On the Adams-Stolz Deconvolution LES Models

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Abstract

We consider a family of LES models with arbitrarily high consistency error $O(\delta^{2N+2})$ for $N = 1, 2, 3..$ that are based on the van Cittert deconvolution procedure. The family has been proposed and tested for LES with success by Adams and Stolz in a series of papers e.g.[2], [1]. We show that these models have an interesting and quite strong stability property. Using this property we prove an energy equality, existence, uniqueness and regularity of strong solutions and give a rigorous bound on the modeling error $||\overline{\mathbf{u}} - \mathbf{w}||$ where $\mathbf{w}$ is the model's solution and $\overline{\mathbf{u}}$ the true flow averages.

Key words : large eddy simulation, scale similarity models, deconvolution, approximate deconvolution models

1 Introduction

We consider the problem of modeling the motion of the large structures in a turbulent fluid. This involves the interaction of many complex decisions made in the simulation. To isolate some effects, we study herein the correctness of the ADM(approximate deconvolution modeling) approach to closure pioneered by Adams and Stolz, e.g.[2], [1].

The pointwise velocity and pressure, $\mathbf{u}, p$ in an incompressible viscous flow satisfy the Navier-Stokes equations

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$$\begin{aligned} \mathbf{u}_t + \nabla\cdot(\mathbf{u}\mathbf{u}^T) - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \\ \nabla\cdot\mathbf{u} &= 0 \\ \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}) \end{aligned} \tag{1}$$

We study (1) subject to periodic boundary conditions (with zero mean)

$$\mathbf{u}(\mathbf{x}+L\mathbf{e},t) = \mathbf{u}(\mathbf{x},t) \tag{2}$$

for $\mathbf{x}\in\mathbb{R}^3, 0<t\le T$.

Periodic boundary conditions separate the hard problem of closure for the interior equations from another hard problem of wall laws and near wall models in turbulence.

Let overbar denote a local spacial averaging associated with a length scale δ which commutes with differentiation. Averaging the Navier-Stokes equations gives the nonclosed equations for $\overline{\mathbf{u}}, \overline{p}$

$$\begin{aligned} \overline{\mathbf{u}}_t + \nabla\cdot(\overline{\mathbf{u}\mathbf{u}^T}) - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} &= \overline{\mathbf{f}} \\ \nabla\cdot\overline{\mathbf{u}} &= 0 \end{aligned} \tag{3}$$

Let the averaging operation $\mathbf{u}\to\overline{\mathbf{u}}$ be denoted formally by G so $\overline{\mathbf{u}} = \mathrm{G}\mathbf{u}$. In most interesting cases G is not invertible. Nevertheless, the closure problem (of replacing $\overline{\mathbf{u}\mathbf{u}^T}$ by a tensor depending only on $\overline{\mathbf{u}}$) is solved once the approximate deconvolution problem (of approximating the action of G^{-1}) is solved.

The van Cittert approximation to G^{-1} can be developed in various ways (see [3] and section 2 for a precise definition of it). The simplest is to find an approximation to $\mathbf{u}$ by extrapolating from the resolved scales of $\overline{\mathbf{u}}$ to those of $\mathbf{u}$. The first three examples are

$$\begin{aligned} \mathbf{u} \approx G_0\overline{\mathbf{u}} &:= \overline{\mathbf{u}} && \text{(constant extrapolation in } \delta) \\ \mathbf{u} \approx \mathrm{G}_1\overline{\mathbf{u}} &:= 2\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}} && \text{(linear extrapolation in } \delta) \\ \mathbf{u} \approx \mathrm{G}_2\overline{\mathbf{u}} &:= 3\overline{\mathbf{u}} - 3\overline{\overline{\mathbf{u}}} + \overline{\overline{\overline{\mathbf{u}}}} && \text{(quadratic extrapolation in } \delta) \end{aligned} \tag{4}$$

Let $\mathrm{G}_N\overline{\mathbf{u}}$ denote the analogous N^{th}degree accurate approximate inverse (section 2). Calling $(\mathbf{w}, q)$ the approximations that result when this is used in (3) to treat the closure problem we are inevitably led to the fundamentally important question of how well the solution $\mathbf{w}$ of the resulting model

$$\begin{aligned} \mathbf{w}_t + \nabla\cdot\overline{(\mathrm{G}_N\mathbf{w}(\mathrm{G}_N\mathbf{w})^T)} - \nu\Delta\mathbf{w} + \nabla\overline{q} &= \overline{\mathbf{f}} \\ \nabla\cdot\mathbf{w} &= 0 \end{aligned} \tag{5}$$

matches the behaviour of the true flow averages $\overline{\mathbf{u}}$. This question has obvious theoretical and experimental components. We consider herein the theoretical parts of the question for the whole family of models. Our analysis is based on a delicate skew symmetry property that the model's nonlinear interaction terms have when the averaging operator is the differential filter $\varphi \to \overline{\varphi}$ (as studied by Germano [5]). Here for given $\varphi \in L^2(Q)$, $\overline{\varphi}$ is defined to be the unique periodic solution of

$$-\delta^2 \Delta \overline{\varphi} + \overline{\varphi} = \varphi \tag{6}$$

in Q where Q denotes the d-dimensional cube of size $L > 0$, $Q = (0, L)^d$.

Our analysis is for periodic boundary conditions. We believe that many of the results presented in this work can be extended to nonperiodic boundary conditions with further research. Indeed, the basic model (5) does not increase the order of the differential operator so the model makes perfect sense coupled with any of the well posed boundary conditions used for the Navier-Stokes equations.

Remark 1.1 *The model (5) using* G_0 *was considered recently in [7] and [8]. On the other hand practical calculations of Adams and Stolz in [2] and [1] have stressed the superiority of models of order 4, 5 and higher in practical tests.*

Herein we show that a single, unified mathematical theory is possible for the entire family of models building on the analysis in [7] and [8].

2 Deconvolution Models

It has been pointed out by Germano(presented well in [6]) that with the differential filter $\overline{\varphi} := (-\delta^2\Delta + I)^{-1}\varphi$ it seems that no decovolution is necessary; one can write exactly $\varphi := (-\delta^2\Delta + I)\overline{\varphi}$.

This leads to the exact model for $\overline{\mathbf{u}}$ given by

$$\overline{\mathbf{u}}_t + \nabla \cdot \overline{((-\delta^2\Delta + \mathrm{I})\overline{\mathbf{u}}[(-\delta^2\Delta + \mathrm{I})\overline{\mathbf{u}}]^T} - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} = \overline{\mathbf{f}} \tag{7}$$

subject to the periodic boundary conditions. One criticism with using exact deconvolution model (7) to predict $\overline{\mathbf{u}}$ is that going from the Navier-Stokes equations to (7) no information is lost.

Thus there is no reason to believe that (7) can be approximated with fewer degrees of freedom than the NSE itself. Another difficulty with (7) is that any model that increases the order of the differential equation must be supplied with extra boundary conditions. Thus for nonperiodic problems models like (7) shift the essential difficulty from interior closure to the harder problem of specifying as boundary conditions the higher derivatives of turbulent velocities at walls. Thus approximate deconvolution which will lose information is necessary. The van Cittert method of approximate deconvolution, see [3], constructs a family G_N of inverses to G as follows: Writing $\mathrm{G} = \mathrm{I} - (\mathrm{I} - \mathrm{G})$ a formal inverse to G can be writen as the nonconvergent power series

$$\mathrm{G}^{-1} = \Sigma_{n=0}^{\infty}(\mathrm{I} - \mathrm{G})^n$$

Truncating this series gives

$$\mathrm{G}_N = \Sigma_{n=0}^{N}(\mathrm{I} - \mathrm{G})^n \tag{8}$$

The first three approximations are given in (4).

Lemma 2.1 *The operator* $\mathrm{G}_N : L^2(Q) \to L^2(Q)$ *is compact, selfadjoint and positive.*

Proof: The operator $\mathrm{G} : L^2(Q) \to L^2(Q)$ is compact, selfadjoint. Multiplying (6) by $\overline{\varphi}$ and integrating over Q gives

$$\delta^2||\nabla\overline{\varphi}||^2 + ||\overline{\varphi}||^2 = (\varphi, \overline{\varphi}) \leq \frac{1}{2}||\varphi||^2 + \frac{1}{2}||\overline{\varphi}||^2$$

It follows that G is positive and $||\mathrm{G}|| \leq 1$. Let $h_N(x) = \Sigma_{k=0}^{N}(1-x)^k$. By the definition of G_N

$$\mathrm{G}_N = h_N(\mathrm{G})$$

and consequently G_N is also compact selfadjoint operator. Because h_N is positive on $[0, 1]$ which contains the spectrum of G it follows also that G_N is positive. ■

Lemma 2.2 *The operators* $\{\mathrm{G}_N\}_N$ *satisfy the following recursion:*

$$(\mathrm{I} - \delta^2\Delta)\mathrm{G}_N\mathbf{u} = -\delta^2\Delta\mathrm{G}_{N-1}\mathbf{u} + (\mathrm{I} - \delta^2\Delta)\mathbf{u} \tag{9}$$

Proof: Using the definition of G_N we write

$$\mathrm{G}_N u = (\mathrm{I} - \mathrm{G})\mathrm{G}_{N-1} + \mathrm{I}$$

for any $N > 0$ and multiplying to the left by $(\mathrm{I} - \delta^2\Delta)$ we get

$$(\mathrm{I} - \delta^2\Delta)\mathrm{G}_N\mathbf{u} = (\mathrm{I} - \delta^2\Delta)(\mathrm{I} - \mathrm{G})\mathrm{G}_{N-1}\mathbf{u} + (\mathrm{I} - \delta^2\Delta)\mathbf{u} =$$

$$(\mathrm{I} - \delta^2\Delta)\mathrm{G}_{N-1}\mathbf{u} - (\mathrm{I} - \delta^2\Delta)\mathrm{G}\mathrm{G}_{N-1}\mathbf{u} + (\mathrm{I} - \delta^2\Delta)\mathbf{u} = -\delta^2\Delta\mathrm{G}_{N-1}\mathbf{u} + (\mathrm{I} - \delta^2\Delta)\mathbf{u}$$

i.e

$$(\mathrm{I} - \delta^2\Delta)\mathrm{G}_N\mathbf{u} = -\delta^2\Delta\mathrm{G}_{N-1}\mathbf{u} + (\mathrm{I} - \delta^2\Delta)\mathbf{u}$$

■

Lemma 2.3 *For smooth* $\mathbf{u}$ *the approximate deconvolution (8) has consistency error* $O(\delta^{2N+2})$

$$\mathbf{u} - \mathrm{G}_N\overline{\mathbf{u}} = (-1)^{N+1}\delta^{2N+2}\Delta^{N+1}\overline{\mathbf{u}} \tag{10}$$

locally in Q *and also*

$$||\mathbf{u} - \mathrm{G}_N\overline{\mathbf{u}}|| \leq \delta^{2N+2}||\overline{\mathbf{u}}||_{H^{2N+2}(Q)}$$

Proof: This is a simple algebraic argument. Let $\mathrm{A} := (\mathrm{I} - \mathrm{G})$ and note that

$$\mathrm{A}\varphi = (\mathrm{I} - \mathrm{G})\varphi = (-\delta^2\Delta + \mathrm{I})^{-1}(-\delta^2\Delta + \mathrm{I} - \mathrm{I})\varphi = -\delta^2\Delta\overline{\varphi}$$

Then with $\mathbf{e} := \mathbf{u} - \mathrm{G}_N\overline{\mathbf{u}}$ we have (by the definition of G_N)

$$\mathbf{u} = \overline{\mathbf{u}} + \mathrm{A}\overline{\mathbf{u}} + ... + \mathrm{A}^N\overline{\mathbf{u}} + \mathbf{e}$$

Applying to both sides the operator A and substracting gives(since $\mathrm{I} - \mathrm{A} = \mathrm{G}$)

$$\mathrm{G}\mathbf{u} = \overline{\mathbf{u}} - \mathrm{A}^{N+1}\overline{\mathbf{u}} + \mathrm{G}\mathbf{e}$$

or as $\mathrm{G}\mathbf{u} = \overline{\mathbf{u}}$

$$\mathrm{G}\mathbf{e} = \overline{\mathbf{e}} = \mathrm{A}^{N+1}\overline{\mathbf{u}}$$

Applying $(-\delta^2\Delta + \mathrm{I})$ to both sides implies $\mathbf{e} = \mathrm{A}^{N+1}\mathbf{u}$ which gives (10). ∎

Lemma (2.3) shows that $\mathrm{G}_N\overline{\mathbf{u}}$ gives an approximation to $\mathbf{u}$ to accuracy $O(\delta^{2N+2})$ in the smooth flow regions. Thus it is justified to use it for the closure approximation

$$\nabla \cdot (\overline{\mathbf{u}\mathbf{u}^T}) \approx \nabla \cdot (\overline{\mathrm{G}_N\overline{\mathbf{u}}(\mathrm{G}_N\overline{\mathbf{u}})^T}) + O(\delta^{2N+2})$$

If μ denotes the usual subfilter scale stress tensor $\mu(\mathbf{u}, \mathbf{u}) := \overline{\mathbf{u}\mathbf{u}^T} - \overline{\mathbf{u}}\,\overline{\mathbf{u}}^T$ then the closure approximation is equivalent to the closure model

$$\mu(\mathbf{u}, \mathbf{u}) \approx \mu_N(\overline{\mathbf{u}}, \overline{\mathbf{u}}) := \overline{\mathrm{G}_N\overline{\mathbf{u}}(\mathrm{G}_N\overline{\mathbf{u}})^T} - \overline{\mathbf{u}}\,\overline{\mathbf{u}}^T \tag{11}$$

Definition 2.1 *A tensor function* $\mu(\mathbf{u}, \mathbf{v})$ *of two vector variables is reversible if*

$$\mu(-\mathbf{u}, -\mathbf{v}) = \mu(\mathbf{u}, \mathbf{v}).$$

The tensor μ *is galilean invariant if for any divergence free periodic vector field* $\mathbf{w}(\mathbf{x})$ *and any constant vector* U

$$\nabla \cdot \mu(\mathbf{w} + \mathrm{U}, \mathbf{w} + \mathrm{U}) = \nabla \cdot \mu(\mathbf{w}, \mathbf{w})$$

The interest in reversibility and Galilean invariance is that the true subgrid stress tensor $\mu(\mathbf{u},\mathbf{u})$ is both reversible and Galilean invariant, Sagaut [9]. Thus many feel that appropriate closure models should at least, to leading order effects, share these two properties. We next show that the model (11) is both reversible and Galilean invariant.

Lemma 2.4 *For each $N = 0, 1, 2..$ the closure model (11) is reversible and Galilean invariant.*

Proof: Reversibility is immediate.Galilean invariance also follows easily once it is noted that $\overline{\mathrm{U}\mathbf{w}^T} = \mathrm{U}\overline{\mathbf{w}}^T$ so $\mathrm{G}_N(\mathrm{U}\overline{\mathbf{u}}^T) = \mathrm{U}\mathrm{G}_N(\overline{\mathbf{u}})^T$. Using these and other analogous properties gives

$$\nabla\cdot\mu(\overline{\mathbf{u}}+\mathrm{U},\overline{\mathbf{u}}+\mathrm{U}) = \nabla\cdot[\overline{\mathrm{G}_N(\overline{\mathbf{u}})\mathrm{G}_N(\overline{\mathbf{u}})^T}+$$

$$+\mathrm{U}\overline{\mathrm{G}_N(\overline{\mathbf{u}})}^T+\overline{\mathrm{G}_N(\overline{\mathbf{u}})}\mathrm{U}^T+\overline{\mathrm{U}\mathrm{U}^T}-(\overline{\mathbf{u}}+\mathrm{U})(\overline{\mathbf{u}}+\mathrm{U})^T] =$$

$$= \nabla\cdot[(\overline{\mathrm{G}_N(\overline{\mathbf{u}})\mathrm{G}_N(\overline{\mathbf{u}})^T}-\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T]+\nabla\cdot(\overline{\mathrm{G}_N(\overline{\mathbf{u}})}\mathrm{U}+\mathrm{U}\nabla\cdot(\overline{\mathrm{G}_N(\overline{\mathbf{u}})}-\nabla\cdot(\overline{\mathbf{u}})\mathrm{U}-\mathrm{U}\nabla\cdot(\overline{\mathbf{u}}) =$$

$$= \nabla\cdot[\overline{\mathrm{G}_N(\overline{\mathbf{u}})\mathrm{G}_N(\overline{\mathbf{u}})^T}-\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T] = \nabla\cdot\mu_N(\overline{\mathbf{u}},\overline{\mathbf{u}})$$

since $\nabla\cdot\overline{\mathbf{u}} = \nabla\cdot\mathrm{G}_N(\overline{\mathbf{u}}) = \nabla\cdot\overline{\mathrm{G}_N(\overline{\mathbf{u}})} = 0$ and $\overline{\mathrm{U}\mathrm{U}^T} = \mathrm{U}\mathrm{U}^T$. ■

3 Variational Spaces

Q denotes the d-dimensional cube of size $L > 0$

$$Q = (0, L)^d.$$

Let

$$H^m(Q) = \{\mathbf{u}\in H^m_{loc}(\mathbb{R}^n)|\mathbf{u} \text{ periodic with period } Q\}$$

and

$$\overline{H}^m(Q) = \{\mathbf{u}\in H^m(Q)|\int_Q \mathbf{u}d\mathbf{x} = 0\}$$

For the variational formulation of the scale similarity model with periodic boundary conditions we consider the spaces of divergence free functions

$$V = \{\mathbf{u}\in H^1(Q)\ ,\nabla\cdot\mathbf{u} = 0 \text{ in } \mathbb{R}^d\}$$

and

$$H = \{\mathbf{u}\in L^2(Q)\ ,\nabla\cdot\mathbf{u} = 0 \text{ in } \mathbb{R}^d\}$$

as in R.Temam [10].

$D(Q)$ is defined as

$$D(Q) = \{\psi \in C^\infty(\mathbb{R}^d) | \psi \text{ is periodic of period } Q\}.$$

and

$$D(Q_T) = \{\psi \in C^\infty([0,T)\times\mathbb{R}^d) | \text{ for } t \in [0,T), \psi(\cdot,t) \text{ is periodic of period } Q \text{ and}$$
$$\psi \text{ has compact support in variable } t \in [0,T)\}.$$

The space of vector valued functions $\mathbb{D}(Q)$ is defined as

$$\mathbb{D}(Q) = D(Q)^d \tag{12}$$

The other spaces $\mathbb{D}(Q_T)$, $\mathbb{H}$, $\overline{\mathbb{H}}^p(Q)$, $\mathbb{V}$, $\mathbb{L}^2(Q)$ are defined accordingly.

Remark 3.1 *Because the inclusion $\overline{H}^2(Q) \to H$ is compact, the inverse of the Laplacian operator $(-\Delta)^{-1} : H \to \overline{H}^2(Q) \subset H$ is a bounded, selfadjoint, compact operator. This implies that there exist an orthonormal basis $(\mathbf{w}_j)_{j\in\mathbb{N}}$ of H consisting of eigenfunctions of the Laplacian operator.*

4 The models and the existence of weak solutions

Definition 4.1 *The strong form of the scale similarity model is:*

Find $(\mathbf{w}, q)$ such that

$$\begin{array}{ll} \mathbf{w} \in \left(\overline{H}^2(Q) \cap H\right)^d & \textit{for a.e } t \in [0,T] \\ \mathbf{w} \in \left(H^1((0,T))\right)^d & \textit{for a.e.}\mathbf{x} \in \overline{Q} \\ q \in H^1(Q) \cap \mathrm{L}_0^2(Q) & \textit{if } t \in (0,T]. \end{array} \tag{13}$$

and

$$\begin{array}{rcll} \mathbf{w}_t - \nu\Delta\mathbf{w} + \nabla\cdot\overline{((\mathrm{G}_N\mathbf{w})(\mathrm{G}_N\mathbf{w})^T)} + \nabla\overline{q} & = & \overline{\mathbf{f}} & \textit{in } (0,T)\times Q \\ \nabla\cdot\mathbf{w} & = & 0 & \textit{in } (0,T]\times Q \\ \mathbf{w}\mid_{t=0} & = & \overline{\mathbf{u}_0} & \textit{in } Q \\ \int_Q q\mathrm{d}x & = & 0 & \textit{in } (0,T]. \end{array} \tag{14}$$

Definition 4.2 *Let* $\mathbf{f} \in L^2(0,T;\mathbb{V}')$ *and* $\mathbf{w}_0 \in \overline{\mathbb{H}}^2(Q)$. *A measurable function* $\mathbf{w} : [0,T] \times Q \to \mathbb{R}^d$ *is a weak solution of (14) if*

$$\mathbf{w} \in L^2(0,T,\overline{\mathbb{H}}^1(Q)) \cap L^\infty(0,T;\mathbb{H}) \tag{15}$$

and

$$\int_0^\infty [(\mathbf{w}, \frac{\partial \varphi}{\partial t}) - \nu(\nabla \mathbf{w}, \nabla \varphi) - (\nabla \cdot \overline{((\mathrm{G}_N\mathbf{w})(\mathrm{G}_N\mathbf{w})^T)}, \varphi)]\mathrm{d}t = \tag{16}$$

$$-\int_0^\infty (\mathbf{f}, \varphi)\mathrm{d}t - (\mathbf{w}_0, \varphi(0))$$

for all $\varphi \in \mathbb{D}(Q_T)$.

The following lemma gives an energy inequality satisfied by the strong solutions of the Adams-Stolz models. We mention here that the same argument is used to derive an energy inequality for the approximate solutions in the proof of existence of weak solutions to the Adams-Stolz models.

Lemma 4.1 *If* $\mathbf{w}$ *is a strong solution of (14) as in Definition (4.1) then* $\mathbf{w}$ *satisfies the following energy inequality*

$$\frac{1}{2}(||\mathbf{w}(t)||^2 + \delta^2||\nabla \mathbf{w}(t)||^2) + \frac{\nu}{2}\int_0^t ||\nabla \mathbf{w}(s)||^2 + \delta^2||\Delta \mathbf{w}(s)||^2 \mathrm{d}s \leq \tag{17}$$

$$K(\int_0^t ||\mathbf{f}(s)||^2_{V'}\mathrm{d}s + ||\mathbf{w}_0||^2 + \delta^2||\nabla \mathbf{w}_0||^2)$$

for all $t \in [0,T]$ *with* $K = max\{\frac{2||\mathrm{G}_N||^2_{L^2(Q)}}{\nu}, \frac{1}{2}\delta^2, \frac{1}{2}, \frac{\delta^2}{2}||\mathrm{G}_{N-1}||_{L^2(Q)}\}$.

Proof:

We multiply (14) by the test function $\varphi = (-\delta^2\Delta + \mathrm{I})\mathrm{G}_N\mathbf{w}$ and integrate on Q. Because the weak form of the nonlinear term will vanish

$$(\nabla \cdot (\overline{(\mathrm{G}_N\mathbf{w})(\mathrm{G}_N\mathbf{w})^T}), (-\delta^2\Delta + \mathrm{I})\mathrm{G}_N\mathbf{w}) = \tag{18}$$

$$= (\nabla \cdot ((\mathrm{G}_N\mathbf{w})(\mathrm{G}_N\mathbf{w})^T), \overline{(-\delta^2\Delta + \mathrm{I})\mathrm{G}_N\mathbf{w}}) = (\nabla \cdot ((\mathrm{G}_N\mathbf{w})(\mathrm{G}_N\mathbf{w})^T), \mathrm{G}_N\mathbf{w}) = 0 \tag{19}$$

we obtain the following energy equality:

$$\frac{1}{2}\frac{d}{dt}(\mathbf{w}, (-\delta^2\Delta + \mathrm{I})\mathrm{G}_N\mathbf{w}) + \nu(\Delta \mathbf{w}, (-\delta^2\Delta + \mathrm{I})\mathrm{G}_N\mathbf{w}) = (\overline{\mathbf{f}}, (-\delta^2\Delta + \mathrm{I})\mathrm{G}_N\mathbf{w}) \tag{20}$$

In the equality above all terms $(-\delta^2\Delta + \mathrm{I})\mathrm{G}_N\mathbf{w}$ are replaced using Lemma (2.2) leading to

$$\frac{1}{2}\frac{d}{dt}(\mathbf{w}, (-\delta^2\Delta + \mathrm{I})\mathbf{w}) + \frac{1}{2}\frac{d}{dt}(\mathbf{w}, -\delta^2\Delta\mathrm{G}_{N-1}\mathbf{w}) - \nu(\Delta\mathbf{w}, (-\delta^2\Delta + \mathrm{I})\mathbf{w}) +$$

$$+\nu\delta^2(\Delta\mathbf{w}, \delta^2\Delta\mathrm{G}_{N-1}\mathbf{w}) = (\mathbf{f}, \mathrm{G}_N\mathbf{w})$$

Using integration by parts and the commutation property of the operator G_{N-1} with differentiation gives

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}||^2 + \frac{1}{2}\delta^2\frac{d}{dt}||\nabla\mathbf{w}||^2 + \frac{\delta^2}{2}\frac{d}{dt}(\nabla\mathbf{w}, \mathrm{G}_{N-1}\nabla\mathbf{w}) +$$

$$+\nu||\nabla\mathbf{w}||^2 + \nu\delta^2||\Delta\mathbf{w}|| + \nu\delta^4(\Delta\mathbf{w}, \mathrm{G}_{N-1}\Delta\mathbf{w}) = (\mathbf{f}, \mathrm{G}_N\mathbf{w})$$

We then integrate on $[0, t]$ and obtain

$$\frac{1}{2}||\mathbf{w}(t)||^2 + \frac{1}{2}\delta^2||\nabla\mathbf{w}(t)||^2 + \frac{\delta^2}{2}(\nabla\mathbf{w}(t), \mathrm{G}_{N-1}\nabla\mathbf{w}(t)) +$$

$$+\nu\int_0^t ||\nabla\mathbf{w}(s)||^2 \mathrm{d}s + \nu\delta^2\int_0^t ||\Delta\mathbf{w}(s)||^2 \mathrm{d}s + \nu\delta^4\int_0^t (\Delta\mathbf{w}(s), \mathrm{G}_{N-1}\Delta\mathbf{w}(s))\mathrm{d}s =$$

$$= \int_0^t (\mathbf{f}(s), \mathrm{G}_N\mathbf{w}(s))\mathrm{d}s + \frac{1}{2}||\mathbf{w}_0||^2 + \frac{1}{2}\delta^2||\nabla\mathbf{w}_0||^2 + \frac{\delta^2}{2}(\nabla\mathbf{w}_0, \mathrm{G}_{N-1}\nabla\mathbf{w}_0)$$

We use the positivity of the operators $(\mathrm{G}_N)_N$ in the inequality above to get

$$\frac{1}{2}||\mathbf{w}(t)||^2 + \frac{1}{2}\delta^2||\nabla\mathbf{w}(t)||^2 + \nu\int_0^t ||\nabla\mathbf{w}(s)||^2 \mathrm{d}s + \nu\delta^2\int_0^t ||\Delta\mathbf{w}(s)||^2 \mathrm{d}s \tag{21}$$

$$\leq \int_0^t (\mathbf{f}(s), \mathrm{G}_N\mathbf{w}(s))\mathrm{d}s + \frac{1}{2}\delta^2||\mathbf{w}_0||^2 + \frac{1}{2}||\nabla\mathbf{w}_0||^2 + \frac{\delta^2}{2}(\nabla\mathbf{w}_0, \mathrm{G}_{N-1}\nabla\mathbf{w}_0)$$

An application of Cauchy's inequality on the first term on the right hand side above gives

$$\int_0^t (\mathbf{f}(s), \mathrm{G}_N\mathbf{w}(s))\mathrm{d}s \leq \int_0^t ||\mathbf{f}(s)||_{V'}||\mathrm{G}_N||_{L^2(Q)}||\nabla\mathbf{w}(s)||_{L^2(Q)}\mathrm{d}s \leq$$

$$\frac{2||\mathrm{G}_N||^2_{L^2(Q)}}{\nu}\int_0^t ||\mathbf{f}(s)||^2_{V'}\mathrm{d}s + \frac{\nu}{2}\int_0^t ||\nabla\mathbf{w}(s)||^2_{L^2(Q)}\mathrm{d}s$$

We use this inequality in (21) to obtain

$$\frac{1}{2}||\mathbf{w}(t)||^2 + \frac{1}{2}\delta^2||\nabla\mathbf{w}(t)||^2 + \nu\delta^2\int_0^t ||\Delta\mathbf{w}(s)||^2 \mathrm{d}s + \frac{\nu}{2}\int_0^t ||\nabla\mathbf{w}(s)||^2 \mathrm{d}s$$

$$\leq \frac{2||\mathrm{G}_N||^2_{L^2(Q)}}{\nu}\int_0^t ||\mathbf{f}(s)||^2_{V'}\mathrm{d}s + \frac{1}{2}\delta^2||\mathbf{w}_0||^2 + \frac{1}{2}||\nabla\mathbf{w}_0||^2 + \frac{\delta^2}{2}||\mathrm{G}_{N-1}||_{L^2}||\nabla\mathbf{w}_0||^2_{L^2}$$

■

Proposition 4.1 *Let $T > 0$. Then for $\mathbf{w}_0 \in \overline{\mathbb{H}}^2(Q) \cap \mathbb{H}$, and $\mathbf{f} \in L^2(0, T; \mathbb{V}')$ there exists a weak solution $\mathbf{w}$ of (14)in the sense of Definition (4.2). This solution $\mathbf{w}$ belongs to the space $L^2(0, T, \overline{\mathbb{H}}^2(Q)) \cap L^\infty(0, T; \mathbb{V})$, it is L^2-weakly continuous and it satisfies the following energy inequality*

$$\frac{1}{2}(||\mathbf{w}(t)||^2 + \delta^2||\nabla\mathbf{w}(t)||^2) + \delta^2\nu\int_0^t ||\Delta\mathbf{w}(s)||^2\mathrm{d}s \leq \tag{22}$$

$$K(\int_0^t ||\mathbf{f}(s)||^2_{V'}\mathrm{d}s + ||\mathbf{w}_0||^2 + ||\nabla\mathbf{w}_0||^2)$$

for all $t \in [0, T]$ with $K = max\{\frac{2||\mathrm{G}_N||^2_{L^2(Q)}}{\nu}, \frac{1}{2}\delta^2, \frac{1}{2}, \frac{\delta^2}{2}||\mathrm{G}_{N-1}||_{L^2(Q)}\}$

Proof: The proof uses the Faedo-Galerkin method. We will use Galdi [4] as a reference and we will only point out the differences between the proof of existence of the weak solution of the Navier-Stokes equations and the proof of existence for our models. We pick an orthonormal basis $\{\psi_j\}_j \in \mathbb{D}(Q)$ of $\mathbb{H}$ consisting of eigenfunctions of the Laplacian operator as in Remark (3.1). Let

$$\mathbf{w}_k(x, t) = \Sigma_{r=1}^k \eta_{kr}(t)\psi_r(x) \tag{23}$$

for $k \in \mathbb{N}$ the solution of the following ODE system

$$(\frac{\partial\mathbf{w}_k}{\partial t}, \psi_r) + \nu(\nabla\mathbf{w}_k, \nabla\psi_r) + (\nabla\cdot\overline{(\mathrm{G}_N\mathbf{w}_k)(\mathrm{G}_N\mathbf{w}_k)^T}, \psi_r) = (\mathbf{f}, \psi_r) \tag{24}$$

for all $r = 1..k$ with initial condition

$$(\mathbf{w}_k(0), \psi_r) = (\mathbf{w}_0, \psi_r)$$

for all $r = 1..k$. It follows that the coefficients η_{kr} satisfy the following ODE system

$$\frac{d\eta_{kr}}{dt} + \Sigma_{i=1}^k a_{ir}\eta_{ki} + \Sigma_{i,j=1}^k a_{ijr}\eta_{ki}\eta_{kj} = f_r \tag{25}$$

for $r = 1..k$ with the initial condition

$$\eta_{kr}(0) = C_{0r}, \quad \text{for } r = 1..k$$

where $a_{ir} = \nu(\nabla\psi_i, \nabla\psi_r)$, $a_{ijr} = (\nabla \cdot \overline{((\mathrm{G}_N\psi_i)\ (\mathrm{G}_N\psi_j)^T)}, \psi_r)$,
$f_r = (\mathbf{f}, \psi_r)$, $C_{0r} = (bw_0, \psi_r)$.

The function f_r belongs to $L^2([0,T))$ for any r and consequently (25) has a unique solution near 0,

$$\eta_{kr} \in W^{1,2}(0, T_k)$$

where $T_k \leq T$. Because $\mathbf{w}_0 \in \overline{\mathbb{H}}^2(Q) \cap \mathbb{H}$ there exists $\mathbf{u}_0 \in \mathbb{H}$ such that $\overline{\mathbf{u}_0} = \mathbf{w}_0$ For the ODE defined above we have $(\mathbf{w}_{k0}, \psi_r) = (\mathbf{w}_0, \psi_r)$ for all $r = 1..k$. This gives

$$(\mathbf{w}_{k0}, \psi_r) = (\overline{\mathbf{u}_0}, \psi_r) \tag{26}$$

for all $r = 1..k$. But $\mathbf{w}_{k,0} \in \mathrm{G}_k = \mathrm{span}\{\psi_j\}_{j=1..k}$ and G_k is an invariant subspace of the Laplacian operator. Consequently we can replace in formula (26) ψ_r with $(\mathrm{I} - \delta^2\Delta)\mathbf{w}_{k,0}$ to get

$$(\mathbf{w}_{k0}, (\mathrm{I} - \delta^2\Delta)w_{k,0}) = (\overline{\mathbf{u}_0}, (\mathrm{I} - \delta^2\Delta)\mathbf{w}_{k,0}) = (\mathbf{u}_0, \mathbf{w}_{k,0}) \tag{27}$$

Integrating by parts the first term above and using Cauchy 's inequality in the second we get

$$||\mathbf{w}_{k0}||^2 + \delta^2||\nabla\mathbf{w}_{k0}||^2 = (\mathbf{u}_0, \mathbf{w}_{k,0}) \leq \frac{1}{2}(||\mathbf{u}_0||^2 + ||\mathbf{w}_{k0}||^2) \tag{28}$$

which gives the following estimate

$$\frac{1}{2}||\mathbf{w}_{k0}||^2 + \delta^2||\nabla\mathbf{w}_{k0}||^2 \leq \frac{1}{2}||\mathbf{u}_0||^2 \tag{29}$$

We want to prove that we can pick $T_k = T$. In equation (24) we replace ψ_r with $(\mathrm{I} - \delta^2\Delta)\mathrm{G}_N\mathbf{w}_k$. One can do this since $(\mathrm{I} - \delta^2\Delta)\mathrm{G}_N\mathbf{w}_k(t) \in \mathrm{G}_k = \mathrm{span}\{\psi_j\}_{j=1..k}$ for any $t \in [0,T)$. In the same way in which the energy inequality (17) for strong solutions was derived we obtain

$$\frac{1}{2}(||\mathbf{w}_k(t)||^2 + \delta^2||\nabla\mathbf{w}_k(t)||^2) + \delta^2\nu\int_0^t ||\boldsymbol{\Delta}\mathbf{w}_k(s)||^2 \mathrm{d}s + \frac{\nu}{2}\int_0^t ||\Delta\mathbf{w}_k(s)||^2 \mathrm{d}s \leq M \tag{30}$$

where

$$M := K(\int_0^T ||\mathbf{f}(s)||^2_{V'}\mathrm{d}s + ||\mathbf{w}_{k0}||^2 + \delta^2||\nabla\mathbf{w}_{k0}||^2) \tag{31}$$

with $K = max\{\frac{||2\mathrm{G}_N||^2_{L^2(Q)}}{\nu}, \frac{1}{2}\delta^2, \frac{1}{2}, \frac{\delta^2}{2}||\mathrm{G}_{N-1}||_{L^2}\}$.

M does not depend on t and using (29) also M does not depend on k. Due to orthonormality of the family $\{\psi_j\}_j$ in H we get that á priori the coefficients η_{kr} satisfy

$$|\eta_{kr}|^2 \leq 2M^{\frac{1}{2}}$$

for any $t \in [0,T)$, $r = 1..k$ and $k \in \mathbb{N}$. This implies that for any k there exists global solution (that is, on $[0,T)$)

$$\eta_{kr} \in W^{1,2}([0,T))$$

$r = 1..k$ of the ODE system (24).

In the same way as in Galdi [4] one can show using the estimate (30) that there exists a subsequence of $\mathbf{w}_k$ (which is redenoted $\mathbf{w}_k$) which converges weakly in V uniformly in t to a function $\mathbf{w} \in L^\infty(0,T,\mathbb{V})$. From estimate (30) we infer that the sequence $\mathbf{w}_k$ is bounded in $L^2(0,T,\overline{\mathbb{H}}^2(Q))$ consequently it contains a sebsequence(which is redenoted $\mathbf{w}_k$)which is weakly convergent to a function $\mathbf{w}' \in L^2(0,T,\overline{\mathbb{H}}^2(Q))$. One can show taking limits of $\mathbf{w}_k$ in the space $L^2(0,T,\mathbb{L}^2(Q))$ that $\mathbf{w} = \mathbf{w}'$. It follows that $\mathbf{w} \in \left(\overline{\mathbb{H}}^2(Q) \cap \mathbb{H}\right)^d$.

We can show that $\mathbf{w}$ satisfies the variational equality (16) in the same way as in Galdi [4] taking the limits of $\mathbf{w}_k$ in the equality (24). In the case of Adams-Stolz models when taking limits the nonlinear term is handled in the following way: One needs to show that for given eigenfunction ψ_r

$$\int_0^t (\overline{\mathrm{G}_N\mathbf{w}_k \cdot \nabla \mathrm{G}_N\mathbf{w}_k}, (\mathrm{I} - \delta^2\Delta)^{-1}\psi_r) - (\overline{\mathrm{G}_N\mathbf{w} \cdot \nabla \mathrm{G}_N\mathbf{w}}, (\mathrm{I} - \delta^2\Delta)^{-1}\psi_r)\mathrm{d}s \to 0$$

But

$$|\int_0^t (\overline{\mathrm{G}_N\mathbf{w}_k \cdot \nabla \mathrm{G}_N\mathbf{w}_k}, (\mathrm{I} - \delta^2\Delta)^{-1}\psi_r) - (\overline{\mathrm{G}_N\mathbf{w} \cdot \nabla \mathrm{G}_N\mathbf{w}}, (\mathrm{I} - \delta^2\Delta)^{-1}\psi_r)\mathrm{d}s| =$$

$$= |\int_0^t (\mathrm{G}_N\mathbf{w}_k \cdot \nabla \mathrm{G}_N\mathbf{w}_k, \psi_r) - (\mathrm{G}_N\mathbf{w} \cdot \nabla \mathrm{G}_N\mathbf{w}, \psi_r)\mathrm{d}s| \leq$$

$$\leq |\int_0^t (\mathrm{G}_N(\mathbf{w}_k - \mathbf{w}) \cdot \nabla \mathrm{G}_N\mathbf{w}_k, \psi_r)\mathrm{d}s| + |\int_0^t (\mathrm{G}_N\mathbf{w} \cdot \nabla \mathrm{G}_N(\mathbf{w}_k - \mathbf{w}), \psi_r)\mathrm{d}s| \leq$$

$$\leq ||\mathrm{G}_N||^2_{L^2(Q)}||\mathbf{w}_k - \mathbf{w}||_{L^2(0,T,L^2)}||\psi_r||_\infty||\nabla\mathbf{w}_k||_{L^2(0,T,L^2)} +$$

$$|\int_0^t (\mathrm{G}_N\mathbf{w} \cdot \mathrm{G}_N(\nabla(\mathbf{w}_k - \mathbf{w})), \psi_r)\mathrm{d}s|$$

The first term on the right hand side above converges to 0 since $\mathbf{w}_k \to \mathbf{w}$ in $L^2(0,T,\mathbb{L}^2(Q))$ and the second converges to 0 because $\nabla \mathbf{w}_k \to \nabla \mathbf{w}$ weakly in $L^2(0,T,\mathbb{L}^2(Q))$ and the operator G_N is selfadjoint. The energy inequality (22) is obtained the same way as in the case of Navier-Stokes equations taking limits in (30). ∎

Lemma 4.2 *The weak solution* $\mathbf{w}$ *that was constructed in the previous theorem is also a strong solution of (14).*

Proof: This follows directly from Definition (16), the regularity proven for the solution and an integration by parts. ∎

Lemma 4.3 *The weak solution* $\mathbf{w}$ *of (14) constructed in Proposition (4.1) is the unique weak solution of (14).*

Proof: This is a consequence of the regularity of $\mathbf{w}$. The proof is the same as in the case of the NSE. ∎

5 An á priori Estimate of the Modelling Error.

Our goal here is to give an á priori estimate of the modeling error $||\overline{\mathbf{u}} - \mathbf{w}||$. In this direction there are several fundamental problems. First, in $3d$ there is no proof of uniqueness of weak solutions $\mathbf{u}$ of the Navier-Stokes equations. Thus for $\mathbf{u}$ a general weak solution of the Navier-Stokes equations the best result attainable in the usual norms with present technique seems to be the following:

Proposition 5.1 *Let* $\mathbf{w} = \mathbf{w}(\delta)$ *be the unique strong solution of the model (14). Then there is a subsequence* $\delta_j \to 0$ *as* $j \to \infty$ *and a weak solution* $\mathbf{u}$ *of the Navier-Stokes equations such that* $\mathbf{w}(\delta_j) \to \mathbf{u}$ *in* $L^\infty(0,T,\mathbb{L}^2(Q)) \cap L^2(0,T,\mathbb{H}^1(Q))$*,*

Proof: This proof follows that of Theorem 3.1 of Layton and Lewandowski [7] ∎

The second question concerns the right norm. Obviously if we are restricting attention to general weak solutions the right norm must be a very weak norm for which the modelling residual $||\mathbf{u}\mathbf{u}^T - \mathrm{G}_N\overline{\mathbf{u}}(\mathrm{G}_N\overline{\mathbf{u}})^T||$ is not only well defined but also vanishes as $\delta \to 0$. The answer to this question is still unknown, see, e.g. Layton and Lewandowski [7] for first steps. The third question concerns extracting a rate of convergence for $||\overline{\mathbf{u}} - \mathbf{w}||$ which gives some insight into the model's accuracy on the laminar regions. This problem is much simpler. It reduces to proving the highest posible rate of convergence for $||\overline{\mathbf{u}} - \mathbf{w}|| \to 0$ for very smooth solution $\mathbf{u}$.

In the remainder of this subsection we give the answer : The modeling error is á priori $O(\delta^{2N+2})$ for smooth $\mathbf{u}$.

Proposition 5.2 *Assume* $\mathbf{u}$ *is a weak solution of the Navier-Stokes equations and* $\nabla\mathbf{u} \in L^4(0,T,\mathbb{L}^2(Q))$. *For* $\mathbf{w} \in L^2(0,T,\overline{\mathbb{H}}^2(Q)) \cap L^\infty(0,T;\mathbb{V})$ *a weak solution of (14) and* $\tau := \mathbf{u}\mathbf{u}^T - \mathrm{G}_N\overline{\mathbf{u}}(\mathrm{G}_N\overline{\mathbf{u}})^T$ *there exists a positive constant* $P = P(\nu, N, ||\nabla\mathbf{u}||_{L^4(0,T,\mathbb{L}^2)}) \geq 0$ *such that*

$$||\overline{\mathbf{u}} - \mathbf{w}||^2_{L^\infty(0,T,\mathbb{L}^2)} + ||\nabla(\overline{\mathbf{u}} - \mathbf{w})||^2_{L^2(0,T,\mathbb{L}^2)} \leq P(\nu, N, ||\nabla\mathbf{u}||_{L^4(0,T,\mathbb{L}^2)})||\tau||^2_{L^2(0,T,\mathbb{L}^2)} \tag{32}$$

Proof: To begin we derive an equation for $\varphi := \overline{\mathbf{u}} - \mathbf{w}$. First we note that $\mathbf{w}$ is a unique strong solution of the model and under stated regularity asumptions on $\mathbf{u}$, $\mathbf{u}$ is a unique strong solution of the Navier-Stokes equations, see Remark 3.3 in [10]. Thus there are no subtelties in the derivation of the error equation. Equality (3) can be rewritten as

$$\begin{aligned} \overline{\mathbf{u}}_t + \nabla\cdot(\overline{\mathrm{G}_N\overline{\mathbf{u}}\mathrm{G}_N\overline{\mathbf{u}}^T}) - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} &= \overline{\mathbf{f}} + \nabla\cdot(\overline{\mathrm{G}_N\overline{\mathbf{u}}\mathrm{G}_N\overline{\mathbf{u}}^T - \mathbf{u}\mathbf{u}^T}) \\ \nabla\cdot\overline{\mathbf{u}} &= 0 \end{aligned} \tag{33}$$

Substraction gives the equation for $\varphi := \overline{\mathbf{u}} - \mathbf{w}$

$$\begin{array}{rcll} \overline{\varphi}_t + \nabla\cdot(\overline{\mathrm{G}_N\overline{\mathbf{u}}\mathrm{G}_N\overline{\mathbf{u}}^T - \mathrm{G}_N\mathbf{w}\mathrm{G}_N\mathbf{w}^T}) - \nu\Delta\varphi + \nabla\overline{p-q} &=& -\nabla\cdot\overline{\tau} & \text{in } (0,T)\times\mathbb{R}^d \\ \nabla\cdot\varphi &=& 0 & \text{in } (0,T]\times\mathbb{R}^d \\ \varphi\,|_{t=0} &=& 0 & \text{in } \mathbb{R}^d \\ \int_Q p - q\,\mathrm{d}x &=& 0 & \text{in } (0,T]. \end{array} \tag{34}$$

We multiply the first equation in (34) by $(\mathrm{I}-\delta^2\Delta)^{-1}\mathrm{G}_N\varphi$ and then integrate on Q. Following exactly the same computations as in Lemma (4.1) gives

$$\frac{1}{2}\frac{d}{dt}||\varphi||^2 + \frac{1}{2}\delta^2\frac{d}{dt}||\nabla\varphi||^2 + \frac{\delta^2}{2}\frac{d}{dt}(\nabla\varphi, \mathrm{G}_{N-1}\nabla\varphi) + \tag{35}$$

$$+\nu||\nabla\varphi||^2 + \nu\delta^2||\Delta\varphi|| + \nu\delta^4(\Delta\varphi, \mathrm{G}_{N-1}\Delta\varphi) = -(\nabla\cdot\tau, \mathrm{G}_N\varphi) + b(\mathrm{G}_N\varphi, \mathrm{G}_N\overline{\mathbf{u}}, \mathrm{G}_N\varphi)$$

where b is the standard trilinear form

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = ((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w})$$

The first term on the right hand side is bounded as follows:

$$|(\nabla\cdot\tau, \mathrm{G}_N\varphi)| = |(\tau, \mathrm{G}_N\nabla\varphi)| \leq$$

$$\leq ||\tau|| ||\mathrm{G}_N||_{L^2} ||\nabla\varphi|| \leq \frac{2||\mathrm{G}_N||^2_{L^2}}{\nu} ||\tau||^2 + \frac{1}{2}\nu ||\nabla\varphi||^2$$

To bound the second term we use Young's inequality

$$ab \leq \epsilon a^4 + \frac{3}{4}(4\epsilon)^{-1/3} b^{4/3}$$

together with the standard estimate for the trilinear form

$$|b(\mathrm{G}_N\varphi, \mathrm{G}_N\overline{\mathbf{u}}, \mathrm{G}_N\varphi)| \leq C(Q) ||\nabla\overline{\mathbf{u}}|| ||\varphi||^{1/2} ||\nabla\varphi||^{3/2}$$

to obtain that for any $\epsilon > 0$

$$|b(\mathrm{G}_N\varphi, \mathrm{G}_N\overline{\mathbf{u}}, \mathrm{G}_N\varphi)| \leq \epsilon ||\mathrm{G}_N||^2 ||\nabla\varphi||^2 + \frac{3}{4}(4\epsilon)^{-1/3} ||\mathrm{G}_N\nabla\overline{\mathbf{u}}||^4 ||\mathrm{G}_N\varphi||^2$$

Picking in the above inequality $\epsilon = \frac{\nu}{2||\mathrm{G}_N||^2}$ we get that

$$|b(\mathrm{G}_N\varphi, \mathrm{G}_N\overline{\mathbf{u}}, \mathrm{G}_N\varphi)| \leq \frac{\nu}{2} ||\nabla\varphi||^2 + \frac{3}{4}\left(\frac{2\nu}{||\mathrm{G}_N||^2}\right)^{-1/3} ||\mathrm{G}_N||^4 ||\nabla\overline{\mathbf{u}}||^4 ||\varphi||^2$$

Using the last two inequalities in (35) gives

$$\frac{1}{2}\frac{d}{dt}||\varphi||^2 + \frac{1}{2}\delta^2 \frac{d}{dt}||\nabla\varphi||^2 + \frac{\delta^2}{2}\frac{d}{dt}(\nabla\varphi, \mathrm{G}_{N-1}\nabla\varphi) +$$

$$\nu\delta^2 ||\Delta\varphi|| + \nu\delta^4(\Delta\varphi, \mathrm{G}_{N-1}\Delta\varphi) \leq \frac{2||\mathrm{G}_N||^2_{L^2}}{\nu} ||\tau||^2 + \frac{3}{4}(2\nu)^{-1/3} ||\mathrm{G}_N||^{10/3} ||\nabla\overline{\mathbf{u}}||^4 ||\mathbf{w}||^2$$

Gronwal's inequality and positivity of the operators $(\mathrm{G}_N)_N$ gives

$$||\varphi||^2 \leq \int_0^t e^{-2\int_s^t (\frac{3}{4}(2\nu)^{-1/3} ||\mathrm{G}_N||^{10/3} ||\nabla\overline{\mathbf{u}}||^4 \mathrm{d}s'} \frac{2||\mathrm{G}_N||^2_{L^2}}{\nu} ||\tau||^2_{L^2} \mathrm{d}s$$

For fixed N we have that

$$||\mathrm{G}_N|| \leq 1 + (1 + ||\mathrm{G}||) + (1 + ||\mathrm{G}||)^2 + ... + (1 + ||\mathrm{G}||)^N$$

and since for every $\delta \quad ||\mathrm{G}|| \leq 1$ it follows

$$||\mathrm{G}_N|| \leq 2^{N+1} - 1$$

uniformly in δ. Under the assumption that $\nabla\mathbf{u} \in L^4(0, T, L^2)$ we infer the existence of a constant $M = M(\nu, N, ||\nabla\mathbf{u}||_{L^4(0,T,L^2)})$ such that

$$||\varphi||^2_{L^\infty(0,T,L^2)} \leq M(\nu, N, ||\nabla \mathbf{u}||_{L^4(0,T,L^2)}) \int_0^T ||\tau||^2_{0,T,L^2} \tag{36}$$

To estimate $||\nabla\varphi||^2_{L^2(0,T,L^2)}$ we integrate (35) from 0 to t and using inequality (36) we obtain

$$||\nabla\varphi||^2_{L^2(0,T,L^2)} \leq R(\nu, N, ||\nabla \mathbf{u}||_{L^4(0,T,L^2)}) \int_0^T ||\tau||^2_{0,T,L^2}$$

for positive constant $R = R(\nu, N, ||\nabla \mathbf{u}||_{L^4(0,T,L^2)})$. Consequently, there exists a constant $P = P(\nu, N, ||\nabla \mathbf{u}||_{L^4(0,T,L^2)})$ such that

$$||\varphi||^2_{L^\infty(0,T,L^2)} + ||\nabla\varphi||^2_{L^2(L^2)} \leq P(\nu, N, ||\nabla \mathbf{u}||_{L^4(0,T,L^2)})||\tau||^2_{L^2(0,T,L^2)} \tag{37}$$

■

Proposition 5.3 *Under the conditions of the previous theorem if* $\mathbf{u} \in \mathbb{H}^{N+1}(Q)$ *there exists* $P = P(\nu, N, \mathbf{u}) \geq 0$ *such that*

$$||\overline{\mathbf{u}} - \mathbf{w}||^2_{L^\infty(0,T,L^2)} + ||\nabla(\overline{\mathbf{u}} - \mathbf{w})||^2_{L^2(0,T,L^2)} \leq P(\nu, N, \mathbf{u})\delta^{2N+2} \tag{38}$$

Proof: An aplication of Lemma (2.3) gives

$$||\tau||^2_{L^2(0,T,L^2)} \leq C(\mathbf{u})\delta^{2N+2}$$

(38) will follow then from (37).

■

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