Linear Algebra Preliminary Exam
January 15 2023

1. Let A be a 3×2 matrix and B a 2×3 matrix, and suppose that
$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$
Prove that
$$BA = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}.$$
(Hint: Show that $(AB)^2 = 9AB$.)

2. Let A, B be 3×3 complex matrices such that $A, B \neq 0$ but $A^2 = B^2 = 0$. Show that A, B are similar matrices.

3. Let X be a finite dimensional real vector space and $P_1, P_2 \in L(X, X)$ are both projections. Suppose that $P_1 - P_2$ is also a projection, show that $R_{P_2} \subset R_{P_1}$. Here R_{P_1}, R_{P_2} are ranges of P_1, P_2 respectively.

4. Let A, B be a pair of linear operators on a finite dimensional complex vector space such that $A^2 = B^2 = I$. Prove that $AB - BA$ is invertible **if and only if** A, B have no common eigenvector.

5. Let A be an $n \times n$ complex invertible matrix such that $\|A\| = \|A^{-1}\| = 1$ (where $\|\cdot\|$ denotes the operator norm). Show that A is unitary.

6. Let N be an $n \times n$ complex nilpotent matrix. Show that there exists a **UNIQUE** nilpotent matrix M such that
$$(I + M)^2 = I + N.$$