \times f||^2 ds \\
&\leq e^{A(t)}(1 + \frac{1}{2}t)\epsilon^2,
\end{aligned}$$

where $A(t) = \int_0^t \frac{1}{2} + ||\nabla w||(s)ds \quad (< \infty)$.
In 2-d, the same argument can be used but the term $(\omega \cdot \nabla w, \omega)$ does not appear on the RHS. Thus, the 2-d estimate does not display exponential growth. □

Remark: An Open Problem. There is one more case for which the question can be formulated mathematically. In 2-d, the true vorticity equation is:

$$(\nabla \times u)_t + u \cdot \nabla(\nabla \times u) - \nu\Delta(\nabla \times u) = \nabla \times f.$$

This equation satisfies a maximum principle. Thus, when $\nabla \times u_0$ and $\nabla \times f$ have one sign, $\nabla \times u$ must also have one sign. Thus, it is reasonable to ask the question: if the initial condition and the body force exert only a counterclockwise rotation force on the flow, does the LES model (correctly) predict that only a counterclockwise rotation of the flow results?
The above proofs can be attempted to be combined with generalized maximum principle arguments. Unfortunately, the obvious combination fails (for a subtle reason we describe below). The mathematical treatment of this last question is an open problem.
To understand the point of failure, define $\omega_- := -\max\{-\omega, 0\}$. Taking the inner product of (4.2) with $A\omega_-$ (and ignoring boundary terms) gives (following the above proofs)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{||\omega_-||^2 + \delta^2||\nabla\omega_-||^2\} + \nu||\nabla\omega_-||^2\\ &+\nu\delta^2(\Delta\omega, \Delta\omega_-) = (\nabla \times f, \omega_-) \leq 0.\end{aligned}$$

Unfortunately, $(\Delta\omega, \Delta\omega_-)$ does not have one sign. A calculation of $(\Delta\omega, \Delta\omega_-)$ from the definition of $\Delta\omega_-$ as a distribution in $H^{-1}(\Omega)$ gives

$$(\Delta\omega, \Delta\omega_-) = \int_\Omega |\Delta\omega_-|^2 dx - \int_{\partial\mathrm{supp}(\omega_-)} (\Delta\omega)(s)\nabla\omega_- \cdot \hat{n}(s) ds.$$

The last term on the RHS can have any sign. Thus, direct proof fails.

Acknowledgements

The author William Layton thanks D.D. Holm and E. Titi for a stimulating discussion which lead to Remark 1.1 and Corollary 1.1.

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