MATH 240 Final Exam

December 12, 2025

Mock exam

1. The final exam covers the entire course (see list of topics in separate document for details). The focus will be on material covered after the second midterm (vector calculus).
2. There will be ten problems.
3. No calculators or cheat sheets allowed.
4. You may use formulas for the length of a circle, area of a disk, surface area of a sphere, volume of a ball, area of a triangle, parallelogram etc without proof. Where these apply, you can use them directly instead of solving integrals (and you can use them to find e.g. the area of an annulus or surface area of a hemi-sphere etc).

Problem 1

1. Find a scalar equation of the plane containing the points $P(1, 0, 0)$, $Q(0, 2, 0)$ and $R(2, 2, 3)$.
2. Find the area of the parallelogram spanned by the vectors $\vec{a} = \langle 1, 1, 1\rangle$ and $b = \langle 1, 1, 0\rangle$.
3. Find the volume of the parallelepiped spanned by $\vec{a}, \vec{b}$ and $\vec{c} = \langle 1, 3, 2\rangle$.

Problem 1

1. Find the distance of the point $P(4, 2, 3)$ from the plane $x + y - z = 0$.
2. Find the cosine of the angle between the planes $x + y + z = 1$ and $x - 2y + 2z = 0$.
3. Compute the vector projection $P_{\vec{w}}(\vec{v})$ of $\vec{v} = \langle 1, 0, 0\rangle$ onto $\vec{w} = \langle 1, 1, 2\rangle$.

Problem 2

1. Parametrize the ellipse $x^2 + \frac{y^2}{4} = 1$ in the form $\vec{r}(t) = \langle a\ \cos t, b \sin t \rangle$ for some a, b.

2. Compute the unit tangent $\vec{T}(t)$ to the ellipse.

3. Compute the curvature $\kappa(t) = \frac{\|\vec{r}''(t) \times \vec{r}'(t)\|}{\|\vec{r}'(t)\|^3}$ of the ellipse and write it as $\kappa(t) = \frac{d}{\left(e+f\ \cos^2(t)\right)^{3/2}}$ for some d, e, f. *For the cross product, consider the ellipse as a curve in the plane $z = 0$ in three dimensions.*

4. At which point(s) (x, y) is the curvature greatest?

Problem 2

Consider the helix C parametrized by $\vec{r}(t) = \langle t,\ 2\cos t,\ 2\sin t \rangle$.

1. Compute unit tangent vector function $\vec{T}(t)$.

2. Compute the unit normal vector function $\vec{N}(t)$.

3. Compute the curvature function $\kappa(t)$.

Problem 3

Compute $f(2,3)$ and use linear approximation to compute $f(2.1, 2.8)$ for $f(x,y) = \sqrt{4x^2 + 3y}$.

Problem 3

Compute $f(2,1)$ and use linear approximation to compute $f(1.9, 1.1)$ for $f(x,y) = ln\big(x + 2y - 3\big)$.
Here ln *denotes the natural logarithm.*

Problem 3

Verify that the point $P(1,1,1)$ lies on the surface given by $\sqrt{x}+y+z^2 = 3$ and compute the tangent plane at P.

Problem 3

Verify that the point $P(1,1,1)$ lies on the surface given by $xy^2z^3 = 1$ and compute the tangent plane at P.

Problem 4

Find the maximum values and minimum values that $f(x, y, z) = x^3 + y^3 + z^3$ takes on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 4

Find the maximum values and minimum values that $f(x, y, z) = x^4 + y^4 + z^4$ takes on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 4

Find all critical points of the function $f(x, y) = x^4 - 2x^2 + y^2 + 1$ and decide whether they are local maxima, minima, or saddle points.

Problem 4

Find all critical points of the function $f(x, y) = x^4 - 2x^2 - 2y^2 + y^4 + 1$ and decide whether they are local maxima, minima, or saddle points.

Problem 5

Compute the double integral $\iint_T x + y \, dA$ where T is the triangle with vertices $P(0,0)$, $Q(1,1)$ and $R(1,2)$.

Problem 5

Compute the double integral $\iint_D x^2 + y^2 \, dA$ over the region which arises if the disk $x^2 + y^2 \leq 1$ is removed from the rectangle with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$.

Hint: It may be easier to write the integral over this region as the difference of two integrals.

Problem 5

Find the mass and center of mass of the region D where $1/9 < x^2 + y^2 < 1$ and $y < 0$, assuming constant density 1.

Remark: One of the components of the center of mass is easy to find by symmetry.

Problem 5

Consider the triangle T with vertices $P(0,0)$, $Q(1,-1)$ and $R(1,1)$.

1. Is T a type I region? If so, write $\iint_T f \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx$ for suitable values of a, b and functions g_1, g_2.
2. Is T a type II region? If so, write $\iint_T f \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy$ for suitable values of c, d and functions h_1, h_2.
3. Compute $\iint f \, dA$ for $f(x,y) = x$.

Problem 6

Compute the volume of the region which is both inside the sphere $x^2 + y^2 + z^2 < 2$ and the cylinder $x^2 + y^2 < 1$.

Hint: Spherical coordinates do not work well here.

Problem 6

Compute the volume and center of mass of the region which is both inside the sphere $x^2+y^2+z^2 < 1$ and inside the cone $z > \sqrt{x^2 + y^2}$, assuming constant density 1. *By symmetry, you may assume that* $\bar{x} = \bar{y} = 0$.

Problem 6

Compute the mass and center of mass of the 'spherical cap' which is both inside the sphere $x^2 + y^2 + z^2 < 2$ above the plane $z = 1$, assuming constant density 1. *By symmetry, you may assume that* $\bar{x} = \bar{y} = 0$.

Hint: Spherical coordinates do not work well here.

Problem 6

Compute the integral $\iiint_D 1 + z \, dV$ where D is the hollowed hemi-sphere given by $1/16 < x^2 + y^2 + z^2 < 1$ and $z > 0$.

Problem 7

1. Is the vector field $\vec{F}(x,y,z) = \langle yz, xz, xy \rangle$ conservative? If so, find a potential function.
2. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the straight line segment from $P(1,0,2)$ to $Q(4,5,1)$.

Problem 7

Compute the line integral $\int_C f\,ds$ where C is the helix segment $\vec{r}(t) = (2\cos t, 2\sin t, t)$ with t between 0 and 5 and $f(x,y,z) = 1 + z$.

Problem 7

1. Is the vector field $\vec{F}(x,y,z) = \langle x+y, 0, 0 \rangle$ conservative?
2. If so, find a potential function. If not, explain why not.
3. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the straight line segment from $P(1,0,2)$ to $Q(4,5,1)$.

Problem 7

Compute the line integrals $\int_C x\,ds$ and $\int_C x\,dy$ where C is the segment of the circle connecting $(1,0)$ to $(0,1)$.

Remark: If you do not know how to solve an integral, give the integral as answer for partial credit.

Problem 8

Evaluate the surface integral $\iint\limits_S f\, dS$ where S is the ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 1$ and

$$f(x, y, z) = \frac{z^2}{\sqrt{1 + 3x^2 + 3y^2}}.$$

One strategy to solve this is imitating spherical polar coordinates to parametrize the ellipsoid.

Problem 8

Evaluate the surface integral $\iint\limits_S f\, dS$ where the surface S is the graph of the function $g(x, y) = xy$ over the disk $x^2 + y^2 < 4$ and $f(x, y, z) = 1$.

Problem 8

Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = (\sin(x^2) + y)\vec{i}$ and C is made up of three straight line segments: From $(0, 0)$ to $(2, 0)$, from $(2, 0)$ to $(1, 2)$, and from $(1, 2)$ to $(0, 0)$.

Problem 8

Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = (e^x - e^y)\vec{j}$ and C is made up of four straight line segments: From $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(1, 1)$, from $(1, 1)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

Bonus/challenge problem

Fix some radius $R > 0$ and take C the circle of radius R around the origin, oriented counterclockwise. For the vector field $\vec{F}(x, y) = \frac{-y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j}$:

- Working from the definition, show that $\int_C \vec{F} \cdot d\vec{r} = 2\pi$ independently of R.
- Try to compute the same integral using Green's Theorem. You should get a different result.
- Explain why Green's Theorem does not apply here.

Problem 9

An ellipsoid is parametrized by $\vec{r}(\theta, \phi) = \langle \sin\phi\, \cos\theta, \sin\phi\, \sin\theta, 2\cos\phi \rangle$. A curve C on the ellipsoid in physical space corresponds to the curve in parameter space going along straight line segments from $(\theta_0, \phi_0) = (0, 0)$ to $(\theta_1, \phi_1) = (0, \pi/2)$, from here to $(\theta_2, \phi_2) = (\pi/2, \pi/2)$, and finally to $(\theta_3, \phi_3) = (\pi/2, 0)$.

1. Argue that C is in a closed curve in physical space.
2. Use Stokes' Theorem to compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle x + 1, 2y + 3, 4z + 5 \rangle$.

Problem 9

Consider the curve C on the hyperbolic paraboloid $z = x^2 - y^2$ which is parametrized by $0 \le t < 2\pi$ using $\vec{r}(t) = \langle \cos t, \sin t, \cos^2 t - \sin^2 t \rangle = \langle \cos t, \sin t, \cos 2t \rangle$. Use Stokes' Theorem to compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x, y, z) = \langle x, x, z \rangle$.

Problem 9

Consider the curve parametrized by $\vec{r}(t) = \langle \cos t,\ \sin t,\ g(\cos t, \sin t) \rangle$ where $g(x, y) = xy - x - y$. Use Stokes' Theorem to compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle 2\cos(x^2 + z^2)\, x,\ -x, 2\cos(x^2 + z^2)\, z \rangle$.

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x,y,z) = \langle x, 2y, 3z \rangle$$

where S is the boundary surface of the hemisphere $x^2 + y^2 + z^2 < 1$ and $z > 0$, oriented by the exterior unit normal.

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x,y,z) = \langle x+2,\ 3,\ z+z^2 \rangle$$

where S is the boundary surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x,y,z) = \langle 0, 1, z \rangle$$

where S is the boundary surface of the region between the surfaces $z = \sqrt{x^2+y^2}$ and $z = 4$.

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x,y,z) = \langle x, y, x^2+y^2 \rangle$$

where S is the boundary surface of the region between the surfaces $z = \sqrt{x^2+y^2}$ and $z = \sqrt{4-x^2-y^2}$.

Bonus/challenge problem

Fix some radius $R > 0$ and take S to be the sphere of radius R centered at the origin, oriented by the exterior unit normal. For the vector field

$$\vec{F}(x,y) = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle :$$

- Show that $\iint_S \vec{F} \cdot d\vec{S} = 4\pi$ independently of R. *You can use symmetry and scaling or parametrize the sphere, e.g. in spherical polar coordinates.*
- Try computing the same integral by the divergence theorem. You should get a different result.
- Explain what goes wrong here and why the divergence theorem doesn't work.

Remark: The field $\vec{F}$ looks similar to the gravitational field for good reason – this example will re-appear if you take a class on electromagnetism.