Analysis of a Discontinuous Finite Element Method for the Coupled Stokes and Darcy Problems

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Abstract

The coupled Stokes and Darcy flows problem is solved by the locally conservative discontinuous Galerkin method. Optimal error estimates for the fluid velocity and pressure are derived.

1 Introduction

Modeling the interaction between surface and subsurface flow is a challenging environmental problem. One such example is the simulation of transport of contaminants through rivers into the aquifers. Mathematically, this complex problem can be modeled by the coupled system of Stokes and Darcy equations.

The aim of this paper is to formulate and analyze a discontinuous finite element method for the coupled Stokes and Darcy problems. The physical domain is decomposed into two regions: the region filled with an incompressible fluid modeled by the Stokes equations and the porous medium region modeled by Darcy's law. The interface conditions consist of the Beavers-Joseph-Saffman condition, the continuity of flux and the balance of forces. The unknowns, namely the fluid velocity and pressure in the fluid region, and the fluid pressure in the porous medium, are approximated by totally discontinuous polynomials of different order. The discontinuous Galerkin (DG) methods considered here, are based on the Non-symmetric Interior Penalty Galerkin (NIPG) method [16, 17] and the Symmetric Interior Penalty Galerkin (SIPG) method [20, 2]. DG methods are attractive methods because they are element-wise conservative, they are high-order methods, and they are easily implementable on unstructured meshes. A

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few algorithms for the coupling of Stokes and Darcy can be found in the literature. In [13], the existence and uniqueness of a weak solution to the coupled system is proved and the proposed scheme combines the continuous finite element method for the Stokes problem with the mixed finite element method for the Darcy equations. The study of continuous finite element methods for both regions can be found in [7]. In a more recent work [18], the DG method is used in the Stokes region while the mixed finite element method is used for Darcy region. Finally, the reader can refer to [4, 15, 14, 8] for analysis of similar coupled models.

The outline of the paper is as follows. Section 2 contains the model problem and notation. The numerical scheme is introduced in Section 3. The a priori error estimates are proved in Section 4 and followed by some concluding remarks.

2 Model problem and notation

Let Ω be a domain in $\mathbb{R}^2$, subdivided into two subdomains Ω_1, Ω_2, with interface $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. Define $\Gamma_i = \partial\Omega_i - \Gamma_{12}$ for $i = 1, 2$. The physical quantities are the fluid velocity $\boldsymbol{u}$ and pressure p. Denote $\boldsymbol{u}_i = \boldsymbol{u}|_{\Omega_i}$ and $p_i = p|_{\Omega_i}$. We assume that the flow satisfies the Stokes equations on Ω_1 and the single phase equations on Ω_2.

$$-\nabla \cdot (2\mu \boldsymbol{D}(\boldsymbol{u}_1) - p_1 \boldsymbol{I}) = \boldsymbol{f}_1, \quad \text{in} \quad \Omega_1, \tag{1}$$

$$\nabla \cdot \boldsymbol{u}_1 = 0, \quad \text{in} \quad \Omega_1, \tag{2}$$

$$\boldsymbol{u}_2 = -\boldsymbol{K}\nabla p_2, \quad \text{in} \quad \Omega_2, \tag{3}$$

$$\nabla \cdot \boldsymbol{u}_2 = f_2, \quad \text{in} \quad \Omega_2. \tag{4}$$

Here, $\boldsymbol{f}_1$ and f_2 are external forces acting on the fluid, $\mu > 0$ is the constant fluid viscosity, $\boldsymbol{D}(\boldsymbol{u})$ is the strain tensor defined by $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ and $\boldsymbol{K}$ is the permeability tensor. We assume that $\boldsymbol{K}$ is symmetric, positive definite tensor, bounded below and above uniformly, and that the force satisfies the solvability condition $\int_{\Omega_2} f_2 = 0$. The boundary conditions are

$$\boldsymbol{u}_1 = \boldsymbol{0}, \quad \text{on} \quad \Gamma_1, \tag{5}$$

$$-\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma_2, \tag{6}$$

where $\boldsymbol{n}$ is the outward normal to the boundary $\partial\Omega$. As the pressure is unique up to an additive constant, we assume that

$$\int_\Omega p = 0. \tag{7}$$

Let $\boldsymbol{n}_{12}$ (resp. $\boldsymbol{\tau}_{12}$) be the unit normal (resp. tangential) vector to Γ_{12} outward of Ω_1. The conditions at the interface Γ are:

$$\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12} = \boldsymbol{u}_2 \cdot \boldsymbol{n}_{12}, \tag{8}$$

$$p_1 - 2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12}) = p_2, \tag{9}$$

$$\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12} = -2G(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12}. \tag{10}$$

Interface conditions (8) and (9) represent the mass conservation and the balance of forces respectively, across the interface. The Beaver-Joseph-Saffman law (10) is the most accepted condition (see [3, 19, 12]) and includes a friction constant $G > 0$ that can be determined experimentally. The existence of a unique weak solution of (1)-(10) was shown in [13]. We assume here that the solution $(\boldsymbol{u}, p)$ is regular enough, and is a strong solution of (1)-(10).

We now define the functional spaces. For $i = 1, 2$, let $\mathcal{E}_h^i$ be a non-degenerate quasi-uniform subdivision of Ω_i, let Γ_h^i be the set of interior edges and let h_i denote the maximum diameter of elements in $\mathcal{E}_h^i$. For any non-negative integer k and number $r \geq 1$, the classical Sobolev space [1] on a domain $\mathcal{O}$ is denoted by $W^{k,r}(\mathcal{O}) = \{v \in L^r(\mathcal{O}) : D^m v \in L^r(\mathcal{O}), \quad \forall |m| \leq k\}$, where $D^m v$ are the partial derivatives of v of order m. The associated Sobolev norm is denoted by $\|\cdot\|_{k,r,\mathcal{O}}$, or by $\|\cdot\|_{k,\mathcal{O}}$ if $r = 2$. We use the usual notation $H^k(\mathcal{O})$ for $W^{k,2}(\mathcal{O})$ and $L_0^2(\mathcal{O})$ for the space of square-integrable functions with zero average. The L^2 inner-product will be denoted by $(\cdot, \cdot)$. Throughout the paper, C will denote a generic positive constant whose value may vary but will but will be independent of the mesh sizes h_1 and h_2. Our scheme requires that the trace of p_1 and the trace of the normal derivatives of $\boldsymbol{u}_1$ and p_2 are well defined, and are square-integrable. Therefore, we define the following functional spaces:

$$\boldsymbol{X}^1 = \{\boldsymbol{v}_1 \in (L^2(\Omega_1))^2 : \quad \forall E \in \mathcal{E}_h^1, \quad \boldsymbol{v}_1|_E \in (W^{2,4/3}(E))^2\}, \tag{11}$$

$$M^1 = \{q_1 \in L^2(\Omega_1) : \quad \forall E \in \mathcal{E}_h^1, \quad q_1|_E \in W^{1,4/3}(E)\}, \tag{12}$$

$$M^2 = \{q_2 \in L^2(\Omega_2) : \quad \forall E \in \mathcal{E}_h^2, \quad q_2|_E \in W^{2,2}(E)\}. \tag{13}$$

We associate to these spaces the following norms $\|\cdot\|_{X^1}, \|\cdot\|_{M^1}$ and semi-norm $\|\cdot\|_{M^2}$:

$$\|\boldsymbol{v}_1\|_{X^1}^2 = \|\nabla \boldsymbol{v}_1\|_{0,\Omega_1}^2 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\boldsymbol{v}_1]\|_{0,e}^2 + \frac{1}{G} \sum_{e \in \Gamma_{12}} \|\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\|_{0,e}^2,$$

$$\|q_1\|_{M^1}^2 = \|q_1\|_{0,\Omega_1}^2,$$

$$\|q_2\|_{M^2}^2 = \|\nabla q_2\|_{0,\Omega_2}^2 + \sum_{e \in \Gamma_h^2} \frac{\sigma_{2,e}}{|e|} \|[q_2]\|_{0,e}^2.$$

Here, $|e|$ denotes the measure of each edge e, the parameters $\sigma_{1,e}$ and $\sigma_{2,e}$ are positive constants that may vary on each edge, and that will be defined in the next section. Finally, the $|||\cdot|||$ norm is the usual "broken" norm: $|||w|||^2_{m,\Omega_i} = \sum_{E\in\mathcal{E}_h^i} \|w\|^2_{m,E}$ for $i = 1, 2$ and for w any scalar or vector function. Given a fixed normal vector $\boldsymbol{n}_e$ on each edge, the average $\{w\}$ and jump $[w]$ of function w can be defined:

$$\begin{aligned} \{w\} &= \tfrac{1}{2}(w|_{E_e^1}) + \tfrac{1}{2}(w|_{E_e^2}), \quad [w] = (w|_{E_e^1}) - (w|_{E_e^2}), \quad \forall e = \partial E_e^1 \cap \partial E_e^2, \\ \{w\} &= w|_{E_e^1}, \qquad\qquad\qquad [w] = w|_{E_e^1}, \quad \forall e = \partial E_e^1 \cap \partial\Omega. \end{aligned}$$

The aim of this paper is to formulate an algorithm that uses totally discontinuous approximating spaces. Let k_1, k_2 be two positive integers and let the finite-dimensional subspaces $\boldsymbol{X}_h^1 \subset \boldsymbol{X}^1$, $M_h^1 \subset M^1$ and $M_h^2 \subset M^2$, with the induced norms, be defined as follows

$$\begin{aligned} \boldsymbol{X}_h^1 &= \{\boldsymbol{v}_1 \in \boldsymbol{X}^1 : \quad \forall E \in \mathcal{E}_h^1, \quad \boldsymbol{v}_1 \in (I\!P_{k_1}(E))^2\}, \\ M_h^1 &= \{q_1 \in M^1 : \quad \forall E \in \mathcal{E}_h^1, \quad q_1 \in I\!P_{k_1-1}(E)\}, \\ M_h^2 &= \{q_2 \in M^2 : \quad \forall E \in \mathcal{E}_h^2, \quad q_2 \in I\!P_{k_2}(E)\}. \end{aligned}$$

We assume that the discrete spaces satisfy the optimal approximation properties. In particular, there exist an operator $R_h \in \mathcal{L}(H^1(\Omega_1)^2; \boldsymbol{X}_h^1)$ such that for any $E \in \mathcal{E}_h^1$,

$$\forall \boldsymbol{v} \in H^1(\Omega_1)^2, \quad \forall q_1 \in I\!P_{k_1-1}(E), \quad \int_E q_1 \nabla\cdot(R_h(\boldsymbol{v}) - \boldsymbol{v}) = 0, \tag{14}$$

$$\forall \boldsymbol{v} \in H^1(\Omega_1)^2, \quad \forall e \in \Gamma_h^1, \quad \forall \boldsymbol{q}_1 \in I\!P_{k_1-1}(e)^2, \quad \int_e \boldsymbol{q}_1 \cdot [R_h(\boldsymbol{v})] = 0, \tag{15}$$

$$\forall \boldsymbol{v} \in H_0^1(\Omega_1)^2, \quad \forall e \in \partial\Omega_1, \quad \forall \boldsymbol{q}_1 \in I\!P_{k_1-1}(e)^2, \quad \int_e \boldsymbol{q}_1 \cdot R_h(\boldsymbol{v}) = 0, \tag{16}$$

$$\forall \boldsymbol{v} \in H^1(\Omega_1)^2, \quad \forall e \in \Gamma_h^1 \cup \Gamma^1 \cup \Gamma_{12}, \int_e (R_h(\boldsymbol{v}) - \boldsymbol{v}) = 0, \tag{17}$$

$$\forall \boldsymbol{v} \in H_0^1(\Omega_1)^2, \quad \|\boldsymbol{v} - R_h(\boldsymbol{v})\|_{X^1} \le C|\boldsymbol{v} - R_h(\boldsymbol{v})|_{1,\Omega_1}, \tag{18}$$

$$\forall \boldsymbol{v} \in W^{s,t}(\Omega_1)^2, \forall s \in [1, k_1+1], \quad |\boldsymbol{v} - R_h(\boldsymbol{v})|_{W^{m,t}(E)} \le C h_E^{s-m} |\boldsymbol{v}|_{W^{s,t}(\Delta_E)}, \tag{19}$$

where Δ_E is a suitable macro-element containing E. Note that property (18) is an easy consequence of (17) (see [10]). There exists also an operator $r_h \in \mathcal{L}(L^2(\Omega); M_h^1 \times M_h^2)$ such that for $s = k_1 - 1, k_2$ and for any $z \in L^2(\Omega) \cap W^{s+1,2}(\Omega)$

$$\forall q \in I\!P_s(E), \int_E q(r_h(z) - z) = 0, \quad \forall E \in \mathcal{E}_h^1 \cup \mathcal{E}_h^2, \tag{20}$$

$$\|z - r_h(z)\|_{m,E} \le C h_E^{s+1-m} |z|_{s+1,E}, \quad \forall E \in \mathcal{E}_h^1 \cup \mathcal{E}_h^2, \quad m = 0, 1 \tag{21}$$

Note that the existence of R_h for $k_1 = 1, 2$ or 3 follows from [6, 9, 5], and the operator r_h is the well-known L^2 projection operator.

We recall a result proved in [10] that generalize a Sobolev imbedding. For any real number $s \in [2, \infty)$, there is a constant C independent of h such that

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h^1, \quad \|\boldsymbol{v}_1\|_{L^s(\Omega_1)} \leq C \|\boldsymbol{v}_1\|_{X^1}. \tag{22}$$

We now finish this section with some trace and inverse inequalities needed for the analysis. Let E be a mesh element with diameter h_E. Then, there exists a constant C independent of h_E such that

$$\forall \phi \in H^1(E), \quad \forall e \subset \partial E, \quad \|\phi\|_{0,e}^2 \leq C(h_E^{-1}\|\phi\|_{0,E}^2 + h_E|\phi|_{1,E}^2), \tag{23}$$

$$\forall \phi \in H^2(E), \quad \forall e \subset \partial E, \quad \|\nabla\phi \cdot \boldsymbol{n}_e\|_{0,e}^2 \leq C(h_E^{-1}|\phi|_{1,E}^2 + h_E|\phi|_{2,E}^2), \tag{24}$$

$$\forall \phi \in I\!P_k(E), \quad \forall e \subset \partial E, \quad \|\nabla\phi \cdot \boldsymbol{n}_e\|_{0,e} \leq C h_E^{-1/2}|\phi|_{1,E}. \tag{25}$$

3 Scheme

We introduce the following bilinear forms $a_1 : \boldsymbol{X}^1 \times \boldsymbol{X}^1 \to I\!R$, $b : \boldsymbol{X}^1 \times M^1 \to I\!R$ and $a_2 : M^2 \times M^2 \to I\!R$:

$$\begin{aligned} a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) = 2 \sum_{E \in \mathcal{E}_h^1} \int_E \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \int_e [\boldsymbol{u}_1] \cdot [\boldsymbol{v}_1] \\ -2 \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\nabla \boldsymbol{u}_1 \boldsymbol{n}_e\} \cdot [\boldsymbol{v}_1] + 2\epsilon_1 \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\nabla \boldsymbol{v}_1 \boldsymbol{n}_e\} \cdot [\boldsymbol{u}_1] \\ + \frac{1}{G} \sum_{e \in \Gamma_{12}} \int_e (\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12})(\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}), \end{aligned} \tag{26}$$

$$b_1(\boldsymbol{v}_1, p_1) = -\sum_{E \in \mathcal{E}_h} \int_E p_1 \nabla \cdot \boldsymbol{v}_1 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{p_1\}[\boldsymbol{v}_1] \cdot \boldsymbol{n}_e, \tag{27}$$

$$\begin{aligned} a_2(p_2, q_2) = \sum_{E \in \mathcal{E}_h^2} \int_E \boldsymbol{K} \nabla p_2 \cdot \nabla q_2 + \sum_{e \in \Gamma_h^2} \frac{\sigma_{2,e}}{|e|} \int_e [p_2][q_2] \\ - \sum_{e \in \Gamma_h^2} \int_e \{\boldsymbol{K} \nabla p_2 \cdot \boldsymbol{n}_e\}[q_2] + \epsilon_2 \sum_{e \in \Gamma_h^2} \int_e \{\boldsymbol{K} \nabla q_2 \cdot \boldsymbol{n}_e\}[p_2]. \end{aligned} \tag{28}$$

By introducing the parameters ϵ_1, ϵ_2 that take the value ± 1, we allow for non-symmetric or symmetric bilinear forms a_1 and a_2. Throughout the paper, we assume that:
Hypothesis A: if on one hand $\epsilon_1 = 1$, then the parameter $\sigma_{1,e}$ needs only to

be positive, and can simply be equal to 1 for the resulting NIPG method. On the other hand, if $\epsilon_1 = -1$, the parameter $\sigma_{1,e}$ cannot be arbitrary, and it must be bounded below by a sufficiently large positive value. This is a standard constraint for the SIPG method. The same assumptions hold true for ϵ_2 and the parameter $\sigma_{2,e}$.

We next define a bilinear form $\Lambda : M^2 \times \boldsymbol{X}^1 \to \mathbb{R}$ acting on the interface Γ_{12}.

$$\Lambda(q_2, \boldsymbol{v}_1) = \sum_{e \in \Gamma_{12}} \int_e q_2 \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}, \quad \forall (q_2, \boldsymbol{v}_1) \in M^2 \times \boldsymbol{X}^1. \tag{29}$$

With these forms, we propose the following variational problem of (1)-(10): Find $(\boldsymbol{u}_1, p_1, p_2) \in \boldsymbol{X}^1 \times M^1 \times M^2$, solution of

$$\mu a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) + b(\boldsymbol{v}_1, p_1) + \Lambda(p_2, \boldsymbol{v}_1) = (\boldsymbol{f}_1, \boldsymbol{v}_1), \quad \forall \boldsymbol{v}_1 \in X_h^1, \tag{30}$$

$$b(\boldsymbol{u}_1, q_1) = 0, \quad \forall q_1 \in M_h^1, \tag{31}$$

$$a_2(p_2, q_2) - \Lambda(q_2, \boldsymbol{u}_1) = (f_2, q_2), \quad \forall q_2 \in M_h^2, \tag{32}$$

$$\int_{\Omega_1} p_1 + \int_{\Omega_2} p_2 = 0. \tag{33}$$

Lemma 3.1 If $(\boldsymbol{u}_1, p_1, p_2)$ is the solution of the coupled Stokes-single phase flow problem (1)-(4), then $(\boldsymbol{u}_1, p_1, p_2)$ is the solution of (30)-(32).

Proof

Multiply the Stokes equation (1) by $\boldsymbol{v}_1$, integrate by parts over one element $E \in \mathcal{E}_h^1$, and sum over all elements in $\mathcal{E}_h^1$:

$$\sum_{E \in \mathcal{E}_h^1} \int_E (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)) : \nabla \boldsymbol{v}_1 - \sum_{e \in \Gamma_h^1} \int_e [(-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)) \boldsymbol{n}_e \cdot \boldsymbol{v}_1]$$
$$- \sum_{e \in \Gamma_{12}} \int_e (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)) \boldsymbol{n}_{12} \cdot \boldsymbol{v}_1 - \sum_{e \in \Gamma_1} \int_e (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)) \boldsymbol{n} \cdot \boldsymbol{v}_1$$
$$= \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1.$$

Noting that $\boldsymbol{D}(\boldsymbol{u}_1) : \nabla \boldsymbol{v}_1 = \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1)$ and $\boldsymbol{I} : \nabla \boldsymbol{v}_1 = \nabla \cdot \boldsymbol{v}_1$, we can rewrite the equation

$$\sum_{E \in \mathcal{E}_h^1} \int_E (2\mu \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) - p_1 \nabla \cdot \boldsymbol{v}_1) - \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{(-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)) \boldsymbol{n}_e\} \cdot [\boldsymbol{v}_1]$$
$$- \sum_{e \in \Gamma_h^1} \int_e [(-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)) \boldsymbol{n}_e] \cdot \{\boldsymbol{v}_1\} - \sum_{e \in \Gamma_{12}} \int_e (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)) \boldsymbol{n}_{12} \cdot \boldsymbol{v}_1$$
$$= (\boldsymbol{f}_1, \boldsymbol{v}_1).$$

By regularity of the true solution and with the boundary condition (5), we have

$$\sum_{E\in\mathcal{E}_h^1}\int_E (2\mu \boldsymbol{D}(\boldsymbol{u}_1):\boldsymbol{D}(\boldsymbol{v})-p_1\nabla\cdot\boldsymbol{v}_1)+\sum_{e\in\Gamma_h^1}\int_e \{p_1\}[\boldsymbol{v}_1]\cdot\boldsymbol{n}_e$$
$$-2\mu\sum_{e\in\Gamma_h^1\cup\Gamma^1}\int_e\{\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_e\}\cdot[\boldsymbol{v}_1]+2\mu\epsilon_1\sum_{e\in\Gamma_h^1\cup\Gamma^1}\int_e\{2\mu\boldsymbol{D}(\boldsymbol{v}_1))\boldsymbol{n}_e\}\cdot[\boldsymbol{u}_1]$$
$$-\sum_{e\in\Gamma_{12}}\int_e(-p_1\boldsymbol{I}+2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12}\cdot\boldsymbol{v}_1=(\boldsymbol{f}_1,\boldsymbol{v}_1). \tag{34}$$

By decomposing $\boldsymbol{v}_1$ and $2\mu\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}$ into their normal and tangential components, the interface integral is reduced to

$$-\sum_{e\in\Gamma_{12}}\int_e(-p_1\boldsymbol{I}+2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12}\cdot\boldsymbol{v}_1=$$
$$\sum_{e\in\Gamma_{12}}\int_e(p_1-2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12})(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})$$
$$-\sum_{e\in\Gamma_{12}}\int_e 2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{\tau}_{12}(\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}).$$

With the interface conditions (9) and (10), the integral becomes:

$$-\sum_{e\in\Gamma_{12}}\int_e(-p_1\boldsymbol{I}+2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12}\cdot\boldsymbol{v}_1=$$
$$\sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})+\frac{\mu}{G}\sum_{e\in\Gamma_{12}}\int_e(\boldsymbol{u}_1\cdot\boldsymbol{\tau}_{12})(\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}). \tag{35}$$

The continuity of $\boldsymbol{u}_1$ and equations (34) and (35) yield:

$$\mu a_1(\boldsymbol{u}_1,\boldsymbol{v}_1)+b(\boldsymbol{v}_1,p_1)+\Lambda(p_2,\boldsymbol{v}_1)=(\boldsymbol{f}_1,\boldsymbol{v}_1).$$

The equation (31) is a consequence of (2), (5) and the fact that $[\boldsymbol{u}]\cdot\boldsymbol{n}_e=0$ on each edge e. Now for the single phase flow part, we repeat the process with (4): multiply by a test function q_2, and integrate by parts and sum over all elements in $\mathcal{E}_h^2$. The definition (3) of $\boldsymbol{u}_2$, the regularity of p_2 and

the boundary condition (6) give

$$\sum_{E\in\mathcal{E}_h^2}\int_E \boldsymbol{K}\nabla p_2\cdot\nabla q_2 - \sum_{e\in\Gamma_h^2\cup\Gamma_{12}}\int_e\{\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_e\}[q_2]$$
$$+\epsilon_2\sum_{e\in\Gamma_h^2}\int_e\{\boldsymbol{K}\nabla q_2\cdot\boldsymbol{n}_e\}\cdot[p_2] + \sum_{e\in\Gamma_h^2}\frac{\sigma_e}{|e|}\int_e[p_2][q_2]$$
$$-\sum_{e\in\Gamma_{12}}\int_e(\boldsymbol{u}_2\cdot\boldsymbol{n}_{12})q_2 = (f_2,q_2).$$

Using the interface condition (8) in the equation above, gives (32). Finally, the equation (33) is just the condition (7).

The discrete scheme is: Find $(\boldsymbol{U}_1,P_1,P_2)\in\boldsymbol{X}_h^1\times M_h^1\times M_h^2$ such that

$$\mu a_1(\boldsymbol{U}_1,\boldsymbol{v}_1)+b(\boldsymbol{v}_1,P_1)+\Lambda(P_2,\boldsymbol{v}_1)=(\boldsymbol{f}_1,\boldsymbol{v}_1),\quad\forall\boldsymbol{v}_1\in X_h^1, \tag{36}$$
$$b(\boldsymbol{U}_1,q_1)=0,\quad\forall q_1\in M_h^1, \tag{37}$$
$$a_2(P_2,q_2)-\Lambda(q_2,\boldsymbol{U}_1)=(f_2,q_2),\quad\forall q_2\in M_h^2, \tag{38}$$
$$\int_{\Omega_1}P_1+\int_{\Omega_2}P_2=0. \tag{39}$$

Before addressing the existence and uniqueness of a solution to the numerical scheme, we recall the fact that the bilinear forms a_1 and a_2 are coercive with respect to $\|\cdot\|_{X^1}$ and $\|\cdot\|_{M^2}$ respectively. The proof of the following lemma is straightforward for the nonsymmetric bilinear forms, and can be found in [20] for the symmetric bilinear forms.

Lemma 3.2 Under hypothesis A, there exist two positive constants C_1 and C_2 such that

$$C_1\|\boldsymbol{v}_1\|_{X^1}^2\le a_1(\boldsymbol{v}_1,\boldsymbol{v}_1),\quad\forall\boldsymbol{v}_1\in\boldsymbol{X}^1. \tag{40}$$
$$C_2\|q_2\|_{M^2}^2\le a_2(q_2,q_2),\quad\forall q_2\in M^2. \tag{41}$$

Lemma 3.3 The discrete scheme has a unique solution.

Proof
Since (36)-(39) is a square system of linear equations in finite dimension, it suffices to prove that $(\boldsymbol{f}_1,f_2)=(\boldsymbol{0},0)$ implies $(\boldsymbol{U}_1,P_1,P_2)=(\boldsymbol{0},0,0)$. We choose $\boldsymbol{v}_1=\boldsymbol{U}_1$ in (36), $q_1=P_1$ in (37) and $q_2=P_2$ in (38). Adding the resulting equations give

$$a_1(\boldsymbol{U}_1,\boldsymbol{U}_1)+a_2(P_2,P_2)=0.$$

From Lemma 3.2, this implies $\|\boldsymbol{U}_1\|_{X^1} = \|P_2\|_{M^2} = 0$, which means that $\boldsymbol{U}_1 = 0$ and P_2 is a global constant over Ω_2. The equation (36), with (39) becomes

$$b(\boldsymbol{v}_1, P_1) - \frac{1}{|\Omega_2|}(\int_{\Omega_1} P_1) \sum_{e \in \Gamma_{12}} \int_e \boldsymbol{v}_1 \cdot \boldsymbol{n}_e = 0, \quad \forall \boldsymbol{v}_1 \in \boldsymbol{X}_h.$$

We now write $P_1 = \bar{P}_1 + \tilde{P}_1$, where $\bar{P}_1 = \frac{1}{|\Omega_1|}\int_{\Omega_1} P_1$. The equation above becomes

$$b(\boldsymbol{v}_1, \tilde{P}_1) + b(\boldsymbol{v}_1, \bar{P}_1) - \frac{|\Omega_1|}{|\Omega_2|}\bar{P}_1 \sum_{e \in \Gamma_{12}} \int_e \boldsymbol{v}_1 \cdot \boldsymbol{n}_e = 0, \quad \forall \boldsymbol{v}_1 \in \boldsymbol{X}_h.$$

Note that

$$b(\boldsymbol{v}_1, \bar{P}_1) = -\bar{P}_1 \sum_{e \in \Gamma_{12}} \int_e \boldsymbol{v}_1 \cdot \boldsymbol{n}_e.$$

So that, we have

$$b(\boldsymbol{v}_1, \tilde{P}_1) - (1 + \frac{|\Omega_1|}{|\Omega_2|})\bar{P}_1 \sum_{e \in \Gamma_{12}} \int_e \boldsymbol{v}_1 \cdot \boldsymbol{n}_e = 0, \quad \forall \boldsymbol{v}_1 \in \boldsymbol{X}_h. \qquad (42)$$

Since $\tilde{P}_1$ belongs to $L^2_0(\Omega_1)$ and the spaces $H^1_0(\Omega_1)^2, L^2_0(\Omega_1)$ satisfy the exact inf-sup condition (see for example [11]), there exists $\tilde{\boldsymbol{v}} \in H^1_0(\Omega_1)^2$ such that $-\nabla \cdot \tilde{\boldsymbol{v}} = \tilde{P}_1$. Choose $\boldsymbol{v}_1 = R_h(\tilde{\boldsymbol{v}})$ in (42), then we have from properties (14)-(16)

$$\|\tilde{P}_1\|^2_{0,\Omega_1} = 0.$$

This implies that $\tilde{P}_1 = 0$, and from (42), we have that $\bar{P}_1 = 0$, which also means that $P_2 = 0$.

Remark: The approximation $\boldsymbol{U}_2$ of the Darcy velocity $\boldsymbol{u}_2$ is a discontinuous piecewise polynomial vector of degree $k_2 - 1$ and is directly obtained from the discontinuous approximation P_2, by the formula $\boldsymbol{U}_2 = -\boldsymbol{K}\nabla P_2$.

4 Error estimates

In this section, we derive first optimal error estimates in the energy norm for the Stokes velocity and in the L^2 norm for the Darcy pressure. A second estimate gives an optimal convergence rate for the L^2 norm of the pressure in the Stokes region.

Theorem 4.1 Let $(\boldsymbol{u}, p)$ be the solution of the coupled problem (1)-(4), such that $\boldsymbol{u}|_{\Omega_1} \in H^{k_1+1}(\Omega_1)$, $p|_{\Omega_1} \in H^{k_1}(\Omega_1)$ and $p|_{\Omega_2} \in H^{k_2+1}(\Omega_2)$. The discrete solution $(\boldsymbol{U}, P)$ of (36)-(39) satisfies the error estimate:

$$\begin{aligned}\|\boldsymbol{u}_1 - \boldsymbol{U}_1\|_{X^1} + \|p_2 - P_2\|_{M^2} \leq \quad & Ch_1^{k_1}(|\boldsymbol{u}_1|_{k_1+1,\Omega_1} + |p|_{k_1,\Omega_1}) \\ & + Ch_2^{k_2}|p_2|_{k_2+1,\Omega_2},\end{aligned}$$

where C is a constant independent of h_1 and h_2.

Proof
Let $\tilde{\boldsymbol{u}}_1 = R_h(\boldsymbol{u}_1), \tilde{p}_1 = r_h(p_1)$ and $\tilde{p}_2 = r_h(p_2)$ be interpolants of $\boldsymbol{u}_1, p_1$ and p_2 respectively. Define $\boldsymbol{\chi} = \boldsymbol{U}_1 - \tilde{\boldsymbol{u}}_1$ and $\xi = P_2 - \tilde{p}_2$. From (30)-(32) and (36)-(38), the error equations are:

$$\begin{aligned}\mu a_1(\boldsymbol{\chi}, \boldsymbol{v}_1) + b(\boldsymbol{v}_1, P_1 - \tilde{p}_1) + \Lambda(\xi, \boldsymbol{v}_1) &= \mu a_1(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1, \boldsymbol{v}_1) \\ + b(\boldsymbol{v}_1, p_1 - \tilde{p}_1) + \Lambda(p_2 - \tilde{p}_2, \boldsymbol{v}_1), \quad \forall \boldsymbol{v}_1 \in X_h^1, &\qquad (43)\\ b(\boldsymbol{\chi}, q_1) = b(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1, q_1), \quad \forall q_1 \in M_h^1, &\qquad (44)\\ a_2(\xi, q_2) - \Lambda(q_2, \boldsymbol{\chi}) = a_2(p_2 - \tilde{p}_2, q_2) - \Lambda(q_2, \boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1), \quad \forall q_2 \in M_h^2, &\qquad (45)\end{aligned}$$

Choosing $\boldsymbol{v}_1 = \boldsymbol{\chi}$ in (43), $q_1 = P_1 - \tilde{p}_1$ in (44), $q_2 = \xi$ in (45) and summing the resulting equations gives:

$$\begin{aligned}\mu a_1(\boldsymbol{\chi}, \boldsymbol{\chi}) + a_2(\xi, \xi) = \mu a_1(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1, \boldsymbol{\chi}) + a_2(p_2 - \tilde{p}_2, \xi) + b(\boldsymbol{\chi}, p_1 - \tilde{p}_1) \\ - b(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1, P_1 - \tilde{p}_1) + \Lambda(p_2 - \tilde{p}_2, \boldsymbol{\chi}) - \Lambda(\xi, \boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1). \qquad (46)\end{aligned}$$

We now bound each of the terms in the right-hand side of (46). We first rewrite $a_1(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1, \boldsymbol{\chi})$:

$$\mu a_1(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1, \boldsymbol{\chi}) = 2\mu \sum_{E \in \mathcal{E}_h^1} \int_E \boldsymbol{D}(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1) : \boldsymbol{D}(\boldsymbol{\chi})$$

$$-2\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{\chi}] + 2\mu\epsilon_1 \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{\chi})\}\boldsymbol{n}_e \cdot [\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1]$$

$$+ \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \int_e [\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1] \cdot [\boldsymbol{\chi}] + \frac{\mu}{G} \sum_{e \in \Gamma_{12}} \int_e ((\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1) \cdot \boldsymbol{\tau}_{12})(\boldsymbol{\chi} \cdot \boldsymbol{\tau}_{12})$$

$$= T_1 + \cdots + T_5.$$

Using Cauchy-Schwarz inequality, and the approximation result (19), we have

$$\begin{aligned}T_1 \leq C\mu \sum_{E \in \mathcal{E}_h^1} \|\nabla(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1)\|_{0,E} \|\nabla \boldsymbol{\chi}\|_{0,E} &\leq \frac{1}{8}\|\nabla \boldsymbol{\chi}\|_{0,\Omega_1}^2 + C\|\nabla(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1)\|_{0,\Omega_1}^2 \\ &\leq \frac{1}{8}\|\nabla \boldsymbol{\chi}\|_{0,\Omega_1}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2. \qquad (47)\end{aligned}$$

Let $L_h(\boldsymbol{u})$ denote the standard Lagrange interpolant of degree k_1 defined in Ω_1 and let us insert it in the second integral term. For e a segment of $\Gamma_h^1 \cup \Gamma_1$, we have

$$\int_e \{\nabla(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{\chi}] = \int_e \{\nabla(\boldsymbol{u}_1 - L_h(\boldsymbol{u}_1))\}\boldsymbol{n}_e \cdot [\boldsymbol{\chi}] + \int_e \{\nabla(L_h(\boldsymbol{u}_1) - \tilde{\boldsymbol{u}}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{\chi}].$$

Expanding the first integral, we obtain from the trace inequality (24) and from the fact that the Lagrange interpolant satisfies the optimal approximation result (19)

$$\begin{aligned} &\sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\nabla(\boldsymbol{u} - L_h(\boldsymbol{u}))\}\boldsymbol{n}_e \cdot [\boldsymbol{\chi}] \\ &\leq \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_{1,e}^{1/2}}{|e|^{1/2}} \|[\boldsymbol{\chi}]\|_{0,e} \frac{|e|^{1/2}}{\sigma_{1,e}^{1/2}} \|\{\nabla(\boldsymbol{u} - L_h(\boldsymbol{u}))\}\boldsymbol{n}_e\|_{0.e} \\ &\leq \frac{1}{8} \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\boldsymbol{\chi}]\|_{0,e}^2 + C h_1^{2k_1} |\boldsymbol{u}|_{k_1+1,\Omega_1}^2. \end{aligned}$$

Similarly, if we denote by E_e^{12} the elements sharing the edge e, and we use the trace inequality (25), a triangle inequality and the approximation results, then we have

$$\begin{aligned} \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\nabla(L_h(\boldsymbol{u}) - \tilde{\boldsymbol{u}})\}\boldsymbol{n}_e \cdot [\boldsymbol{\chi}] &\leq \frac{1}{8} \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\boldsymbol{\chi}]\|_{0,e}^2 \\ &+ \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{|e|}{\sigma_{1,e}} h_e^{-1} |\tilde{\boldsymbol{u}} - L_h(\boldsymbol{u})|_{1,E_e^{12}}^2 \\ &\leq \frac{1}{8} \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\boldsymbol{\chi}]\|_{0,e}^2 + C h_1^{2k_1} |\boldsymbol{u}|_{k_1+1,\Omega_1}^2. \end{aligned}$$

Therefore,

$$T_2 \leq \frac{1}{4} \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\boldsymbol{\chi}]\|_{0,e}^2 + C h_1^{2k_1} |\boldsymbol{u}|_{k_1+1,\Omega_1}^2. \tag{48}$$

The third term vanishes because of the properties (15), (16) satisfied by $\tilde{\boldsymbol{u}}$.

$$T_3 = 0. \tag{49}$$

Using Cauchy-Schwarz inequality, the jump term T_4 is bounded by virtue of (23) and (19):

$$\begin{aligned} T_4 &\leq \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + C\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{e,1}}{|e|}\|[\boldsymbol{u}-\tilde{\boldsymbol{u}}]\|_{0,e}^2 \\ &\leq \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{e,1}}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k+1,\Omega_1}^2. \end{aligned} \tag{50}$$

Finally, the fifth term is bounded using the trace inequality (23) and the bound (19)

$$\begin{aligned} T_5 &\leq \frac{\mu}{G}\sum_{e\in\Gamma_1}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{0,e}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e} \\ &\leq \frac{\mu}{2G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 + C\sum_{e\in\Gamma_1}(h_e^{-1}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{0,E}^2 + h_e|\boldsymbol{u}-\tilde{\boldsymbol{u}}|_{1,E}^2) \\ &\leq \frac{\mu}{2G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2. \end{aligned} \tag{51}$$

Combining (47), (48), (49), (50) and (51), we have

$$\begin{aligned} a_1(\boldsymbol{u}_1-\tilde{\boldsymbol{u}}_1,\boldsymbol{\chi}) &\leq \frac{1}{8}\|\nabla\boldsymbol{\chi}\|_{0,\Omega_1}^2 + \frac{3}{4}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 \\ &+\frac{\mu}{2G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2. \end{aligned} \tag{52}$$

Let us now expand $a_2(p_2-\tilde{p}_2,\xi)$.

$$\begin{aligned} a_2(p_2-\tilde{p}_2,\xi) &= \sum_{E\in\mathcal{E}_h^2}\int_E \boldsymbol{K}\nabla(p_2-\tilde{p}_2)\cdot\nabla\xi + \sum_{e\in\Gamma_h^2}\frac{\sigma_{2,e}}{|e|}\int_e[p_2-\tilde{p}_2][\xi] \\ &-\sum_{e\in\Gamma_h^2}\int_e\{\boldsymbol{K}\nabla(p_2-\tilde{p}_2)\cdot\boldsymbol{n}_e\}[\xi] + \epsilon_2\sum_{e\in\Gamma_h^2}\int_e\{\boldsymbol{K}\nabla\xi\cdot\boldsymbol{n}_e\}[p_2-\tilde{p}_2]. \end{aligned}$$

Clearly, these terms are bounded in a similar fashion as the terms $T_1,\ldots,T_4$, using in particular the approximation result (21).

$$a_2(p_2-\tilde{p}_2,\xi) \leq \frac{1}{8}\|\nabla\xi\|_{0,\Omega_2}^2 + \frac{3}{4}\sum_{e\in\Gamma_h^2}\frac{\sigma_{2,e}}{|e|}\int_e[\xi]^2 + Ch_2^{2k_2}|p_2|_{k_2+1,\Omega_2}^2. \tag{53}$$

Using property (20) of the operator r_h, the third term in the right-hand side of (46) reduces to

$$b(\boldsymbol{\chi},p_1-\tilde{p}_1) = \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{p_1-\tilde{p}_1\}[\boldsymbol{\chi}]\cdot\boldsymbol{n}_e,$$

which is bounded using Cauchy-Schwarz inequality, trace inequality (23) and the estimate (21) by:

$$b(\boldsymbol{\chi}, p_1 - \tilde{p}_1) \leq \frac{1}{8} \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\boldsymbol{\chi}]\|_{0,e}^2 + C h_1^{2k_1} |p|_{k_1,\Omega_1}^2. \tag{54}$$

The term $b(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1, P_1 - \tilde{p}_1)$ vanishes because of the properties (14), (15) and (16) of $\tilde{\boldsymbol{u}}_1$. It suffices then to bound the interface terms $\Lambda(p_2 - \tilde{p}_2, \boldsymbol{\chi})$ and $\Lambda(\xi, \boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1)$ in (46). Using the trace inequality (23), the approximation result (21) and the bound (22), we obtain

$$\begin{aligned}
\Lambda(p_2 - \tilde{p}_2, \boldsymbol{\chi}) &\leq \sum_{e \in \Gamma_{12}} \|p_2 - \tilde{p}_2\|_{0,e} \|\boldsymbol{\chi}\|_{0,e} \\
&\leq C \|\boldsymbol{\chi}\|_{0,\Omega_1} h^{k_2} |p_2|_{k_2+1,\Omega_2} \\
&\leq \frac{\mu}{8} \|\boldsymbol{\chi}\|_{X^1}^2 + C h^{2k_2} |p_2|_{k_2+1,\Omega_2}^2.
\end{aligned} \tag{55}$$

For each edge e belonging to the interface, we associate the constant $c_e = \frac{1}{|e|} \int_e \xi$. From the property (17) and (19), we can write

$$\begin{aligned}
\Lambda(\xi, \boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1) &= \sum_{e \in \Gamma_{12}} \int_e (\xi - c_e)(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1) \cdot \boldsymbol{n}_{12} \\
&\leq \sum_{e \in \Gamma_{12}} \|\xi - c_e\|_{0,e} \|\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1\|_{0,e} \\
&\leq C h_1^{k_1+1/2} h_2^{1/2} \|\nabla \xi\|_{0,\Omega_2} |\boldsymbol{u}_1|_{k_1+1,\Omega_1} \\
&\leq \frac{1}{8} \|\xi\|_{M^2}^2 + C h_1^{2k_1} |\boldsymbol{u}_1|_{k_1+1,\Omega_1}^2.
\end{aligned} \tag{56}$$

Collecting the bounds (52), (53), (54), (55) and (56) and using Lemma 3.2 yields

$$\|\boldsymbol{\chi}\|_{X^1}^2 + \|\xi\|_{M^2}^2 \leq C h_1^{2k_1} (|\boldsymbol{u}_1|_{k_1+1,\Omega_1}^2 + |p|_{k_1,\Omega_1}^2) + C h_2^{2k_2} |p_2|_{k_2+1,\Omega_2}^2.$$

The final result is obtained with a triangle inequality and approximation results.

Remark: Clearly, Theorem 4.1 gives an estimate of the error in the L^2 norm of the Darcy velocity. The convergence rate is optimal.

Theorem 4.2 Under the assumptions and notation of Theorem 4.1, we have

$$\|p_1 - P_1\|_{0,\Omega_1} \leq C h_1^{k_1} (|\boldsymbol{u}_1|_{k_1+1,\Omega_1} + |p_1|_{k_1,\Omega_1}) + C h_2^{k_2} |p_2|_{k_2+1,\Omega_2}.$$

where C is a constant independent of h_1 and h_2.

Proof
Subtracting (30) from (36) gives

$$\mu a_1(\boldsymbol{U}_1 - \boldsymbol{u}_1, \boldsymbol{v}_1) + b(\boldsymbol{v}_1, P_1 - p_1) + \Lambda(P_2 - p_2, \boldsymbol{v}_1) = 0, \quad \forall \boldsymbol{v}_1 \in \boldsymbol{X}_h^1.$$

As in Theorem 4.1, let $\tilde{p}_1 = r_h(p_1), \tilde{p}_2 = r_h(p_2), \zeta = P_1 - \tilde{p}_1$ and $\xi = P_2 - \tilde{p}_2$. The error equation becomes

$$\begin{aligned} -b(\zeta, \boldsymbol{v}_1) = \mu a_1(\boldsymbol{U}_1 - \boldsymbol{u}_1, \boldsymbol{v}_1) + \Lambda(\xi, \boldsymbol{v}_1) \\ - \Lambda(p_2 - \tilde{p}_2, \boldsymbol{v}_1) - b(\boldsymbol{v}_1, p_1 - \tilde{p}_1), \quad \forall \boldsymbol{v}_1 \in \boldsymbol{X}_h^1. \end{aligned}$$

If we decompose $\zeta = \bar{\zeta} + \tilde{\zeta}$, where $\bar{\zeta} = \frac{1}{|\Omega_1|}\int_{\Omega_1} \zeta$, and $\xi = \bar{\xi} + \tilde{\xi}$, where $\bar{\xi} = \frac{1}{|\Omega_2|}\int_{\Omega_2} \xi$, then (39) and (33) with property (20) yield:

$$|\Omega_1|\bar{\zeta} + |\Omega_2|\bar{\xi} = 0.$$

The error equation is then rewritten

$$\begin{aligned} -b(\tilde{\zeta}, \boldsymbol{v}_1) + (1 + \frac{|\Omega_1|}{|\Omega_2|})\bar{\zeta} \sum_{e \in \Gamma_{12}} \int_e \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12} = \mu a_1(\boldsymbol{U}_1 - \boldsymbol{u}_1, \boldsymbol{v}_1) \\ + \Lambda(\tilde{\xi}, \boldsymbol{v}_1) - \Lambda(p_2 - \tilde{p}_2, \boldsymbol{v}_1) - b(\boldsymbol{v}_1, p_1 - \tilde{p}_1), \quad \forall \boldsymbol{v}_1 \in \boldsymbol{X}_h^1. \end{aligned} \tag{57}$$

Since $\tilde{\zeta}$ belongs to $L_0^2(\Omega_1)$, there exists $\tilde{\boldsymbol{v}} \in H_0^1(\Omega_1)^2$ such that $-\nabla \cdot \tilde{\boldsymbol{v}} = \tilde{\zeta}$ and $|\tilde{\boldsymbol{v}}|_{1,\Omega_1} \leq C\|\tilde{\zeta}\|_{0,\Omega_1}$. Choose $\boldsymbol{v}_1 = R_h(\tilde{\boldsymbol{v}})$ in (57). From properties (14), (15) and (16), (57) is reduced to:

$$\|\tilde{\zeta}\|_{0,\Omega_1}^2 = \mu a_1(\boldsymbol{U}_1 - \boldsymbol{u}_1, \boldsymbol{v}_1) + \Lambda(\tilde{\xi}, \boldsymbol{v}_1) - \Lambda(p_2 - \tilde{p}_2, \boldsymbol{v}_1) - b(\boldsymbol{v}_1, p_1 - \tilde{p}_1). \tag{58}$$

We expand and bound each term in the right-hand side of (58).

$$\begin{aligned} a(\boldsymbol{U}_1 - \boldsymbol{u}_1, \boldsymbol{v}_1) = 2\mu \sum_{E \in \mathcal{E}_h^1} \int_E \boldsymbol{D}(\boldsymbol{U}_1 - \boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \int_e [\boldsymbol{U}_1 - \boldsymbol{u}_1] \cdot [\boldsymbol{v}_1] \\ -2\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{U}_1 - \boldsymbol{u}_1)\boldsymbol{n}_e\} \cdot [\boldsymbol{v}_1] + 2\mu\epsilon \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{v}_1)\boldsymbol{n}_e\} \cdot [\boldsymbol{U}_1 - \boldsymbol{u}_1] \\ + \frac{\mu}{G} \sum_{e \in \Gamma_{12}} \int_e (\boldsymbol{U}_1 - \boldsymbol{u}_1) \cdot \boldsymbol{\tau}_{12} (\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}) \\ = Q_1 + \cdots + Q_5. \end{aligned}$$

The bounds for Q_1, Q_2, Q_4, Q_5 are easily obtained:

$$Q_1 + Q_2 + Q_4 + Q_5 \leq C\|\boldsymbol{U}_1 - \boldsymbol{u}_1\|_{X^1}\|\boldsymbol{v}_1\|_{X^1}.$$

The remaining term is bounded by introducing the interpolant $\tilde{\boldsymbol{u}}_1 = R_h(\boldsymbol{u}_1)$.

$$Q_3 = -2\mu \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{U}_1-\tilde{\boldsymbol{u}}_1)\boldsymbol{n}_e\}\cdot[\boldsymbol{v}_1] - 2\mu \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\boldsymbol{D}(\tilde{\boldsymbol{u}}_1-\boldsymbol{u}_1)\boldsymbol{n}_e\}\cdot[\boldsymbol{v}_1].$$

The first part vanishes because of (15) and (16) and the second part is bounded like T_2 in Theorem 4.1. Therefore, we have

$$a_1(\boldsymbol{U}_1-\boldsymbol{u}_1,\boldsymbol{v}_1) \le \|\boldsymbol{v}_1\|_{X^1}(\|\boldsymbol{U}_1-\boldsymbol{u}_1\|_{X^1} + Ch_1^{k_1}|\boldsymbol{u}_1|_{k_1+1,\Omega_1}).$$

Let us denote by E_e the element in $\mathcal{E}_h^2$ that shares the edge e of the interface Γ_{12}. Define also the constant $c_e = \frac{1}{|e|}\int_e \tilde{\xi}$. From property (16) of the operator R_h, we have

$$\begin{aligned}\sum_{e\in\Gamma_{12}} \int_e \tilde{\xi}\boldsymbol{v}_1\cdot\boldsymbol{n}_{12} &= \sum_{e\in\Gamma_{12}} \int_e (\tilde{\xi}-c_e)\boldsymbol{v}_1\cdot\boldsymbol{n}_{12}\\ &\le C\sum_{e\in\Gamma_{12}} |e|^{1/2}\|\nabla\tilde{\xi}\|_{0,E_e}\|\boldsymbol{v}_1\|_{0,e}\\ &\le Ch_2^{1/2}\|\nabla\xi\|_{0,\Omega_2}(\sum_{e\in\Gamma_{12}}\|\boldsymbol{v}_1\|_{0,e}^2)^{1/2}.\end{aligned}$$

By (21) and (23), we have

$$\begin{aligned}\sum_{e\in\Gamma_{12}} \int_e (p_2-\tilde{p}_2)\boldsymbol{v}_1\cdot\boldsymbol{n}_{12} &\le \sum_{e\in\Gamma_{12}} \|p_2-\tilde{p}_2\|_{0,e}\|\boldsymbol{v}_1\|_{0,e}\\ &\le Ch_2^{k_2+1/2}|p_2|_{k_2+1,\Omega_2}(\sum_{e\in\Gamma_{12}}\|\boldsymbol{v}_1\|_{0,e}^2)^{1/2}.\end{aligned}$$

Now, since $\boldsymbol{v}_1$ satisfies $\int_e \boldsymbol{v}_1 = 0$ for any edge e in Γ_{12}, we have

$$(\sum_{e\in\Gamma_{12}}\|\boldsymbol{v}_1\|_{0,e}^2)^{1/2} \le Ch_1^{1/2}\|\nabla\boldsymbol{v}_1\|_{0,\Omega_1} \le Ch_1^{1/2}\|\boldsymbol{v}_1\|_{X^1}.$$

Therefore, the resulting bound is

$$\Lambda(\tilde{\xi},\boldsymbol{v}_1)-\Lambda(p_2-\tilde{p}_2,\boldsymbol{v}_1) \le Ch_1^{1/2}h_2^{1/2}\|\boldsymbol{v}_1\|_{X^1}(\|P_2-\tilde{p}_2\|_{M^2}+h_2^{k_2}|p_2|_{k_2+1,\Omega_2}).$$

Finally, the bound for $b(\boldsymbol{v}_1, p_1-\tilde{p}_1)$ is similar to the one obtained in (54):

$$b(\boldsymbol{v}_1,p_1-\tilde{p}_1) \le C\|\boldsymbol{v}_1\|_{X^1}h_1^{k_1}|p_1|_{k_1,\Omega_1}.$$

From properties (19) and (18), we have

$$\|\boldsymbol{v}_1\|_{X^1} \le \|R_h(\tilde{\boldsymbol{v}})-\tilde{\boldsymbol{v}}\|_{X^1} + \|\tilde{\boldsymbol{v}}\|_{X^1} \le C|\tilde{\boldsymbol{v}}|_{1,\Omega_1} \le C\|\tilde{\zeta}\|_{0,\Omega_1}.$$

Combining all the bounds above give

$$\begin{aligned}\|\tilde{\zeta}\|_{0,\Omega_1} &\leq C\|\boldsymbol{U}_1-\boldsymbol{u}_1\|_{X^1}+C\|P_2-\tilde{p}_2\|_{M^2}\\ &\quad +Ch_1^{k_1}(|\boldsymbol{u}_1|_{k_1+1,\Omega_1}+|p_1|_{k_1,\Omega_1})+Ch_2^{k_2}|p_2|_{k_2+1,\Omega_2}. \quad (59)\end{aligned}$$

To finish the proof of the theorem, it remains to bound $\|\bar{\zeta}\|_{0,\Omega_1}$. This is accomplished by choosing a particular test function $\boldsymbol{v}_1$ in (57). Let $\boldsymbol{\rho}$ be a function in $\mathcal{C}^2(\bar{\Omega})^2$, with compact support in Ω such that

$$\sum_{e\in\Gamma_{12}}\int_e \boldsymbol{\rho}\cdot\boldsymbol{n}_{12}=1.$$

Denote $\bar{\boldsymbol{v}}=|\Omega_1|\bar{\zeta}\boldsymbol{\rho}$ and choose $\boldsymbol{v}_1=R_h(\bar{\boldsymbol{v}})$. Note that it suffices to take the Crouzeix-Raviart operator [6], so that $\boldsymbol{v}_1$ is piecewise linear. We first show that $\|\boldsymbol{v}_1\|_{X^1}$ is bounded by $\|\bar{\zeta}\|_{0,\Omega_1}$. By property (18) and (19), we have:

$$\|\boldsymbol{v}_1\|_{X^1}\leq\|R_h(\bar{\boldsymbol{v}})-\bar{\boldsymbol{v}}\|_{X^1}+\|\bar{\boldsymbol{v}}\|_{X^1}\leq C|\bar{\boldsymbol{v}}|_{1,\Omega_1}+\|\bar{\boldsymbol{v}}\|_{X^1}.$$

But,

$$\|\bar{\boldsymbol{v}}\|_{X^1}^2=|\bar{\boldsymbol{v}}|_{1,\Omega_1}^2+\frac{|\Omega_1|^2|\bar{\zeta}|^2}{G}\sum_{e\in\Gamma_{12}}\int_e(\boldsymbol{\rho}\cdot\boldsymbol{\tau}_{12})^2\leq C|\Omega_1|^{1/2}\|\bar{\zeta}\|_{0,\Omega_1}^2.$$

Thus,

$$\|\boldsymbol{v}_1\|_{X^1}\leq C\|\bar{\zeta}\|_{0,\Omega_1}. \quad (60)$$

With that test function $\boldsymbol{v}_1$, the error equation (57) becomes

$$\begin{aligned}(1+\frac{|\Omega_1|}{|\Omega_2|})\|\bar{\zeta}\|_{0,\Omega_1}^2&=\mu a_1(\boldsymbol{U}_1-\boldsymbol{u}_1,\boldsymbol{v}_1)\\ +\Lambda(\tilde{\xi},\boldsymbol{v}_1)-\Lambda(p_2-\tilde{p}_2,\boldsymbol{v}_1)-b(\boldsymbol{v}_1,p_1-\tilde{p}_1)-b(\tilde{\zeta},\boldsymbol{v}_1). \quad (61)\end{aligned}$$

Except for the last term, all the terms in the right-hand side of (61) are bounded exactly as in (58). We now rewrite the last term.

$$b(\tilde{\zeta},\boldsymbol{v}_1)=b(\tilde{\zeta},R_h(\bar{\boldsymbol{v}})-\bar{\boldsymbol{v}})+b(\tilde{\zeta},\bar{\boldsymbol{v}}).$$

Clearly,

$$b(\tilde{\zeta},\bar{\boldsymbol{v}})=-|\Omega_1|\bar{\zeta}\sum_{E\in\mathcal{E}_h^1}\int_E\tilde{\zeta}\nabla\cdot\boldsymbol{\rho}\leq C|\Omega_1|^{1/2}\|\bar{\zeta}\|_{0,\Omega_1}\|\tilde{\zeta}\|_{0,\Omega_1}.$$

And it is easy to check that

$$b(\tilde{\zeta},R_h(\bar{\boldsymbol{v}})-\bar{\boldsymbol{v}})\leq C\|\tilde{\zeta}\|_{0,\Omega_1}\|R_h(\bar{v})-\bar{v}\|_{0,\Omega_1}.$$

Finally, we obtain from (61) and the bounds above:

$$\begin{aligned}\|\bar{\zeta}\|_{0,\Omega_1} \quad &\leq \quad C\|\boldsymbol{U}_1 - \boldsymbol{u}_1\|_{X^1} + C\|P_2 - \tilde{p}_2\|_{M^2} + C\|\tilde{\zeta}\|_{0,\Omega_1} \\ &\quad + Ch_1^{k_1}(|\boldsymbol{u}_1|_{k_1+1,\Omega_1} + |p_1|_{k_1,\Omega_1}) + Ch_2^{k_2}|p_2|_{k_2+1,\Omega_2}. \quad (62)\end{aligned}$$

The bounds (59), (62), (21) and Theorem 4.1 give the final result.
Remark: The results stated in Theorems 4.1 and 4.2 hold true in the case where $k_2 \geq 2$ and the parameter $\sigma_{1,e}$ is equal to zero, for all edges e in $\Gamma_h^1 \cup \Gamma^1$. In this case, the proof differs in the choice of the interpolant $\tilde{p}_2$, which now has to satisfy special flux properties (see [17] for further details).

5 Conclusion

In this paper, a discontinuous Galerkin method is formulated for the coupled Stokes and Darcy equations. Both symmetric and non-symmetric cases are considered. Optimal convergence rates are obtained for the fluid velocity and pressure.

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