Preliminary Exam in Analysis, May 2022

Problem 1. *Let $[0,1] \subset \mathbb{R}$ be the unit closed interval. For a continuous function $f : [0,1] \mapsto \mathbb{R}$ and $0 < \alpha < 1$ we define*

$$[f]_\alpha := \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha},$$

and

$$\|f\|_\alpha := \sup_{x\in[0,1]} |f(x)| + [f]_\alpha.$$

We now define

$$X_\alpha := \{f \in C^0([0,1]);\ \|f\|_\alpha < +\infty\}; \quad \forall f,g \in X_\alpha \quad d_\alpha(f,g) := \|f-g\|_\alpha.$$

(a) *Prove that the metric space (X_α, d_α) is complete. You do not need to prove that d_α is a metric.*
(b) *Let f_k be a bounded sequence in (X_α, d_α). Prove that there exists a subsequence of f_k which is uniformly converging. Prove moreover that the limit belongs to X_α.*
(c) *Let $f_0(x) = \sqrt{x}$. Prove that $f_0 \in X_\alpha$ if and only if $0 < \alpha \leq 1/2$.*

Problem 2. *Let (X,d) be a metric space and let $K \subset X$ be a compact set. Let for all $x \in X$:*

$$d(x,K) := \inf_{z\in K} d(x,z)$$

Prove that

$$\forall x,y \in X \quad |d(x,K) - d(y,K)| \leq d(x,y).$$

Problem 3. *Let S be the unit sphere in $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^3 function which vanishes on S. Assume that for a constant $c > 0$*

$$f(x) \geq c\,\mathrm{dist}^2(x,S)$$

where

$$\mathrm{dist}(x,S) := \inf_{y\in S} |x-y|.$$

Prove that for all $x_0 \in S$, $v \in \mathbb{R}^3$, we have

$$v \cdot D^2 f(x_0) v \geq 2c\,|v\cdot x_0|^2,$$

where $D^2 f = [\frac{\partial^2 f}{\partial x_i \partial x_j}]$ is the Hessian matrix of the 2nd derivatives.

Problem 4. (1) *Let $\vec{F}$ be a smooth vector field in $\mathbb{R}^n$. Let $B^n(x_0,r)$ denote the ball centered at $x_0 \in \mathbb{R}^n$ with radius $r > 0$, let $S^{n-1}(x_0,r)$ denote the sphere centered at x_0 of radius r, an let $\vec{n}$ be the outer unit normal in $S^{n-1}(x_0,r)$. Let $|B^n(x_0,r)|$ denote the n-dimensional volume of the ball $B^n(x_0,r)$. Prove that we have*

$$(div\,\vec{F})(x_0) = \lim_{r\to 0} \frac{1}{|B^n(x_0,r)|} \int_{S^{n-1}(x_0,r)} \langle \vec{F}(y), \vec{n}(y)\rangle\, d\sigma(y).$$

(2) *Given fixed unit vector $\nu \in \mathbb{R}^3$, let $D(x_0,r)$ be the 2-dimensional disk centered at x_0 with radius r and perpendicular to ν. Let $\vec{t}$ be the unit tangent vector to $\partial D(x_0,r)$. Prove that we have*

$$\langle (curl\,\vec{F})(x_0), \nu\rangle = \lim_{r\to 0} \frac{1}{\pi r^2} \int_{\partial D(x_0,r)} \langle \vec{F}(y), \vec{t}(y)\rangle\, ds(y).$$

Problem 5. *Let* $f\colon [a,b] \mapsto \mathbb{R}$ *be a Riemann integrable function. Define the coefficients*

$$a_n(f) = \int_a^b f(x)\sin(nx)\,dx.$$

Show that

$$\lim_{n\to\infty} a_n(f) = 0.$$

HINT: *First prove it for the characteristic function of an interval* $[\alpha,\beta] \subset [a,b]$. *Then prove it for a (finite) linear combination of characteristic functions of intervals (these are called simple functions). Since* f *is Riemmann integrable, given* $\epsilon > 0$ *use a lower sum to show that* f *can be approximated by a simple function*

$$g = \sum_{\textit{finite}} c_i \chi_{I_i} \le f$$

for certain numbers c_i *and intervals* I_i *in the following sense*

$$0 \le \int_a^b (f-g)\,dx < \epsilon.$$

Deduce the statements for f *from the statements for* g.

Problem 6. *Let* $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ *be a* C^1 *function such that* $f(0) = 0$. *Let* $G\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ *be a* C^1 *mapping such that* $G(0) = 0$ *and* $DG(0)$ *is invertible. Prove that there exists an open neighborhood* U *of the origin in* $\mathbb{R}^n$ *and a continuous mapping* $H\colon U \mapsto \mathbb{R}^n$ *such that*

$$f(x) = \langle G(x), H(x)\rangle = \sum_{i=1}^n G^i(x)H^i(x)$$

for every $x \in U$.

HINT: Use the inverse function theorem to reduce the problem to showing that any function h *which is* C^1 *and satisfies* $h(0) = 0$ *can be written in the form*

$$h(y) = \langle y, K(y)\rangle,$$

for a continuous mapping K *defined in a neighborhood of zero.*