Existence of Weak Solutions for a Scale-Similarity Model of the Motion of Large Eddies in Turbulent Flow

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Abstract

In turbulent flow the normal procedure has been seek means $\overline{u}$ of the fluid velocity u rather than the velocity itself. In large eddy simulation, we use an averaging operator which allows for the separation of large and small length scales in the flow field. $\overline{u}$ denotes the eddies of size $O(\delta)$ and larger. Applying local spatial averaging operator with averaging radius δ to the Navier-Stokes equations gives a new system of equation governing the large scales. However, it has the well-known problem of closure. One approach to the closure problem which arises from averaging the nonlinear term is use of a scale-similarity hypothesis. We consider one such scale similarity model. We prove existence of weak solutions for the resulting system.

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1 Introduction

The turbulent flow of an incompressible fluid is modelled by solution (u, p) of the incompressible Navier-Stokes equations:

$$\begin{aligned}
&u_t + \nabla \cdot (uu) - Re^{-1}\Delta u + \nabla p = f, \text{ in } \Omega, \text{ for } 0 < t \leq T, \\
&\nabla \cdot u = 0, \text{ in } \Omega, \text{ for } 0 < t \leq T, \\
&u(x, 0) = u_0(x), \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, \text{ for } 0 < t \leq T, \text{ and } \int_\Omega p dx = 0
\end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$), $u : \Omega \times [0, T] \to \mathbb{R}^d$ is the fluid velocity, $p : \Omega \to \mathbb{R}$ is the fluid pressure f(x,t) is the (known) body force, $u_0(x)$ the initial flow field and Re the Reynolds number. There are numerous approaches to the simulation of turbulent flows in practical settings. One of the most promising current approaches is large eddy simulation (LES) in which approximations to local spatial averages of u are calculated. In large eddy simulation (LES), the filtered quantities and fluctuations are defined as

$$\overline{u}(x, t) = g_\delta * u = \int_{\mathbb{R}^3} g_\delta(x - x')u(x', t)dx' \quad u' = u - \overline{u}$$

where

$$g_\delta = \delta^{-3}g(x/\delta).$$

and g is the filter function of characteristic width δ. Applying the filtering operator to the Navier-Stokes equations gives :

$$\overline{u}_t + \nabla \cdot (\overline{uu}) - Re^{-1}\Delta\overline{u} + \nabla\overline{p} = \overline{f}, \quad \nabla \cdot \overline{u} = 0 \text{ in } \Omega \times (0, T]. \tag{1.1}$$

The governing equation (1.1) may be rewritten as

$$\overline{u}_t + \nabla \cdot (\overline{u}\,\overline{u}) - Re^{-1}\Delta\overline{u} + \nabla\overline{p} + \nabla\overline{\mathbb{T}} = \overline{f}, \quad \nabla \cdot \overline{u} = 0 \text{ in } \Omega \times (0, T].$$

where $\mathbb{T}$ denotes the subgrid tensor,

$$\mathbb{T} := \overline{uu} - \overline{u}\,\overline{u} \tag{1.2}$$

which must be modelled. On general approach to closure in LES based on the scale similarity hypothesis, introduced in 1980 by Bardina, Ferziger and

Reynolds [1]. The idea of scale similarity can be thought of as a sort of extrapolation from the resolved scales to the unresolved scales. The original Bardina model is given by

$$\overline{uu} - \overline{u}\,\overline{u} \cong \overline{\overline{u}\,\overline{u}} - \overline{\overline{u}}\,\overline{\overline{u}}.$$

This model has proven to be highly consistent [15], [3], but stability problems have been reported in various test of the Bardina model. These have led to various extensions of Bardina model such as Layton proposed in [12], the Liu, Maneveau, Katz model [16], Horuitu's filtered Bardina model [6] and many 'mixed' models. In this report we consider a model proposed in [12] which is another realization of the idea of scale similarity seeking a clear kinetic energy balance. The model is based on the following three modelling steps. The nonlinear term is written as [14]

$$\overline{uu} = \overline{\overline{u}\,\overline{u}} + \overline{\overline{u}u' + u'\overline{u}} + \overline{u'u'}.$$

Step1 :The cross terms are modelled by scale similarity:

$$\overline{\overline{u}u' + u'\overline{u}} = \overline{\overline{u}(u - \overline{u}) + (u - \overline{u})\overline{u}} \sim \overline{\overline{\overline{u}}(\overline{u} - \overline{\overline{u}}) + (\overline{u} - \overline{\overline{u}})\overline{\overline{u}}}. \tag{1.3}$$

Step2 :The resolved term $\overline{\overline{u}\,\overline{u}}$ is modelled with a Boussinesq type assumption

$$\overline{\overline{u}\,\overline{u}} \sim \overline{\overline{u}}\,\overline{\overline{u}} + \textit{dissipative mechanism on } O(\delta) \textit{ scales},$$

where

$$\nabla \cdot (\overline{\overline{u}\,\overline{u}}) \sim \nabla \cdot (\overline{\overline{u}}\,\overline{\overline{u}}) - A(\delta)\overline{u}. \tag{1.4}$$

The operator $A(\delta)w$ takes the general form $A(\delta)w = R^*\nabla \cdot \mathbb{T}_F(Rw)$, where R is a restriction operator to the finest resolved scales. It is defined by the use of its variational representation

$$-(A(\delta)w, v) = (\nu_F(\delta)\mathbb{D}(w - \overline{w}), \mathbb{D}(v - \overline{v})),$$

where $\nu_F(\delta)$ is the fine scale fluctuation coefficient. This simplifies to $A(\delta)w \sim \nabla \cdot (\nu_F(\delta)\mathbb{D}(w - \overline{w}) - (w - \overline{\overline{w}}))$ where $\mathbb{D}(w) := \frac{1}{2}(\nabla w + \nabla w^t)$.
Step3:The $\overline{u'u'}$ term are modelled by a Boussinesq hypothesis that

$$\overline{u'u'} \sim -\nu_T(\delta, \overline{u})(\nabla\overline{u} + \nabla\overline{u}^t) \tag{1.5}$$

where $\nu_T(\delta, \overline{u})$ is called turbulent viscosity coefficient. Using (1.3), (1.4) and (1.5) in (1.1), the model written below, (w, q) denotes as usual the approximations to $(\overline{u}, \overline{p})$,

$$w_t + \nabla \cdot (\overline{w}\,\overline{w}) + \nabla \cdot (\overline{\overline{w}(w - \overline{w}) + (w - \overline{w})\overline{w}}) - \nabla \cdot (\nu_T(\delta, w)(\nabla w + \nabla w^t))$$
$$- \nabla q - Re^{-1}\Delta w - A(\delta)w = \overline{f}, \quad \nabla \cdot w = 0 \text{ in } \Omega \times (0, T] \tag{1.6}$$

where $w, f : \Omega \times [0, T] \to R^d$, $q : \Omega \to R$. Boundary and zero mean conditions must be imposed on (1.6). There are several possibilities for the *turbulent viscosity* coefficient. The most common ones used in computational practice are a bulk viscosity $\nu_T = \nu_T(\delta)$, the viscosity of [7], $\nu_T = (0.17)\delta\,|w - \overline{w}|$ and the Smagorinsky model, see [17, 9, 2, 11]

$$\nu_T(\delta, w) = (c_s\delta)^2 \left|\nabla w + \nabla w^t\right|. \tag{1.7}$$

We shall assume $\nu_T = 0$ namely, there is no extra viscosity terms. With (1.7) or $\nu_T = \nu_T(\delta)$ our results can be easily extended. Before starting to prove the existence of weak solution for the model, we will give a proof that the model, is given by (1.6), is Galilean invariant. It has been shown that the filtered form of Navier-Stokes equation are Galilean invariant [18]. Thus it is enough to show

$$\nabla \cdot (\widetilde{\mathbb{T}}(w + W)) = \nabla \cdot \widetilde{\mathbb{T}}(w)$$

for any constant vector W. To this end we will give the following Lemma.

Lemma 1.1. *Let consider the model of the subgrid tensor*

$$\mathbb{T} = \overline{uu} - \overline{u}\,\overline{u} \sim \overline{w}\,\overline{w} + \overline{\overline{w}(w - \overline{w}) + (w - \overline{w})\overline{w}} - (c_s\delta)^2 \left|\nabla w + \nabla w^t\right| (\nabla w + \nabla w^t)$$
$$- (\nu_F(\delta)\mathbb{D}(w - \overline{w}) - (w - \overline{\overline{w}}) - ww = \widetilde{\mathbb{T}}(w),$$

$\nabla \cdot \widetilde{\mathbb{T}}(w + W) = \nabla \cdot \widetilde{\mathbb{T}}(w)$ *for any constant vector W.*

Proof.

$$\begin{aligned}\widetilde{\mathbb{T}}(w + W) = {} & (\overline{w + W})(\overline{w + W}) + \overline{(\overline{w + W})(w + W - (\overline{w + W}))} \\ & + \overline{(w + W - (\overline{w + W}))(\overline{w + W})} \\ & - (c_s\delta)^2 \left|\nabla(w + W) + \nabla(w + W)^t\right| (\nabla(w + W) + \nabla(w + W)^t) \\ & - (\nu_F(\delta)\mathbb{D}(w + W - (\overline{w + W})) - (w + W - \overline{\overline{w + W}})) - (w + W)(w + W)\end{aligned}$$

Since W is a constant vector, $\overline{\overline{W}} = W$, $\overline{W} = W$, $\overline{Ww} = W\overline{w}$, $\overline{wW} = \overline{w}W$. Thus,

$$\begin{aligned}\widetilde{\mathbb{T}}(w+W) &= \overline{w}\,\overline{w} + \overline{\overline{w}(w-\overline{w}) + (w-\overline{w})\overline{w}} - (c_s\delta)^2 \left|\nabla w + \nabla w^t\right| (\nabla w + \nabla w^t) \\ &- (\nu_F(\delta)\mathbb{D}(w-\overline{w}) - (w - \overline{\overline{w}}) - ww + (\overline{w} - w)W + W(\overline{w} - w) \\ &+ \overline{W(w-\overline{w})} + \overline{(w-\overline{w})W}.\end{aligned}$$

Hence we have

$$\nabla\cdot\widetilde{\mathbb{T}}(w+W) = \nabla\cdot\widetilde{T}(w) + \nabla\cdot(\overline{w}-w)W + \nabla\cdot(W(\overline{w}-w)) + \nabla\cdot((\overline{w}-\overline{\overline{w}})W) + \nabla\cdot(W(\overline{w}-\overline{\overline{w}})).$$

Since the averaging preserves incompressibility [18], that is $\nabla\cdot w = \nabla\cdot\overline{w} = 0$, so we have

$$\nabla\cdot\widetilde{\mathbb{T}}(w+W) = \nabla\cdot\widetilde{\mathbb{T}}(w).$$

This complete the proof. □

2 Existence of Solutions

In this section we consider the question of existence of weak solutions to the following systems. Thus, we seek (w, q) satisfying

$$\begin{aligned}& w_t + \nabla\cdot(\overline{w}\,\overline{w}) + \nabla\cdot(\overline{\overline{w}(w-\overline{w}) + (w-\overline{w})\overline{w}}) - \nabla q - Re^{-1}\Delta w \\ & - A(\delta)w = \overline{f},\ \ \nabla\cdot w = 0, \text{in } \Omega\times(0,T]\end{aligned} \tag{2.1}$$

$$w(x,0) = g_\delta * u_0(x),\ \text{ in } \Omega, \tag{2.2}$$

$$w(x_j + L, t) = w(x_j, t) \ \text{ and } \int_\Omega \overline{u}_0 dx = 0, \int_\Omega \overline{f} dx = 0, \int_\Omega \overline{q} dx = 0. \tag{2.3}$$

We shall begin by giving the definition of weak solution. Let $D(\Omega) = \{\psi \in C_0^\infty(\Omega) : \nabla.\psi = 0 \text{ in } \Omega\}$, $H(\Omega)$ be the completion of $D(\Omega)$ in $L^2(\Omega)$ $H^1(\Omega)$ be the completion of $D(\Omega)$ in $W^{1,2}(\Omega)$ and $\psi \in D(\Omega)$.

Definition 2.1. *Let $u_0 \in H(\Omega)$, $f \in L^2(\Omega_T)$. A measurable function $w : \Omega_T \longrightarrow \mathbb{R}^n$ is a weak solution of the problem (2.1)- (2.2) in Ω_T if*

a)$w \in V_T = \ L^2(0,T;H^1) \cap \ L^\infty(0,T;H)$ *;*
b)w *verifies*

$$\int_0^t [-Re^{-1}(\nabla w, \nabla\psi) + (\overline{w}\,\overline{w}, \nabla\psi) + (\overline{w}(w-\overline{w}) + (w-\overline{w})\overline{w}, \nabla\overline{\psi})$$
$$- \nu_F(\delta)(\mathbb{D}(w-\overline{w}), \mathbb{D}(\psi-\overline{\psi}))]ds = -\int_0^t (\overline{f},\psi)ds + (w(t),\psi) - (w_0,\psi)$$

where for $T \in (0,\infty)$, $\Omega_T = \Omega \times [0,T]$.

Before we prove of the existence of weak solutions of (2.1)- (2.3) we give the following Lemma. It is proved [12]. Here we shall give this proof briefly. This Lemma gives a useful result about the following (nonstandard) trilinear form.

Lemma 2.1. *Let b(u,v,w) denote the (nonstandard) trilinear form:*

$$b(u,v,w) := \int_\Omega \overline{u}\,\overline{v} : \nabla w + [\overline{u}(v-\overline{v}) + (u-\overline{u})\overline{v}] : \nabla\overline{w}dx.$$

Suppose the averaging used in $L^2(\Omega)$ *self-adjoint and commutes with differentiation,* $w \in L^2(\Omega)$ *and* $\nabla w \in L^2(\Omega)$ *are periodic with zero mean. Then*

$$I = \int_\Omega \nabla.[\overline{w}\,\overline{w} + \overline{\overline{w}(w-\overline{w}) + (w-\overline{w})\overline{w}}].wdx = 0.$$

Proof. Integration by parts and using the properties of the averaging operator yields

$$I = \int_\Omega [\overline{w}\,\overline{w} + \overline{\overline{w}(w-\overline{w}) + (w-\overline{w})\overline{w}}] : \nabla wdx$$

$$= \int_\Omega [\overline{w}\,\overline{w} : \nabla w + \overline{w}w : \nabla\overline{w} - \overline{w}\,\overline{w} : \nabla\overline{w} + w\overline{w} : \nabla\overline{w} - \overline{w}\,\overline{w} : \nabla\overline{w}]dx.$$

An easy index calculation shows that

$$\int_\Omega uv : \nabla wdx = \int_\Omega u.(\nabla w)vdx$$

which is the more familiar trilinear form. Making this change gives

$$I = \int_\Omega [\overline{w}.(\nabla w)\overline{w} + \overline{w}.(\nabla \overline{w})w + w.(\nabla \overline{w})\overline{w} - 2\overline{w}.(\nabla \overline{w})\overline{w}]dx.$$

Since $\nabla \cdot w = 0$, the third term vanishes. By the assumption on the averaging process, $\nabla \cdot \overline{w} = 0$, so the last term vanishes. We use the usual skew symmetry property we obtain

$$\int_\Omega \overline{w}.(\nabla w)\overline{w} + \overline{w}.(\nabla \overline{w})w = 0.$$

Thus I=0. □

Theorem 2.1. *Let $T > 0$, and Ω be any domain in $\mathbb{R}^d$. Then for any given $u_0 \in L^2(\Omega)$ $f \in L^2(\Omega_T)$ there exist at least one weak solution to (2.1)- (2.3) in Ω_T.*

Proof. We shall use the Faedo-Galerkin method following the presentation of Galdi in the Navier-Stokes case [4]. Let $D(\Omega) =: \{\psi \in C_0^\infty : \nabla.\psi = 0 \text{ in}\Omega\}$, $H(\Omega)$ be the completion of $D(\Omega)$ in $L^2(\Omega)$ $H^1(\Omega)$ be the completion of $D(\Omega)$ in $W^{1,2}(\Omega)$ $\{\psi_r\} \subset D(\Omega)$ be the orthonormal basis of $H(\Omega)$. We shall look for approximating solutions v^k of the problem (2.1)- (2.3) which have the form

$$v^k(x,t) = \sum_{r=1}^{k} c_{kr}(t)\psi_r(x), \quad k \in \mathbb{N}. \tag{2.4}$$

In (2.1) we set $w = v^k$, multiply by ψ_r and integrate over Ω we obtain

$$\frac{d}{dt}(v^k, \psi_r) - (\overline{v}^k\overline{v}^k, \nabla\psi_r) + Re^{-1}(\nabla v^k, \nabla\psi_r) + \nu_F(\delta)(\mathbb{D}(v^k - \overline{v}^k), \mathbb{D}(\psi_r - \overline{\psi}_r))$$
$$- ((\overline{v}^k(v^k - \overline{v}^k) + (v^k - \overline{v}^k)\overline{v}^k, \nabla\overline{\psi}_r) = (\overline{f}, \psi_r).$$

Note that since $\nabla \cdot u = 0$, it follows $\Delta u = 2\nabla \cdot \mathbb{D}(u)$. The symmetry of deformation tensor yields

$$\frac{1}{2}(\nabla u, \nabla v) = (\mathbb{D}(u), \mathbb{D}(v)).$$

Thus, we obtain the following equality.

$$\frac{d}{dt}(v^k, \psi_r) - (\overline{v}^k\overline{v}^k, \nabla\psi_r) + Re^{-1}(\nabla v^k, \nabla\psi_r) + \frac{\nu_F(\delta)}{2}(\nabla(v^k - \overline{v}^k), \nabla(\psi_r - \overline{\psi}_r))$$
$$- (\overline{v}^k(v^k - \overline{v}^k) + (v^k - \overline{v}^k)\overline{v}^k, \nabla\overline{\psi}_r) = (\overline{f}, \psi_r). \tag{2.5}$$

If we write (2.4), in (2.5) this represent a system of ordinary differential equations of the form

$$
\begin{aligned}
&\frac{d}{dt}c_{kr}(t) - \sum_{i,j=1}^{k} c_{ki}c_{kj}((g_\delta * \psi_i)(g_\delta * \psi_j), \nabla\psi_r) + Re^{-1}\sum_{i=1}^{k} c_{ki}(\nabla\psi_i, \nabla\psi_r) \\
&+ \frac{\nu_F(\delta)}{2}\sum_{j=1}^{k} c_{kj}(\nabla(\psi_j - g_\delta * \psi_j), \nabla(\psi_r - g_\delta * \psi_r) \\
&- \sum_{i=1}^{k} c_{ki}c_{kj}[(g_\delta * \psi_i)(\psi_j - g_\delta * \psi_j) + (\psi_j - g_\delta * \psi_j)(g_\delta * \psi_i), \nabla(g_\delta * \psi_r)] \\
&= (\overline{f}, \psi_r) = \overline{f}_r \quad r = 1, \cdots, k \qquad (2.6)
\end{aligned}
$$

with the initial condition

$$c_{kr}(0) = c_{0r} = (v_0, \psi_r). \qquad (2.7)$$

Since $\overline{f}_r \in L^2(0,T)$ for all $r = 1, \cdots, k$, from the elementary theory of ordinary differential equations we know the problem admits a unique solution $c_{kr} \in W^{1,2}(0, T_k)$ where $T_k \leq T$.

Multiplying (2.6) by c_{kr} and summing over r from 1 to k we get:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|v_t^k\right\|_2^2 - (\overline{v}^k\overline{v}^k, \nabla v^k) + \frac{\nu_F(\delta)}{2}\left\|\nabla(v^k - \overline{v}^k)\right\|_2^2 - (\overline{v}^k(v^k - \overline{v}^k) + \\
&(v^k - \overline{v}^k)\overline{v}^k, \nabla\overline{v}^k) + Re^{-1}\left\|\nabla v^k\right\|_2^2 = (\overline{f}, v^k).
\end{aligned}
$$

We integrate this equality, we obtain

$$
\begin{aligned}
&\left\|v^k\right\|_2^2 + 2Re^{-1}\int_0^t \left\|\nabla v^k\right\|_2^2 ds - 2\int_0^t (\overline{v}^k\overline{v}^k, \nabla v^k)ds \\
&+ \nu_F(\delta)\int_0^t \left\|\nabla(v^k - \overline{v}^k)\right\|_2^2 ds - 2\int_0^t (\overline{v}^k(v^k - \overline{v}^k) + (v^k - \overline{v}^k)\overline{v}^k, \nabla\overline{v}^k)ds \\
&= 2\int_0^t (\overline{f}, v^k)ds + \|v_{0k}\|_2^2
\end{aligned}
$$

with $v_{0k} = v_k(0)$. We consider third and last term in the left hand side of the above equality. Let us write these two terms nonstandard trilinear form:

$$b(v^k, v^k, v^k) = -2\int_\Omega [\overline{v}^k\nabla v^k\overline{v}^k + \overline{v}^k\nabla\overline{v}^k v^k - \overline{v}^k\nabla\overline{v}^k\overline{v}^k + v^k\nabla\overline{v}^k\overline{v}^k - \overline{v}^k\nabla\overline{v}^k\overline{v}^k]dx. \qquad (2.8)$$

From Lemma 2.1 $I = b(v^k, v^k, v^k) = 0$. In the last equality, we use I=0, Schwarz inequality, Poincaré-Friedrichs inequality, and since $\|v_{0k}\| \leq \|v_0\|$, we obtain,

$$\|v_k\|_2^2 + Re^{-1} \int_0^t \left\|\nabla v^k\right\|_2^2 ds + \nu_F(\delta) \int_0^t \left\|\nabla(v^k - \overline{v}^k)\right\|_2^2 ds$$
$$\leq CRe \int_0^t \left\|\overline{f}\right\|_2^2 ds + \|v_0\|_2^2 .$$

where C is a constant. Then we easily deduce the following bound

$$\|v_k\|_2^2 + Re^{-1} \int_0^t \left\|\nabla {v_k}^2\right\|_2^2 ds \leq M \text{ for all } t \in [0, T] \tag{2.9}$$

with M independent of t and k. We shall now investigate the properties of convergence of the sequence $\{v_k\}$ when $k \to \infty$. To this end we begin to show that, for any fixed $r \in \mathbb{N}$ the sequence of functions

$$G_k^r(t) \equiv (v_k(x,t), \psi_r)$$

is uniformly bounded and uniformly continuous in $t \in [0, T]$. The uniform boundness follows at once from (2.9). To show the uniform continuity, integrating (2.5) with respect to t from s to t and using Schwarz inequality we obtain

$$|G_k^r(t) - G_k^r(s)| = \left|(v^k(x,t) - v^k(x,s), \psi_r)\right| \leq \int_s^t \left|b(v^k, v^k, \psi_r)\right| d\tau$$
$$+ \frac{\nu_F(\delta)}{2} \int_s^t \left\|\nabla(v^k - \overline{v}^k)\right\| \left\|\nabla(\psi_r - \overline{\psi}_r)\right\| d\tau + Re^{-1} \int_s^t \left\|\nabla v^k\right\| \|\nabla \psi_r\| d\tau$$
$$+ \int_s^t \left\|\overline{f}\right\| \|\psi_r\| d\tau. \tag{2.10}$$

On the other hand an easy index calculation shows that

$$\int_\Omega uv : \nabla w dx = \int_\Omega u.(\nabla w) v dx$$

which is more familiar trilinear form. Making this change in the following formula

$$b(v^k, v^k, \psi_r) := \int_\Omega (\overline{v}^k \overline{v}^k : \nabla \psi_r + (\overline{v}^k (v^k - \overline{v}^k) + (v^k - \overline{v}^k) \overline{v}^k) : \nabla \overline{\psi}_r) dx.$$

it gives

$$b(v^k, v^k, \psi_r) = \int\limits_\Omega \overline{v}^k.\nabla\psi_r\overline{v}^k + \overline{v}^k.\nabla\overline{\psi}_r v^k + v^k.\nabla\overline{\psi}_r\overline{v}^k - 2\overline{v}^k.\nabla\overline{\psi}_r\overline{v}^k.$$

By the usual skew symmetry property of this trilinear form, we obtain

$$b(v^k, v^k, \psi_r) = \int\limits_\Omega -\overline{v}^k.\nabla\overline{v}^k\psi_r - \overline{v}^k.\nabla v^k\overline{\psi}_r - v^k.\nabla\overline{v}^k\overline{\psi}_r + 2\overline{v}^k.\nabla\overline{v}^k\overline{\psi}_r$$

Using Cauchy-Schwarz inequality and Young inequality for convolutions we get

$$\int_s^t \left|b(v^k, v^k, \psi_r)\right| \leq s_1 \max_t \left\|v^k(x,t)\right\| \sqrt{t-s}(\int_s^t \left\|\nabla v^k\right\|^2)^{\frac{1}{2}}$$
$$+ s_2 \max_t \left\|v^k(x,t)\right\| \sqrt{t-s}(\int_s^t \left\|\nabla v^k\right\|^2)^{\frac{1}{2}}$$

where $s_1 = max_{x\in\Omega}\left|\psi_r(x)\right|$ and $s_2 = 4max_{x\in\Omega}\left|\overline{\psi}_r(x)\right|$. Now we use this inequality and triangle inequality in (2.10) we obtain

$$|G_k^r(t) - G_k^r(s)| \leq \max_t \left\|v_k(x,t)\right\| \sqrt{t-s}\left\{s_1(\int_s^t \left\|\nabla v_k\right\|^2)^{\frac{1}{2}} + s_2(\int_s^t \left\|\nabla v^k\right\|^2)^{\frac{1}{2}})\right\}$$
$$\frac{\nu_F(\delta)}{2}s_3\sqrt{t-s}(\int_s^t \left\|\nabla v^k\right\|^2)^{\frac{1}{2}} + Re^{-1}\left\|\nabla\psi_r\right\|\sqrt{t-s}(\int_s^t \left\|\nabla v^k\right\|^2)^{\frac{1}{2}}$$
$$+ \max_{x\in\Omega}\left\|\psi_r\right\|\sqrt{t-s}(\int_s^t \left\|f\right\|^2)^{\frac{1}{2}}$$

where $s_3 = 2\left\|\nabla\psi_r\right\|$. Because of (2.9), the right hand side of this inequality converges to zero uniformly as $t \to s$. $G_k^r(t)$ is equicontinuity. By the Ascoli-Arzela theorem, from the sequence $\{G_k^r(t)\}_{k\in\mathbb{N}}$ we may then select a subsequence which we continue to denote by $\{G_k^r(t)\}_{k\in\mathbb{N}}$ uniformly converging to a continuous function $G^r(t)$. The selected sequence $\{G_k^r(t)\}_{k\in\mathbb{N}}$ may depend on r. However using Cantor diagonalization method, we end up by with a sequence again denoted by $\{G_k^r(t)\}_{k\in\mathbb{N}}$ converging to G^r for all $r \in \mathbb{N}$ uniformly in $t \in [0,T]$. This information together with (2.9) and the weak compactness of the space H, allows us to infer the existence of $v(t) \in H(\Omega)$ such that

$$\lim_{k\to\infty}(v_k(t)-v(t), \psi_r) = 0 \text{ uniformly in } t \in [0,T] \; and \; for \; all \; r \in \mathbb{N}. \quad (2.11)$$

$v_k(t)$ converges weakly in L^2 to $v(t)$, uniformly in $t \in [0,T]$ that is

$$\lim_{k\to\infty}(v_k(t)-v(t),u)=0 \text{ uniformly } in\ t\in[0,T] \ \ for\ all\ u\in L^2(\Omega). \quad (2.12)$$

In view of (2.9) $v \in L^\infty(0,T;H(\Omega))$. Again from (2.9) by the weak of compactness of the space $L^2(\Omega_T)$

$$\lim_{k\to\infty}\int_0^t(\partial_m(v_k-v),w)ds=0 \text{ for all } w\in L^2(\Omega_T)\ m=1,\cdots,n \qquad (2.13)$$

(with $\partial_m=\frac{\partial}{\partial x_m}$) $v\in L^2(0,T;H^1(\Omega))$ [4]. It is shown that (2.11) imply the strong convergence of $\{v_k\}$ to v in $L^2(w\times[0,T])$ for all $w\subset\Omega$, that is

$$\lim_{k\to\infty}\int_0^T\|v_k(t)-v(t)\|^2_{2,Q}dt=0 \qquad (2.14)$$

in [4] where Q is a cube in $\mathbb{R}^n$. Now with the help (2.12)- (2.14), we shall now show that v is a weak solution to (2.1)- (2.2). Since we already proved that $v\in V_T$, it remains to show v satisfy (2.3). Integrating (2.5) from 0 to $t<T$ we find

$$\begin{aligned}-Re^{-1}\int_0^t(\nabla v^k,\nabla\psi_r)ds+\int_0^t(\overline{v}^k(v^k-\overline{v}^k)+(v^k-\overline{v}^k)\overline{v}^k,\nabla\overline{\psi}_r)ds\\+\int_0^t(\overline{v}^k\overline{v}^k,\nabla\psi_r)-\frac{\nu_F(\delta)}{2}\int_0^t(\nabla(v^k-\overline{v}^k),\nabla(\psi_r-\overline{\psi}_r))ds\\=-\int_0^t(\overline{f},\psi_r)ds+(v^k(t),\psi_r)-(v_o,\psi_r)\end{aligned} \qquad (2.15)$$

Now we consider second and third terms of the left hand side of the equation (2.15) by the usual skew symmetry property we write

$$b(v^k,v^k,\psi_r)=\int_0^t\int_\Omega[-\overline{v}^k\nabla\overline{v}^k\psi_r-\overline{v}^k\nabla v^k\overline{\psi}_r\ -v^k\nabla\overline{v}^k\overline{\psi}_r+2\overline{v}^k\nabla\overline{v}^k\overline{\psi}_r]dx.$$

From (2.12) and (2.13) we get

$$\lim_{k\to\infty}(v^k(t)-v(t),\psi_r)=0,\lim_{k\to\infty}\int_0^t(\nabla v^k(s)-\nabla v(s),\nabla\psi_r)ds=0. \qquad (2.16)$$

Furthermore let Q be a cube containing the support of ψ_r, then we have

$$\left|\int_0^t [(\overline{v}^k\nabla\overline{v}^k,\psi_r)-(\overline{v}\nabla\overline{v},\psi_r)]ds\right| \leq \left|\int_0^t ((\overline{v}^k-\overline{v})\nabla\overline{v}^k,\psi_r)_Q ds\right| + \left|\int_0^t (\overline{v}\nabla(\overline{v}^k-\overline{v}),\psi_r)_Q ds\right| \tag{2.17}$$

We consider first term of the right hand side and using Cauchy-Schwarz inequality we obtain

$$\left|\int_0^t ((\overline{v}^k-\overline{v})\nabla\overline{v}^k,\psi_r)_Q\right| \leq \int_0^t \left\|\overline{v}^k-\overline{v}\right\| \left\|\nabla\overline{v}^k\right\| \max_{x\in Q}|\psi_r(x)|\,.$$

Setting $s_1 : \max_{x\in Q}|\psi_r(x)|$ and using (2.9) and Young inequality for convolution, we have

$$\left|\int_o^t ((\overline{v}^k-\overline{v})\nabla\overline{v}^k,\psi_r)_Q\right| \leq Cs_1M^{\frac{1}{2}}(\int_0^t \left\|v^k-v\right\|_{2,Q}^2)^{\frac{1}{2}}$$

where C is a constant. Thus using (2.14) we get:

$$\lim_{k\to\infty}\left|\int_0^t (\overline{v}^k-\overline{v})\nabla\overline{v}^k,\psi_r)_Q ds\right| = 0 \tag{2.18}$$

We also have:

$$\begin{aligned}\left|\int_0^t (\overline{v}\nabla(\overline{v}^k-\overline{v}),\psi_r)_Q ds\right| &\leq \sum_{m=1}^n \left|\int_0^t (\partial_m(\overline{v}^k-\overline{v}),\overline{v}_i m\psi_r)_Q ds\right| \\ &\leq \sum_{m=1}^n \left|\int_0^t (\partial_m(v^k-v),g_\delta*((g_\delta*v_i)\psi_r))_Q ds\right|\end{aligned}$$

and since $g_\delta*((g_\delta*v_i)\psi_r)\in L^2(\Omega_T)$, (2.13) implies

$$\lim_{k\to\infty}\left|\int_0^t (\overline{v}\nabla(\overline{v}^k-\overline{v}),\psi_r)_Q ds\right| = 0. \tag{2.19}$$

Relation (2.18)- (2.19) yield:

$$\lim_{k\to\infty}\left|\int_0^t (\overline{v}^k\nabla\overline{v}^k,\psi_r)-(\overline{v}\nabla\overline{v},\psi_r))ds\right| = 0. \tag{2.20}$$

Now we consider the second term of $b(v^k, v^k, \psi_r)$. Again let Q be a cube containing the support of ψ_r,then we have

$$\left|\int_0^t ((\overline{v}^k\nabla v^k, \overline{\psi}_r) - (\overline{v}\nabla v, \overline{\psi}_r))ds\right| \leq \left|\int_0^t ((\overline{v}^k - \overline{v})\nabla v^k, \overline{\psi}_r)_Q ds\right| + \left|\int_0^t (\overline{v}\nabla(v^k - v), \overline{\psi}_r)_Q ds\right|. \qquad (2.21)$$

We use Cauchy-Schwarz, the first term of the right hand side of (2.21) we obtain

$$\left|\int_0^t (\overline{v}^k - \overline{v})\nabla v^k, \overline{\psi}_r)_Q ds\right| \leq s_2(\int_0^t \left\|\overline{v}^k - \overline{v}\right\|_{2,Q}^2 ds)^{\frac{1}{2}}(\int_0^t \left\|\nabla v^k\right\|_{2,Q}^2 ds)^{\frac{1}{2}}$$

Using (2.9) and Young inequality, we get

$$\left|\int_0^t (\overline{v}^k - \overline{v})\nabla v^k, \overline{\psi}_r)_Q ds\right| \leq C s_2 M^{\frac{1}{2}}(\int_0^t \left\|v^k - v\right\|_{2,Q}^2 ds)^{\frac{1}{2}}$$

Thus using (2.14), we obtain:

$$\lim_{k\to\infty}\left|\int_0^t ((\overline{v}^k - \overline{v})\nabla v^k, \overline{\psi}_r)_Q ds\right| = 0 \qquad (2.22)$$

Now we consider the second term of the right hand side of the equation (2.21) we write

$$\left|\int_0^t (\overline{v}^k\nabla(v^k - v), \overline{\psi}_r)_Q\right| \leq \sum_{m=1}^n \left|\int_0^t (\partial_m(v^k - v), \overline{v}_m\overline{\psi}_r)_Q ds\right|$$

and since $\overline{v}_m\overline{\psi}_r \in L^2(\Omega_T)$ (2.13) implies

$$\lim_{k\to\infty}\left|\int_0^t (\overline{v}^k\nabla(v^k - v), \overline{\psi}_r)_Q ds\right| = 0 \qquad (2.23)$$

Relation (2.22)- (2.23) yield:

$$\lim_{k\to\infty}\left|\int_0^t ((\overline{v}^k\nabla v^k, \overline{\psi}_r) - (\overline{v}\nabla v, \overline{\psi}_r))ds\right| = 0 \qquad (2.24)$$

Similarly we consider third term of $b(v^k, v^k, \psi_r)$ we write

$$\left|\int_0^t [(v_k \nabla \overline{v}^k, \overline{\psi}_r) - (v\nabla\overline{v}, \overline{\psi}_r)]ds\right| \leq \left|\int_0^t ((v^k - v)\nabla\overline{v}^k, \overline{\psi}_r)_Q ds\right| + \left|\int_0^t (v\nabla(\overline{v}^k - \overline{v}), \overline{\psi}_r)_Q\right|. \tag{2.25}$$

Again using Cauchy-Schwarz, Young inequality, and (2.9) in the first term of the right hand side of the equation (2.25) we get:

$$\left|\int_0^t ((v^k - v)\nabla\overline{v}^k, \overline{\psi}_r)_Q ds\right| \leq Cs_2 M^{\frac{1}{2}}(\int_0^t \left\|v^k - v\right\|_{2,Q}^2 ds)^{\frac{1}{2}}.$$

Using (2.14) we get:

$$\lim_{k\to\infty}\left|\int_0^t ((v^k - v)\nabla\overline{v}^k, \overline{\psi}_r)_Q ds\right| = 0. \tag{2.26}$$

Now we consider the second term of the right hand side of (2.25)

$$\left|\int_0^t (v\nabla(\overline{v}^k - \overline{v}, \overline{\psi}_r)_Q ds\right| \leq \sum_{m=1}^n \left|\int_0^t (\partial_m(\overline{v}^k - \overline{v}), v_m\overline{\psi}_r)_Q ds\right|$$

We use the properties of convolutions, we obtain

$$\left|\int_0^t (v\nabla(\overline{v}^k - \overline{v}, \overline{\psi}_r)_Q ds\right| \leq \sum_{m=1}^n \left|\int_0^t \partial_m(v^k - v), g_\delta * (v_m\overline{\psi}_r)_Q ds\right|$$

since $g_\delta * (v_m\overline{\psi}_r) \in L^2(\Omega_T)$, (2.13) implies

$$\lim_{k\to\infty}\left|\int_0^t (v\nabla(\overline{v}^k - \overline{v}, \overline{\psi}_r)_Q ds\right| = 0. \tag{2.27}$$

Relation (2.26)- (2.27) yield:

$$\lim_{k\to\infty}\left|\int_0^t [(v_k\nabla\overline{v}^k, \overline{\psi}_r) - (v\nabla\overline{v}, \overline{\psi}_r)]ds\right| = 0 \tag{2.28}$$

Now we consider last term of $b(v^k, v^k, \psi_r)$ again we can write

$$\left|\int_0^t [(\overline{v}_k\nabla\overline{v}^k, \overline{\psi}_r) - (\overline{v}\nabla\overline{v}, \overline{\psi}_r)]ds\right| \leq \left|\int_0^t ((\overline{v}^k - \overline{v})\nabla\overline{v}^k, \overline{\psi}_r)_Q ds\right| + \left|\int_0^t (\overline{v}\nabla(\overline{v}^k - \overline{v}), \overline{\psi}_r)_Q ds\right|. \tag{2.29}$$

Similarly, using Cauchy-Schwarz, Young inequality, (2.9) and (2.14) in the first term of the right hand side of (2.29) we get:

$$\lim_{k\to\infty}\left|\int_0^t ((\overline{v}^k-\overline{v})\nabla\overline{v}^k,\overline{\psi}_r)_Q ds\right| = 0, \tag{2.30}$$

Besides this, we get the following inequality for the second term of the equation (2.29)

$$\left|\int_0^t (\overline{v}\nabla(\overline{v}^k-\overline{v}),\overline{\psi}_r)_Q ds\right| \leq \sum_{m=1}^n \left|\int_0^t (\partial_m(\overline{v}^k-\overline{v}),\overline{v}_m\overline{\psi}_r)_Q\right|.$$

From the properties of convolution we write

$$\left|\int_0^t (\overline{v}\nabla(\overline{v}^k-\overline{v}),\overline{\psi}_r)_Q ds\right| \leq \sum_{m=1}^n \left|\int_0^t (\partial_m(v^k-v),g_\delta*(\overline{v}_m\overline{\psi}_r))_Q\right|$$

and since $g_\delta*(\overline{v}_m\overline{\psi}_r)\in L^2(\Omega_T)$. From (2.13) we obtain

$$\lim_{k\to\infty}\left|\int_0^t (\overline{v}\nabla(\overline{v}^k-\overline{v}),\overline{\psi}_r)_Q ds\right| = 0. \tag{2.31}$$

Thus relation (2.30)- (2.31) yield:

$$\lim_{k\to\infty}\left|\int_0^t [(\overline{v}_k\nabla\overline{v}^k,\overline{\psi}_r)-(\overline{v}\nabla\overline{v},\overline{\psi}_r)]ds\right| = 0 \tag{2.32}$$

Finally, we consider fourth term the left hand side of (2.15). Again let Q be a cube containing the support of ψ_r, then we have

$$\left|\int_0^t [(\nabla(v^k-v),\nabla(\psi_r-\overline{\psi}_r))-(\nabla(\overline{v}^k-\overline{v}),\nabla(\psi_r-\overline{\psi}_r))]ds\right|$$
$$\leq \left|\int_0^t (\nabla(v^k-v),\nabla(\psi_r-\overline{\psi}_r))_Q ds\right| + \left|\int_0^t (\nabla(\overline{v}^k-\overline{v}),\nabla(\psi_r-\overline{\psi}_r))_Q ds\right|.$$

Since $\nabla(\psi_r-\overline{\psi}_r)\in L^2(\Omega_T)$ and using (2.13) we get

$$\lim_{k\to\infty}\left|\int_0^t (\nabla(v^k-v),\nabla(\psi_r-\overline{\psi}_r))_Q\right| = 0. \tag{2.33}$$

Similarly since $g_\delta * \nabla(\psi_r - \overline{\psi}_r) \in L^2(\Omega_T)$ and using (2.13) it gives

$$\lim_{k\to\infty}\left|\int_0^t (\nabla(\overline{v}^k - \overline{v}), \nabla(\psi_r - \overline{\psi}_r))_Q ds\right| = 0. \tag{2.34}$$

Using (2.33) and (2.34) we get

$$\lim_{k\to\infty}\left|\int_0^t (\nabla(v^k - \overline{v}^k) - \nabla(v - \overline{v}), \nabla(\psi_r - \overline{\psi}_r)) ds\right| = 0 \tag{2.35}$$

Therefore taking the limit over $k \to \infty$ in (2.15) and using (2.16), (2.20), (2.24), (2.28), (2.32), (2.35), we get

$$\begin{aligned}
&-Re^{-1}\int_0^t (\nabla v, \nabla\psi_r) + \int_0^t (\overline{v}(v - \overline{v}) + (v - \overline{v})\overline{v}, \nabla\overline{\psi}_r) ds + \int_0^t (\overline{v}\,\overline{v}, \nabla\psi_r) ds\\
&-\frac{\nu_F(\delta)}{2}\int_0^t (\nabla(v - \overline{v}), \nabla(\psi_r - \overline{\psi}_r)) = -\int_0^t (\overline{f}, \psi_r) ds + (v(t), \psi_r) - (v_0, \psi_r)
\end{aligned} \tag{2.36}$$

However, from Lemma 2.3 in [4] we know that every function $\psi \in D(\Omega)$ can be uniformly approximated in $C^2(\overline{\Omega})$ by functions of the form

$$\psi_N(x) = \sum_{r=1}^{N} \gamma_r \psi_r(x) \;\; N \in \mathbb{N}, \gamma_r \in \mathbb{R}$$

so writing (2.36) with ψ_N in place of ψ_r and we may pass to the limit $N \to \infty$ in this new relation and use the fact that $v \in L^2(0, T; H^1) \cap L^\infty(0, T; H)$ to show v is a weak solution of (2.1)- (2.2). $\square$

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