1. Let $A$ be a $3 \times 2$ matrix and $B$ a $2 \times 3$ matrix, and suppose that

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$

Prove that

$$BA = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}.$$

(Hint: Show that $(AB)^2 = 9AB$.)

2. Let $A, B$ be $3 \times 3$ complex matrices such that $A, B \neq 0$ but $A^2 = B^2 = 0$. Show that $A, B$ are similar matrices.

3. Let $X$ be a finite dimensional real vector space and $P_1, P_2 \in L(X, X)$ are both projections. Suppose that $P_1 - P_2$ is also a projection, show that $R_{P_2} \subset R_{P_1}$. Here $R_{P_1}, R_{P_2}$ are ranges of $P_1, P_2$ respectively.

4. Let $A, B$ be a pair of linear operators on a finite dimensional complex vector space such that $A^2 = B^2 = I$. Prove that $AB - BA$ is invertible if and only if $A, B$ have no common eigenvector.

5. Let $A$ be an $n \times n$ complex invertible matrix such that $\|A\| = \|A^{-1}\| = 1$ (where $\|\cdot\|$ denotes the operator norm). Show that $A$ is unitary.

6. Let $N$ be an $n \times n$ complex nilpotent matrix. Show that there exists a UNIQUE nilpotent matrix $M$ such that

$$(I + M)^2 = I + N.$$