Preliminary Exam in Analysis, January 2023

Problem 1.

(1) Let $\vec{F}$ be a smooth vector field in $\mathbb{R}^n$. Let $B^n(x_0, r)$ denote the ball centered at $x_0 \in \mathbb{R}^n$ with radius $r > 0$, let $S^{n-1}(x_0, r)$ denote the sphere centered at x_0 of radius r, an let $\vec{n}$ be the outer unit normal in $S^{n-1}(x_0, r)$. Let $|B^n(x_0, r)|$ denote the n-dimensional volume of the ball $B^n(x_0, r)$. Use the Divergence Theorem to give a rigorous proof of the formula

$$(\operatorname{div} \vec{F})(x_0) = \lim_{r \to 0} \frac{1}{|B^n(x_0, r)|} \int_{S^{n-1}(x_0,r)} \langle \vec{F}(y), \vec{n}(y) \rangle \, d\sigma(y).$$

(2) Given fixed unit vector $\nu \in \mathbb{R}^3$, let $D(x_0, r)$ be the 2-dimensional disk centered at x_0 with radius r and perpendicular to ν. Let $\vec{t}$ be the unit tangent vector to $\partial D(x_0, r)$. Use Stokes' Theorem in $\mathbb{R}^3$ to give a rigorous proof of the formula

$$\langle (\operatorname{curl} \vec{F})(x_0), \nu \rangle = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{\partial D(x_0,r)} \langle \vec{F}(y), \vec{t}(y) \rangle \, ds(y).$$

Problem 2. Take for granted that the volume of $\mathbb{B}$, the unit ball in $\mathbb{R}^n$, is

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)},$$

where Γ is Euler's gamma function. Let $A \colon \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear mapping whose matrix with respect to the canoninal basis is $A = (a_{i,j})_{i,j=1}^n$. Compute the volume of the image of $\mathbb{B}$ under the mapping A

$$\operatorname{vol}(A(\mathbb{B}))$$

and use your formula to find the volume of the ellipsoid E given by

$$\frac{x_1^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} + \cdots + \frac{x_n^2}{\lambda_n^2} = 1.$$

Here $x = (x_1, x_1, \ldots, x_n)$ is a point in $\mathbb{R}^n$ and $\lambda_i \neq 0$ for $i = 1, \ldots, n$. Note that some of the λ_i could be negative.

Problem 3. For a fixed $r > 0$, find the supremum of the function

$$f(x, y, z) = \log x + 2 \log y + 3 \log z$$

on $D = \{(x, y, z) : x, y, z > 0 \text{ and } x^2 + y^2 + z^2 \leq 6r^2\}$. Then prove that for $a, b, c > 0$

$$ab^2c^3 \leq 108 \left(\frac{a + b + c}{6} \right)^6.$$

Hint: Lagrange multipliers.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be Hölder continuous with Hölder exponent $\alpha \in (0, 1]$, that is,

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq C$$

for some constant $C > 0$. Suppose that $\lim_{x\to\infty} |x|^\alpha f(x)$ and $\lim_{x\to-\infty} |x|^\alpha f(x)$ both exist, this is to say that f behaves like $1/|x|^\alpha$ at infinity. Define the inversion of f to be

$$g(x) = \begin{cases} f(1/x) & \text{if } \ x \neq 0, \\ 0 & \text{if } \ x = 0. \end{cases}$$

Prove that

(1) there exists some constant $K > 0$ such that $|g(x)| \le K|x|^\alpha$;

(2) $g \in C^{\alpha/3}(\mathbb{R})$.

Problem 5. Denote the set

$$K := \{f \in C^0([-1,1]) : f(-1) = 0,\ f(0) = -2\,f(1) = 1\}$$

and the function $\mathcal{F} : K \to \mathbb{R} \cup \{+\infty\}$

$$\mathcal{F}(f) := \sup_{x\neq y\in[-1,1]} \frac{|f(x)-f(y)|}{|x-y|^{1/2}} + \sqrt{\sup_{x\in[0,1]} |f(x)|} \in [0,\infty]$$

Show that their exists a function $\bar{f} \in K$ such that

$$\mathcal{F}(\bar{f}) = \inf_{f\in K} \mathcal{F}(f).$$

To show this, proceed as follows (some of the following statements need only a one line argument, others need a longer proof)

(1) Show that
$$\inf_{f\in K} \mathcal{F}(f) < \infty.$$

(2) Show that there is a sequence $(f_k)_{k\in\mathbb{N}} \subset K$ such that
$$\mathcal{F}(f_k) \xrightarrow{k\to\infty} \inf_{f\in K} \mathcal{F}(f).$$

(3) Show that there is a subsequence $(f_{k_i})_{i=0}^\infty$ that converges to some $\bar{f} \in K$ and this convergence is uniform in $[-1,1]$.

(4) Show that
$$\sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{1/2}} \le \liminf_{i\to\infty} \sup_{x\neq y} \frac{|f_{k_i}(x)-f_{k_i}(y)|}{|x-y|^{1/2}}.$$

(5) Conclude that $\bar{f}$ is the function we are looking for.

Problem 6. Let $A_{ij} : \mathbb{R}^n \to \mathbb{R}$ for $1 \le i,j \le N$ be a continuously differentiable $\mathbb{R}^{N\times N}$ matrix. Assume that at $x_0 \in \mathbb{R}^n$ the matrix $A(x_0) = (A_{ij}(x_0))_{i,j=1}^N$ has N distinct real eigenvalues, say

$$\mu_1 < \mu_2 < \ldots < \mu_N.$$

That is we assume we have

$$\det(A(x_0) - \mu_i I) = 0, \quad i = 1,\ldots,N,$$

where $I \in \mathbb{R}^{N\times N}$ denots the identity matrix.

Show that there exists a small open neighborhood $U \subset \mathbb{R}^n$ of x_0 and continuously differentiable maps $\lambda_i : U \to \mathbb{R}$, $i = 1,\ldots,N$, such that $\lambda_i(x_0) = \mu_i$ and for each $x \in U$ and each $i = 1,\ldots,N$ we have that $\lambda_i(x)$ is an eigenvalue of $A(x)$, that is

$$\det(A(x) - \lambda_i(x)I) = 0, \quad \forall\, i = 1,\ldots,N, \quad x \in U.$$

Hint: Implicit Function Theorem.