Weak Imposition Of Boundary Conditions For The Navier-Stokes Equations *

Atife Çağlar †

Ph.D. Student

Department of Mathematics

University of Pittsburgh

Pittsburgh, PA 15260, USA

November 23, 2002

Abstract

We prove the convergence of a finite element method for the Navier-Stokes equations in which the no-slip condition, $\mathbf{u}\cdot\boldsymbol{\tau}_i = 0$ on Γ for $i = 1, 2$ is imposed by a penalty method and the no-penetration condition, $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ, is imposed by Lagrange multipliers. This approach has been studied for the Stokes problem in [2]. In most flows the Reynolds number is not

*This work partially supported by NSF grants DMS9972622, INT9814115, and INT9805563.

†atcst11@pitt.edu, http://www.math.pitt.edu/∼atife

negligable so the $\mathbf{u}\cdot\nabla\mathbf{u}$ inertial effects are important. Thus the extension beyond the Stokes problem to the Navier-Stokes equations is critical. We show existence and uniqueness of the approximate solution and optimal order of convergence can be achieved if the computational mesh follows the real boundary. Our results for the (nonlinear) Navier-Stokes equations improve known results for this approach for the Stokes problem.

Keywords. Navier-Stokes equations, Lagrange multiplier, slip with friction.

AMS Subject classifications. 65N30.

1 Introduction

The problem of predicting the equilibrium flow of a viscous incompressible fluid is one of solving approximately the stationary, incompressible Navier-Stokes equations:

$$\begin{aligned} -2Re^{-1}\nabla\cdot(\mathcal{D}(\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \quad \text{on} \quad \Gamma = \partial\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \end{aligned} \tag{1.1}$$

in a bounded polyhedral domain $\Omega \subset \mathbb{R}^d$ d=2,3, where $\mathcal{D}$ is the **deformation tensor** given by

$$\mathcal{D}_{i\,j}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for} \quad 1 \le i, j \le d.$$

The boundary Γ is assumed to be the union of k flat parts Γ_j:

$$\Gamma = \cup_{j=1}^{k}\Gamma_j.$$

The boundary condition $\mathbf{u} = 0$ is decomposed of two separate conditions:

$$\text{“no} - \text{penetration}'' : \qquad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma,$$

$$\text{“no} - \text{slip}'' : \qquad \mathbf{u} \cdot \boldsymbol{\tau}_i = 0 \quad \text{on} \quad \Gamma,$$

where $\mathbf{n}$ be the unit normal vector on Γ and $\boldsymbol{\tau}_i$, $i = 1, \ldots, d-1$, a system of orthonormal tangential vectors.

There are physical and computational reasons why these two boundary conditions should sometimes be separated and imposed using different techniques see, e.g. the discussions in [2]. For example, the Lagrange multiplier implementation of the no-penetration condition allows slight, but locally balancing in and out flow capturing some aspects of surface roughness. For the no-slip condition, the penalty formulation (as we shall see) is equivalent to the Navier slip law. Thus, it is natural to use it for problems in which large tangential stress might occur. The numerical analysis of techniques for imposing essential boundary conditions weakly was begun in [3], see also [4], [5, 6], [7], [2] and the references therein. In many flows inertial effects are important and thus the extension beyond the Stokes problem, of [7,10] to the Navier-Stokes equations is important. The purpose of this report is to begin this extension. Our analysis both extends the work of [2] and [7] from the (linear) Stokes problem to nonlinear Navier-Stokes problem and improves the basic results of [2] even in the linear case.

2 The Continuous Problem

For our mathematical formulation we introduce the following spaces:

$$X = [H^1(\Omega)]^d, \qquad X_0 = [H_0^1(\Omega)]^d,$$

$$V = \{\mathbf{v} \in X \mid \quad \langle \nabla \cdot \mathbf{v}, q \rangle_\Omega = 0, \quad \forall\, q \in L_0^2(\Omega)\},$$

$$V_0 = \left\{\mathbf{v} \in X_0 \mid \quad \langle \nabla \cdot \mathbf{v}, q \rangle_\Omega = 0, \quad \forall\, q \in L_0^2(\Omega)\right\},$$

$$Y = L_0^2(\Omega) = \left\{q \in L^2(\Omega) \mid \quad \langle q, 1 \rangle_\Omega = 0\right\},$$

$$Z = \prod_{j=1}^{k} H^{-1/2}(\Gamma_j),$$

where k is the number of edges in 2d or faces in 3d of the boundary Γ. $\langle ., . \rangle_\Omega$ is the usual L^2 inner product, $H^k(\Omega)$ the usual $W^{k,2}(\Omega)$ Sobolev space with norm $\|.\|_{k,\Omega}$, and the space $H^{-k}(\Omega)$ is the dual of $H_0^k(\Omega)$, the space of functions in $H^k(\Omega)$ that vanish on Γ. The spaces $H^{k-1/2}(\Gamma)$ consist of the traces of all functions in $H^k(\Omega)$. Analogously, we denote by $H^{-(k-1/2)}(\Gamma)$ the dual space of $H^{k-1/2}(\Gamma)$ with $\langle ., . \rangle_\Gamma$ being the duality pairing.

A norm $\|.\|_\Gamma$ of a function $\varphi \in \prod_{j=1}^{k} H^{1/2}(\Gamma_j)$ is defined by

$$\|\varphi\|_\Gamma = \left(\sum_{j=1}^{k} \|\varphi\|^2_{1/2,\Gamma_j} \right)^{1/2}$$

with the dual norm $\|.\|_\Gamma^* =: \|.\|_Z$. In addition, we set $\|.\|_{\Gamma_j} = \|.\|_{H^{1/2}(\Gamma_j)}$ and $\|.\|^*_{\Gamma_j} = \|.\|_{H^{-1/2}(\Gamma_j)}$.

The most common formulation of the Navier-Stokes equations in (X_0, Y) is given by (see e.g. [8], [9]) : *Find* $(\mathbf{u}, p) \in (X_0, Y)$ *such that*:

$$a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) + a_2(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega, \ \forall \mathbf{v} \in X_0,$$

$$a_2(\mathbf{u}, q) = 0, \forall q \in Y,$$

where

$$\begin{aligned} a_0(\mathbf{u}, \mathbf{v}) &= 2Re^{-1} \int_\Omega \mathcal{D}(\mathbf{u}) : \mathcal{D}(\mathbf{v}) \, dx, \\ a_1(\mathbf{u}; \mathbf{v}, \mathbf{w}) &= \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{v}) \cdot \mathbf{w} \, dx, \\ b(\mathbf{u}; \mathbf{v}, \mathbf{w}) &= \frac{1}{2}[a_1(\mathbf{u}; \mathbf{v}, \mathbf{w}) - a_1(\mathbf{u}; \mathbf{w}, \mathbf{v})], \\ a_2(\mathbf{u}, p) &= -\int_\Omega p(\nabla \cdot \mathbf{u}) \, dx. \end{aligned}$$

Lemma 2.1 For $\mathbf{u} \in V_0$ and $\mathbf{v}, \mathbf{w} \in X_0$, $b(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \langle \mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w} \rangle_\Omega$. Since $a_1(\mathbf{u}; \mathbf{v}, \mathbf{w}) = -a_1(\mathbf{u}; \mathbf{w}, \mathbf{v})$ then $b(\mathbf{u}; \mathbf{v}, \mathbf{v}) = 0$.

Proof It follows from integral by parts.

If we impose the boundary conditions weakly we must seek a formulation with velocities in X rather than X_0. Using Green's formula, the definition of deformation and stress tensor we arrive at the following weak formulation of (1.1) in (X, Y):

Find $(\mathbf{u}, p) \in (X, Y)$ *such that*:

$$a_0(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}; \mathbf{u}, \mathbf{v}) + a_2(\mathbf{v}, p) - \sum_{j=1}^{k} \langle \mathbf{n}_j \cdot \Im(\mathbf{u}, p), \mathbf{v} \rangle_{\Gamma_j} = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega, \ \forall \mathbf{v} \in X,$$

$$a_2(\mathbf{u}, q) = 0, \ \forall q \in Y. \tag{2.1}$$

Here, the tensor $\Im(.,.)$ is defined by

$$\Im_{ik}(\mathbf{u}, p) = -p\delta_{ik} + 2Re^{-1}\mathcal{D}_{ik}(\mathbf{u}) + \frac{1}{2}\mathbf{u}_i\mathbf{u}_k, \text{for} \quad 1 \leq i, k \leq d,$$

and for all $\mathbf{v} \in X$, $q \in Y$.

Remark: $\Im(.,.)$ is a modification of the stress tensor where the pressure p is replaced by the Bernoulli pressure $p\delta_{ik} + \frac{1}{2}\mathbf{u}_i\mathbf{u}_k$

Split the fourth term of the left hand side of equation (2.1) to its normal and tangential parts to obtain Lagrange multipliers:

$$\begin{aligned} \rho|_{\Gamma_j} &:= -\mathbf{n}_j \cdot \Im(\mathbf{u}, p) \cdot \mathbf{n}_j, \\ \lambda_1|_{\Gamma_j} &:= -\mathbf{n}_j \cdot \Im(\mathbf{u}, p) \cdot \boldsymbol{\tau}_1^{(j)}, \\ \lambda_2|_{\Gamma_j} &:= -\mathbf{n}_j \cdot \Im(\mathbf{u}, p) \cdot \boldsymbol{\tau}_2^{(j)}, \end{aligned}$$

Thus, $\rho, \lambda_1, \lambda_2$ are the individual components of the normal stress on Γ_j and $1 \le j \le k$, where k is the number of boundary pieces of Γ. Then we get

$$\begin{aligned} 2Re^{-1}\int_\Omega \mathcal{D}(\mathbf{u}) : \mathcal{D}(\mathbf{v})\,dx + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\mathbf{v}\,dx - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{v})\mathbf{u}\,dx \\ - \int_\Omega p(\nabla\cdot\mathbf{v})\,dx + \sum_{j=1}^{k}\int_{\Gamma_j} \rho\,(\mathbf{v}\cdot\mathbf{n}_j)\,ds + \sum_{j=1}^{k}\int_{\Gamma_j} \lambda_1\,(\mathbf{v}\cdot\boldsymbol{\tau}_1^{(j)})\,ds \\ + \sum_{j=1}^{k}\int_{\Gamma_j} \lambda_2\,(\mathbf{v}\cdot\boldsymbol{\tau}_2^{(j)})\,ds = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx, \\ - \int_\Omega q(\nabla\cdot\mathbf{u})\,dx = 0, \end{aligned} \tag{2.2}$$

Define $c(.;.,.,.,.)$ on $X \times Y \times Z \times Z \times Z$ as:

$$c(\mathbf{u}; p, \rho, \lambda_1, \lambda_2) := \langle p, \nabla\cdot\mathbf{u}\rangle_\Omega - \sum_{j=1}^{k}\langle \rho, \mathbf{u}\cdot\mathbf{n}_j\rangle_{\Gamma_j} - \sum_{i=1}^{2}\sum_{j=1}^{k}\langle \lambda_i, \mathbf{u}\cdot\boldsymbol{\tau}_i^{(j)}\rangle_{\Gamma_j}.$$

The corresponding weak formulation of (2.1) is :

$$\begin{aligned} a(\mathbf{u};\mathbf{u},\mathbf{v}) - c(\mathbf{v}; p, \rho, \lambda_1, \lambda_2) &= \langle \mathbf{f}, \mathbf{v}\rangle_\Omega, \\ c(\mathbf{u}; q, \sigma, \chi_1, \chi_2) &= 0, \end{aligned} \tag{2.3}$$

for all $\mathbf{v} \in X, q \in Y, \sigma, \chi_1, \chi_2 \in Z$, where $a(\mathbf{u};\mathbf{u},\mathbf{v}) = a_0(\mathbf{u};\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v})$.

Define the space $\mathcal{K}$ by

$$\mathcal{K} = \{\mathbf{u} \in X \mid \quad c(\mathbf{u}; p, \rho, \lambda_1, \lambda_2) = 0 \quad \forall\, p \in Y, \quad \forall\, \rho, \lambda_1, \lambda_2 \in Z\}.$$

Lemma 2.2 The multi-linear form $c(\cdot;\cdot,\cdot,\cdot,\cdot)$ is continuous on $X \times Y \times Z \times Z$.

Proof Define $|||.|||$ as:

$$|||\mathbf{u}||| := \left[\|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\cdot\mathbf{n}_j\|_\Gamma^2 + \|\mathbf{u}\cdot\boldsymbol{\tau}_1\|_\Gamma^2 + \|\mathbf{u}\cdot\boldsymbol{\tau}_2\|_\Gamma^2\right]^{1/2}$$

which is equal to $\|.\|_1$ on X, and apply Cauchy-Schwartz inequality

to $c(. : ., ., ., .)$ we obtain:

$$\begin{aligned}
&c(\mathbf{u}; p, \rho, \lambda_1, \lambda_2) \\
&\quad \le \|p\|\|\nabla\cdot\mathbf{u}\| + \sum_{j=1}^{k} \|\rho\|_Z \|\mathbf{u}\cdot\mathbf{n}_j\|_{\Gamma_j} + \sum_{i=1}^{2}\sum_{j=1}^{k} \|\lambda_i\|_Z \|\mathbf{u}\cdot\boldsymbol{\tau}_1^j\|_{\Gamma_j} \\
&\quad \le \|p\|\|\nabla\mathbf{u}\| + \left(\sum_{j=1}^{k}\|\rho\|_Z^2\right)^{1/2} \left(\sum_{j=1}^{k}\|\mathbf{u}\cdot\mathbf{n}_j\|_{\Gamma_j}^2\right)^{1/2} \\
&\quad + \sum_{i=1}^{2}\left(\sum_{j=1}^{k}\|\lambda_i\|_Z^2\right)^{1/2} \left(\sum_{j=1}^{k}\|\mathbf{u}\cdot\boldsymbol{\tau}_i^j\|_{\Gamma_j}^2\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&= \|p\|\|\nabla\mathbf{u}\| + \|\rho\|_Z\|\mathbf{u}\cdot\mathbf{n}_j\|_\Gamma + \sum_{i=1}^{2}\|\lambda_i\|_Z\|\mathbf{u}\cdot\boldsymbol{\tau}_i\|_\Gamma \\
&= \langle(\|p\|, \|\rho\|_Z, \|\lambda_1\|_Z, \|\lambda_2\|_Z), (\|\nabla\mathbf{u}\|, \|\mathbf{u}\cdot\mathbf{n}_j\|_\Gamma, \|\mathbf{u}\cdot\boldsymbol{\tau}_1\|_\Gamma, \|\mathbf{u}\cdot\boldsymbol{\tau}_2\|_\Gamma)\rangle \\
&\le \left[\|p\|^2 + \|\rho\|_Z^2 + \sum_{i=1}^{2}\|\lambda_i\|_Z^2\right]^{1/2} \left[\|\nabla\mathbf{u}\|^2 + \|\mathbf{u}\cdot\mathbf{n}_j\|_\Gamma^2 + \sum_{i=1}^{2}\|\mathbf{u}\cdot\boldsymbol{\tau}_i\|_\Gamma^2\right]^{1/2} \\
&= \left[\|p\|^2 + \|\rho\|_Z^2 + \sum_{i=1}^{2}\|\lambda_i\|_Z^2\right]^{1/2} |||\mathbf{u}|||.
\end{aligned}$$

Lemma 2.3 The space $\mathcal{K}$ is a closed subspace of the Hilbert space X.

Proof This follows provided $c(.;.,.,.)$ is continuous on $X \times Y \times Z \times Z \times Z$ which was proven in Lemma 2.2

Lemma 2.4 For $\mathbf{u} \in \mathcal{K}$, Korn's inequality

$$\|\nabla \mathbf{u}\| \leq C_K(\Omega)\|\mathcal{D}(\mathbf{u})\|$$

and the Poincaré inequality

$$\|\mathbf{u}\| \leq C_P(\Omega)\|\nabla \mathbf{u}\|$$

hold.

Proof If $\mathbf{u} \in \mathcal{K}$ then $c(\mathbf{v}, q, \rho, \lambda_1, \lambda_2) = 0$ for all $q \in Y, \rho, \lambda_1, \lambda_2 \in Z$. If we pick $\rho = 0, q = 0$, then $\mathbf{u}$ satisfies $\sum_{i=1}^{2} \sum_{j=1}^{k} \langle \lambda_i, \mathbf{u} \cdot \boldsymbol{\tau}_i \rangle_{\Gamma_j} = 0$ for $\lambda_1, \lambda_1 \in Z$. This means that $\mathbf{u} \cdot \boldsymbol{\tau}_i = 0$ in $H^{1/2}(\Gamma_j)$ for all $j = 1, ..., k$, and thus $\mathbf{u} \cdot \boldsymbol{\tau}_j = 0$ a.e on Γ_j. Similarly, $\mathbf{u} \cdot \mathbf{n} = 0$ a.e on each Γ_j. These imply for all $1 \leq j \leq k$ so that $\mathbf{u} = 0$ a.e on Γ_j with $meas(\Gamma_j)$ is strictly positive. By Korn's inequalities see [10], [11], [9] or [12] there exist $C_K > 0$ such that

$$\|\nabla \mathbf{u}\| \leq C_K(\Omega)(\gamma(\mathbf{u}) + \|\mathcal{D}(\mathbf{u})\|)$$

for all $\mathbf{u} \in X$. In particular, for all $\mathbf{u} \in \mathcal{K}$, where $\gamma(\mathbf{u})$ is a seminorm on $L^2(\Omega)$ which is a norm on the constants. Since measure of Γ_j are strictly positive, define $\gamma(\mathbf{u}) = \|\mathbf{u}\|_{\Gamma_j}$ then, we get $\gamma(\mathbf{u}) = 0$ if $\mathbf{u}$ is constant on Ω. The result thus follows. By the same argument, for the same $\gamma(\mathbf{u})$ we get:

$$\|\mathbf{u}\| \leq C_P(\Omega)\|\nabla(\mathbf{u})\|$$

for all $\mathbf{u} \in \mathcal{K}$.

Corollary 2.1 The bilinear form $a_0(\cdot, \cdot)$ is coercive on $\mathcal{K}$:

$$\alpha \|\mathbf{u}\|_1^2 \leq a_0(\mathbf{u}, \mathbf{u})$$

with $\alpha = Re^{-1} \min\{C_K^{-2}(\Omega), C_K^{-2}(\Omega) C_P^{-2}(\Omega)\}$.

Proof This follows from the Korn and Poincaré inequalities on $\mathcal{K}$.

Also note that the problem associated with problem (2.3) has the following form in $\mathcal{K}$. *Find* $\mathbf{u} \in \mathcal{K}$ *such that*:

$$a(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega, \quad \forall\, \mathbf{v} \in \mathcal{K}. \tag{2.4}$$

Remark: The problem (2.3) is different than the standard weak formulation of the Navier Stokes equations. Thus, we need to guarantee that the weak formulation is correct.

Lemma 2.5 (**Existence**) There exists a solution to the formulation (2.4) in $\mathcal{K}$.

Proof By the abstract theory developed in [8], the formulation (2.4) has a solution in $\mathcal{K}$ provided:

1) The form $a(\mathbf{u}; \mathbf{u}, \mathbf{v})$ is coercive in $\mathcal{K}$ but since $b(\mathbf{u}; \mathbf{u}, \mathbf{u}) = 0$ it is sufficient to show that $a_0(\mathbf{u}, \mathbf{v})$ is coercive. This was done in the Corollary 1.1.

2) The space $\mathcal{K}$ is separable and $\mathbf{u} \to a(\mathbf{u}; \mathbf{u}, \mathbf{v})$ is weakly continuous.The space $\mathcal{K}$ is separable since it is a closed subset of the Hilbert space X. Next, we prove that $\mathbf{u} \to a(\mathbf{u}; \mathbf{u}, \mathbf{v})$ is weakly continuous in $\mathcal{K}$. Let $\mathbf{u}$ be a function in $\mathcal{K}$

and let $\mathbf{u}_m$ be a sequence in $\mathcal{K}$ so that $\mathbf{u}_m \to \mathbf{u}$ weakly as $m \to \infty$. It is known by Rellich's theorem that $H^1(\Omega)$ can be embedded compactly in $L^2(\Omega)$. Then $\mathbf{u}_m$ converges strongly to $\mathbf{u} \in L^2(\Omega)^d$ as $m \to \infty$. Let $\mathbf{v}$ be in a dense subset L of $\mathcal{K}$, $L = \{\mathbf{w} \in D(\Omega) \subset V : c(\mathbf{v}, q, \rho, \lambda_1, \lambda_2) = 0, \text{for all } \lambda_1, \lambda_2, \rho \in Z\}$.

$$b(\mathbf{u}_m; \mathbf{u}_m, \mathbf{v}) = \frac{1}{2}\langle \mathbf{u}_m \cdot \nabla \mathbf{u}_m, \mathbf{v}\rangle_\Omega - \frac{1}{2}\langle \mathbf{u}_m \cdot \nabla \mathbf{v}, \mathbf{u}_m\rangle_\Omega.$$

Since $\mathbf{u}_m \in \mathcal{K}$ so is in V. Thus, we have

$$\langle \mathbf{u}_m \cdot \nabla \mathbf{u}_m, \mathbf{v}\rangle_\Omega = -\langle \mathbf{u}_m \cdot \nabla \mathbf{v}, \mathbf{u}_m\rangle_\Omega.$$

Hence we can actually write $b(.;.,.)$ as:

$$b(\mathbf{u}_m; \mathbf{u}_m, \mathbf{v}) = -\langle \mathbf{u}_m \cdot \nabla \mathbf{v}, \mathbf{u}_m\rangle_\Omega = -\sum_{i,j=1}^{d} \int_\Omega \mathbf{u}_{mi} \mathbf{u}_{mj} \mathbf{v}_{i,j} dx.$$

$\mathbf{v}_{i,j} \in L^\infty(\Omega)$ since $\mathbf{v}$ is infinitely many times differentiable. This implies that

$$\lim_{m\to\infty} \mathbf{u}_{mi} \mathbf{u}_{mj} = \mathbf{u}_i \mathbf{u}_j \in L^1(\Omega).$$

Thus,

$$\begin{aligned}\lim_{m\to\infty} b(\mathbf{u}_m; \mathbf{u}_m, \mathbf{v}) &= -\sum_{i,j=1}^{d} \int_\Omega \mathbf{u}_i \mathbf{u}_j \mathbf{v}_{i,j} dx \\ &= -b(\mathbf{u}; \mathbf{v}, \mathbf{u}).\end{aligned}$$

Since $\mathbf{u} \in \mathcal{K} \subset V$,

$$-b(\mathbf{u}; \mathbf{v}, \mathbf{u}) = \langle \mathbf{u}; \nabla \mathbf{u}, \mathbf{v}\rangle_\Omega.$$

The form $a_0(.,.)$ is continuous and $\mathbf{u}_m \to \mathbf{u}$ in $L^2(\Omega)$ implies that

$$\lim_{m\to\infty} a_0(\mathbf{u}_m, \mathbf{v}) = a_0(\mathbf{u}, \mathbf{v})$$

So we get

$$\lim_{m\to\infty} a(\mathbf{u}_m; \mathbf{u}_m, \mathbf{v}) = a(\mathbf{u}; \mathbf{u}, \mathbf{v})$$

for all $\mathbf{v} \in L$. By density of L in $\mathcal{K}$ and continuity of the forms $a_0(.,.)$ and $b(.;.,.)$ the result follows. □

In order to prove uniqueness we follow the approach of [8]. We define the finite constant as

$$N := \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathcal{K}} \frac{b(\mathbf{u};\mathbf{v},\mathbf{w})}{|\mathbf{u}|_1|\mathbf{v}|_1|\mathbf{w}|_1}.$$

Lemma 2.6 (**Uniqueness**) Suppose that the form $a_0(.,.)$ is coercive and the form $b(.;.,.)$ is locally Lipschitz continuous in $\mathcal{K}$. Then, under the condition

$$\frac{1}{\alpha^2}N|\mathbf{f}|^* < 1 \quad where \quad |\mathbf{f}|^* := \sup_{\mathbf{v}\in\mathcal{K}} \frac{\langle \mathbf{f},\mathbf{v}\rangle_\Omega}{|\mathbf{v}|_1}$$

problem (2.4) has a unique solution $\mathbf{u} \in \mathcal{K}$.

Proof Suppose there are two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ to the problem (2.4) then

$$a_0(\mathbf{u}_1,\mathbf{v}) + b(\mathbf{u}_1;\mathbf{u}_1,\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle_\Omega, \forall\, \mathbf{v} \in \mathcal{K}$$

and

$$a_0(\mathbf{u}_2,\mathbf{v}) + b(\mathbf{u}_2;\mathbf{u}_2,\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle_\Omega, \quad \forall\, \mathbf{v} \in \mathcal{K}.$$

Adding, subtracting the term $b(\mathbf{u}_1;\mathbf{u}_2,\mathbf{v})$ and setting $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ we obtain:

$$a_0(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)+b(\mathbf{u}_1;\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)+b(\mathbf{u}_1-\mathbf{u}_2;\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2) = 0. \quad (2.7)$$

The second term in (2.7) vanishes by the skew-symmetric property of the form $b(.;.,.)$. Thus

$$a_0(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2) \leq |b(\mathbf{u}_1-\mathbf{u}_2;\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)|. \quad (2.8)$$

Coercivity of the form $a_0(.,.)$ and the definition of N imply that

$$\alpha|\mathbf{u}_1-\mathbf{u}_2|_1^2 \leq N|\mathbf{u}_2|_1|\mathbf{u}_1-\mathbf{u}_2|_1^2.$$

Since the form is coercive and continuous in $\mathcal{K}$ by Lax-Milgram lemma we have

$$|\mathbf{u}_2|_1 \leq \frac{1}{\alpha}|\mathbf{f}|^*.$$

Substituting (2.8) in (2.7) we obtain:

$$\alpha|\mathbf{u}_1 - \mathbf{u}_2|_1^2 \leq \frac{N}{\alpha}|\mathbf{f}|^*|\mathbf{u}_1 - \mathbf{u}_2|_1^2.$$

This implies that:

$$(\alpha - \frac{N}{\alpha}^2 |\mathbf{f}|^*)|\mathbf{u}_1 - \mathbf{u}_2|_1^2 \leq 0.$$

Thus, $\mathbf{u}_1 = \mathbf{u}_2$ if $\left(1 - \frac{N}{\alpha^2}|\mathbf{f}|^*\right) > 0$. □

We have shown that the problem (2.4) has a unique solution in $\mathcal{K}$.

Theorem 2.1 Assume that the multilinear form $c(.;.,.,.)$ satisfies the inf-sup condition. There is a constant $\beta > 0$ such that:

$$\inf_{\substack{\chi_1,\chi_2,\sigma\in Z\\ p\in Y}} \sup_{\mathbf{v}\in X} \frac{c(\mathbf{v}; q, \sigma, \chi_1, \chi_2)}{|||\mathbf{v}|||\left(\|p\|_0^2 + \|\chi_1\|_Z^2 + \|\chi_2\|_Z^2 + \|\sigma\|_Z^2\right)^{1/2}} \geq \beta. \tag{2.9}$$

Then, for each solution $\mathbf{u}$ of problem (2.4), there exist a unique $p \in Y, \rho, \lambda_1, \lambda_2 \in Z$ such that $(\mathbf{u}, p, \rho, \lambda_1, \lambda_2)$ is a solution of problem (2.3).

Proof It follows from [13]. For error analysis we need the following lemma.

Lemma 2.7 There is a constant $\beta'' > 0$ such that:

$$\inf_{\sigma,\chi_1,\chi_2\in Z} \sup_{\mathbf{v}\in V} \frac{\sum_{j=1}^k \left(\langle\chi_1, \mathbf{v}\cdot\boldsymbol{\tau}_1^{(j)}\rangle_{\Gamma_j} + \langle\chi_2, \mathbf{v}\cdot\boldsymbol{\tau}_2^{(j)}\rangle_{\Gamma_j} + \langle\sigma, \mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j}\right)}{\|\mathbf{v}\|_1\left(\|\chi_1\|_Z^2 + \|\chi_2\|_Z^2 + \|\sigma\|_Z^2\right)^{1/2}} \geq \beta''.$$

Proof See [2].

3 Penalty-Lagrange Multiplier Method

It has been observed by [14] that slip with friction boundary conditions match the experimental behavoir of real fluids at higher Reynolds number better than no-slip condition. Consequently, we propose to impose $\mathbf{u}\cdot\boldsymbol{\tau}_i^{(j)}\mid_{\Gamma_j}=0$ for $i=1,2$ using a penalty technique that is equivalent to such a condition. We choose small, positive penalty parameters ϵ_1,ϵ_2 whose selection will be guided by the following analysis. We consider the following problem:

Find $\mathbf{u}_\epsilon \in V$ *and* $\rho_\epsilon \in Z$ *such that*:

$$a_0(\mathbf{u}_\epsilon,\mathbf{v})+b(\mathbf{u}_\epsilon;\mathbf{u}_\epsilon,\mathbf{v})+\epsilon_1^{-1}\sum_{j=1}^{k}\langle\ \mathbf{u}_\epsilon,\mathbf{v}\cdot\boldsymbol{\tau}_1^{(j)}\rangle_{\Gamma_j}$$

$$+\epsilon_2^{-1}\sum_{j=1}^{k}\langle\mathbf{u}_\epsilon,\mathbf{v}\cdot\boldsymbol{\tau}_2^{(j)}\rangle_{\Gamma_j}+\sum_{j=1}^{k}\langle\rho_\epsilon,\mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j}=\langle\mathbf{f},\mathbf{v}\rangle,\ \forall\mathbf{v}\in V, \tag{3.1}$$

$$\sum_{j=1}^{k}\langle\sigma,\mathbf{u}_\epsilon\cdot\mathbf{n}_j\rangle_{\Gamma_j}=0,\ \forall\sigma\in Z.$$

Suppose that instead of considering penalty methods to weaken the condition $\mathbf{u}\cdot\boldsymbol{\tau}_i^{(j)}\mid_{\Gamma_j}=0$, we consider slip with non-linear resistance to slip, i.e.:

$$-2Re^{-1}\nabla\cdot\mathcal{D}(\mathbf{u}_\epsilon)+\mathbf{u}_\epsilon\cdot\nabla\mathbf{u}_\epsilon+\nabla p_\epsilon=\mathbf{f}\text{ in }\Omega, \tag{3.2}$$

$$\nabla\cdot\mathbf{u}_\epsilon=0\text{ in }\Omega,$$

$$\sum_{j=1}^{k}\int_{\Gamma_j}\mathbf{u}_\epsilon\cdot\mathbf{n}_j\,ds=0\text{ on }\Gamma=\partial\Omega,$$

$$\mathbf{u}_\epsilon\cdot\boldsymbol{\tau}_i^{(j)}+\epsilon_i\mathbf{n}_j\cdot\left(\mathcal{D}(\mathbf{u}_\epsilon)-\frac{1}{2}|\mathbf{u}_\epsilon|\mathbf{u}_\epsilon\right)\cdot\boldsymbol{\tau}_i^{(j)}=0\text{ on }\Gamma_j,\ i=1,2,\ 1\le j\le k.$$

Lemma 3.1 The variational formulation of equation (3.2) is the same as equation (3.1).

Proof We multiply equation in (3.2) by $\mathbf{v} \in X, \zeta \in Y, \rho \in Z$, respectively, and integrate by parts yield:

$$2Re^{-1}\langle \mathcal{D}(\mathbf{u}_\epsilon) : \mathcal{D}(\mathbf{v})\rangle_\Omega + 1/2\langle \mathbf{u}_\epsilon \cdot \nabla \mathbf{u}_\epsilon, \mathbf{v}\rangle_\Omega - 1/2\langle \mathbf{u}_\epsilon \cdot \nabla \mathbf{v}, \mathbf{u}_\epsilon\rangle_\Omega +$$

$$+\langle \mathcal{D}(\mathbf{u}_\epsilon)\cdot \mathbf{n}_j, \mathbf{v}\rangle_{\Gamma_j} - \langle p_\epsilon, \nabla\cdot \mathbf{v}\rangle_\Omega + \sum_{i=1}^{2}\sum_{j=1}^{k} \epsilon_i^{-1}\langle\ \mathbf{u}_\epsilon \cdot \boldsymbol{\tau}_i^{(j)}, \mathbf{v}\cdot \boldsymbol{\tau}_i^{(j)}\rangle_{\Gamma_j} +$$

$$+\sum_{j=1}^{k}\langle \rho_\epsilon, \mathbf{v}\cdot \mathbf{n}_j\rangle_{\Gamma_j} = \langle \mathbf{f}, \mathbf{v}\rangle_\Omega,\ \forall \mathbf{v} \in X.$$

$$\sum_{j=1}^{k}\langle \sigma, \mathbf{u}_\epsilon \cdot \mathbf{n}_j\rangle_{\Gamma_j} = 0,\ \forall \sigma \in Z.$$

$$\langle \zeta, \nabla\cdot \mathbf{u}_\epsilon\rangle_\Omega = 0,\ \forall \zeta \in Y.$$

□

The lemma implies that the penalty method captures the idea of slip with non-linear resistance to slip.To simplify the notation we define:

$$\lambda_1^\epsilon\ |_{\Gamma_j} := \epsilon_1^{-1}\mathbf{u}_\epsilon \cdot \boldsymbol{\tau}_1^{(j)},$$

$$\lambda_2^\epsilon\ |_{\Gamma_j} := \epsilon_2^{-1}\mathbf{u}_\epsilon \cdot \boldsymbol{\tau}_2^{(j)};$$

where $\lambda_1^\epsilon, \lambda_2^\epsilon \in Z$. Then equation (3.1) takes the following form:

$$a_0(\mathbf{u}_\epsilon, \mathbf{v}) + b(\mathbf{u}_\epsilon; \mathbf{u}_\epsilon, \mathbf{v}) + \sum_{j=1}^{k}\langle \lambda_1^\epsilon, \mathbf{v}\cdot \boldsymbol{\tau}_1^{(j)}\rangle_{\Gamma_j} +$$

$$+\sum_{j=1}^{k}\langle \lambda_2^\epsilon, \mathbf{v}\cdot \boldsymbol{\tau}_2^{(j)}\rangle_{\Gamma_j} + \sum_{j=1}^{k}\langle \rho_\epsilon, \mathbf{v}\cdot \mathbf{n}_j\rangle_{\Gamma_j} = \langle \mathbf{f}, \mathbf{v}\rangle, \quad \forall\, \mathbf{v} \in V,$$

$$\sum_{j=1}^{k}\langle \sigma, \mathbf{u}_\epsilon \cdot \mathbf{n}_j\rangle_{\Gamma_j} = 0, \quad \forall\, \sigma \in Z, \tag{3.3}$$

$$\sum_{j=1}^{k}\langle \mathbf{u}_\epsilon \cdot \boldsymbol{\tau}_1^{(j)}, \chi_1\rangle_{\Gamma_j} = \epsilon_1 \sum_{j=1}^{k}\langle \lambda_1^\epsilon, \chi_1\rangle_{\Gamma_j}, \quad \forall\, \chi_1 \in Z,$$

$$\sum_{j=1}^{k}\langle \mathbf{u}_\epsilon \cdot \boldsymbol{\tau}_2^{(j)}, \chi_2\rangle_{\Gamma_j} = \epsilon_2 \sum_{j=1}^{k}\langle \lambda_2^\epsilon, \chi_2\rangle_{\Gamma_j}, \quad \forall\, \chi_2 \in Z.$$

Proposition 3.1 Let $(\mathbf{u}, \lambda_1, \lambda_2, \rho)$ be the solution of the Navier-Stokes equations (2.1) and let $(\mathbf{u}_\epsilon, \lambda_1^\epsilon, \lambda_2^\epsilon, \rho_\epsilon)$ be the solution to (3.2) Then:

$$e_\epsilon(\lambda_1, \lambda_2, \rho) + \|\mathbf{u} - \mathbf{u}_\epsilon\|_{1,\Omega} \leq C[\epsilon_1^2 \|\lambda_1\|_\Gamma^2 + \epsilon_2^2 \|\lambda_2\|_\Gamma^2]^{1/2}, \qquad (3.4)$$

where

$$e_\epsilon(\lambda_1, \lambda_2, \rho) := [\|\lambda_1 - \lambda_1^\epsilon\|_Z^2 + \|\lambda_2 - \lambda_2^\epsilon\|_Z^2 + \|\rho - \rho_\epsilon\|_Z^2]^{1/2}$$

and C depends on Re, α, β'', and $|f|^*$.

Proof Subtracting equation (3.3) from equation (2.3) yields:

$$\begin{aligned} a_0(\mathbf{u} - \mathbf{u}_\epsilon, \mathbf{v}) + b(\mathbf{u} - \mathbf{u}_\epsilon; \mathbf{u}_\epsilon, \mathbf{v}) + b(\mathbf{u}; \mathbf{u} - \mathbf{u}_\epsilon, \mathbf{v}) + \\ \sum_{i=1}^{2} \sum_{j=1}^{k} \langle \lambda_i - {\lambda_i}^\epsilon, \mathbf{v} \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j} + \sum_{j=1}^{k} \langle \rho - \rho_\epsilon, \mathbf{v} \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0, \quad \forall\, \mathbf{v} \in V, \\ \sum_{j=1}^{k} \langle \sigma, (\mathbf{u} - \mathbf{u}_\epsilon) \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0, \quad \forall\, \sigma \in Z, \\ \sum_{j=1}^{k} \langle (\mathbf{u} - \mathbf{u}_\epsilon) \cdot \boldsymbol{\tau}_i^{(j)}, \chi_i \rangle_{\Gamma_j} = -\epsilon_i \sum_{j=1}^{k} \langle \lambda_i^\epsilon, \chi_i \rangle_{\Gamma_j}, \quad \forall\, \chi_i \in Z, \end{aligned} \qquad (3.5)$$

for $i = 1, 2$. From Lemma 2.7 given $\lambda_1, \lambda_2, \rho \in Z$ we have:

$$\begin{aligned} &\beta'' e_\epsilon(\lambda_1, \lambda_2, \rho) \\ &\quad \leq \sup_{0 \neq \mathbf{v} \in V} \frac{a_0(\mathbf{u} - \mathbf{u}_\epsilon, \mathbf{v}) + b(\mathbf{u} - \mathbf{u}_\epsilon; \mathbf{u}_\epsilon, \mathbf{v}) + b(\mathbf{u}; \mathbf{u} - \mathbf{u}_\epsilon, \mathbf{v})}{\|\mathbf{v}\|_1} \\ &\quad \leq \frac{2Re^{-1}|\mathbf{u} - \mathbf{u}_\epsilon|_1 |\mathbf{v}|_1 + M|\mathbf{u} - \mathbf{u}_\epsilon|_1 |\mathbf{u}_\epsilon|_1 |\mathbf{v}|_1 + M|\mathbf{u} - \mathbf{u}_\epsilon|_1 |\mathbf{v}|_1 |\mathbf{u}|_1}{\|\mathbf{v}\|_1} \\ &\quad \leq |\mathbf{u} - \mathbf{u}_\epsilon|_1 (2Re^{-1} + 2M\alpha^{-1}|\mathbf{f}|^*) \\ &\quad \leq 2(Re^{-1} + \alpha)\|\mathbf{u} - \mathbf{u}_\epsilon\|_1 \end{aligned} \qquad (3.6)$$

where

$$M := \sup_{0\neq \mathbf{u},\mathbf{v},\mathbf{w}\in V} \frac{|b(\mathbf{u};\mathbf{v},\mathbf{w})|}{|\mathbf{u}|_1|\mathbf{v}|_1|\mathbf{w}|_1}.$$

In equation (3.5) we take $\mathbf{v}=\mathbf{u}-\mathbf{u}_\epsilon$. Then,

$$\begin{aligned} & a_0(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon)+b(\mathbf{u}-\mathbf{u}_\epsilon;\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon) \\ & \quad -\epsilon_i\sum_{j=1}^{k}\langle\lambda_i^\epsilon,\lambda_i-\lambda_i^\epsilon\rangle_{\Gamma_j}+\sum_{j=1}^{k}\langle\rho-\rho_\epsilon,(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\mathbf{n}_j\rangle_{\Gamma_j}=0. \end{aligned} \tag{3.7}$$

Thus,

$$a_0(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon)=b(\mathbf{u}-\mathbf{u}_\epsilon;\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u}_\epsilon)+\sum_{i=1}^{2}\epsilon_i\sum_{j=1}^{k}\langle\lambda_i^\epsilon,\lambda_i-\lambda_i^\epsilon\rangle_{\Gamma_j}.$$

Add and subtract the terms $\sum_{i=1}^{2}\epsilon_i\sum_{j=1}^{k}\langle\lambda_i,\lambda_i-\lambda_i^\epsilon\rangle_{\Gamma_j}$, and dropping the positive term $\sum_{i=1}^{2}\epsilon_i\sum_{j=1}^{k}\langle\lambda_i-\lambda_i^\epsilon,\lambda_i-\lambda_i^\epsilon\rangle_{\Gamma_j}$ we get:

$$a_0(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon)\le b(\mathbf{u}-\mathbf{u}_\epsilon;\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u}_\epsilon)+\sum_{i=1}^{2}\epsilon_i\sum_{j=1}^{k}\langle\lambda_i,\lambda_i-\lambda_i^\epsilon\rangle_{\Gamma_j}.$$

Applying Cauchy-Schwarz inequality and definition of M we get:

$$\begin{aligned} a_0(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon) &\le b(\mathbf{u}-\mathbf{u}_\epsilon;\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon)+\sum_{i=1}^{2}\epsilon_i\sum_{j=1}^{k}\langle\lambda_i,\lambda_i-\lambda_i^\epsilon\rangle_{\Gamma_j} \\ &\le M\alpha^{-1}|\mathbf{f}|^*|\mathbf{u}-\mathbf{u}_\epsilon|_1^2+\epsilon_i\|\lambda_i-\lambda_i^\epsilon\|_Z\|\lambda_i\|_\Gamma. \end{aligned}$$

By the arithmetic geometric mean inequality for $n=2$, we have

$$\begin{aligned} & a_0(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon) \\ & \le M\alpha^{-1}|\mathbf{f}|^*|\mathbf{u}-\mathbf{u}_\epsilon|_1^2+\left(\sum_{i=1}^{2}\|\lambda_i-\lambda_i^\epsilon\|_\Gamma^2\right)^{1/2}\left(\sum_{i=1}^{2}\epsilon_i^2\|\lambda_i\|_\Gamma^2\right)^{1/2}. \end{aligned}$$

Using the coercivity of $a_0(.,.)$ and inequality (3.6) we get

$$\|\mathbf{u}-\mathbf{u}_\epsilon\|_1\le\left[\frac{2(Re^{-1}+\alpha)}{\beta''\alpha(1-M\alpha^{-2}|\mathbf{f}|^*)}\right]\left(\sum_{i=1}^{2}\epsilon_i^2\|\lambda_i\|_\Gamma^2\right)^{1/2}.$$

Thus

$$e_\epsilon(\lambda_1, \lambda_2, \rho) + \|\mathbf{u} - \mathbf{u}_\epsilon\|_1 \leq \frac{2(Re^{-1} + \alpha)}{\beta''} \|\mathbf{u} - \mathbf{u}_\epsilon\|_1 + \|\mathbf{u} - \mathbf{u}_\epsilon\|_1.$$

Combining the latter inequalities we obtain:

$$e_\epsilon(\lambda_1, \lambda_2, \rho) + \|\mathbf{u} - \mathbf{u}_\epsilon\|_1 \leq C \left(\sum_{i=1}^{2} \epsilon_i^2 \|\lambda_i\|_\Gamma^2 \right)^{1/2} . \tag{3.8}$$

□

4 Finite Element Spaces

The polyhedral domain Ω is subdivided into d-simplices with sides of length less than h with $\mathcal{T}^h$ being the family of partitions. We will assume that $\mathcal{T}^h$ satisfies the usual regularity assumptions, see e.g. [15] that:

1. Each vertex of Ω is a vertex of a $T \in \mathcal{T}^h$,
2. Each $T \in \mathcal{T}^h$ has at least one vertex in the interior of Ω,
3. Any two d-simplices $T, T' \in \mathcal{T}^h$ may meet in a vertex, a whole edge, or a whole face,

The constants c_0, c_1 denote different constants which are independent of h.

1. Each $T \in \mathcal{T}^h$ contains a ball with radius $c_0 h$ and is contained in a ball with radius $c_1 h$.

Denote by $\mathcal{O}_j^h$ the partition of Γ_j which is induced by $\mathcal{T}^h$.

Let $X^h \subset X,\ Y^h \subset Y,\ Z^h \subset Z$. Also let

$$X_0^h = \left\{ \mathbf{u}^h \in X^h \mid \quad \mathbf{u}^h = 0 \quad \text{on} \quad \Gamma \right\}.$$

The spaces X^h and Y^h are assumed to satisfy the following properties:

I. There is a constant $\tilde{\beta} > 0$ independent of h for which:

$$\inf_{0\neq p^h\in Y^h} \sup_{0\neq \mathbf{u}^h\in X_0^h} \frac{\int_\Omega p^h \text{div}\mathbf{u}^h \, dx}{\|p^h\|_{0,\Omega}\|\mathbf{u}^h\|_{1,\Omega}} \geq \tilde{\beta}, \tag{4.1}$$

II.

$$\inf_{p^h\in Y^h} \|p - p^h\|_{0,\Omega} \leq ch\|p\|_{1,\Omega}, \quad \forall\, p \in H^1(\Omega),$$

III. There exists a continuous linear operator $\Pi^h : H^1(\Omega)^d \to X^h$ for which:

$$\Pi^h(H_0^1(\Omega)^d) \subset X_0^h,$$

$$\|\mathbf{u} - \Pi^h\mathbf{u}\|_{s,\Omega} \leq ch^{t-s}\|\mathbf{u}\|_{t,\Omega}, \quad \forall\, \mathbf{u} \in H^t(\Omega) \quad \text{with } s = 0, 1 \text{ and } t = 1, 2,$$

$$\|\mathbf{u} - \Pi^h\mathbf{u}\|_{0,\Gamma} \leq ch^{1/2}\|\mathbf{u}\|_{1,\Omega},$$

where $\|.\|_{0,\Gamma} = (\sum_{j=1}^k \|.\|_{0,\Gamma_j})^{1/2}$. Assumption I balances the influence of the constraint $\text{div}\mathbf{u} = 0$ and also implies that the space:

$$V_0^h = \left\{\mathbf{v}^h \in X_0^h \mid \quad \langle q^h, \nabla\cdot\mathbf{v}^h\rangle = 0, \quad \forall\, q^h \in Y^h\right\},$$

$$V^h = \left\{\mathbf{v}^h \in X^h \mid \quad \langle q^h, \nabla\cdot\mathbf{v}^h\rangle = 0, \quad \forall\, q^h \in Y^h\right\} \supset V_0^h,$$

is also not empty. As usual V^h is not a subset of V and in particular, the functions of V^h are not divergence free. Hence we introduce anti-symmetric form $b(.;.,.)$,

$$N^h = \sup_{0\neq \mathbf{u}^h,\mathbf{v}^h,\mathbf{w}^h\in X^h} \frac{b(\mathbf{u}^h;\mathbf{v}^h,\mathbf{w}^h)}{|\mathbf{u}^h|_1|\mathbf{v}^h|_1|\mathbf{w}^h|_1} \tag{4.2}$$

and

$$\|\mathbf{f}\|_h^* = \sup_{\mathbf{v}^h\in V^h} \frac{\langle \mathbf{f}, \mathbf{v}^h\rangle_\Omega}{|\mathbf{v}^h|_1}.$$

With the above notation, the discrete analogue of problem (2.3) is:

Find $\mathbf{u}^h \in X^h,\ p^h \in Y^h, \rho^h, \lambda_1^h, \lambda_2^h \in Z^h$ *such that*:

$$a(\mathbf{u}^h;\mathbf{u}^h,\mathbf{v}^h) - c(\mathbf{v}^h;\ p^h,\rho^h,\lambda_1^h,\lambda_2^h) = \langle \mathbf{f},\mathbf{v}^h\rangle_\Omega \tag{4.3}$$

$$c(\mathbf{u}^h;\ q^h,\sigma^h,\chi_1^h,\chi_2^h) = 0$$

for all $\mathbf{v}^h \in X^h, q^h \in Y^h, \sigma^h, \chi_1^h, \chi_2^h \in Z^h$.

By assumption I this is equivalent to the following problem in V^h:

Find $\mathbf{u}^h \in X^h,\ p^h \in Y^h, \rho^h, \lambda_1^h, \lambda_2^h \in Z^h$ *such that*:

$$a(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h) = \langle \mathbf{f},\mathbf{v}^h\rangle_\Omega \tag{4.4}$$

By the abstract theory developed in [8] the discrete problem will have unique solution provided:

$$\alpha^{-2}N^h|\mathbf{f}|_h^* < 1 \tag{4.5}$$

and, there is a constant $\widehat{\beta} > 0$, independent of h such that

$$\inf_{\substack{\rho^h,\lambda_1^h,\lambda_2^h\in Z^h\\ p^h\in Y^h}} \sup_{0\neq\mathbf{v}^h\in X^h} \frac{c(\mathbf{v}^h;p^h,\rho^h,\lambda_1^h,\lambda_2^h)}{\|\mathbf{v}^h\|_1\left[\|\lambda_1^h\|_Z^2+\|\lambda_2^h\|_Z^2+\|\rho^h\|_Z^2+\|p^h\|_{0,\Omega}^2\right]^{1/2}} \geq \widehat{\beta}. \tag{4.6}$$

Now we need to find right spaces X^h, Y^h, Z^h so that they hold (4.6). An example of spaces X^h, Y^h, Z^h satisfying (4.6) and the classical inf-sup condition are given in [7].

Lemma 4.1 There exists a constant $\overline{\beta} > 0$ independent of h, such that:

$$\inf_{\substack{\rho^h\in Z^h\\ \lambda_1^h,\lambda_2^h\in Z^h}} \sup_{0\neq\mathbf{v}^h\in X^h} \frac{\sum_{j=1}^k\left(\langle\lambda_1^h,\mathbf{v}^h\cdot\boldsymbol{\tau}_1^{(j)}\rangle_{\Gamma_j}+\langle\lambda_2^h,\mathbf{v}^h\cdot\boldsymbol{\tau}_2^{(j)}\rangle_{\Gamma_j}+\langle\rho^h,\mathbf{v}^h\cdot\mathbf{n}_j\rangle_{\Gamma_j}\right)}{\|\mathbf{v}^h\|_1\left[\|\lambda_1^h\|_Z^2+\|\lambda_2^h\|_Z^2+\|\rho^h\|_Z^2\right]^{1/2}} \geq \overline{\beta}. \tag{4.7}$$

Lemma 4.2 There is a constant $\widehat{\beta} > 0$, independent of h such that:

$$\inf_{\substack{\rho^h,\lambda_1^h,\lambda_2^h \in Z^h \\ p^h \in Y^h}} \sup_{0 \neq \mathbf{v}^h \in X^h} \frac{c(\mathbf{v}^h; p^h, \rho^h, \lambda_1^h, \lambda_2^h)}{\|\mathbf{v}^h\|_1 \left[\|\lambda_1^h\|_Z^2 + \|\lambda_2^h\|_Z^2 + \|\rho^h\|_Z^2 + \|p^h\|_{0,\Omega}^2\right]^{1/2}} \geq \widehat{\beta}. \quad (4.8)$$

Proof The last two lemmas are proven in [2].

5 The Discrete Penalty-Lagrange Multiplier Method

We can now write the discrete analogue of the equation (3.1) as:

Find $\mathbf{u}_\epsilon^h \in X^h$ *satisfying:*

$$a_0(\mathbf{u}_\epsilon^h, \mathbf{v}^h) + b(\mathbf{u}_\epsilon^h; \mathbf{u}_\epsilon^h, \mathbf{v}^h) + \sum_{i=1}^{2} \epsilon_i^{-1} \sum_{j=1}^{k} \langle \mathbf{u}_\epsilon^h \cdot \boldsymbol{\tau}_i^{(j)}, \mathbf{v}^h \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j} +$$

$$+ \sum_{j=1}^{k} \langle \rho_\epsilon^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} - \langle p_\epsilon^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega = \langle \mathbf{f}, \mathbf{v}^h \rangle, \quad (5.1)$$

$$\langle q^h, \nabla \cdot \mathbf{u}_\epsilon^h \rangle_\Omega = 0,$$

$$\sum_{j=1}^{k} \langle \sigma^h, \mathbf{u}_\epsilon^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0,$$

for all $\mathbf{v}^h \in X^h,\ q^h \in Y^h,\ \sigma^h \in Z^h$. This is equivalent to finding $\mathbf{u}_\epsilon^h \in V^h$ such that:

$$a_0(\mathbf{u}_\epsilon^h, \mathbf{v}^h) + b(\mathbf{u}_\epsilon^h; \mathbf{u}_\epsilon^h, \mathbf{v}^h) + \sum_{i=1}^{2} \epsilon_i^{-1} \sum_{j=1}^{k} \langle \mathbf{u}_\epsilon^h \cdot \boldsymbol{\tau}_i^{(j)}, \mathbf{v}^h \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j} + \quad (5.2)$$

$$+ \sum_{j=1}^{k} \langle \rho_\epsilon^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = \langle \mathbf{f}, \mathbf{v}^h \rangle,$$

$$\sum_{j=1}^{k} \langle \sigma^h, \mathbf{u}_\epsilon^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0,$$

for all $\mathbf{v}^h \in V^h, \sigma^h \in Z^h$.

Since $\lambda_i^{\epsilon,h}$ for $i=1,2$ are not necessarily in Z^h equation (5.2) is not equivalent to (5.3). In fact

$$\lambda_i^{\epsilon,h} \in \tilde{Z}_i^h := X^h \cdot \boldsymbol{\tau}_i^{(j)} |_{\Gamma_j}, \quad i=1,2,\ 1 \le j \le k.$$

$$\begin{aligned} a_0(\mathbf{u}_\epsilon^h, \mathbf{v}^h) + b(\mathbf{u}_\epsilon^h; \mathbf{u}_\epsilon^h, \mathbf{v}^h) + \sum_{i=1}^{2}\sum_{j=1}^{k} \langle \lambda_i^{\epsilon,h}, \mathbf{v}^h \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j} + \\ + \sum_{j=1}^{k} \langle \rho_\epsilon^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = \langle \mathbf{f}, \mathbf{v}^h \rangle, \quad \forall\, \mathbf{v}^h \in V^h, \\ \sum_{j=1}^{k} \langle \sigma^h, \mathbf{u}_\epsilon^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0, \quad \forall\, \sigma^h \in Z^h, \\ \sum_{j=1}^{k} \langle \mathbf{u}_\epsilon^h \cdot \boldsymbol{\tau}_i^{(j)}, \chi_i^h \rangle_{\Gamma_j} = \epsilon_i \sum_{j=1}^{k} \langle \lambda_i^{\epsilon,h}, \chi_i^h \rangle_{\Gamma_j}, \quad \forall\, \chi_i^h \in Z^h, \end{aligned} \tag{5.3}$$

for $i=1,2$. The latter problem will be equivalent if we make the following assumption: There exists $\ddot{\beta} > 0$

$$\inf_{\substack{\lambda_i^h \in \tilde{Z}_i^h \\ \rho^h \in Z^h}} \sup_{0 \ne \mathbf{v}^h \in X^h} \frac{\sum_{j=1}^{k} \langle \lambda_1^{\epsilon,h}, \mathbf{v}^h \cdot \boldsymbol{\tau}_1^{(j)} \rangle_{\Gamma_j} + \langle \lambda_2^{\epsilon,h}, \mathbf{v}^h \cdot \boldsymbol{\tau}_2^{(j)} \rangle_{\Gamma_j} + \langle \rho^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j}}{\|\mathbf{v}^h\|_1 \left[\|\lambda_1^{\epsilon,h}\|_Z^2 + \|\lambda_2^{\epsilon,h}\|_Z^2 + \|\rho^h\|_Z^2 \right]^{1/2}} \ge \ddot{\beta}. \tag{5.4}$$

We shall now study the error $\mathbf{u} - \mathbf{u}_\epsilon^h$ in $1-$ norm where $(\mathbf{u}, \lambda_1, \lambda_2, p, \rho)$ is the solution of (2.3) and $(\mathbf{u}_\epsilon^h, {\lambda_1}^{\epsilon,h}, {\lambda_2}^{\epsilon,h}, p^h, {\rho_\epsilon}^h)$ is the solution of the finite element problem (5.3). We estimate the error in two cases. In the first case the computational boundary Γ^h does not follow the fluid boundary Γ. For example, suppose the domain Ω has smooth boundary and we use quadratic elements for our approximation. Then by approximating the fluid velocity $\mathbf{u}$ by penalized velocity $\mathbf{u}_\epsilon$ and $\mathbf{u}_\epsilon$ by discrete penalized velocity $\mathbf{u}_\epsilon^h$. We will conclude that the error in

$1-$ norm has order h convergence. This requires that the penalty parameter ϵ should be scaled by h. In the second case, Γ^h follows Γ exactly. Our domain is polygonal (in $2d$) with the choice of quadratic elements we approximate $\mathbf{u}$ by discrete velocity $\mathbf{u}^h$ and $\mathbf{u}^h$ by $\mathbf{u}^h_\epsilon$. This leads order h^2 convergence in the error $1-$ norm. This implies scaling ϵ by h^2. Thus, it has improved the basic results of [2] even in the linear case.

CASE I: In this case, the approximation $\mathbf{u}^h_\epsilon$ satisfies (5.3). We need similar equations for the continous $\mathbf{u}_\epsilon$ in the velocity space. For the reason we multiply by $\mathbf{v} \in V$ to obtain (3.1), instead we multiply by $\mathbf{v}^h \in V^h$. It gives:

$$\begin{aligned} a_0(\mathbf{u}_\epsilon, \mathbf{v}^h) + b(\mathbf{u} - \mathbf{u}_\epsilon; \mathbf{u}_\epsilon, \mathbf{v}^h) + \langle p_\epsilon - q^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega + \\ + \sum_{i=1}^{2} \epsilon_i^{-1} \sum_{j=1}^{k} \langle \mathbf{u}_\epsilon \cdot \boldsymbol{\tau}_i^{(j)}, \mathbf{v}^h \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j} + \sum_{j=1}^{k} \langle \rho_\epsilon, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = \langle \mathbf{f}, \mathbf{v}^h \rangle, \\ \sum_{j=1}^{k} \langle \sigma^h, \mathbf{u}_\epsilon \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0, \end{aligned} \tag{5.5}$$

for all $\mathbf{v}^h \in V^h$ and $\sigma^h \in Z^h$, since $Z^h \subset Z$. Subtracting (5.2) from (5.5) yields:

$$\begin{aligned} a_0(\mathbf{u}_\epsilon - \mathbf{u}^h_\epsilon, \mathbf{v}^h) + b(\mathbf{u}^h_\epsilon; \mathbf{u}_\epsilon - \mathbf{u}^h_\epsilon, \mathbf{v}^h) + b(\mathbf{u}_\epsilon - \mathbf{u}^h_\epsilon; \mathbf{u}_\epsilon, \mathbf{v}^h) - \langle p_\epsilon - q^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega + \\ \sum_{i=1}^{2} \epsilon_i^{-1} \sum_{j=1}^{k} \langle (\mathbf{u}_\epsilon - \mathbf{u}^h_\epsilon) \cdot \boldsymbol{\tau}_i^{(j)}, \mathbf{v}^h \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j} + \sum_{j=1}^{k} \langle \rho_\epsilon - \rho^h_\epsilon, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0, \end{aligned} \tag{5.6}$$

for all $\mathbf{v}^h \in V^h$. We write:

$$\mathbf{u}_\epsilon - \mathbf{u}^h_\epsilon = (\mathbf{u}_\epsilon - \mathbf{v}^h) - (\mathbf{u}^h_\epsilon - \mathbf{v}^h), \tag{5.7}$$

with $\mathbf{v}^h$ being the best approximation of $\mathbf{u}_\epsilon$ in V^h. Let

$$\eta := \mathbf{u}_\epsilon - \mathbf{v}^h,$$

$$\phi^h := \mathbf{u}^h_\epsilon - \mathbf{v}^h.$$

Then equation (5.6) becomes:

$$a_0(\eta-\phi^h,\mathbf{v}^h)+b(\eta-\phi^h;\mathbf{u}_\epsilon,\mathbf{v}^h)+b(\mathbf{u}_\epsilon^h;\eta-\phi^h,\mathbf{v}^h)-\langle p_\epsilon-q^h,\nabla\cdot\mathbf{v}^h\rangle_\Omega$$
$$+\sum_{i=1}^{2}\epsilon_i^{-1}\sum_{j=1}^{k}\langle(\eta-\phi^h)\cdot\boldsymbol{\tau}_i^{(j)},\mathbf{v}^h\cdot\boldsymbol{\tau}_i\rangle_{\Gamma_j}+\sum_{j=1}^{k}\langle\rho_\epsilon-\rho_\epsilon^h,\mathbf{v}^h\cdot\mathbf{n}_j\rangle_{\Gamma_j}=0, \quad (5.8)$$

for all $\mathbf{v}^h\in V^h$. Equation (5.8) can be rewritten as:

$$a_0(\phi^h,\mathbf{v}^h)+b(\phi^h;\mathbf{u}_\epsilon,\mathbf{v}^h)+b(\mathbf{u}_\epsilon^h;\phi^h,\mathbf{v}^h)+\sum_{i=1}^{2}\epsilon_i^{-1}\sum_{j=1}^{k}\langle\phi^h\cdot\boldsymbol{\tau}_i,\mathbf{v}^h\cdot\boldsymbol{\tau}_i^{(j)}\rangle_{\Gamma_j}$$
$$=a_0(\eta,\mathbf{v}^h)+b(\eta;\mathbf{u}_\epsilon,\mathbf{v}^h)+b(\mathbf{u}_\epsilon^h;\eta,\mathbf{v}^h)-\langle p_\epsilon-q^h,\nabla\cdot\mathbf{v}^h\rangle_\Omega \quad (5.9)$$
$$+\sum_{j=1}^{k}\langle\rho_\epsilon-\rho_\epsilon^h,\mathbf{v}^h\cdot\mathbf{n}_j\rangle_{\Gamma_j}+\sum_{i=1}^{2}\epsilon_i^{-1}\sum_{j=1}^{k}\langle\eta\cdot\boldsymbol{\tau}_i^{(j)},\mathbf{v}^h\cdot\boldsymbol{\tau}_i^{(j)}\rangle_{\Gamma_j}$$

Set $\mathbf{v}^h=\phi^h$. Then:

$$2Re^{-1}\|\mathcal{D}(\phi^h)\|_0^2+b(\phi^h;\mathbf{u}_\epsilon^h,\phi^h)+\sum_{i=1}^{2}\epsilon_i^{-1}\left\|\phi^h\cdot\boldsymbol{\tau}_i\right\|_{0,\Gamma}^2=$$
$$=a_0(\eta,\phi^h)+b(\mathbf{u}_\epsilon^h;\eta,\phi^h)+b(\eta;\mathbf{u}_\epsilon,\phi^h)-\langle p_\epsilon-q^h,\nabla\cdot\phi^h\rangle_\Omega$$
$$+\sum_{i=1}^{2}\epsilon_i^{-1}\sum_{j=1}^{k}\langle\eta\cdot\boldsymbol{\tau}_i^{(j)},\phi^h\cdot\boldsymbol{\tau}_i^{(j)}\rangle_{\Gamma_j}+\sum_{j=1}^{k}\langle\rho_\epsilon-\rho_\epsilon^h,\phi^h\cdot\mathbf{n}_j\rangle_{\Gamma_j}. \quad (5.10)$$

But for each j, $\phi^h\cdot\mathbf{n}_j\perp Z^h$ since $Z^h\subset Z$. Thus we can replace ρ_ϵ^h in (5.10) with any test function, say $\sigma^h\in Z^h$. Then we have

$$2Re^{-1}\|\mathcal{D}(\phi^h)\|^2+\sum_{i=1}^{2}\epsilon_i^{-1}\|\phi^h\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2$$
$$\le 2Re^{-1}|\eta|_1|\phi^h|_1+N^h|\mathbf{u}_\epsilon{}^h|_1|\phi^h|_1|\eta|_1+N^h|\mathbf{u}_\epsilon||\phi^h|_1|\eta|_1+\sum_{i=1}^{2}\frac{\epsilon_i^{-1}}{2}\|\eta\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2$$
$$+\sum_{i=1}^{2}\frac{\epsilon_i^{-1}}{2}\|\phi^h\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2+\|\rho_\epsilon-\sigma^h\|_Z|\phi^h|_1+\|p_\epsilon-q^h\||\phi^h|_1 \quad (5.11)$$

$$Re^{-1}\|\mathcal{D}(\phi^h)\|^2 + \sum_{i=1}^{2}\frac{\epsilon_i^{-1}}{2}\|\phi^h\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2$$
$$\leq Re^{-1}|\eta|_1^2 + N^h|\phi^h|_1|\eta|_1(|\mathbf{u}_\epsilon{}^h|_1 + |\mathbf{u}|_1) + \|p_\epsilon - q^h\|^2 + \frac{|\phi^h|_1^2}{4}$$
$$+\sum_{i=1}^{2}\frac{\epsilon_i^{-1}}{2}\|\eta\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2 + \|\rho_\epsilon - \sigma^h\|_Z^2 + \frac{|\phi^h|_1^2}{4}. \tag{5.12}$$

By Korn's inequality we have:

$$C_1\|\phi^h\|_1^2 \leq Re^{-1}\|\mathcal{D}(\phi^h)\|^2 + \sum_{i=1}^{2}\frac{\epsilon_i^{-1}}{2}\|\phi^h\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2$$
$$\leq Re^{-1}|\eta|_1^2 + (N^h|f|^*\alpha^{-1} + |\mathbf{u}_\epsilon|_1)(|\eta|_1^2 + \frac{|\phi^h|_1^2}{4}) + \frac{1}{C_1}\|p_\epsilon - q^h\|^2$$
$$+\sum_{i=1}^{2}\frac{\epsilon_i^{-1}}{2}\|\eta\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2 + \frac{1}{C_1}\|\rho_\epsilon - \sigma^h\|_Z^2 + C_1\frac{|\phi^h|_1^2}{2}. \tag{5.13}$$

where C_1 depends on Re, α, and ϵ

Since $N|f|^*\alpha^{-2} < 1$, we can choose h sufficiently small so that:

$$N^h|f|^*{}_h\alpha^{-1} < \alpha.$$

Using this we obtain:

$$C_1\|\phi^h\|_1^2 \leq Re^{-1}|\eta|_1^2 + (\alpha + |\mathbf{u}_\epsilon|_1)\frac{|\phi^h|_1^2}{4} + \frac{1}{C_1}\|p_\epsilon - q^h\|^2$$
$$+\sum_{i=1}^{2}\frac{\epsilon_i^{-1}}{2}\|\eta\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2 + \frac{1}{C_1}\|\rho_\epsilon - \sigma^h\|_0^2 + C_1\frac{|\phi^h|_1^2}{2}. \tag{5.14}$$

Thus we get:

$$C_2\|\phi^h\|_1^2 \leq C_3|\eta|_1^2 + \frac{1}{C_1}(\|p_\epsilon - q^h\|^2 + \|\rho_\epsilon - \sigma^h\|^2) + \sum_{i=1}^{2}\frac{\epsilon_i^{-1}}{2}\|\eta\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2 \tag{5.15}$$

where C_2 and C_3 are depend on Re, ϵ, α and $\mathbf{u}_\epsilon$ but not on h.

Adding and subtracting η in every norm of the left hand side of equation (5.15) yields:

$$\|\eta - \phi^h\|_1^2 \leq C_4 |\eta|_1^2 + C_5 \sum_{i=1}^{2} \epsilon_i^{-1} \|\eta \cdot \boldsymbol{\tau}_i\|_{0,\Gamma}^2 + C_6(\|\rho_\epsilon - \sigma^h\|_Z^2 + \|p_\epsilon - q^h\|^2).$$

Applying the triangle inequality and taking infima we have:

$$\begin{aligned} &\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^h\|_1^2 \leq \\ &\leq C_4 \inf_{0 \neq \mathbf{v}^h \in V^h} |\mathbf{u}_\epsilon - \mathbf{v}^h|_1^2 + C_5 \sum_{i=1}^{2} \epsilon_i^{-1} \inf_{0 \neq \mathbf{v}^h \in V^h} \|(\mathbf{u}_\epsilon - \mathbf{v}^h) \cdot \boldsymbol{\tau}_i\|_{0,\Gamma}^2 + \\ &+ C_6 (\inf_{0 \neq \sigma^h \in Z^h} \|\rho_\epsilon - \sigma^h\|_Z^2 + \inf_{0 \neq q^h \in Y^h} \|p_\epsilon - q^h\|^2), \end{aligned}$$

where, for ϵ_1, ϵ_2 small. Now we have to derive a bound for the error in the Lagrange multiplier. From Lemma 4.1 we take $\lambda_1^h = \lambda_2^h = 0$:

$$\begin{aligned} &\inf_{0 \neq \rho^h \in Z^h} \sup_{0 \neq \mathbf{v}^h \in X^h} \frac{\sum_{j=1}^{k} \langle \rho^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j}}{\|\mathbf{v}^h\|_1 \, \|\rho^h\|_Z} = \\ &\inf_{\substack{\rho^h \in Z^h \\ \lambda_1^h, \lambda_2^h \in Z^h}} \sup_{0 \neq \mathbf{v}^h \in X^h} \frac{\sum_{j=1}^{k} \langle \lambda_1^h, \mathbf{v}^h \cdot \boldsymbol{\tau}_1^{(j)} \rangle_{\Gamma_j} + \langle \lambda_2^h, \mathbf{v}^h \cdot \boldsymbol{\tau}_2^{(j)} \rangle_{\Gamma_j} + \langle \rho^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j}}{\|\mathbf{v}^h\|_1 \left[\|\lambda_1^h\|_Z^2 + \|\lambda_2^h\|_Z^2 + \|\rho^h\|_Z^2 \right]^{1/2}} \geq \overline{\beta}. \end{aligned} \tag{5.16}$$

Equation (5.16) implies that:

$$\overline{\beta} \left\| \rho_\epsilon^h - \sigma^h \right\|_Z \leq \sup_{\mathbf{v}^h \in X^h} \frac{\sum_{j=1}^{k} \langle \rho_\epsilon^h - \sigma^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j}}{\|\mathbf{v}^h\|_1}. \tag{5.17}$$

From (5.6) we have:

$$\sum_{j=1}^{k}\langle\rho_\epsilon^h-\sigma^h,\mathbf{v}^h\cdot\mathbf{n}_j\rangle_{\Gamma_j}=-a_0(\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h,\mathbf{v}^h)-b(\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h;\mathbf{u}_\epsilon,\mathbf{v}^h)-$$

$$-\,b(\mathbf{u}_\epsilon^h;\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h,\mathbf{v}^h)+\langle p_\epsilon-q^h,\nabla\cdot\mathbf{v}^h\rangle_\Omega+$$

$$+\sum_{i=1}^{2}\epsilon_i^{-1}\sum_{j=1}^{k}\langle(\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h)\cdot\boldsymbol{\tau}_i^{(j)},\mathbf{v}^h\cdot\boldsymbol{\tau}_i^{(j)}\rangle_{\Gamma_j}+\sum_{j=1}^{k}\langle\rho_\epsilon-\sigma^h,\mathbf{v}^h\cdot\mathbf{n}_j\rangle_{\Gamma_j}$$

$$\le 2Re^{-1}\left|\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h\right|_1\left|\mathbf{v}^h\right|_1+2\alpha\left|\mathbf{v}^h\right|_1\left|\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h\right|_1+\left\|p_\epsilon-q^h\right\|_0\left|\mathbf{v}^h\right|_1+$$

$$+\sum_{i=1}^{2}\epsilon_i^{-1}\|(\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h)\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}|\mathbf{v}^h|_1+\|\rho_\epsilon-\sigma^h\|_Z|\mathbf{v}^h|_1$$

$$\le (C|\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h|_1+\|p_\epsilon-q^h\|+\sum_{i=1}^{2}\epsilon_i^{-1}\|(\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h)\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}+\|\rho_\epsilon-\sigma^h\|_Z)|v^h|_1 \tag{5.18}$$

where $C:=2(\alpha+Re^{-1})$.

Thus dividing (5.18) by $\left|\mathbf{v}^h\right|_1$, combining it with (5.17), applying the triangle inequality and taking infima yields:

$$\left\|\rho_\epsilon-\rho_\epsilon^h\right\|_Z\le\frac{C}{\overline{\beta}}\left|\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h\right|_1+\frac{1}{\overline{\beta}}\inf_{q^h\in Y^h}\left\|p_\epsilon-q^h\right\|+$$
$$+\sum_{i=1}^{2}\frac{\epsilon_i^{-1}}{\overline{\beta}}\left\|(\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h)\cdot\boldsymbol{\tau}_i\right\|_{0,\Gamma}+\frac{1+\overline{\beta}}{\overline{\beta}}\inf_{\sigma^h\in Z^h}\left\|\rho_\epsilon-\sigma^h\right\|_Z. \tag{5.19}$$

Squaring both sides of (5.19) and bounding the mixed terms yields:

$$\|\rho_\epsilon-\rho_\epsilon^h\|_Z^2\le\frac{5C_*^2}{\overline{\beta}^2}\left|\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h\right|_1^2+\frac{5}{\overline{\beta}^2}\inf_{q^h\in Y^h}\left\|p_\epsilon-q^h\right\|^2+$$
$$+\sum_{i=1}^{2}\frac{5\epsilon_i^{-2}}{\overline{\beta}^2}\left\|(\mathbf{u}_\epsilon-\mathbf{u}_\epsilon^h)\cdot\boldsymbol{\tau}_i\right\|_{0,\Gamma}^2+\frac{10}{\overline{\beta}^2}\inf_{\sigma^h\in Z^h}\left(\|\rho_\epsilon-\sigma^h\|_Z\right)^2. \tag{5.20}$$

Combining (5.20) with (**??**) yields the following theorem for the total error.

Theorem 5.1 Assume that the discrete spaces satisfy condition (4.7). Then the error in the discrete solution of the penalty-Lagrange multiplier method is as follows:

$$\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^h\|_1^2 \le C_1 \inf_{0\neq \mathbf{v}^h \in V^h} \left|\mathbf{u}_\epsilon - \mathbf{v}^h\right|_1^2 + \sum_{i=1}^{2} C_2 \epsilon_i^{-1} \inf_{0\neq \mathbf{v}^h \in V^h} \left\|\left(\mathbf{u}_\epsilon - \mathbf{v}^h\right)\cdot \boldsymbol{\tau}_i\right\|_{0,\Gamma}^2$$
$$+ C_3 \left(\inf_{0\neq \sigma^h \in Z^h} \|\rho_\epsilon - \sigma^h\|_Z^2 + \inf_{0\neq q^h \in Y^h} \left\|p_\epsilon - q^h\right\|_0^2 \right). \tag{5.21}$$

Incorporating (3.6) into (5.21) yields the following corollary for the total error for the penalty-Lagrange multiplier method:

Corollary 5.1 Assume that the discrete spaces satisfy condition (4.7). Then the error in the discrete solution of the penalty-Lagrange multiplier method is as follows:

$$\left\|\mathbf{u} - \mathbf{u}_\epsilon^h\right\|_1^2 \le C_1 \inf_{\mathbf{v}^h \in V^h} \left|\mathbf{u} - \mathbf{v}^h\right|_{1,\Omega}^2 + \sum_{i=1}^{2} \frac{C_2}{\epsilon_i} \inf_{\mathbf{v}^h \in V^h} \left\|(\mathbf{u} - \mathbf{v}^h)\cdot \boldsymbol{\tau}_i\right\|_{0,\Gamma}^2$$
$$+ C_3 \left(\inf_{\sigma^h \in Z^h} \|\rho - \sigma^h\|_Z^2 + \inf_{q^h \in Y^h} \left\|p - q^h\right\|_0^2 \right)$$
$$+ C_1 \left|\mathbf{u} - \mathbf{u}_\epsilon\right|_{1,\Omega}^2 + \sum_{i=1}^{2} \frac{C_2}{\epsilon_i} \|(\mathbf{u} - \mathbf{u}_\epsilon)\cdot \boldsymbol{\tau}_i\|_{0,\Gamma}^2 + \tag{5.22}$$
$$C_3 \|\rho - \rho_\epsilon\|_Z^2 + \|p - p_\epsilon\|_0^2 + C_4 \sum_{i=1}^{2} \epsilon_i^2 \|\lambda_i\|_\Gamma^2 .$$

Proof Squaring both sides of (3.4) yields the following:

$$\|\mathbf{u} - \mathbf{u}_\epsilon\|_{1,\Omega}^2 + \sum_{i=1}^{2} \|\lambda_i - \lambda_i^\epsilon\|_Z^2 + \|\rho - \rho_\epsilon\|_Z^2 \le C_1 \sum_{i=1}^{2} \epsilon_i^2 \|\lambda_i\|_\Gamma^2 . \tag{5.23}$$

In (5.21) we replace the $\|.\|_{0,\Gamma}$ norm on the left hand side by $\|.\|_Z$, add $(-\mathbf{u}+\mathbf{u})$, $(-p+p)$, $(-\rho+\rho)$ in the appropriate terms of the right hand side and add it to (5.22) to get the stated result.

Remark: The error bound that includes the difference in the value of the stress vector between the true and the discrete penalty-Lagrange multiplier solutions is as follows:

$$\begin{aligned}
&\left\|\mathbf{u}-\mathbf{u}_\epsilon^h\right\|_1^2+\sum_{i=1}^{2}\|\lambda_i-\lambda_i^{\epsilon,h}\|_Z^2+\|\rho-\rho_\epsilon^h\|_Z^2\le\\
&\qquad\le\frac{1}{\min_i\epsilon_i}\left\{C_1\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1^2+\sum_{i=1}^{2}\frac{C_2}{\epsilon_i}\inf_{\mathbf{v}^h\in V^h}\left\|(\mathbf{u}-\mathbf{v}^h)\cdot\boldsymbol{\tau}_i\right\|_{0,\Gamma}^2\right\}+\\
&\qquad\left(C_3\min_i\epsilon_i+C_0\right)\left(\inf_{\sigma^h\in Z^h}\|\rho-\sigma^h\|_Z^2+\inf_{q^h\in Y^h}\left\|p-q^h\right\|_0^2\right)+\\
&\qquad C_1\left|\mathbf{u}-\mathbf{u}_\epsilon\right|_1^2+\left(C_3\min_i\epsilon_i+C_0\right)\left(\|\rho-\rho_\epsilon\|_Z^2+\|p-p_\epsilon\|_0^2\right)+\\
&\qquad\sum_{i=1}^{2}\frac{C_2}{\epsilon_i}\|(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2+C_4\sum_{i=1}^{2}{\epsilon_i}^2\|\lambda_i\|_\Gamma^2\,.
\end{aligned}$$

□

To obtain a bound on $\|p-p_\epsilon\|_0$ we use (1.12) with $\chi_i=\lambda_i-\lambda_i^\epsilon$, $\sigma=\rho-\rho_\epsilon$:

$$\begin{aligned}
&\beta\left(\|\rho-\rho_\epsilon\|_Z^2+\|p-p_\epsilon\|_0^2+\sum_{i=1}^{2}\|\lambda_i-\lambda_i^\epsilon\|_Z^2\right)^{1/2}\\
&\le\sup_{\mathbf{v}\in X}\frac{\langle p-p_\epsilon,\nabla\cdot\mathbf{v}\rangle_\Omega-\sum_{i=1}^{2}\sum_{j=1}^{k}\langle\lambda_i-\lambda_i^\epsilon,\mathbf{v}\cdot\boldsymbol{\tau}_i^{(j)}\rangle_{\Gamma_j}-\sum_{j=1}^{k}\langle\rho-\rho_\epsilon,\mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j}}{|||\mathbf{v}|||}\\
&\le\sup_{\mathbf{v}\in X}\frac{a_0(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{v})+b(\mathbf{u}-\mathbf{u}_\epsilon;\mathbf{u}_\epsilon,\mathbf{v})}{|||\mathbf{v}|||}\le C\left|\mathbf{u}-\mathbf{u}_\epsilon\right|_1,
\end{aligned}$$

where C depends on Re and α. Thus:

$$\|p-p_\epsilon\|_0^2\le\frac{C}{\beta^2}\left|\mathbf{u}-\mathbf{u}_\epsilon\right|_{1,\Omega}^2\,. \tag{5.24}$$

To get a bound on $\|(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2$ we set $\chi_i=(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\boldsymbol{\tau}_i^{(j)}$ in (2.8):

$$\|(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}\le\epsilon_i\,\|\lambda_i^\epsilon\|_{0,\Gamma}\,. \tag{5.25}$$

Add and subtract λ_i in the norm in the right hand side of (5.25) to yield:

$$\begin{aligned}\|(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\boldsymbol{\tau}_i\|_{0,\Gamma} &\leq \epsilon_i\left(\|\lambda_i\|_{0,\Gamma}+\|\lambda_i-\lambda_i^\epsilon\|_{0,\Gamma}\right)\\ &\leq \epsilon_i\left(\|\lambda_i\|_{\Gamma}+\|\lambda_i-\lambda_i^\epsilon\|_{0,\Gamma}\right),\end{aligned}$$

or

$$\sum_{i=1}^{2}\frac{C_2}{\epsilon_i}\|(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2 \leq \sum_{i=1}^{2} C_2\left(\|\lambda_i\|_{\Gamma}^2+\|\lambda_i-\lambda_i^\epsilon\|_{0,\Gamma}^2\right). \tag{5.26}$$

Now we must bound $\|\lambda_i-\lambda_i^\epsilon\|_{0,\Gamma}^2$. Take $\mathbf{v}=\mathbf{u}-\mathbf{u}_\epsilon$ in (3.7):

$$a_0(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon)-b(\mathbf{u}-\mathbf{u}_\epsilon;\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon)-\sum_{i=1}^{2}\epsilon_i\sum_{j=1}^{k}\langle\lambda_i-\lambda_i^\epsilon,\lambda_i^\epsilon\rangle_{\Gamma_j}=0.$$

Add $\sum_{i=1}^{2}\epsilon_i\sum_{j=1}^{k}\langle\lambda_i-\lambda_i^\epsilon,\lambda_i\rangle_{\Gamma_j}$ to both sides:

$$\sum_{i=1}^{2}\epsilon_i\|\lambda_i-\lambda_i^\epsilon\|_{0,\Gamma}^2 \leq \sum_{i=1}^{2}\epsilon_i\|\lambda_i\|_{\Gamma}\|\lambda_i-\lambda_i^\epsilon\|_Z + a_0(\mathbf{u}-\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon)+b(\mathbf{u}-\mathbf{u}_\epsilon;\mathbf{u}_\epsilon,\mathbf{u}-\mathbf{u}_\epsilon).$$

Definition of N and the Cauchy-Schwarz inequality:

$$\sum_{i=1}^{2}\epsilon_i\|\lambda_i-\lambda_i^\epsilon\|_{0,\Gamma}^2 \leq \sum_{i=1}^{2}\frac{\epsilon_i}{2}\left(\|\lambda_i\|_{1/2,\Gamma}^2+\|\lambda_i-\lambda_i^\epsilon\|_Z^2\right).$$

Thus overall:

$$\sum_{i=1}^{2}\frac{C_2}{\epsilon_i}\|(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\boldsymbol{\tau}_i\|_{0,\Gamma}^2 \leq \sum_{i=1}^{2} C_2\epsilon_i\left(\|\lambda_i\|_{\Gamma}^2+\|\lambda_i-\lambda_i^\epsilon\|_Z^2\right). \tag{5.27}$$

Under the approximation assumption:

$$\begin{aligned}\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1+\left(\inf_{q^h\in Y^h}\left\|p-q^h\right\|_0^2+\inf_{\rho^h\in Z^h}\|\rho-\rho^h\|_Z^2\right)^{1/2} &\leq\\ \leq Ch^k\max\left\{\|\mathbf{f}\|_{-1},\|\mathbf{u}\|_2\right\}, &\end{aligned} \tag{5.28}$$

where k is the degree of the polynomial space we are using. If we let $\epsilon=\epsilon_1=\epsilon_2$ then corollary 5.1 indicates that proper choice of ϵ in (5.22) is $\epsilon=h^{k/2}$.

Theorem 5.2 Assume that the discrete spaces satisfy condition (4.7) and let $\epsilon = h^{k/2}$. Then the error in the solution of the penalty-Lagrange multiplier method and the discrete solution of the penalty-Lagrange multiplier method is as follows:

$$\|\mathbf{u} - \mathbf{u}_\epsilon^h\|_1 \leq C\epsilon.$$

CASE II: Let $(\mathbf{u}, \lambda_1, \lambda_2, \rho, p)$ be the solution of (2.3) and $(\mathbf{u}^h, \lambda_1^h, \lambda_2^h, \rho^h, p^h)$ be the finite element approximation of (2.3), that is (4.3) then substracting (4.3) from (2.3) we get:

$$a_0(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{u}; \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{u} - \mathbf{u}^h; \mathbf{u}^h, \mathbf{v}^h) - \langle p - p^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega$$
$$+ \sum_{i=1}^{2} \sum_{j=1}^{k} \langle (\mathbf{u} - \mathbf{u}^h) \cdot \boldsymbol{\tau}_i^{(j)}, \mathbf{v}^h \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j} + \sum_{j=1}^{k} \langle \rho - \rho^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0, \quad (5.29)$$

for all $\mathbf{v}^h \in V^h$. We write:

$$\mathbf{u} - \mathbf{u}^h = (\mathbf{u} - \mathbf{v}^h) - (\mathbf{u}^h - \mathbf{v}^h),$$

with $\mathbf{v}^h$ being the best approximation of $\mathbf{u}$ in V^h. Let

$$\eta := \mathbf{u} - \mathbf{v}^h,$$

$$\phi^h := \mathbf{u}^h - \mathbf{v}^h.$$

Since $\Gamma^h = \Gamma$ error terms which come from approximating boundary vanish, then equation (5.29) becomes:

$$a_0(\eta - \phi^h, \mathbf{v}^h) + b(\mathbf{u}; \eta - \phi^h, \mathbf{v}^h) + b(\eta - \phi^h; \mathbf{u}^h, \mathbf{v}^h) - \langle p - p^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega = 0, \quad (5.31)$$

for all $\mathbf{v}^h \in V^h$.

Equation (5.31) can be rewritten as:

$$a_0(\phi^h, \mathbf{v}^h) + b(\mathbf{u}; \phi^h, \mathbf{v}^h) + b(\phi^h; \mathbf{u}^h, \mathbf{v}^h) = a_0(\eta, \mathbf{v}^h)$$
$$+ b(\mathbf{u}; \eta, \mathbf{v}^h) + b(\eta; \mathbf{u}^h, \mathbf{v}^h) + \langle p - p^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega \tag{5.32}$$

Setting $\mathbf{v}^h = \phi^h$ in (5.32), applying Cauchy-Schwartz inequality and definition of N^h. We obtain:

$$2Re^{-1}\|\mathcal{D}(\phi^h)\|^2 \leq 2Re^{-1}\|\mathcal{D}(\eta)\|\|\mathcal{D}(\phi^h)\| + 2N^h|f|_h^*\alpha^{-1}|\phi^h|_1|\eta|_1 +$$
$$N^h|f|_h^*\alpha^{-1}|\phi^h|_1^2 + \|p - p^h\||\phi^h|_1 \tag{5.33}$$

By Korn's inequality we have:

$$2Re^{-1}\|\mathcal{D}(\phi^h)\| \leq 2Re^{-1}\|\mathcal{D}(\eta)\| + 2\alpha\|\mathcal{D}(\eta)\| + \alpha\|\mathcal{D}(\phi^h)\| + \|p - p^h\|$$

Thus we get:

$$C_1\|\mathcal{D}(\phi^h)\| \leq C_2\|\mathcal{D}(\eta)\| + \|p - p^h\|. \tag{5.34}$$

where C_1 and C_2 are depend on Re, α,and $|f|^*$. Adding and substracting η from the left hand side of (5.34) we get:

$$\|\mathcal{D}(\mathbf{u} - \mathbf{u}^h)\| \leq (1 + C_1/C_2)\|\mathcal{D}(\mathbf{u} - \mathbf{v}^h)\| + 1/C_1\|p - p^h\|.$$

Taking infima we get:

$$\|\mathcal{D}(\mathbf{u} - \mathbf{u}^h)\| \leq (1 + C_1/C_2) \inf_{0 \neq \mathbf{v}^h \in V^h} \|\mathcal{D}(\mathbf{u} - \mathbf{v}^h)\| + 1/C_1 \inf_{0 \neq p^h \in Y^h} \|p - p^h\|. \tag{5.35}$$

Under the approximation assumption (5.30) we get $\|\mathbf{u} - \mathbf{u}^h\|_1 \sim h^k$. In order to bound $\|\mathbf{u} - \mathbf{u}_\epsilon^h\|_1$ besides (5.35) we also need a bound on $\|\mathbf{u}^h - \mathbf{u}_\epsilon^h\|_1$. For the reason we substract equation (5.3) from (4.3) then we set $\mathbf{v}^h = \mathbf{u}^h - \mathbf{u}^h_\epsilon$ and we obtain:

$$a_0(\mathbf{u}^h - \mathbf{u}^h_\epsilon, \mathbf{u}^h - \mathbf{u}^h_\epsilon) + b(\mathbf{u}^h - \mathbf{u}^h_\epsilon; \mathbf{u}^h_\epsilon, \mathbf{u}^h - \mathbf{u}^h_\epsilon) + \sum_{i=1}^{2} \epsilon_i \sum_{j=1}^{k} \langle \lambda_i^h, \lambda_i - \lambda_i^{\epsilon,h} \rangle_{\Gamma_j}$$
$$+ \sum_{j=1}^{k} \langle \rho^h - \rho^h_\epsilon, (\mathbf{u}^h - \mathbf{u}^h_\epsilon) \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0. \qquad (5.36)$$

From Lemma 3.1 we have:

$$e_\epsilon{}^h(\lambda_1^{\epsilon,h}, \lambda_2^{\epsilon,h}, \rho_\epsilon{}^h) \le \frac{2(Re^{-1} + \alpha)}{\widehat{\beta}} \left\| \mathbf{u}^h - \mathbf{u}^h_\epsilon \right\|_1 \qquad (5.37)$$

where

$$e_\epsilon{}^h(\lambda_1^{\epsilon,h}, \lambda_2^{\epsilon,h}, \rho^h_\epsilon) := \left[\|\lambda_1^h - \lambda_1^{\epsilon,h}\|_Z^2 + \|\lambda_2^h - \lambda_2^{\epsilon,h}\|_Z^2 + \|\rho^h - \rho^h_\epsilon\|_Z^2 \right]^{1/2}.$$

Since $\Gamma^h = \Gamma$ adding and substracting

$$\sum_{i=1}^{2} \epsilon_i \sum_{j=1}^{k} \langle \lambda_i, \lambda_i - \lambda_i^{\epsilon,h} \rangle_{\Gamma_j}, \text{and } \sum_{j=1}^{k} \langle \rho, (\mathbf{u}^h - \mathbf{u}^h_\epsilon) \cdot \mathbf{n}_j \rangle_{\Gamma_j}$$

in (5.36) gives:

$$a_0(\mathbf{u}^h - \mathbf{u}^h_\epsilon, \mathbf{u}^h - \mathbf{u}^h_\epsilon) + b(\mathbf{u}^h - \mathbf{u}^h_\epsilon; \mathbf{u}^h_\epsilon, \mathbf{u}^h - \mathbf{u}^h_\epsilon) \qquad (5.38)$$
$$+ \sum_{i=1}^{2} \epsilon_i \sum_{j=1}^{k} \langle \lambda_i, \lambda_i - \lambda_i^{\epsilon,h} \rangle_{\Gamma_j} + \sum_{j=1}^{k} \langle \rho - \rho^h_\epsilon, (\mathbf{u}^h - \mathbf{u}^h_\epsilon) \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0.$$

By the same argument as in the proof of Proposition 2.1 we obtain:

$$\|\mathbf{u}^h - \mathbf{u}^h_\epsilon\|_1 \le C_1 \left[\epsilon_1^2 \|\lambda_1\|_\Gamma^2 + \epsilon_2^2 \|\lambda_2\|_\Gamma^2 \right]^{1/2}, \qquad (5.39)$$

where C_1 depends on Re, α, $\widehat{\beta}$, and $|f|^*$. Then combining (5.37) and (5.39). We get:

$$\|\mathbf{u}^h - \mathbf{u}_\epsilon{}^h\|_1 \sim \epsilon$$

Thus from (5.35) and (5.38)

$$\|\mathbf{u} - \mathbf{u}_\epsilon{}^h\|_1 \sim h^k + \epsilon. \tag{5.40}$$

where C depends on C_1 and $\|\lambda_i\|_\Gamma$ for $i = 1, 2$. Thus (5.40) indicates that the proper choice of ϵ in (5.40) is $\epsilon = h^k$.

Theorem 5.3 Assume that the discrete spaces satisfy condition (4.7) and let $\epsilon = h^k$. Then the error in the solution of the penalty-Lagrange multiplier method and the discrete solution of the penalty-Lagrange multiplier method is as follows:

$$\|\mathbf{u} - \mathbf{u}_\epsilon^h\|_1 \leq C\epsilon.$$

Conclusion. Weak imposition of essential boundary conditions by penalty-Lagrange multipliers method lessens the ill-effects of domains with non-smooth boundaries. In this paper we have shown that the optimal order of convergence can be achieved if the computational boundary follows the real flow boundary exactly. We did not include numerical results because of the difficulty in implementation of the Lagrange multipliers method. Traditionally, Lagrange multipliers method is used as a model problem. However, we have done similar analysis by using the penalty- penalty method and verified our results numerically in [20].

Acknowledgments. I would like to thank Prof. W. J. Layton and Dr. T. Liakos for their valuable contributions to this paper.

References

[1] R. A. ADAMS *Sobolev Spaces.* Academic Press, 1975.

[2] T. LIAKOS. *Weak imposition of boundary conditions in the Stokes problem.* Ph.D. Thesis, Univ. of Pittsburgh, PA, 1999

[3] I. BABUŠKA. *The finite element method with lagrangian multipliers.* Numer. Math., **20** 1973.

[4] J. PITKÄRANTA. *The finite element method with lagrange multipliers for domain with corners.* Math. Comput., **37**:13-30 (1981).

[5] R. VERFÜRTH. *Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition.* Numer. Math., **50**:697-721 (1987).

[6] R. VERFÜRTH. *Finite element approximation of incompressible Navier-Stokes equations with slip boundary conditionII.* Numer. Math., **59**:615-636 (1991).

[7] W. J. LAYTON. *Weak Imposition of no-slip conditions in finite element methods.* Computers and Mathematics with Applications, **38**:129-142 (1999).

[8] V. GIRAULT, P.A. RAVIART. *Finite Element Approximation of The Navier-Stokes equations*, Lecture notes in Mathematics **v.749**, Springer-Verlag, Berlin, Hiedelberg, New York,1979, p.106.

[9] R. Temam. *Navier-stokes Equations: Theory and Numerical Analysis.* North-Holland, Amsterdam, 1984.

[10] P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations I.*, Volume**38** of Springer Tracts in Natural Philosophy. Springer, 1994.

[11] J. Gobert. *Sur une inégalité de coercivité,* J. Math. Anal. Appl. **36**:518-528 (1971).

[12] G. Fichera. *Existence Theorems in elasticity,* Volume IV. a/2 of Handbook de Physik. Springer, Berlin, 1972.

[13] V. Girault, P.A. Raviart. *Finite Element Methods for Navier-Stokes equations;* Theory and Algorithms. Springer Series in Computtaional Mathematics,**v.5**, Springer-Verlag, Berlin, Hiedelberg, New York,1986,p.108.

[14] J. Serrin *Mathematical Principles of Classical Fluid Mechanics.* Encyclopedia of Physics VII/1. Springer, Berlin, 1959.

[15] P. G. Ciarlet. *Finite Element Methods For Eliptic Problems.* North Holland, Amsterdam, 1978.

[16] I. Babuška. *The finite element method with penalty.* Math. Comp. **27**:221-228(1973).

[17] R. S. Falk, J. T. King. *A penalty and extrapolation method for the stationary Stokes equations.* SIAM J. Numer. Anal., **13** No.5:814-829(1976).

[18] G. F. Carey, R. Krishnan. *Continuation techniques for a penalty approximation of the Navier-Stokes equations.* Comput. Methods Appl. Mech. Engrg. **60** No.1:1-29(1987).

[19] D. Arnold,F. Brezzi ,and M. Fortin. *A stable element for the Stokes equations.*, Calcolo **21**:337-344(1984).

[20] A. Çağlar, V. John. *Weak imposition of boundary conditions for the Navier-Stokes equations by a Penalty-Penalty Method.* 2002.(submitted)