1. (*10 points*)

Solve the following initial value problems:

(a) $y' = \frac{te^t}{y}$, $\quad y(1) = 1.$ Separable

$y\,dy = te^t dt$, $\int y\,dy = \int te^t dt$

$\frac{1}{2}y^2 = (t-1)e^t + C_1$, $y^2 = 2(t-1)e^t + C$

$y(1) = 1 \iff 1 = C$

$y(t) = \left[2(t-1)e^t + 1\right]^{1/2}$

[Note: $y(t) = \pm\left[2(t-1)e^t+1\right]^{1/2}$ is a wrong answer. Why?]

(b) $y' = y^2 x$, $\quad y(3) = 1.$ Separable.

$y^{-2}dy = x\,dx$, $\int y^{-2}dy = \int x\,dx$

$-y^{-1} = \frac{1}{2}x^2 + C_1$ $\quad y = \left(C - \frac{1}{2}x^2\right)^{-1}$

$y(3) = \left(C - \frac{9}{2}\right)^{-1} = 1$, $C = \frac{11}{2} = 5\frac{1}{2}$

$y(x) = \left(\frac{11 - x^2}{2}\right)^{-1}$ or $y(x) = \frac{2}{11 - x^2}$

2. (*10 points*)

Solve the following differential equations. If an explicit solution cannot be found, leave the solution in an implicit form:

(a)

$$e^t y y' = e^{-y} + e^{-2t-y}$$

$e^t y y' = e^{-y}(e^{-2t}+1)$, $y e^y dy = (e^{-2t}+1)e^t dt$

$\int y e^y dy = \int (e^t + e^{-t}) dt$, $(y-1)e^y = e^t - e^{-t} + C$

(b)

$$y' + 3t^2 y = t^2$$

Linear equation, integrating factor is

$u = e^{\int 3t^2 dt} = e^{t^3}$

$(e^{t^3} y)' = t^2 e^{t^3}$, $e^{t^3} y = \int t^2 e^{t^3} dt = \frac{1}{3}\int e^u du$

$u = t^3$

$e^{t^3} y = \frac{1}{3} e^{t^3} + C$

$y = \frac{1}{3} + C e^{-t^3}$

3. (*10 points*)

(a) Show that the transformation $y(x) = 1/z(x)$ transforms the nonlinear ODE

$$xy' + y = y^2$$

to the linear ODE

$$z' - \frac{z}{x} = -\frac{1}{x}$$

(b) Give the general solution of $y(x)$. Plot the specific solutions for initial conditions $y(1) = 1$ and $y(1) = 2$ for $0 < x < 2$.

(a) $y' = (z^{-1})' = -z^{-2}z'$

$x(-z^{-2}z') + z^{-1} = z^{-2}$, $\quad -xz' + z = 1$, $\quad z' - \frac{z}{x} = -\frac{1}{x}$

(b) $u = e^{\int(-\frac{1}{x})dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$

$x^{-1}z' - x^{-2}z = -x^{-2}$

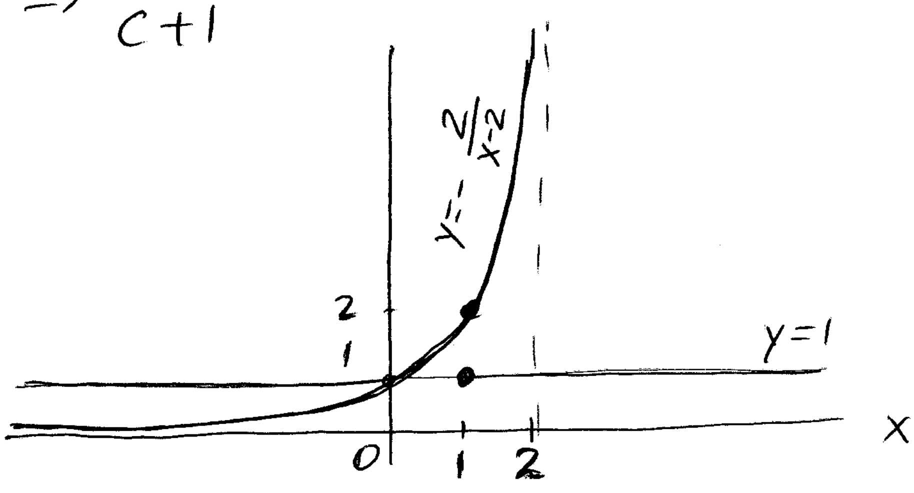
$(x^{-1}z)' = -x^{-2}$, $\quad x^{-1}z = -\int x^{-2}dx$

$x^{-1}z = x^{-1} + C$

$y = \frac{1}{Cx+1} \Leftarrow z = Cx + 1$

$y(1) = 1 \Rightarrow \frac{1}{C+1} = 1 \Rightarrow C = 0 \Rightarrow y = 1$

$y(1) = 2 \Rightarrow \frac{1}{C+1} = 2 \Rightarrow C = -\frac{1}{2} \Rightarrow y = -\frac{2}{x-2}$



char. eq. : $\lambda^2-2\lambda+1=(\lambda-1)^2=0$, $\lambda=1$

FSS: $y_1=e^t$, $y_2=te^t$

4. (*10 points*)

Use the method of variation of parameters OR the method of undetermined coefficients to find a particular solution, $y_p(t)$, to the following differential equation:

$$y''-2y'+y=e^{-t}\cos t.$$

Variation of param's: $W=\begin{vmatrix} e^t & te^t \\ e^t & e^t+te^t \end{vmatrix}=e^{2t}$

$\frac{1}{W}=e^{-2t}$

$v_1=-\int te^t\cdot e^{-t}\cos t\cdot e^{-2t}\,dt$

$v_1=-\int te^{-2t}\cos t\,dt=\frac{e^{-2t}}{25}\left((10t+3)\cos t-(5t+4)\sin t\right)$

$v_2=\int e^t\cdot e^{-t}\cos t\cdot e^{-2t}\,dt=\int e^{-2t}\cos t\,dt=\frac{e^{-2t}}{5}(\sin t-2\cos t)$

$y_p=v_1y_1+v_2y_2=\frac{e^{-t}}{25}(3\cos t-4\sin t)$

Underterm. coef's: $y_p=e^{-t}\cdot z$, $z=a\cos t+b\sin t$

$z'=-a\sin t+b\cos t$

$z''=-z$

$y_p'=-e^{-t}z+e^{-t}z'$, $y_p''=e^{-t}z-e^{-t}z'-e^{-t}z'-e^{-t}z$

$y_p''=-2e^{-t}z'$

$y_p''-2y_p'+y_p=-2e^{-t}z'+2e^{-t}z-2e^{-t}z'+e^{-t}z$

$=e^{-t}(3z-4z')=e^{-t}\cos t$

$3z-4z'=3a\cos t+3b\sin t+4a\sin t-4b\cos t=\cos t$

$3a-4b=1$ $\quad a=\frac{3}{25}$, $b=-\frac{4}{25}$

$4a+3b=0$

$y_p=e^{-t}\left(\frac{3}{25}\cos t-\frac{4}{25}\sin t\right)=\frac{e^{-t}}{25}(3\cos t-4\sin t)$

Char. eq: $\lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)=0$ $\quad \lambda=-3,\ \lambda=2$

FSS: $y_1=e^{-3t}, \quad y_2=e^{2t}$

5. *(10 points)*

Solve the following initial value problems:

(a)

$$y'' + y' - 6y = 2\sin(2t); \qquad y(0) = 0, y'(0) = 0;$$

$y_p = a\sin 2t + b\cos 2t$

$y_p' = 2a\cos 2t - 2b\sin 2t$

$y_p'' = -4y_p$

$y_p''+y_p'-6y_p = -10y_p+y_p' = 2\sin 2t$

$-(10a+2b)\sin 2t + (2a-10b)\cos 2t = 2\sin 2t$

$a-5b=0,\ 5a+b=-1,\ a=-\frac{5}{26},\ b=-\frac{1}{26}$

Gen. sln: $y(t) = C_1e^{-3t}+C_2e^{2t}-\frac{5}{26}\sin 2t - \frac{1}{26}\cos 2t$

$y'(t) = -3C_1e^{-3t}+2C_2e^{2t}-\frac{5}{13}\cos 2t+\frac{1}{13}\sin 2t$

ICs give: $C_1+C_2=\frac{1}{26}$ $\qquad C_1=-\frac{4}{65}$

$-3C_1+2C_2=\frac{5}{13}$ $\qquad C_2=-\frac{3}{130}$

$y(t) = -\frac{4}{65}e^{-3t}-\frac{3}{130}e^{2t} - \frac{5}{26}\sin 2t - \frac{1}{26}\cos 2t$

(b)

$$y'' + y' - 6y = e^{-3t}; \qquad y(0) = 0, y'(0) = 0;$$

$y_p = ate^{-3t}$ (because e^{-3t} is one of FSS)

$y_p' = ae^{-3t}-3y_p, \quad y_p'' = -3ae^{-3t}-3y_p' = -6ae^{-3t}+9y_p$

$y_p''+y_p-6y_p = -6ae^{-3t}+9y_p+ae^{-3t}-3y_p-6y_p$

$= -5ae^{-3t} = e^{-3t}, \quad a=-\frac{1}{5}$

Gen. sln: $y(t) = C_1e^{-3t}+C_2e^{2t}-\frac{1}{5}te^{-3t}$

$y'(t) = -3C_1e^{-3t}+2C_2e^{2t}-\frac{1}{5}e^{-3t}+\frac{3}{5}te^{-3t}$

ICs: $C_1+C_2=0$ $\qquad C_1=-\frac{1}{25}$

$-3C_1+2C_2-\frac{1}{5}=0$ $\qquad C_2=\frac{1}{25}$

$y(t) = -\frac{1}{25}e^{-3t}+\frac{1}{25}e^{2t} - \frac{1}{5}te^{-3t}$

6. (*10 points*)

My apartment has an ant infestation. The growth rate of the ants is r and the population size is currently estimated to be A_0. I have decided to fumigate my apartment with a fumigation agent that kills F ants per hour. The governing ODE for the ant population $A(t)$ is $A' = rA - F$ with $A(0) = A_0$. What is the minimal strength fumigation agent (i.e the smallest F) that I can use to ensure that after T hours of fumigation all of the ants are dead? Write your solution in terms of r, T, and A_0.

$$A' - rA = -F \quad , \quad u = e^{-\int r\,dt} = e^{-rt}$$

$$(e^{-rt}A)' = -Fe^{-rt} \quad , \quad e^{-rt}A = \int(-Fe^{-rt})\,dt$$

$$e^{-rt}A = \frac{F}{r}e^{-rt} + C \quad , \quad A = \frac{F}{r} + Ce^{rt}$$

$$A(0) = A_0 \quad , \quad \frac{F}{r} + C = A_0 \quad , \quad C = A_0 - \frac{F}{r}$$

$$A(t) = \frac{F}{r} + \left(A_0 - \frac{F}{r}\right)e^{rt}$$

$$A(T) = 0 \quad , \quad \frac{F}{r} + \left(A_0 - \frac{F}{r}\right)e^{rT} = 0$$

$$\frac{F}{r} + A_0e^{rT} - \frac{F}{r}e^{rT} = 0 \quad , \quad \frac{F}{r}(1 - e^{rT}) = -A_0e^{rT}$$

$$F(e^{rT} - 1) = A_0re^{rT}$$

$$F = \frac{A_0re^{rT}}{e^{rT} - 1}$$

(a) $T = -2, \quad D = -15 \qquad T^2 - 4D = 64$

$$\lambda = \frac{1}{2}\left(-2 \pm \sqrt{64}\right) = -1 \pm 4 \qquad \lambda_1 = -5, \ \lambda_2 = 3$$

7. (*10 points*)

Consider the system $\mathbf{y}' = A\mathbf{y}$ where:

$$A = \begin{pmatrix} -3 & -4 \\ -3 & 1 \end{pmatrix}.$$

(a) Find the fundamental solution set for the above system.

(b) Plot the phase portrait of the solution near $\mathbf{y} = (0, 0)^T$.

$$\lambda_1 = -5 : \quad (A - \lambda I)\bar{v}_1 = \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} 2v_1 - 4v_2 = 0 \\ v_1 = 2v_2 \end{matrix}$$

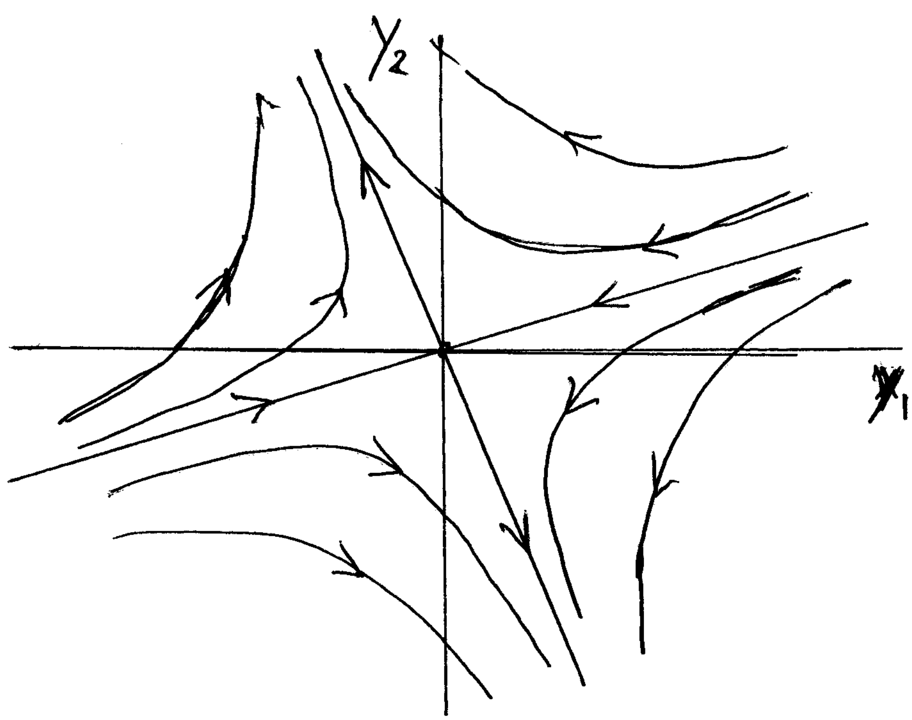
$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3 : \quad (A - \lambda I)\bar{v}_2 = \begin{bmatrix} -6 & -4 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} 3v_1 + 2v_2 = 0 \\ v_1 = 2, \ v_2 = -3 \end{matrix}$$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Fund. sln. set: $\bar{y}_1(t) = \begin{bmatrix} 2e^{-5t} \\ e^{-5t} \end{bmatrix}, \quad \bar{y}_2(t) = \begin{bmatrix} 2e^{3t} \\ -3e^{3t} \end{bmatrix}$

(b)



it is a saddle

$T=-2,\ D=10,\ T^2-4D=-36<0$

$\lambda=\frac{1}{2}(-2\pm 6i) \quad \lambda=-1+3i,\ \bar{\lambda}=-1-3i$

8. (*10 points*)

Consider the system $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}.$$

Find the solution $\mathbf{y}(t)$ with initial condition $\mathbf{y}(0) = (1,0)^T$. Express your solution as a real vector in $\mathbb{R}^2$ (i.e no complex numbers).

$$(A-\lambda I)\vec{w} = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad -3i w_1 + 3w_2 = 0, \quad 3i w_1 = 3w_2$$

Take $w_1=3,\ w_2=3i$. Then $\vec{w} = \begin{bmatrix} 3 \\ 3i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{w} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Gen. sln: $\bar{y}(t) = e^{-t}\left[C_1(\cos 3t\cdot\bar{v}_1 - \sin 3t\cdot\bar{v}_2) + C_2(\sin 3t\cdot\bar{v}_1 + \cos 3t\cdot\bar{v}_2)\right]$

$\bar{y}(0) = C_1\bar{v}_1 + C_2\bar{v}_2 = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_1 = 1,\ C_2 = 0$

Therefore, $\bar{y}(t) = e^{-t}(\cos 3t\cdot\bar{v}_1 - \sin 3t\cdot\bar{v}_2)$

$$\bar{y}(t) = \begin{bmatrix} e^{-t}\cos 3t \\ -e^{-t}\sin 3t \end{bmatrix}$$

9. (*10 points*)

Consider the following nonlinear system:

$$\begin{aligned} x' &= 2(xy-1) \\ y' &= x - y^3. \end{aligned}$$

(a) Find all equilibrium points (hint: there is more than one).

(b) Classify the stability of all equilibrium points.

(a) x-nullcline : $xy-1=0$, $x=\frac{1}{y}$

y-nullcline : $x=y^3$

Intersection : $\frac{1}{y}=y^3$, $y^4=1$, $y=-1$, $y=1$

EPs are $(-1,-1)$ and $(1,1)$

(b) Jacobian : $J=\begin{bmatrix} 2y & 2x \\ 1 & -3y^2 \end{bmatrix}$

$J(-1,-1)=\begin{bmatrix} -2 & -2 \\ 1 & -3 \end{bmatrix}$, $T=-5$, $D=8$, $T^2-4D=-7<0$

There is a spiral sink at $(-1,-1)$

It is stable

$J(1,1)=\begin{bmatrix} 2 & 2 \\ 1 & -3 \end{bmatrix}$, $T=-1$, $D=-8<0$

There is a saddle

at $(1,1)$

It is unstable

10. (*10 points*)

Consider the following model of competition between two animal species:

$$\begin{aligned} X' &= (1 - X - Y) \qquad (X \geq 0) \\ Y' &= (1 - 2Y - X/2) \qquad (Y \geq 0) \end{aligned}$$

(a) Find all equilibrium points, and classify the stability of each equilibrium point.

(b) Draw the qualitative phase portrait of the system.

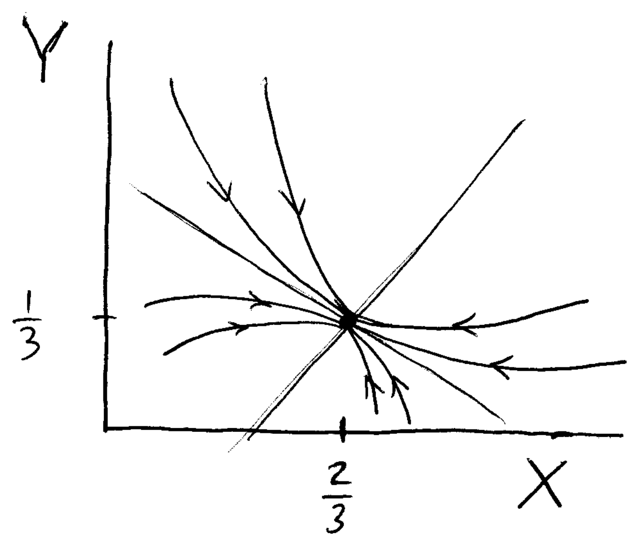
(c) Are there initial conditions which result in one of the populations going extinct?

(a) X-nullcline: $y = -x + 1$, y-nullcline: $y = -\frac{1}{4}x + \frac{1}{2}$

The only EP is $\left(\frac{2}{3}, \frac{1}{3}\right)$

(b) $J = \begin{bmatrix} -1 & -1 \\ -\frac{1}{2} & -2 \end{bmatrix}$ $T = -3$, $D = \frac{3}{2}$, $T^2 - 4D = 3 > 0$

It is a nodal sink



(c)

11. (*10 points*)

Using the Laplace method solve the following initial value problem:

$$y'' + 2y' - 3y = 16e^{-3t}; \qquad y(0) = 1, y'(0) = -3.$$

$$s^2Y - sy(0) - y'(0) + 2(sY - y(0)) - 3Y = \frac{16}{s+3}$$

$$(s^2+2s-3)Y - s + 3 - 2 = \frac{16}{s+3}$$

$$(s-1)(s+3)Y = \frac{16}{s+3} + s - 1$$

$$Y = \frac{16}{(s+3)^2(s-1)} + \frac{1}{s+3}$$

$$\frac{16}{(s+3)^2(s-1)} = \frac{a}{s+3} + \frac{b}{(s+3)^2} + \frac{c}{s-1}$$

$$a(s+3)(s-1) + b(s-1) + c(s+3)^2 = 16$$

$$a(s^2+2s-3) + b(s-1) + c(s^2+6s+9) = 16$$

s^2: $a + c = 0$ $\qquad a = -1$

s: $2a + b + 6c = 0$ $\qquad b = -4$

1: $-3a - b + 9c = 16$ $\qquad c = 1$

$$Y = -\frac{1}{s+3} - \frac{4}{(s+3)^2} + \frac{1}{s-1} + \frac{1}{s+3} = -\frac{4}{(s+3)^2} + \frac{1}{s-1}$$

$$y = L^{-1}[Y] = -4L^{-1}\left[\frac{1}{(s+3)^2}\right] + L^{-1}\left[\frac{1}{s-1}\right] = -4te^{-3t} + e^{+t}$$

12. (*10 points*)

Solve the following initial value problem:

$$y' + y = g(t); \qquad y(0) = 0,$$

with

$$g(t) = \begin{cases} t^2 & \text{if } 0 \le t < 2 \\ 4 & \text{if } t \ge 2. \end{cases}$$

$$g(t) = t^2 H_{02}(t) + 4H_2(t) = t^2(H(t) - H(t-2)) + 4H(t-2)$$

$$= t^2 H(t) - (t^2 - 4)H(t-2)$$

$$t^2 - 4 = (t-2)^2 + 4t - 4 - 4 = (t-2)^2 + 4(t-2)$$

$$g(t) = t^2 H(t) - (t-2)^2 H(t-2) - 4H(t-2)$$

$$L[y' + y] = L[g], \quad (s+1)Y = \frac{2}{s^3} - e^{-2s} \cdot \frac{2}{s^3} - 4e^{-2s}\frac{1}{s}$$

$$Y = \frac{2}{s^3(s+1)}(1 - e^{-2s}) - 4e^{-2s}\frac{1}{s(s+1)}$$

$$Y = 2(1 - e^{-2s})\left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s+1}\right) - 4e^{-2s}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$Y = 2(1 - e^{-2s})\left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right) - \frac{2}{s+1} + \frac{2e^{-2s}}{s+1} - \frac{4e^{-2s}}{s}$$

$$y = L^{-1}[Y] = 2 - t + \frac{1}{3}t^2 - 2e^{-t} - 2H(t-2)\left[1 - \frac{1}{2}(t-2)\right.$$

$$\left. + \frac{1}{6}(t-2)^2\right] + 2e^{-(t-2)}H(t-2) - 4H(t-2)$$

13. (*10 points*)

The convolution of two functions g and h is defined as $g * h(t) = \int_0^t g(u)h(t-u)du$. Let $f(t) = e^{-2t}$.

(a) Evaluate $f * f$.

(b) Use your answer from (a) and verify that:

$$\mathcal{L}(f * f) = (\mathcal{L}(f))^2.$$

(c) Solve the following initial value problem:

$$y' + y = f * f, \qquad y(0) = 0.$$

(a) $f*f = \int_0^t e^{-2u} e^{-2(t-u)} du = \int_0^t e^{-2u} e^{-2t} e^{2u} du$

$= \int_0^t e^{-2t} du = e^{-2t} \cdot u \Big|_0^t = te^{-2t}$

(b) $L[f*f] = L[te^{-2t}] = \frac{1}{(s+2)^2} = \left(\frac{1}{s+2}\right)^2 = \left(L[e^{-2t}]\right)^2$

$= \left(L[f]\right)^2$

(c) $L[y'+y] = L[f*f]$

$(s+1)Y = \frac{1}{(s+2)^2}$, $Y = \frac{1}{(s+1)(s+2)^2} = \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{(s+2)^2}$

$Y = \frac{1}{s+1} - \frac{1}{(s+2)^2} - \frac{1}{s+2}$

$y(t) = L^{-1}[Y] = e^{-t} - te^{-2t} - e^{-2t}$

14. (*10 points*)

Write $f(x)$ as a Fourier series over $-1 \le x \le 1$ where

$$f(x) = \begin{cases} 1+2x & \text{if } -1 \le x < 0 \\ 1-2x & \text{if } 0 \le x \le 1. \end{cases}$$

$$f(-x) = \begin{cases} 1-2x & \text{if } -1 \le -x < 0 \\ 1+2x & \text{if } 0 \le -x \le 1 \end{cases} = \begin{cases} 1+2x, & -1 \le x \le 0 \\ 1-2x, & 0 \le x \le 1 \end{cases} = f(x)$$

$\Rightarrow$ $f(x)$ is even $\Rightarrow$ All $b_n = 0$

$$L = 1, \quad a_n = \frac{2}{1} \int_0^1 (1+2x)\cos(n\pi x)\,dx$$

$$a_n = 2\int_0^1 \cos(n\pi x)\,dx + 4\int_0^1 x\cos(n\pi x)\,dx$$

$$= \frac{2}{n\pi} \sin(n\pi x)\Big|_0^1 + \frac{4}{n\pi} \sin(n\pi x)\Big|_0^1 - \frac{4}{n\pi}\int_0^1 \sin(n\pi x)\,dx$$

$$= 0 + 0 + \frac{4}{n^2\pi^2} \cos(n\pi x)\Big|_0^1 = \frac{4}{n^2\pi^2}\left(\cos(n\pi) - \cos 0\right)$$

$$a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right), \quad a_0 = 0$$

$$(-1)^n - 1 = \begin{cases} 0, & n \text{ is even} \\ -2, & n \text{ is odd} \end{cases} \quad n = 2k+1$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)\cos(n\pi x)$$

$$\left[f(x) = -\frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos((2k+1)\pi x) \right]$$

15. *(10 points)*

Write $f(x)$ as a Fourier series over $-2 \le x \le 2$ where

$$f(x) = \begin{cases} -1 & \text{if } -2 < x < 0 \\ 1 & \text{if } 0 \le x \le 2. \end{cases}$$

$$f(-x) = \begin{cases} -1, & -2 < -x < 0 \\ 1, & 0 \le -x \le 2 \end{cases} = \begin{cases} -1, & 0 < x < 2 \\ 1, & -2 \le x \le 0 \end{cases} = \begin{cases} 1, & -2 \le x \le 0 \\ -1, & 0 < x < 2 \end{cases}$$

Hence $f(-x) = -f(x)$ (except at $x=0$, $x=-2$)

and $f(x)$ is odd $\Rightarrow$ All $a_n = 0$

$$L = 2, \quad b_n = \int_0^2 1 \cdot \sin\left(\frac{n\pi}{2}x\right) dx = \frac{2}{n\pi}\left[-\cos\left(\frac{n\pi}{2}x\right)\right]_0^2$$

$$= \frac{2}{n\pi}\left(-\cos(n\pi) + 1\right) = \frac{2}{n\pi}\left(1 - (-1)^n\right) = \frac{2}{n\pi}\left(1 + (-1)^{n+1}\right)$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}x\right)$$