MATH 240 Final Exam

December 12, 2025

Practice Problems – Solutions

1. The final exam covers the entire course (see list of topics in separate document for details). The focus will be on material covered after the second midterm (vector calculus).
2. There will be ten problems.
3. No calculators or cheat sheets allowed.
4. You may use formulas for the length of a circle, area of a disk, surface area of a sphere, volume of a ball, area of a triangle, parallelogram etc without proof. Where these apply, you can use them directly instead of solving integrals (and you can use them to find e.g. the area of an annulus or surface area of a hemi-sphere etc).
5. **Disclaimer:** Many problems admit multiple solution strategies. Unless you are explicitly instructed to use a particular strategy, you may choose whichever one you like best. In this sense, the solutions in this document may be incomplete and only provide one approach.
   There *shouldn't* be mistakes in the solutions, but this is not a guarantee. This document should be considered a draft – solutions are procedurally correct, but there may yet by typos or numerical errors.
   If you find one, please bring it to the attention of your instructor as soon as possible.

Problem 1

1. Find a scalar equation of the plane containing the points $P(1,0,0)$, $Q(0,2,0)$ and $R(2,2,3)$.
2. Find the area of the parallelogram spanned by the vectors $\vec{a} = \langle 1,1,1 \rangle$ and $b = \langle 1,1,0 \rangle$.
3. Find the volume of the parallellepiped spanned by $\vec{a}, \vec{b}$ and $\vec{c} = \langle 1,3,2 \rangle$.

Solution

1. We find two arbitrary vectors in the plane, for example

   $$\overrightarrow{PR} = \langle 1,2,3 \rangle, \quad \overrightarrow{QR} = \langle 2,0,3 \rangle \qquad \text{and their cross product} \quad \overrightarrow{QR} \times \overrightarrow{PR} = \langle -6,-3,4 \rangle.$$

   The equation is $-6x - 3y + 4z = -6$ where the -6 on the right hand side can be found by plugging the coordinates of any one of the three points into the equation and seeing what the value is.

2. The area is $\|\vec{a} \times \vec{b}\| = \|\langle -1,1,0 \rangle\| = \sqrt{2}$. *Since we are only interested in the length of the vector, it does not matter whether we take $\vec{a} \times \vec{b}$ or $\vec{b} \times \vec{a}$.*

3. The volume is

   $$\left|(\vec{a} \times \vec{b}) \cdot \vec{c}\right| = \left|\langle -1,1,0 \rangle \cdot \langle 1,3,2 \rangle\right| = |-1+3+0| = 2.$$

   Volumes are always positive, so we take the absolute value, so it does not matter whether we take $\vec{a} \times \vec{b}$ or $\vec{b} \times \vec{a}$. We an also decide freely which vectors are cross-producted and which one is dot-producted afterwards due to the triple product identities.

Problem 1

1. Find the distance of the point $P(4, 2, 3)$ from the plane $x + y - z = 0$.
2. Find the cosine of the angle between the planes $x + y + z = 1$ and $x - 2y + 2z = 0$.
3. Compute the vector projection $P_{\vec{w}}(\vec{v})$ of $\vec{v} = \langle 1, 0, 0 \rangle$ onto $\vec{w} = \langle 1, 1, 2 \rangle$.

Solution

1. The distance is
$$d = \frac{|4 + 2 - 3|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$
do not forget the absolute value – distances are always positive.

2. The angle between the planes is the angle between their normal vectors (and we take the acute angle – if necessary, one normal can be replaced by its negative).
$$\cos\alpha = \frac{\langle 1, 1, 1 \rangle \cdot \langle 1, -2, 2 \rangle}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{1 + (-2)^2 + 2^2}} = \frac{1 - 2 + 2}{\sqrt{3}\sqrt{9}} = \frac{1}{3\sqrt{3}}.$$
There is no nice formula for the angle – do not attempt to find α directly when not asked for it.

3. The vector projection is
$$P_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\,\vec{w} = \frac{1 + 0 + 0}{1^2 + 1^1 + 2^2}\,\langle 1, 1, 2 \rangle = \frac{1}{6}\,\langle 1, 1, 2 \rangle.$$
Do not confuse $\vec{v}, \vec{w}$ here! The projection onto $\vec{w}$ has to be parallel to $\vec{w}$.

Problem 2

1. Parametrize the ellipse $x^2 + \frac{y^2}{4} = 1$ in the form $\vec{r}(t) = \langle a \cos t, b \sin t \rangle$ for some a, b.

2. Compute the unit tangent $\vec{T}(t)$ to the ellipse.

3. Compute the curvature $\kappa(t) = \frac{\|\vec{r}''(t) \times \vec{r}'(t)\|}{\|\vec{r}'(t)\|^3}$ of the ellipse and write it as $\kappa(t) = \frac{d}{\left(e+f \cos^2(t)\right)^{3/2}}$ for some d, e, f. *For the cross product, consider the ellipse as a curve in the plane $z = 0$ in three dimensions.*

4. At which point(s) (x, y) is the curvature greatest?

Solution

1.
$$\vec{r}(t) = \langle \cos t,\ 2 \sin t \rangle \qquad \text{since} \quad \cos^2 + \frac{(2\sin)^2}{4} = \cos^2 + \frac{4}{4}\sin^2 = 1.$$

2.
$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle -\sin t,\ 2\cos t\rangle}{\sqrt{(-\sin)^2 + (2\cos)^2}} = \frac{\langle -\sin t,\ 2\cos t\rangle}{\sqrt{\sin^2 + 4\cos^2}} = \frac{\langle -\sin t,\ 2\cos t\rangle}{\sqrt{\sin^2 + 4\cos^2}} = \frac{\langle -\sin t,\ 2\cos t\rangle}{\sqrt{1 + 3\cos^2 t}}.$$

3. For this solution, we add a 'dummy' coordinate $z = 0$: $\vec{r}(t) = \langle \cos t, 2\sin t, 0\rangle$. This does not change the curvature – we are just considering the plane as the xy-coordinate plane in three-dimensional space.

   We could alternatively use the formula $\kappa = \|\vec{T}'\|/\|\vec{r}'\|$, but taking the derivative of $\vec{T}$ takes a long time since $\|\vec{r}'\|$ is not constant.[1] Now

$$\begin{aligned}
\kappa &= \frac{\|\vec{r}'' \times \vec{r}'\|}{\|\vec{r}'\|^3} \\
&= \frac{\|\langle -\sin t, 2\cos t, 0\rangle \times \langle -\cos t, -2\sin t, 0\rangle\|}{\left(1 + 3\cos^2 t\right)^{3/2}} \\
&= \frac{\|\langle 0, 0, 2(\sin^2 t + \cos^2 t)\rangle\|}{\left(1 + 3\cos^2 t\right)^{3/2}} \\
&= \frac{2}{\left(1 + 3\cos^2 t\right)^{3/2}}
\end{aligned}$$

4. Since the enumerator is constant, the curvature κ is largest where the denominator is smallest, which coincides with saying that $\cos^2 t$ has to be as small as possible.

   This occurs when $\cos^2 t = 0$, i.e. $\cos(t) = 0$ and therefore $\sin t = 1$ or $\sin t = -1$. Notably, we do not have to find t, as we have found the points $(0, -2)$ and $(0, 2)$.

[1] This approach is the best solution in the problem below though where $\|\vec{r}'\|$ is constant.

Problem 2

Consider the helix C parametrized by $\vec{r}(t) = \langle t,\, 2\cos t,\, 2\sin t\rangle$.

1. Compute unit tangent vector function $\vec{T}(t)$.
2. Compute the unit normal vector function $\vec{N}(t)$.
3. Compute the curvature function $\kappa(t)$.

Solution

1. $\vec{r}'(t) = \langle 1, -2\sin t, 2\cos t\rangle$ so $\|\vec{r}'(t)\| = \sqrt{1^2 + 4(\sin^2 + \cos^2)} = \sqrt{5}$. Hence $\vec{T}(t) = \frac{1}{\sqrt{5}}\langle 1, -2\sin t, 2\cos t\rangle$.
2. $\vec{T}'(t) = \frac{1}{\sqrt{5}}\langle 0, -2\cos t, -2\sin t\rangle$ and $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \langle 0, -\cos t, -\sin t\rangle$.
3. Since $\|\vec{r}'\|$ is constant, we can easily compute

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\left\|\frac{1}{\sqrt{5}}\langle 0, -2\cos t, -2\sin t\rangle\right\|}{\sqrt{5}} = \frac{\|\langle 0, -2\cos t, -2\sin t\rangle\|}{5} = \frac{2}{5}.$$

Problem 3

Compute $f(2,3)$ and use linear approximation to compute $f(2.1, 2.8)$ for $f(x,y) = \sqrt{4x^2 + 3y}$.

Solution

$f(2,3) = \sqrt{4 \cdot 4 + 3 \cdot 3} = \sqrt{16+9} = \sqrt{25} = 5$ and

$$f(x,y) \approx f(2,3) + \nabla f(2,3) \cdot \langle x-2, y-3 \rangle$$

much like for the 1d first order Taylor formula.[2] The gradient contains

$$\partial_x f(x,y) = \frac{4 \cdot 2x}{2\sqrt{4x^2+3y}} = \frac{4x}{\sqrt{4x^2+3y}}, \qquad \partial_x f(2,3) = \frac{8}{5}$$

and

$$\partial_y f(x,y) = \frac{3}{2\sqrt{4x^2+3y}}, \qquad \partial_y f(2,3) = \frac{3}{10}.$$

so

$$f(x,y) \approx 5 + \frac{1}{10}\langle 16, 3 \rangle \cdot \langle x-2, y-3 \rangle$$

and specifically

$$f(2.1, 2.8) \approx 5 + \frac{1}{10}\,(16 \cdot 0.1 - 3 \cdot 0.2) = 5 + \frac{16-6}{100} = 5 + \frac{10}{100} = 5 + \frac{1}{10} = 5.1.$$

[2] $f(x) \approx f(a) + f'(a)\,(x-a)$

Problem 3

Compute $f(2,1)$ and use linear approximation to compute $f(1.9,1.1)$ for $f(x,y) = ln\big(x+2y-3\big)$.
Here ln *denotes the natural logarithm.*

Solution

$$\begin{aligned}
f(2,1) &= ln(2+2-3) = ln(1) = 0 \\
\nabla f(x,y) &= \left\langle \frac{1}{x+2y-3}, \frac{2}{x+2y-3} \right\rangle \\
\nabla f(2,1) &= \langle 1, 2 \rangle \\
f(1.9,1.1) &\approx 0 + 1\cdot(1.9-2) + 2\cdot(1.1-1) = -0.1 + 0.2 = 0.1.
\end{aligned}$$

Problem 3

Verify that the point $P(1,1,1)$ lies on the surface given by $\sqrt{x}+y+z^2=3$ and compute the tangent plane at P.

Solution

We have $\sqrt{1}+1+1^2=1+1+1=3$, so the point lies on the surface.

$$f(x,y,z)=\sqrt{x}+y+z^2 \qquad \Rightarrow \quad \nabla f(x,y,z)=\left\langle \frac{1}{2\sqrt{x}},\ 1,\ 2z \right\rangle, \qquad \nabla f(1,1,1)=\langle 1/2,\ 1,\ 2\rangle.$$

The normal vector of the tangent place is given by the gradient:

$$\frac{1}{2}x+y+2z=\frac{1}{2}+1+2=3.5 \qquad \text{or} \quad x+2y+4z=7$$

where we find the right hand side by knowing that $(1,1,1)$ lies on the plane. *This can be written down in a number of alternative ways which are equivalent except in notation.*

Problem 3

Verify that the point $P(1, 1, 1)$ lies on the surface given by $xy^2z^3 = 1$ and compute the tangent plane at P.

Solution

$1 \cdot 1^2 \cdot 1^3 = 1$, so the point lies on the surface.

$$f(x, y, z) = xy^2z^3 \qquad \Rightarrow \quad \nabla f(x, y, z) = \langle y^2z^3,\ 2xyz^3,\ 3xy^2z^2 \rangle, \qquad \nabla f(1, 1, 1) = \langle 1, 2, 3 \rangle.$$

The plane is given by

$$x + 2y + 3z = 1 + 2 \cdot 1 + 3 \cdot 1 = 7.$$

Problem 4

Find the maximum values and minimum values that $f(x, y, z) = x^3 + y^3 + z^3$ takes on the sphere $x^2 + y^2 + z^2 = 1$.

Solution

The gradient $\nabla g(x, y) = (2x, 2y, 2z)$ of the constraint function does not vanish on the sphere since $\|\nabla g\|^2 = 4g = 4$, so we can use Lagrange multipliers:

$$3\big(x^2, y^2, z^2\big) = 2\lambda(x, y, z)$$

so we face a dichotomy for each coordinate: Either $x = 0$ or $x = \frac{2\lambda}{3}$, $y = 0$ or $y = \frac{2\lambda}{3}$ and $z = 0$ or $z = \frac{2\lambda}{3}$. Not knowing what λ is, it still tells us that $x, y \neq 0 \Rightarrow x = y$. Up to permutation of dimension, this leaves three possibilities:

1. $x = \pm 1$ and $y = z = 0$ with function value 1 or -1.
2. $x = y = \pm 1/\sqrt{2}$ and $z = 0$ with function value $\pm 2 \cdot 2^{-3/2} = \pm 2^{-1/2}$.
3. $z = y = z = \pm 1/\sqrt{3}$ with function value $\pm 3 \cdot 3^{-3/2} = 3^{-1/2}$.

The maximum and minimum values are therefore -1 and 1 (and we are not interested in enumerate all points).

Problem 4

Find the maximum values and minimum values that $f(x, y, z) = x^4 + y^4 + z^4$ takes on the sphere $x^2 + y^2 + z^2 = 1$.

Solution

The setup is the same as for the previous problem, but we end up with a dichotomy for coordinates $x = 0$ or $x^2 = \frac{\lambda}{2}$, and the same for the other coordinates. The sign of the coordinates is irrelevant for function outputs since x^4 is symmetric, and we are not interested in enumerating maximizers and minimizers. Hence the candidates are

1. $x = 1$ and $y = z = 0$ with function value 1.
2. $|x| = |y| = 1/\sqrt{2}$ and $z = 0$ with function value $2 \cdot 2^{-4/2} = 1/2$.
3. $|x| = |y| = |z| = 1/\sqrt{3}$ with function value $3 \cdot 3^{-4/2} = 1/3$.

The largest value that can be attained is therefore 1 and the lowest value is 1/3.

Problem 4

Find all critical points of the function $f(x, y) = x^4 - 2x^2 + y^2 + 1$ and decide whether they are local maxima, minima, or saddle points.

Solution

We compute

$$\nabla f(x,y) = \langle 4x^3 - 4x, 2y\rangle = \langle 4x(x^2-1),\ 2y\rangle f_{xx} \qquad = 12x^2 - 4 = 4(3x^2-1)$$
$$f_{xy} = 0$$
$$f_{yy} = 2.$$

Critical points require $f_y = 2y = 0$, so $y = 0$. Additionally, we need $\partial_x f = 4x(x^2-1) = 0$, so x is either $-1, 1$ or 0. We have

$$f_{xx} f_{yy} - f_{xy}^2 = 8(3x^2 - 1)$$

which is greater than zero for $x = 1$ and $x = -1$ but negative for $x = 0$. For $|x| = 1$, observe that $f_{xx} = 8 > 0$. Hence, there are three critical points: $(-1, 0)$ and $(0, 0)$ (local minima) and $(0, 0)$ (saddle point).

Problem 4

Find all critical points of the function $f(x, y) = x^4 - 2x^2 - 2y^2 + y^4 + 1$ and decide whether they are local maxima, minima, or saddle points.

Solution

We compute

$$\nabla f(x, y) = \langle 4x^3 - 4x, 4y^3 - 4y \rangle = \langle 4x(x^2 - 1),\ 4y(y^2 - 1) \rangle f_{xx} \qquad = 12x^2 - 4 = 4(3x^2 - 1)$$
$$f_{xy} = 0$$
$$f_{yy} = 4(y^2 - 1).$$

There are nine critical points, associated all possible matchings of $-1, 0, 1$ for both x and y. Compute

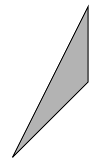
$$f_{xx} f_{yy} - f_{xy}^2 = 16(3x^2 - 1)(3y^2 - 1).$$

This is positive if $3x^2 - 1$ and $3y^2 - 1$ have the same sign, so at $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ where they are both positive, and at $(0, 0)$ where they are both negative. The origin is a local maximum, the four others are local minima. The remaining critical points $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$ are saddle points.

Problem 5

Compute the double integral $\iint_T x + y \, dA$ where T is the triangle with vertices $P(0,0)$, $Q(1,1)$ and $R(1,2)$.

Solution

Draw or visualize the triangle to find the correct integration bounds: 

$$\begin{aligned}
\iint_T x + y \, dA &= \int_0^1 \int_x^{2x} x + y \, dy \, dx \\
&= \int_0^1 x \cdot (2x - x) + \left. \frac{y^2}{2} \right|_{y=x}^{y=2x} dx \\
&= \int_0^1 x^2 + \frac{(2x)^2}{2} - \frac{x^2}{2} \, dx \\
&= \int_0^1 \frac{5}{2} x^2 \, dx \\
&= \frac{5}{6}.
\end{aligned}$$

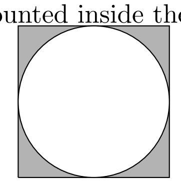
Problem 5

Compute the double integral $\iint_D x^2 + y^2 \, dA$ over the region which arises if the disk $x^2 + y^2 \leq 1$ is removed from the rectangle with vertices $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$.

Hint: It may be easier to write the integral over this region as the difference of two integrals.

Solution

We first integrate over the entire rectangle R, then we subtract the part which we shouldn't have counted inside the disk O.



$$\begin{aligned}
\iint_D x^2 + y^2 \, dA &= \iint_R x^2 + y^2 \, dA - \iint_O x^2 + y^2 \, dA \\
&= \int_{-1}^{1} \int_{-1}^{1} x^2 + y^2 \, dy \, dx - \int_0^1 \int_0^{2\pi} r^2 \, r \, d\theta \, dr \\
&= 4 \int_{-1}^{1} x^2 \, dx - 2\pi \int_0^1 r^3 \, dr \\
&= \frac{8}{3} - \frac{2\pi}{4} \\
&= \frac{8}{3} - \frac{\pi}{2}.
\end{aligned}$$

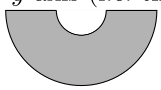
Problem 5

Find the mass and center of mass of the region D where $1/9 < x^2 + y^2 < 1$ and $y < 0$, assuming constant density 1.

Remark: One of the components of the center of mass is easy to find by symmetry.

Solution

The symmetry is only broken by the constraint $y < 0$, but it is symmetric under reflection across the y-axis (i.e. the line $x = 0$). Thus, we have $\bar{x} = 0$.



We get

$$m = \int_0^{\pi} \int_{1/3}^{1} r\, dr\, d\theta = \pi \frac{r^2}{2}\bigg|_{r=1/3}^{r=1} = \frac{\pi}{2}\left(1 - \frac{1}{9}\right) = \frac{\pi}{2}\frac{8}{9} = \frac{4\pi}{9}$$

$$\bar{y} = \int_0^{\pi} \int_{1/3}^{1} \underbrace{r\,\sin\theta}_{y}\, r\, dr\, d\theta = \frac{9}{4\pi}\left(\int_0^{\pi} \sin\theta\, d\theta\right)\left(\int_{1/3}^{1} r^2\, dr\right) = \frac{9}{4\pi} \cdot 2 \cdot \frac{r^3}{3}\bigg|_{r=1/3}^{r=1}$$

$$= \frac{3}{2\pi}\left(1 - \frac{1}{27}\right) = \frac{3 \cdot 26}{2 \cdot 27\,\pi} = \frac{78}{54\pi}.$$

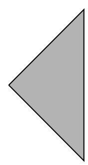
Recall that $r = \sqrt{x^2 + y^2}$, so the lower bound of $1/9$ on r^2 corresponds to a lower bound of $1/3$ on r itself.

Problem 5

Consider the triangle T with vertices $P(0,0)$, $Q(1,-1)$ and $R(1,1)$.

1. Is T a type I region? If so, write $\iint_T f\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ for suitable values of a, b and functions g_1, g_2.
2. Is T a type II region? If so, write $\iint_T f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$ for suitable values of c, d and functions h_1, h_2.
3. Compute $\iint f\,dA$ for $f(x,y) = x$.

Solution

Visualize T: 

This domain is both of type 1 and type 2:

$$\iint_T f\,dA = \int_0^1 \int_{-x}^{x} f(x,y)\,dy\,dx = \int_{-1}^{1} \int_{|y|}^{1} f(x,y)\,dx\,dy.$$

For $f(x,y) = x$, we have

$$\iint_T f\,dA = \int_0^1 \int_{-x}^{x} x\,dy\,dx = \int_0^1 2x^2\,dx = 1.$$

Problem 6

Compute the volume of the region which is both inside the sphere $x^2+y^2+z^2<2$ and the cylinder $x^2+y^2<1$.

Hint: Spherical coordinates do not work well here.

Solution

We use cylindrical coordinates.

$$\begin{aligned} m &= \int_0^1 \int_0^{2\pi} \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r\,dz\,d\theta\,dr \\ &= 2\pi \int_0^1 2\sqrt{2-r^2}r\,dr \\ &= -\frac{4\pi}{3}\int_0^1 \frac{d}{dr}\left(2-r^2\right)^{3/2} dr \\ &= \frac{4\pi}{3}\left(2^{3/2}-1^{3/2}\right). \end{aligned}$$

You can also solve this integral by u-substitution.

Problem 6

Compute the volume and center of mass of the region which is both inside the sphere $x^2+y^2+z^2 < 1$ and inside the cone $z > \sqrt{x^2+y^2}$, assuming constant density 1. *By symmetry, you may assume that* $\bar{x} = \bar{y} = 0$.

Solution

Since it is not given, we simply assume constant density 1 (i.e. mass = volume). We can compute in spherical polar coordinates

$$vol = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 r^2 \sin\phi \, dr\, d\phi\, d\theta = 2\pi\left(\int_0^1 r^2\, dr\right)\left(\int_0^{\pi/4} \sin\phi\, d\phi\right) = \frac{2\pi}{3}\,(-\cos(\pi/4) - (-\cos 0))$$
$$= \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{2\pi(\sqrt{2}-1)}{3\sqrt{2}}.$$

The unknown component of the center of mass is

$$\begin{aligned}
\bar{z} &= \frac{1}{m}\int_0^{2\pi}\int 0^{\pi/4}\int_0^1 \underbrace{r\cos\phi}_{z}\, r^2\sin\phi\, dr\, d\phi\, d\theta \\
&= \frac{3\sqrt{2}}{2\pi(\sqrt{2}-1)}\, 2\pi\left(\int_0^1 r^3\, dr\right)\left(\int_0^{\pi/4} \sin\phi\cos\phi\, d\phi\right) \\
&= \frac{3\sqrt{2}}{\sqrt{2}-1}\cdot\frac{1}{4}\int_0^{\pi/4} \frac{d}{d\phi}\frac{\sin^2\phi}{2}\, d\phi \\
&= \frac{3\sqrt{2}}{4(\sqrt{2}-1)}\cdot\frac{1}{2}\left(\sin^2(\pi/4) - \sin^2(0)\right) \\
&= \frac{3\sqrt{2}}{8(\sqrt{2}-1)}\left(\frac{1}{2} - 0\right) \\
&= \frac{3\sqrt{2}}{16(\sqrt{2}-1)}.
\end{aligned}$$

Unless it is specified differently, you could also use cylindrical coordinates here. As a rule of thumb, cylindrical coordinates will lead to integrals involving square roots, while spherical coordinates will lead to integrals involving trigonometric functions. Neither is always easier to solve.

Problem 6

Compute the mass and center of mass of the 'spherical cap' which is both inside the sphere $x^2 + y^2 + z^2 < 2$ above the plane $z = 1$, assuming constant density 1. *By symmetry, you may assume that $\bar{x} = \bar{y} = 0$.*

Hint: Spherical coordinates do not work well here.

Solution

Since it is not mentioned, we assume constant density 1.

We use cylindrical coordinates. The sphere $x^2 + y^2 + z^2 = 2$ intersects the plane $z = 1$ when $x^2 + y^2 + 1 = 2$, i.e. $r^2 = x^2 + y^2 = 1$. Therefore

$$\begin{aligned}
m &= \int_0^{2\pi} \int_0^1 \int_1^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta \\
&= 2\pi \int_0^1 \sqrt{2 - r^2}\, r - r \, dr \\
&= 2\pi \left(-\frac{1}{3} \int_0^1 \frac{d}{dr} (2 - r^2)^{3/2} \, dr - \frac{1}{2} \right) \\
&= 2\pi \left(\frac{2^{3/2} - 1}{3} - \frac{1}{2} \right) \\
&= 2\pi \frac{2^{5/2} - 2 - 3}{6} = 2\pi \, \frac{2^{5/2} - 5}{6}. \\
\bar{z} &= \frac{1}{m} \int_0^{2\pi} \int_0^1 \int_1^{\sqrt{2-r^2}} zr \, dz \, dr \, d\theta \\
&= \frac{6}{2^{5/2} - 5} \int_0^1 \left. \frac{z^2}{2} \right|_{z=1}^{z=\sqrt{2-r^2}} r \, dr \\
&= \frac{6}{2^{5/2} - 5} \int_0^1 \frac{2 - r^2 - 1}{2} r \, dr \\
&= \frac{3}{2^{5/2} - 5} \int_0^1 r - r^3 \, dr \\
&= \frac{3}{2^{5/2} - 5} \left(\frac{1}{2} - \frac{1}{4} \right) \\
&= \frac{3}{2^{5/2} - 5} \cdot \frac{1}{4} \\
&= \frac{3}{4(2^{5/2} - 5)}
\end{aligned}$$

Problem 6

Compute the integral $\iiint_D 1 + z\,dV$ where D is the hollowed hemi-sphere given by $1/16 < x^2 + y^2 + z^2 < 1$ and $z > 0$.

Solution

We work in spherical polar coordinates and for a general hollow hemi-sphere D_R given by $R^2 < x^2 + y^2 + z^2 < 1$.

$$\iiint_{D_R} 1\,dV = \int_R^1 \int_0^{2\pi} \int_0^{\pi/2} r^2 \sin\phi\,d\phi\,d\theta\,dr = \frac{1-R^3}{3} \cdot 2\pi \cdot \int_0^{\pi/2} \sin\phi\,d\phi = \frac{2\pi}{3}\,(1-R^3).$$

Additionally

$$\begin{aligned}\iiint_{D_R} z\,dV &= \int_R^1 \int_0^{2\pi} \int_0^{\pi/2} r\cos\phi \cdot r^2 \sin\phi\,d\phi\,d\theta\,dr = \frac{1-R^4}{4} \cdot 2\pi \cdot \int_0^{\pi/2} \sin\phi\cos\phi\,d\phi \\ &= \frac{1-R^4}{4} \cdot 2\pi \int_0^{\pi/2} \frac{d}{d\phi}\frac{\sin^2\phi}{2}\,d\phi = 2\pi\,\frac{1-R^4}{8}.\end{aligned}$$

and overall

$$\iiint_{D_R} 1 + z\,dV = 2\pi\left(\frac{1-R^3}{3} + \frac{1-R^4}{8}\right) = 2\pi\left(\frac{8-8R^3+3-3R^4}{24}\right) = \frac{11-8R^3-3R^4}{12}\,\pi.$$

Specifically for $R = 1/4$, this yields

$$\iiint_{D_{1/4}} 1 + z\,dV = \frac{11 - 1/8 - 3/256}{12}\pi.$$

Problem 7

1. Is the vector field $\vec{F}(x, y, z) = \langle yz, xz, xy \rangle$ conservative? If so, find a potential function.
2. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the straight line segment from $P(1, 0, 2)$ to $Q(4, 5, 1)$.

Solution

1. We compute the curl $\nabla \times \vec{F} = \langle x - x, y - y, z - z \langle 0, 0, 0 \rangle$, so we know that $\vec{F}$ is conservative. We can therefore find a potential function:

$$\begin{aligned} \partial_x f &= yz && \Rightarrow && f = xyz + g(y, z) \\ \partial_y f &= xz && \Rightarrow && f = xyz + h(x, z) \\ \partial_z f &= xy && \Rightarrow && f = xyz + k(x, y) \end{aligned}$$

   for some functions g, h, k of two variables. We can match this by choosing $g = h = k$ to just be the zero function, so $f(x, y, z) = xyz$ is a potential function of f.

2. $\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P) = 4 \cdot 5 \cdot 1 - 1 \cdot 0 \cdot 2 = 20 - 0 = 20$ by the fundamental theorem of line integrals.

Problem 7

Compute the line integral $\int_C f\,ds$ where C is the helix segment $\vec{r}(t) = (2\cos t, 2\sin t, t)$ with t between 0 and 5 and $f(x, y, z) = 1 + z$.

Solution

We have $z(t) = t$ and

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 1\rangle, \qquad \|\vec{r}\,'(t)\| = \sqrt{(2\sin)^2 + (2\cos)^2 + 1} = \sqrt{4+1} = \sqrt{5}.$$

Thus

$$\int_C f\,ds = \int_0^5 f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt = \int_0^5 (1+t)\sqrt{5}\,dt = \sqrt{5}\left(5 + \frac{25}{2}\right).$$

Problem 7

1. Is the vector field $\vec{F}(x, y, z) = \langle x + y, 0, 0 \rangle$ conservative?
2. If so, find a potential function. If not, explain why not.
3. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the straight line segment from $P(1, 0, 2)$ to $Q(4, 5, 1)$.

Solution

The vector field $\vec{F}$ is not conservative since

$$\nabla \times \langle x + y, 0, 0 \rangle = \langle 0, \partial_z(x + y), \partial_y(x + y) \rangle = \langle 0, 0, 1 \rangle,$$

but conservative vector fields have curl $\vec{0}$ at all points.

To compute the line integral, we need to go back to the definition:

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle x(t) + y(t), 0, 0 \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

where

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle 1 + 3t, 5t, 2 - t \rangle$$

parametrizes the line segment. Hence

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \langle 1 + 8t, 0, 0 \rangle \cdot \langle 3, 5, -1 \rangle \, dt \\ &= 3 \int_0^1 1 + 8t \, dt = 3 \left(1 + \frac{8}{2} \right) = 3 \cdot 5 = 15.\end{aligned}$$

Problem 7

Compute the line integrals $\int_C x\,ds$ and $\int_C x\,dy$ where C is the segment of the circle connecting $(1,0)$ to $(0,1)$.

Remark: If you do not know how to solve an integral, give the integral as answer for partial credit.

Solution

Parametrize the semi-circle by $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle \cos t, \sin t\rangle$ with t between 0 and π. Then

$$\int_C x\,ds = \int_0^\pi x(t)\,\|\vec{r}\,'(t)\|\,dt = \int_0^\pi \cos t\cdot 1\,dt = \sin\pi - \sin 0 = 0$$
$$\int_C x\,dy = \int_0^\pi x(t)\,\frac{dy}{dt}\,dt = \int_0^\pi x(t)y'(t)\,dt = \int_0^\pi \cos^2 t\,dt = \frac{\pi}{2}.$$

An anti-derivative of $\cos^2$ *is*

$$F(t) = \frac{t+\sin t\cos t}{2} \qquad \mathit{since} F'(t) = \frac{1+\cos^2 t - \sin^2 t}{2} = \frac{1-\sin^2+\cos^2}{2} = \frac{2\cos^2}{2} = \cos^2,$$

but you can argue also simply that

$$\int_0^\pi \sin^2 + \cos^2\ dt = \int_0^\pi 1\,dt = 1$$

and $\int_0^\pi \sin^2\ dt = \int_0^\pi \cos^2\ dt$ *by symmetry – this works over any sector of the unit circle (but not every interval of length* $\pi/2$*). It does work for any interval of length* π *since* $\sin^2$ *and* $\cos^2$ *are* π*-periodic.*

Problem 8

Evaluate the surface integral $\iint\limits_S f\, dS$ where S is the ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 1$ and

$$f(x, y, z) = \frac{z^2}{\sqrt{1 + 3x^2 + 3y^2}}.$$

One strategy to solve this is imitating spherical polar coordinates to parametrize the ellipsoid.

Solution

We parametrize[3]

$$\begin{aligned} x &= \sin\phi\, \cos\theta \\ y &= \sin\phi\, \sin\theta \\ z &= 2\cos\phi \end{aligned}$$

and compute the surface element

$$\begin{aligned} \vec{r}_\phi \times \vec{r}_\theta &= \langle \cos\phi\, \cos\theta, \cos\phi\, \sin\theta,\, -2\sin\phi\rangle \times \langle -\sin\phi\, \sin\theta,\, \sin\phi\, \cos\theta, 0\rangle \\ &= \langle 2\sin^2\phi\, \cos\theta,\, 2\sin^2\phi \sin\theta,\, \sin\phi\, \cos\phi(\cos^2\theta + \sin^2\theta)\rangle \\ &= \sin\phi\, \langle 2\sin\phi\, \cos\theta,\, 2\sin\phi\, \sin\theta,\, \cos\phi\rangle \\ \|\vec{r}_\phi \times \vec{r}_\theta\| &= \sin\phi \sqrt{(2\sin\phi)^2\cos^2\theta + (2\sin\phi)^2\sin^2\theta + \cos^2\phi} \\ &= \sin\phi\, \sqrt{4\sin^2\phi + \cos^2\phi} \\ &= \sin\phi\, \sqrt{3\sin^2\phi + 1} \end{aligned}$$

Additionally

$$f(x, y, z) = \frac{4\cos^2\phi}{\sqrt{1 + 3\sin^2\phi}}$$

and thus

$$\begin{aligned} \iint\limits_S f\, dS &= \int_0^\pi \int_0^{2\pi} \frac{4\cos^2\phi}{\sqrt{1 + 3\sin^2\phi}} \sin\phi\, \sqrt{3\sin^2\phi + 1}\, d\theta\, d\phi \\ &= 8\pi \int_0^\pi \cos^2\phi\, \sin\phi\, d\phi \\ &= -\frac{8\pi}{3} \int_0^\pi \frac{d}{d\phi} \cos^3\phi\, d\phi \\ &= -\frac{8\pi}{3}\, (-1 - 1) \\ &= \frac{16\pi}{3}. \end{aligned}$$

[3] Compare the parametrization of the ellipse in problem 2.

Problem 8

Evaluate the surface integral $\iint\limits_S f\, dS$ where the surface S is the graph of the function $g(x,y) = xy$ over the disk $x^2 + y^2 < 4$ and $f(x,y,z) = 1$.

Solution

We can compute the surface element 'by hand' for the parametrization $\vec{r}(x,y) = \langle x, y, g(x,y)\rangle$, but we also have the simpler formula

$$\iint\limits_S f\, dS = \iint_D f(x,y)\sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2}\, dA$$

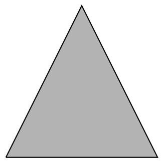
since the (upper) normal direction is given by $\langle -g_x, -g_y, 1\rangle$. Thus

$$\begin{aligned}\iint\limits_S f\, dS &= \iint_D 1\sqrt{1 + y^2 + x^2}\, dA \\ &= \int_0^1 \int_0^{2\pi} \sqrt{1 + r^2}\, r\, d\theta\, dr \\ &= 2\pi \int_0^1 \frac{1}{3}\frac{d}{dr}(1 + r^2)^{3/2}\, dr \\ &= \frac{2\pi}{3}\left(2^{3/2} - 1\right).\end{aligned}$$

Problem 8

Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = (\sin(x^2) + y)\vec{\imath}$ and C is made up of three straight line segments: From $(0, 0)$ to $(2, 0)$, from $(2, 0)$ to $(1, 2)$, and from $(1, 2)$ to $(0, 0)$.

Solution

Visualize/draw: 

This us the positively oriented boundary curve of a triangle T, so we can simply apply Green's Theorem:

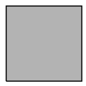
$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_C (\sin(x^2) + y)\, dx \\ &= \iint_T \partial_x 0 - \partial_y (\sin(x^2) + y)\, dA \\ &= -\iint_T 1\, dA \\ &= -2\end{aligned}$$

since the area of the triangle is $\frac{1}{2} 2 \cdot 2 = 2$. *Obviously, if $\partial_x Q - \partial_y P$ is not constant, you may need to compute the double integral here.*

Problem 8

Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = (e^x - e^y)\vec{\jmath}$ and C is made up of four straight line segments: From $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(1, 1)$, from $(1, 1)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

Solution

Visualize/draw:  Again, we have a positively oriented boundary curve and can use Green's Theorem, this time with the square Q:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint_Q \partial_x (e^x - e^y) \, dA \\
&= \int_0^1 \int_0^1 e^x \, dy \, dx \\
&= \int_0^1 e^x \, dx \\
&= e - 1.
\end{aligned}$$

Bonus/challenge problem

Fix some radius $R > 0$ and take C the circle of radius R around the origin, oriented counterclockwise. For the vector field $\vec{F}(x, y) = \frac{-y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j}$:

- Working from the definition, show that $\int_C \vec{F} \cdot d\vec{r} = 2\pi$ independently of R.
- Try to compute the same integral using Green's Theorem. You should get a different result.
- Explain why Green's Theorem does not apply here.

Solution

Working from the definition

$$\int_C \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt,$$

we find that

$$\int_C \vec{F} \cdot d\vec{r} = \frac{1}{R^2} \int_C \langle -y, x\rangle \cdot d\vec{r} = \frac{1}{R^2} \int_0^2 \pi \langle -R\sin t, R\cos t\rangle \cdot \langle -R\sin t, R\cos t\rangle\,dt = \frac{R^2}{R^2} \int_0^{2\pi} 1\,dt = 2\pi.$$

On the other hand, we have

$$\begin{aligned}
\partial_x \frac{x}{x^2+y^2} - \partial_y \frac{-y}{x^2+y^2} &= \partial_x \frac{x}{x^2+y^2} + \partial_y \frac{y}{x^2+y^2} \\
&= \frac{1}{x^2+y^2} + x\partial_x (x^2+y^2)^{-1} + \frac{1}{x^2+y^2} + y\partial_y (x^2+y^2)^{-1} \\
&= \frac{2}{x^2+y^2} - 2x^2(x^2+y^2)^{-2} - 2y^2(x^2+y^2)^{-2} \\
&= \frac{2}{x^2+y^2} - 2\frac{x^2+y^2}{(x^2+y^2)^2} \\
&= \frac{2}{x^2+y^2} - \frac{2}{x^2+y^2} \\
&= 0
\end{aligned}$$

at all points (x, y) *except* at the origin, where we cannot cancel two infinite terms. A naive use of Green's theorem would suggest that the integral should be zero – this is not true since our circle contains a point where $\vec{F}$ blows up due to division by 0.

Green's theorem is applicable though and yields integral 0 for all curves which do not wind around the origin.

Problem 9

An ellipsoid is parametrized by $\vec{r}(\theta, \phi) = \langle \sin\phi \cos\theta, \sin\phi \sin\theta, 2\cos\phi \rangle$. A curve C on the ellipsoid in physical space corresponds to the curve in parameter space going along straight line segments from $(\theta_0, \phi_0) = (0, 0)$ to $(\theta_1, \phi_1) = (0, \pi/2)$, from here to $(\theta_2, \phi_2) = (\pi/2, \pi/2)$, and finally to $(\theta_3, \phi_3) = (\pi/2, 0)$.

1. Argue that C is in a closed curve in physical space.

2. Use Stokes' Theorem to compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle x + 1, 2y + 3, 4z + 5 \rangle$.

Solution

1. The curve is closed since $\vec{r}(0, 0) = (0, 0, 1) = \vec{r}(\pi/2, 0)$: When $\phi = 0$, it does not matter what θ is. If we wanted to, we could add an additional straight line segment from (θ_3, ϕ_3) back to (θ_3, ϕ_4) in the parameter domain, which would just collapse to a single point in physical space.[4]

2. First things first, compute the curl:

$$curl(\vec{F}) = \langle 0, 0, 0 \rangle.$$

We can 'fill in' the curve by the segment S of the sphere given by $0 < \phi < \pi/2$ and $0 < \theta < \pi/2$ and use Stokes' Theorem to see that

$$\int C\vec{F} \cdot d\vec{r} = \iint_S curl(\vec{F}) \cdot d\vec{S} = \iint_S \langle 0, 0, 0 \rangle \cdot d\vec{S} = 0.$$

Conveniently, we do not have to parametrize anything here – it does not matter so much how we fill in the curve, just that we can do so.

This example could also be treated by the fundamental theorem of line integrals: Since $curl(\vec{F}) = \vec{0}$ *everywhere, the field* $\vec{F}$ *is conservative. The line integral along the closed curve* C *is therefore* 0. *Specifically, we have*

$$\vec{F} = \nabla \left(\frac{x^2}{2} + x + y^2 + 3y + 2z^2 + 5z \right).$$

The fundamental theorem and Stokes' Theorems generalize in different directions from this point:

- *The fundamental theorem of line integrals works also for curves which are not closed (integral not 0, but difference of $f(endpoint) - f(initial\ point)$).*

- *Stokes' Theorem, like Green's Theorem, requires closed curves (or even curves that are the boundary of a surface), but can handle vector fields that are not conservative.*

[4] This sort of collapse can be a problem in practice.

Problem 9

Consider the curve C on the hyperbolic paraboloid $z = x^2 - y^2$ which is parametrized by $0 \leq t < 2\pi$ using $\vec{r}(t) = \langle \cos t, \sin t, \cos^2 t - \sin^2 t \rangle = \langle \cos t, \sin t, \cos 2t \rangle$. Use Stokes' Theorem to compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x, y, z) = \langle x, x, z \rangle$.

Solution

$$curl(\vec{F}) = \langle 0, 0, 1 \rangle,$$

so this time we do have to compute an integral.

The paraboloid is already given to us as a graph. The curve C is the boundary of the curved 'disk' S on the paraboloid given by $x^2 + y^2 < 1$ and $z = x^2 - y^2$, which we can use to 'fill in' the curve. It is parametrized by the graph of $g(x, y) = x^2 - y^2$ over the disk D in two dimensions given by $x^2 + y^2 < 1$.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S curl(\vec{F}) \cdot d\vec{S} = \iint_S \langle 0, 0, 1 \rangle \cdot d\vec{S} = \iint_D \langle 0, 0, 1 \rangle \cdot \langle -g_x, -g_y, 1 \rangle \, dA = \iint_D 1 \, dA = \pi.$$

This is the formula for graphs, but you can also derive it from the parametrization $\vec{r}(x, y) = \langle x, y, g(x, y) \rangle$ if you prefer.

This integral is easy to solve directly and does not require Stokes' Theorem. This is by design to make sure the computations are easy – still, if you solve it another way, you will not receive points. You can make computing the curl a little easier by noticing that

$$\vec{F} = \nabla \frac{x^2 + z^2}{2} + \langle 0, x, 0 \rangle$$

and throwing away the gradient which is guaranteed to have curl $\vec{0}$ anyways.

Problem 9

Consider the curve parametrized by $\vec{r}(t) = \langle \cos t,\ \sin t,\ g(\cos t, \sin t)\rangle$ where $g(x, y) = xy - x - y$. Use Stokes' Theorem to compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle 2\cos(x^2 + z^2)\, x,\ -x, 2\cos(x^2 + z^2)\, z\rangle$.

Solution

$$curl(\vec{F}) = \langle 0, 0, -1\rangle.$$

You can simplify the computation by noticing that $\vec{F} = \nabla \sin(x^2 + z^2) + \langle 0, -x, 0\rangle$. Once again, we can fill in the curve by the piece of the graph above the disk $x^2 + y^2 < 1$, and we have

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S curl(\vec{F}) \cdot d\vec{S} = \iint_D \langle 0, 0, -1\rangle \cdot \langle -g_x, -g_y, 1\rangle\, dA = -\iint_D 1\, dA = -\pi.$$

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x, y, z) = \langle x, 2y, 3z \rangle$$

where S is the boundary surface of the hemisphere $x^2 + y^2 + z^2 < 1$ and $z > 0$, oriented by the exterior unit normal.

Solution

Write E for the hemisphere. We have $div(\langle x, 2y, 3z \rangle) = 1 + 2 + 3 = 6$. Thus

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div(\vec{F})\, dV = \iiint_E 6\, dV = 6 \cdot \frac{2}{3}\pi = 4\pi$$

since the volume of the hemisphere is $\frac{1}{2} \cdot \frac{4\pi}{3} = \frac{2\pi}{3}$.

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x, y, z) = \langle x + 2,\ 3,\ z + z^2 \rangle$$

where S is the boundary surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

Solution

We write T for the tetrahedron. We have $div(\vec{F}) = 1 + 0 + (1 + 2z) = 2(1 + z)$, so

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= 2 \iiint_T 1 + z \, dV \\
&= 2 \int_0^1 \int_0^{1-z} \int_0^{1-z-y} 1 + z \, dx \, dy \, dz \\
&= 2 \int_0^1 \int_0^{1-z} (1 + z)(1 - z - y) \, dy \, dz \\
&= 2 \int_0^1 \int_0^{1-z} (1 - z)(1 + z) - (1 + z)y \, dy \, dz \\
&= 2 \int_0^1 (1 - z)^2(1 + z) - (1 + z)\frac{(1 - z)^2}{2} \, dz \\
&= 2 \int_0^1 (1 + z)\frac{(1 - z)^2}{2} \, dz \\
&= \int_0^1 (1 + z)(1 - 2z + z^2) \, dz \\
&= \int_0^1 1 - 2z + z^2 + z - 2z^2 + z^3 \, dz \\
&= \int_0^1 1 - z - z^2 + z^3 \, dz \\
&= 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \\
&= \frac{12 - 6 - 4 + 3}{12} \\
&= \frac{5}{12}.
\end{aligned}$$

Doing the z integral last since there's a dependence on z is often good strategy, but results may vary.

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x,y,z) = \langle 0, 1, z \rangle$$

where S is the boundary surface of the region between the surfaces $z = \sqrt{x^2+y^2}$ and $z = 4$.

Solution

We call the region between the surfaces E and note that

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \iiint_E div(\vec{F})\, dV \\
&= \iiint_E 1\, dV \\
&= \int_0^4 \int_0^{2\pi} \int_r^4 r\, dz\, d\theta\, dr \\
&= 2\pi \int_0^4 (4-r) r\, dr \\
&= 2\pi \left(4 \cdot \frac{4^2}{2} - \frac{4^3}{3} \right) \\
&= 2\pi \left(32 - \frac{64}{3} \right) \\
&= \frac{2\pi}{3} (3 \cdot 32 - 2 \cdot 32) \\
&= \frac{2\pi}{3} \cdot 32 \\
&= \frac{64\pi}{3}.
\end{aligned}$$

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x,y,z) = \langle x, y, x^2 + y^2 \rangle$$

where S is the boundary surface of the region between the surfaces $z = \sqrt{x^2+y^2}$ and $z = \sqrt{4-x^2-y^2}$.

Solution

We compute $div(\vec{F}) = 1 + 1 + 0 = 2$. The surfaces bounding the region which we call E are a cone and sphere of radius 2 respectively. We can solve this in cylindrical coordinates as

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div(\vec{F})\, dV = \iiint_E 2\, dV = \begin{cases} \int_0^2 \int_0^{2\pi} \int_r^{\sqrt{4-r^2}} 2r\, dz\, d\theta\, dr \\ \\ \int_0^2 \int_0^{\pi/4} \int_0^{2\pi} 2r^2 \sin\phi\, d\theta\, d\phi\, dr. \end{cases} = 4\pi \left(1 - \frac{1}{\sqrt{2}}\right) \frac{2^3}{3}.$$

Bonus/challenge problem

Fix some radius $R > 0$ and take S to be the sphere of radius R centered at the origin, oriented by the exterior unit normal. For the vector field

$$\vec{F}(x,y) = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z\rangle :$$

- Show that $\iint_S \vec{F}\cdot d\vec{S} = 4\pi$ independently of R. *You can use symmetry and scaling or parametrize the sphere, e.g. in spherical polar coordinates.*
- Try computing the same integral by the divergence theorem. You should get a different result.
- Explain what goes wrong here and why the divergence theorem doesn't work.

Remark: The field $\vec{F}$ looks similar to the gravitational field for good reason – this example will re-appear if you take a class on electromagnetism.

Solution

This is exceedingly similar to the other bonus problem. Denote the sphere of radius R by $S(R)$. Note that the normal direction to the sphere $x^2+y^2+z^2 = R^2$ is given by (x,y,z). We make this the unit normal by dividing by $\|\langle x,y,z\rangle\| = R$.

$$\begin{aligned}\iint_{S(R)} \vec{F}\cdot d\vec{S} &= \iint_{S(R)} \frac{\langle x,y,z\rangle}{R^3}\cdot\frac{\langle x,y,z\rangle}{R}\,dS = \frac{1}{R^4}\iint_{S(R)} x^2+y^2+z^2\,dS = \frac{1}{R^4}\iint_{S(R)} R^2\,dS\\ &= \frac{area(S(R))}{R^2} = 4\pi\end{aligned}$$

since the surface area of a sphere of radius R is $4\pi R^2$ (derivative of the volume $\frac{4\pi}{3}R^3$ of a ball of radius R). As above, we see that $\vec{F}$ 'blows up' at the origin precisely in the critical scaling which keeps integrals from going to zero or infinity. This is where the divergence theorem will fail for any boundary surface where the bounded domain contains the coordinate origin.

We use short-hand $\vec{r} = \langle x,y,z\rangle$ and $r = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$.

$$\begin{aligned}div(\vec{F}) = div(r^{-3}\,\vec{r}) &= r^{-3}\,div(\vec{r}) + \nabla(r^{-3})\cdot\vec{r}\\ &= 3r^{-3} - 3r^{-4}\,\nabla r\cdot\vec{r}\\ &= 3r^{-3} - 3\,r^{-4}\frac{\vec{r}}{r}\cdot\vec{r}\\ &= 3(r^{-3}-r^{-3}) = 0\end{aligned}$$

where we only have to verify that $\nabla r = \frac{\vec{r}}{r}$, which follows easily as

$$\partial_x r = \partial_x(x^2+y^2+z^2)^{1/2} = \frac{1}{2}\,(x^2+y^2+z^2)^{-1/2}2x = \frac{x}{r}$$

and analogously for y, z.

This in fact works in any dimension d: The vector field $r^{-d}\vec{r}$ is divergence free. It is the gradient of $f(x) = \frac{1}{2-d}\,r^{2-d}$ if $d \geq 3$ and of $f(x) = \log(r)$ in dimension 2. You may encounter this in a class on electromagnetism or partial differential equations, where you may also study more closely how to interpret this failure of the divergence theorem in terms of the δ-distribution (sometimes also less precisely called the δ-function).