A note on chain lengths and the Tutte polynomial

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Abstract

We show that the number of chains of given length in a graph G can be easily found from the Tutte polynomial of G. Hence two Tutte equivalent graphs will have the same distribution of chain lengths. We give two applications of this latter statement.

We also give the dual results for the numbers of multiple edges with given muliplicities.

1 Nomenclature and notation.

In this paper we shall allow graphs to have loops and multiple edges. For convenience we shall assume that all graphs are connected.

By the 'suppression' of a vertex u of degree 2 in a graph we mean the deletion of u followed by the joining by an edge of the two vertices formerly adjacent to u. If this suppression operation is performed as often as possible in a graph G, the result will be a graph with no vertices of degree 2 called the 'homeomorphically reduced graph', or just the 'reduced graph' of G. Call it M.

The reverse operation, that of inserting vertices of degree 2 into the edges of M, or, what amounts to the same thing, replacing each edge of M by a 'chain' of edges in series, will reproduce the original graph G, and also many others, depending on the lengths of the chains which replace the edges of M (the 'length' of a chain is the number of edges in it). Two graphs are said to be 'homeomorphic' if they have the same reduced graph. This is an equivalence relation and under it the graphs having the same reduced graph form a 'homeomorphism class' of equivalent graphs.

Let M be a graph whose edges are labeled, say with the symbols $a, b, c, \ldots$ etc., and let a chain length be associated with each edge. Denote the chain lengths by $n_a, n_b, n_c, \ldots$ etc. Such a 'chain graph' will clearly represent all the graphs in the corresponding homeomorphism class.

In [3] we introduced the 'chain polynomial' as a means of subsuming in one polynomial the chromatic polynomials of all the graphs in a homeomorphism class. For our present purposes we define the chain polynomial of a graph G as follows:

$$Ch(G;\lambda,x;a,b,c,\ldots) = \sum_{(Y,U)} F(Y,\lambda)x^{\nu(U)} \qquad (1.1)$$

where the sum is over all partitions (Y,U) of the edges of G, $F(Y,\lambda)$ denotes the flow polynomial of the graph $\langle Y\rangle$ induced by the edges in Y and $\nu(U)$ denotes the sum of the edge lengths of the edges in the subset U.

Note: In [3] it was convenient to work in terms of another variable $\omega = 1-\lambda$, but there is no advantage in doing that here. Hence λ is the usual variable for the flow polynomial, and (below) for the chromatic polynomial.

In [3] we showed that the chromatic polynomial of a chain graph is

$$\lambda^{-(q-p)}Ch(G;\lambda,1-\lambda;a,b,c,\ldots),$$

and in [4] we showed that the Tutte polynomial is given by

$$t(G;x,y) = \frac{1}{(x-1)^{q-p+1}}Ch(G;(x-1)(y-1),x;a,b,c,\ldots) \qquad (1.2)$$

where, here and elsewhere, p and q denote the numbers of vertices and edges respectively in G.

Note: p and q will often occur together as '$q-p$' which is the same for homeomorphic graphs; when they occur separately they will be taken to refer to the reduced graph.

If two graphs have the same Tutte polynomial we say that they are 'Tutte equivalent'. If no non-isomorphic graph is Tutte equivalent to a graph G we shall say that G is 'Tutte unique'.

2 Three theorems.

Let G be a graph whose reduced graph M is 3-edge-connected, and let the edges of M be labeled $a,b,c,\ldots$. We look at the two highest powers of λ in the chain polynomial of G.

If we take $U=\emptyset$ in (1.1) then $\langle Y\rangle$ is the whole of G and we get a contribution of

$$F(G,\lambda) = \lambda^{q-p+1} - q\lambda^{q-p} + terms\ in\ lower\ powers\ of \lambda$$

to the chain polynomial.

If U contains a single edge, say $U=\{a\}$, then $\langle Y\rangle$ is $G-a$, that is, G with the edge a removed. Because of the 3-edge-connectedness requirement this graph does not have a bridge. Hence its flow polynomial does not vanish. This polynomial is monic and of degree $q-p$, and thus gives contribution

$$\lambda^{q-p}x^{n_a} + terms\ in\ lower\ powers\ of \lambda$$

to the chain polynomial. The other chains in G give similar contributions.

We thus arrive at the following result.

Theorem 1. The coefficient of λ^{q-p} in $Ch(G)$ is

$$x^{n_a} + x^{n_b} + x^{n_c} + \ldots - q.$$

Note: The restriction to 3-edge-connecteness is not as stringent as it might appear. As we showed on page 348 of [3], using the 'bead on a string' principle, a reduced graph that is only 2-edge-connected can be manipulated into a 3-edge-connected graph having the same chain polynomial. With this understanding it suffices for the graph to have no bridges. We shall assume this in what follows.

Following the method used to prove Theorem 1 we see that the coefficient of y^{q-p} in $Ch(G; (x-1)(y-1), a, b, c, \ldots)$ is made up from a contribution

$$-(q-p+1)(x-1)^{q-p+1} - q(x-1)^{q-p}$$

from the leading term, and a term like

$$(x-1)^{q-p} x^{n_a}$$

from each of the terms where U is a single-element set.

It follows from (1.2) that the coefficient of y^{q-p} in $t(G; x, y)$ is

$$C = \frac{\sum x^{n_a} - q}{x-1} - (q-p+1)$$

Now $\sum x^{n_a}$, with like terms collected up, is the (polynomial) generating function for the chain lengths in G. We therefore derive the following result.

Theorem 2. If C is the coefficient of y^{q-p} in $t(G; x, y)$ then

$$f(x) = (x-1)[C + (q-p+1)] + q$$

is the generating function for the chain lengths in G, in the sense that the coefficient of x^k in $f(x)$ is the number of chains in G of length k.

It is well-known that the Tutte polynomial of a given graph can provide information about properties that are not obvious from inspection of the graph (such as the number of spanning trees). That is not the case here; if we are *given* a graph we hardly need the Tutte polynomial to tell us what the chain lengths are! However Theorem 2 can be meaningful in contexts where the graphs are not given, and the following theorem has proved to be useful.

Theorem 3. If G_1 and G_2 are Tutte equivalent, then, for any positive integer k, G_1 and G_2 have the same number of chains of length k.

Proof. This follows immediately from Theorem 2.

3 The dual case.

The dual concept to a chain graph is a 'sheaf' graph, for which the reduced graph, say N, has no multiple edges and the graph is obtained by replacing each edge of N by a 'sheaf' of edges in parallel. Corresponding to the chain polynomial there is a 'sheaf polynomial' in which the role of the flow polynomial in the chain graph is played instead by the chromatic polynomial. For details see [3].

The dual analogs of Theorems 1 and 2 can be obtained by the same proof technique, amounting essentially to the interchange of the roles of x and y in the Tutte polynomial. We spare the reader the details, but note the dual equivalent of the Theorem 3.

Theorem 4. If two loopless graphs G_1 and G_2 are Tutte equivalent, then, for any positive integer k, G_1 and G_2 have the same number of multiple edges of multiplicity k.

4 Two simple applications.

(a) We define an 's-theta-graph' to be a graph consisting of s chains between two distinguished vertices; in other words a graph whose reduced graph is a multiple edge of multiplicity s. Thus the usual 'theta graph' is a 3-theta-graph. If G is an s-theta-graph we ask if some other graph, H say, can be Tutte equivalent to G.

If so, then the reduced graph of H has 2 vertices and s edges. If all the chains in H join the two vertices then H is also an s-theta-graph, and by Theorem 3 it has the same multiset of chain lengths. This makes it isomorphic to G, since the chain lengths obviously determine the graph uniquely.

If one chain does not join these vertices it must be a loop chain. But if one chain is a loop then, by symmetry, they must all be. In this case, however, the plane dual of H is a tree and hence $F(H)$ is a power of $\lambda - 1$. If $s > 2$ then $F(G)$ is not of this form and H cannot therefore have the same Tutte polynomial as G. If $s = 2$ G is a cycle, and cycles are known to be Tutte unique.

We deduce that all s-theta-graphs are Tutte unique.

(b) Consider any homeomorph, G, of K_4 – the complete graph on four vertices.

If some graph H is Tutte equivalent to G then, *a fortiori*, it is chromatically equivalent to G. But it was shown by Chao and Zhao in [1] that any graph chromatically equivalent to a homeomorph of K_4 is itself a homeomorph of K_4. Hence H is a homeomorph of K_4.

By Theorem 3 G and H will have the same multiset of chain lengths. However, Li [2] has shown that if two homeomorphs of K_4 have the same multiset of chain lengths and are chromatically equivalent, then they are isomorphic.

It follows that all homeomorphs of K_4 are Tutte unique.

References

[1] Chao, C.-Y. & Zhao, L.-C., Chromatic polynomials of a family of graphs, *Ars Combinatoria* **15** (1983) 111-129.

[2] Li, W.-M., Almost every K_4 homeomorph is chromatically unique, *Ars Combinatoria* **23** (1987) 13-36.

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[4] Read, R. C. & Whitehead, E. G. Jr., The Tutte polynomial for homeomorphism classes of graphs, *Discrete Math.* **243** (2002) 267-272.