Linear Algebra Preliminary Exam

May 2022

- *Write each problem solution on a separate sheet of paper.*
- *Put down your reference code, page number, and the total number of pages on each sheet.*
- *Justify all steps in your arguments accurately, clearly, and completely.*
- *You can use theorems proved in class or in the textbooks (with proper explanation). If you use a statement from homework or tests, you need to provide a proof.*

Problem 1 Let l and m be linear functionals on a (not necessarily finite dimensional) linear space X such that for all $x \in X$, $l(x) = 0$ if and only if $m(x) = 0$. Show that there exists a nonzero scalar α such that $l = \alpha m$, i.e., $l(x) = \alpha m(x)$ for all $x \in X$.

Problem 2 Let $B \in \mathbb{C}^{n\times n}$ and $L : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ be the linear transformation $L(A) = BA$. Show that

$$\det L = (\det B)^n$$

Problem 3 Let A be an $n \times n$ real nilpotent matrix and $B = \sum_{k=0}^{2022} A^k$.

a) Show that B is invertible.

b) Show that BA is similar to A

Problem 4 Suppose that a matrix $A \in \mathbb{R}^{6\times 6}$ satisfies the following conditions:

- $\operatorname{rank}(A) > 3$
- A^3 is a projection, but $Ax \neq x$ for all nonzero vectors $x \in \mathbb{R}^6$
- There is an A-invariant decomposition $\mathbb{R}^6 = U \oplus V$ such that $\dim U = 2$ and the restriction $A|_U$ is orthogonal, and $\dim V = 4$ and the restriction $A|_V$ is nilpotent

Describe all possible Jordan forms of A (there is more than one) and prove that these are the only possibilities.

Problem 5 Let A be a linear operator on a complex finite-dimensional Euclidean space. Show that if $A^* + A = AA^*$, then A is normal.

Problem 6 Let $A \in \mathbb{C}^{n\times n}$ be such that for all $x \in \mathbb{C}^n$,

$$((A + A^*)x, x) \geq \alpha > 0,$$

where $(.,.)$ is the standard scalar product. Show that A is invertible and the spectral radius of A^{-1} satisfies $\rho(A^{-1}) \leq 2/\alpha$.