Preliminary Exam in Analysis, August 2022

Problem 1. *Let (X, d) be a complete metric space, and $f : X \to X$ be a function. For any $m \in \mathbb{N}$, let f^n denote the n-times iterated composition of the function f with itself. (It is defined recursively by $f^1 := f$, $f^n := f \circ f^{n-1}$ for $n \geq 2$.) Let*

$$\lambda := \liminf_{n\to\infty} \sup_{x,y\in X,\, x\neq y} \frac{d(f^n(x), f^n(y))}{d(x,y)}.$$

Prove that if $\lambda < 1$, then f admits a unique fixed point.

Problem 2. *Let $f, g \colon X \mapsto \mathbb{R}$ be bounded uniformly continuous real functions defined on the metric space (X, d). Prove that the product $x \mapsto f(x)g(x)$ is also uniformly continuous. Show also that the conclusion is false when the boundedness hypothesis is omitted.*

Problem 3. *Let $\{f_n\}_{n\in\mathbb{N}} : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions with*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} |f_n| < \infty.$$

Let

$$g_n(x) := \int_{\mathbb{R}} \eta(x - z)\, f_n(z)\, dz$$

for some $\eta \in C^\infty(\mathbb{R})$ and $\eta \equiv 0$ in $\mathbb{R}\backslash[-1, 1]$.

(1) *Show that the sequence $\{g_n\}_{n\in\mathbb{N}} : \mathbb{R} \to \mathbb{R}$ is uniformly bounded.*
(2) *Show that the sequence $\{g_n\}_{n\in\mathbb{N}} : \mathbb{R} \to \mathbb{R}$ is equicontinuous.*
(3) *Show that the sequence $\{g_n\}_{n\in\mathbb{N}}$ has a subsequence $\{g_{n_i}\}_{i\in\mathbb{N}}$ which is uniformly convergent in any finite interval (a, b)*

Problem 4. *Let $\mathbb{R}^{2\times 2}$ be the four-dimensional vector space of all 2×2 real matrices and define $f : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $f(A) = A^2$.*

(i) *Show that f has a local inverse near the point*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(ii) *Show that f does not have a local inverse near the point*

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Problem 5. *Let $X = \mathbb{R}^{n\times n}$ be the set of $n \times n$ square matrices $M = [M_{ij}]_{i,j=1}^n$. We define the following metric on X:*

$$\forall M = [M_{ij}],\, N = [N_{ij}] \in X \quad d(M, N) := \sum_{i=1}^n \sum_{j=1}^n |M_{ij} - N_{ij}|$$

Let

$$O(n) := \{R \in X; \quad R^T R = \mathrm{Id}\}$$

be the set of orthogonal matrices, where R^T denotes the transpose of R. Let also $f : X \to \mathbb{R}$ be defined by

$$\forall M \in X \quad f(M) = \sum_{i=1}^n \sum_{j=1}^n (M_{ij})^3.$$

Prove that

(a) *f is a continuous function on (X, d).*
(b) *There exists $R_{max} \in O(n)$ such that*

$$\forall R \in O(n) \quad f(R) \leq f(R_{max}).$$

Problem 6.

(1) *Let $\Omega \subset \mathbb{R}^2$ be an open set and $f \colon \Omega \mapsto \mathbb{R}$ be a continuous function. Suppose that for $x_0 \in \Omega$ there exists a sequence of cubes $Q_{\delta_n}(x_0)$ centered at x_0 with side length δ_n such that $\delta_n \to 0$ as $n \to \infty$ and*

$$\int_{Q_{\delta_n}(x_0)} f = 0$$

for all n. Give a very detailed proof ($\epsilon - \delta$ style) that $f(x_0) = 0$.

(2) *Use Fubini's thorem and (a) above to prove that for $f \in C^2(\mathbb{R}^2)$, we always have symmetric second order partial derivatives*

$$\frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial^2 f}{\partial y \, \partial x}.$$