Preliminary Exam in Analysis, May 2022

Problem 1. Let \([0,1] \subset \mathbb{R}\) be the unit closed interval. For a continuous function \(f : [0,1] \rightarrow \mathbb{R}\) and \(0 < \alpha < 1\) we define

\[
[f]_\alpha := \sup_{x,y \in [0,1], x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha},
\]

and

\[
\|f\|_\alpha := \sup_{x \in [0,1]} |f(x)| + [f]_\alpha.
\]

We now define

\(X_\alpha := \{ f \in C^0([0,1]); \|f\|_\alpha < +\infty \}; \forall f, g \in X_\alpha \; d_\alpha(f, g) := \|f - g\|_\alpha.\)

(a) Prove that the metric space \((X_\alpha, d_\alpha)\) is complete. You do not need to prove that \(d_\alpha\) is a metric.

(b) Let \(f_k\) be a bounded sequence in \((X_\alpha, d_\alpha)\). Prove that there exists a subsequence of \(f_k\) which is uniformly converging. Prove moreover that the limit belongs to \(X_\alpha\).

(c) Let \(f_0(x) = \sqrt{x}\). Prove that \(f_0 \in X_\alpha\) if and only if \(0 < \alpha \leq 1/2\).

Problem 2. Let \((X, d)\) be a metric space and let \(K \subset X\) be a compact set. Let for all \(x \in X:\)

\[d(x, K) := \inf_{z \in K} d(x, z)\]

Prove that

\[\forall x, y \in X \; |d(x, K) - d(y, K)| \leq d(x, y).\]

Problem 3. Let \(S\) be the unit sphere in \(\mathbb{R}^3\) and let \(f : \mathbb{R}^3 \rightarrow \mathbb{R}\) be a \(C^3\) function which vanishes on \(S\). Assume that for a constant \(c > 0\)

\[f(x) \geq c \text{dist}^2(x, S)\]

where

\[\text{dist}(x, S) := \inf_{y \in S} |x - y|,\]

Prove that for all \(x_0 \in S, v \in \mathbb{R}^3\), we have

\[v \cdot D^2 f(x_0) v \geq 2c |v \cdot x_0|^2,\]

where \(D^2 f = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]\) is the Hessian matrix of the 2nd derivatives.

Problem 4.

(1) Let \(\vec{F}\) be a smooth vector field in \(\mathbb{R}^n\). Let \(B^n(x_0, r)\) denote the ball centered at \(x_0 \in \mathbb{R}^n\) with radius \(r > 0\), let \(S^{n-1}(x_0, r)\) denote the sphere centered at \(x_0\) of radius \(r\), and let \(\vec{n}\) be the outer unit normal in \(S^{n-1}(x_0, r)\). Let \(|B^n(x_0, r)|\) denote the \(n\)-dimensional volume of the ball \(B^n(x_0, r)\). Prove that we have

\[
(div \vec{F})(x_0) = \lim_{r \rightarrow 0} \frac{1}{|B^n(x_0, r)|} \int_{S^{n-1}(x_0, r)} \langle \vec{F}(y), \vec{n}(y) \rangle \; d\sigma(y).
\]

(2) Given fixed unit vector \(\nu \in \mathbb{R}^3\), let \(D(x_0, r)\) be the 2-dimensional disk centered at \(x_0\) with radius \(r\) and perpendicular to \(\nu\). Let \(\vec{F}\) be the unit tangent vector to \(\partial D(x_0, r)\). Prove that we have

\[
(curl \vec{F})(x_0, \nu) = \lim_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{\partial D(x_0, r)} \langle \vec{F}(y), \tilde{t}(y) \rangle \; ds(y).
\]
Problem 5. Let \( f: [a, b] \to \mathbb{R} \) be a Riemann integrable function. Define the coefficients

\[
a_n(f) = \int_a^b f(x) \sin(nx) \, dx.
\]

Show that

\[
\lim_{n \to \infty} a_n(f) = 0.
\]

HINT: First prove it for the characteristic function of an interval \([\alpha, \beta] \subset [a, b]\). Then prove it for a (finite) linear combination of characteristic functions of intervals (these are called simple functions). Since \( f \) is Riemann integrable, given \( \epsilon > 0 \) use a lower sum to show that \( f \) can be approximated by a simple function

\[
g = \sum_{\text{finite}} c_i \chi_{I_i} \leq f
\]

for certain numbers \( c_i \) and intervals \( I_i \) in the following sense

\[
0 \leq \int_a^b (f - g) \, dx < \epsilon.
\]

Deduce the statements for \( f \) from the statements for \( g \).

Problem 6. Let \( f: \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \) function such that \( f(0) = 0 \). Let \( G: \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^1 \) mapping such that \( G(0) = 0 \) and \( DG(0) \) is invertible. Prove that there exists an open neighborhood \( U \) of the origin in \( \mathbb{R}^n \) and a continuous mapping \( H: U \to \mathbb{R}^n \) such that

\[
f(x) = \langle G(x), H(x) \rangle = \sum_{i=1}^n G^i(x)H^i(x)
\]

for every \( x \in U \).

HINT: Use the inverse function theorem to reduce the problem to showing that any function \( h \) which is \( C^1 \) and satisfies \( h(0) = 0 \) can be written in the form

\[
h(y) = \langle y, K(y) \rangle,
\]

for a continuous mapping \( K \) defined in a neighborhood of zero.