Linear Algebra Preliminary Exam

August 2022

- *Write each problem solution on a separate sheet of paper.*
- *Put down your reference code, page number, and the total number of pages on each sheet.*
- *Justify all steps in your arguments accurately, clearly, and completely.*
- *You can use theorems proved in class or in the textbooks (with proper explanation). If you use a statement from homework or tests, you need to provide a proof.*

Problem 1 For linear operators S and T on an n-dimensional complex linear space V, prove

$$\operatorname{rank}(S) + \operatorname{rank}(T) - n \leq \operatorname{rank}(ST) \leq \min\{\operatorname{rank}(S), \operatorname{rank}(T)\}.$$

Problem 2 Show that if A, B are commuting real matrices, then

$$\det(A^2 + B^2) \geq 0.$$

Problem 3 Let X be the linear vector space of all 3×3 complex matrices over the field of complex numbers, and let $T : X \to X$ be the linear map defined by

$$T(A) = AD + DA$$

where D is a diagonal matrix with numbers $1, 2, 3$ on the main diagonal. Find the minimal polynomial of T.

Problem 4 Let $A \in \mathbb{R}^{n\times n}$ be such that $A^4 = A$. Show that A is diagonalizable.

Problem 5 Let A be an $n \times n$ real matrix such that $A + A^*$ is positive definite. Show that

$$\det(A + A^*) \leq 2^n \det A.$$

Problem 6 Let $X = \mathbb{R}^n$ be a normed space with the standard $|.|_p$ norm where $p \in (1, \infty)$. Show that if $\frac{1}{p} + \frac{1}{q} = 1$ then $|y|_q = \max_{|x|_p=1} \sum_i x_i y_i$ for all $y \in X$.